Topological and metric emergence of continuous maps

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Abstract

We prove that every homeomorphism of a compact manifold with dimension one has zero topological emergence, whereas in dimension greater than one the topological emergence of a C^0-generic homeomorphism is maximal, equal to the dimension of the manifold. We also show that the metric emergence of a continuous self-map on compact metric space has the intermediate value property.

1. *Introduction*

The topological entropy is an invariant by topological conjugation which quantifies to what extend nearby orbits diverge as the dynamical system evolves. On a compact metric space, a Lipschitz map has finite topological entropy. However, if the dynamics is just continuous, the topological entropy may be infinite. Actually, K. Yano proved in [**36**] that, on compact smooth manifolds with dimension greater than one, the set of homeomorphisms having infinite topological entropy are C^0-generic. So the topological entropy is not an effective label to classify them. Bringing together dimension and dynamics, E. Lindenstrauss and B. Weiss [**24**] introduced the notion of upper metric mean dimension of a continuous self-map f of a compact metric space (X, d), which may be thought as a mean upper box-counting dimension. Its value is metric dependent and always upper bounded by the upper box dimension of the space X, defined by

$$\overline{\dim}_{\mathrm{B}}\, X \;=\; \limsup_{\varepsilon \,\to\, 0^+} \frac{\log S_X(\varepsilon)}{-\log \varepsilon}$$

where $S_X(\varepsilon)$ is the maximum cardinality of an $\varepsilon-$separated subset of X (see [**12**, **29**] for more details). Thus, it is natural to ask what is the upper metric mean dimension of a C^0-generic homeomorphism of X, and whether there exists a homeomorphism of X having a prescribed value in the interval $[0, \overline{\dim}_{\mathrm{B}}\, X]$ as its upper metric mean dimension. These questions were partially answered in [**7**], where we proved that there exists a C^0-Baire generic subset of homeomorphisms of any compact smooth manifold with dimension $\dim X \geqslant 2$ whose elements have the highest possible upper metric mean dimension, namely $\dim X$; and that any level set of the metric mean dimension of continuous interval self-maps is C^0-dense.

Recently, Berger and Bochi introduced in [**3**] another concept to quantify the statistical complexity of a system: the topological emergence of a continuous self-map of a compact metric space X, which evaluates the size of the space of Borel $f-$invariant and ergodic probability measures (cf. Subsection 1·1 for the definition and more details). To illustrate its importance, they proved, among other equally interesting general results for diffeomorphisms on surfaces, that within $C^{1+\alpha}$ conformal expanding maps admitting a hyperbolic basic set Λ the topological emergence is the largest possible, that is, equal to the upper box dimension of Λ. This means that, when $\overline{\dim}_{\mathrm{B}}\, \Lambda > 0$, the number of $\varepsilon-$distinguished ergodic probability measures grows super-exponentially with respect to the parameter ε. Moreover, Berger and Bochi also proved that there is an open set $\mathcal{U}$ of $C^\infty-$surface diffeomorphisms and a generic subset $\mathcal{G}$ of $\mathcal{U}$ such that the Lebesgue measure has topological emergence equal to 2 with respect to each element of $\mathcal{G}$ (cf. [**3**, Theorem D]).

Our first aim in this work is to characterize the topological emergence of C^0-generic homeomorphisms acting on compact manifolds.

1·1. *Topological emergence*

We start by recalling the concept of topological emergence which measures the complexity of the space of ergodic probability measures preserved by a map. Given a compact metric space X and a continuous map $f\colon X \to X$, we denote by $\mathfrak{B}$ the $\sigma-$algebra of the Borel subsets of X, by $\mathcal{M}_1(X)$ the space of Borel probability measures on X, by $\mathcal{M}_f(X)$ its subset of $f-$invariant elements, and by $\mathcal{M}_f^{\mathrm{erg}}(X)$ the subset of $f-$invariant and ergodic probability measures.

Definition 1. *Let X be a compact metric space, $f\colon X \to X$ be a continuous map and $\mathcal{D}$ be a distance on the space $\mathcal{M}_1(X)$ such that $(\mathcal{M}_1(X), \mathcal{D})$ is compact. The* topological emergence map *associated to f is the function*

$$\varepsilon \in \,]0, +\infty[\quad \mapsto \quad \mathcal{E}_{\mathrm{top}}(f)(\varepsilon)$$

where $\mathcal{E}_{\mathrm{top}}(f)(\varepsilon)$ denotes the minimal number of balls of radius ε in $(\mathcal{M}_1(X), \mathcal{D})$ necessary to cover the set $\mathcal{M}_f^{\mathrm{erg}}(X)$.

It is clear from the previous definition that the topological emergence depends on the metric we consider in $\mathcal{M}_1(X)$. In what follows, we will always assume that $\mathcal{D}$ is one of the Wasserstein metrics W_p, for some $p \geqslant 1$, or the Lévy-Prokhorov metric LP (both metrics are defined in Subsection 2·1). These metrics induce in $\mathcal{M}_1(X)$ the weak$^*-$topology (cf. [**32**]).

Definition 2. *The* upper and lower metric orders *of a compact metric space (Y, D), defined by Kolmogorov and Tikhomirov* [**20**]*, are given respectively by*

$$\overline{\mathrm{mo}}\,(Y) = \limsup_{\varepsilon \to 0^+} \frac{\log \log S_Y(\varepsilon)}{-\log \varepsilon} \qquad \textit{and} \qquad \underline{\mathrm{mo}}\,(Y) = \liminf_{\varepsilon \to 0^+} \frac{\log \log S_Y(\varepsilon)}{-\log \varepsilon}$$

where $S_Y(\varepsilon)$ denotes the maximal cardinality of an $\varepsilon-$separated subset of Y. In case both quantities coincide we simply denote them by $\mathrm{mo}(Y)$*, the metric order of the set Y. This notions may be extended in a straightforward way to nonempty subsets of Y.*

It is worth referring that Definitions 1 and 2 are related. Indeed, given a compact metric space Y, the minimal number of balls in Y with radius ε necessary to cover a subset $A \subset Y$ is bounded from above by the maximal number of disjoint balls with radius $\varepsilon/2$ that intersect A, and is bounded from below by the maximal number of disjoint balls of radius ε that intersect A.

To define the next concept, we need to select either a Wasserstein metric or the Lévy-Prokhorov metric, but its value does not depend on this choice (cf. [**3**]).

Definition 3. *The* topological emergence *of a continuous map $f\colon X \to X$ on a compact metric space X, which we will denote by $\mathcal{E}_{\mathrm{top}}(f)$, is the upper metric order of the space of Borel $f-$invariant ergodic probability measures on X endowed with either the Wasserstein metric W_p, for some $p \in [1, +\infty[$, or the Lévy-Prokhorov metric (we denote by* LP*).*

We specify that, in what follows, $\log\log 1 = 0$. This way, a uniquely ergodic map f is granted a zero topological emergence, as expected.

Berger and Bochi proved in [**3**, Theorem 1.3]) that, if $f\colon X \to X$ is a continuous map acting on a compact metric space X whose upper and lower box dimensions are $\overline{\dim}_{\mathrm{B}}\, X$ and $\underline{\dim}_{\mathrm{B}}\, X$, respectively, then for any $p \geqslant 1$ one has

$$\underline{\dim}_{\mathrm{B}}\, X \;\leqslant\; \underline{\mathrm{mo}}\,(\mathcal{M}_1(X), \mathrm{W}_p) \;\leqslant\; \overline{\mathrm{mo}}\,(\mathcal{M}_1(X), \mathrm{W}_p) \;\leqslant\; \overline{\dim}_{\mathrm{B}}\, X \tag{1·1}$$

and that similar inequalities hold if we consider $\mathcal{M}_1(X)$ endowed with the distance LP. In particular, this ensures that

$$\limsup_{\varepsilon \to 0^+} \frac{\log\log \mathcal{E}_{\mathrm{top}}(f)(\varepsilon)}{-\log \varepsilon} \;=\; \overline{\mathrm{mo}}\,(\mathcal{M}_f^{\mathrm{erg}}(X), \mathrm{W}_p) \;\leqslant\; \overline{\mathrm{mo}}\,(\mathcal{M}_1(X), \mathrm{W}_p) \;\leqslant\; \overline{\dim}_{\mathrm{B}}\, X. \tag{1·2}$$

1·2. *Metric emergence*

Fix a compact metric space (X, d), a continuous map $f\colon X \to X$, a positive integer n and $x \in X$. The $n^{\mathrm{th}}-$*empirical measure associated to* x is defined by

$$\mathrm{e}_n^f(x) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$$

where δ_z denotes the Dirac probability measure supported on z. We recall that, if μ is an $f-$invariant probability measure, then the Birkhoff's ergodic theorem guarantees that for $\mu-$almost every $x \in X$ the sequence $\left(\mathrm{e}_n^f(x)\right)_{n \in \mathbb{N}}$ converges in the weak$^*-$topology to a unique probability measure (cf. [**35**]), which we denote by $\mathrm{e}^f(x)$ and call *empirical measure associated to x by f*. For instance, given a periodic point P of period k, its orbit supports a unique invariant probability measure, so called *periodic Dirac measure*, defined by $\mu_P = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{f^i(P)}$, which coincides with $\mathrm{e}^f(P)$. Misiurewicz gives in [**25**] an example of a homeomorphism $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ of the $2-$torus that is expansive, has the specification property and such that, for Lebesgue almost every point $x \in \mathbb{T}^2$, the sequence $\left(\mathrm{e}_n^f(x)\right)_{n \in \mathbb{N}}$ accumulates on the whole $\mathcal{M}_f(\mathbb{T}^2)$, which in this example is very large.

Definition 4. *Let (X, d) be a compact metric space, $f\colon X \to X$ be a continuous map and μ be a probability measure on X (not necessarily $f-$invariant). The* metric emergence map *of* μ *with respect to* f *assigns to each* $\varepsilon > 0$ *the minimal number* $\mathcal{E}_\mu(f)(\varepsilon) = N$ *of probability measures* $\mu_1, \ldots, \mu_N$ *on* X *such that*

$$\limsup_{n \to +\infty} \int_X \min_{1 \leqslant i \leqslant N} \mathcal{D}(\mathrm{e}_n^f(x),\, \mu_i)\; d\mu(x) \;\leqslant\; \varepsilon. \tag{1·3}$$

The metric emergence *of* μ *with respect to* f *is the limit, if well defined,*

$$\mathcal{E}_\mu(f) \;=\; \limsup_{\varepsilon \to 0^+} \frac{\log\log \mathcal{E}_\mu(f)(\varepsilon)}{-\log \varepsilon}. \tag{1·4}$$

The previous concepts were introduced in [**2**] when X is a compact manifold and μ is the Lebesgue measure, and generalized in [**3**]. In rough terms, $\mathcal{E}_\mu(f)$ essentially evaluates how far is μ from being ergodic. If μ is $f-$invariant then $(\mathrm{e}_n^f(x))_{n \in \mathbb{N}}$ converges to $\mathrm{e}^f(x)$ at $\mu-$almost every x, and so (1·3) can be replaced by

$$\int_X \min_{1 \leqslant i \leqslant N} \mathcal{D}(\mathrm{e}^f(x),\, \mu_i)\; d\mu(x) \;\leqslant\; \varepsilon. \tag{1·5}$$

So, if μ is $f-$invariant and ergodic, its metric emergence map is minimal, equal to 1.

By [**3**, Proposition 3.14], it is known that, if $f\colon X \to X$ is a continuous map of a compact metric space X and $\mu \in \mathcal{M}_f(X)$, then

$$\mathcal{E}_\mu(f)(\varepsilon) \;\leqslant\; \mathcal{E}_{\mathrm{top}}(f)(\varepsilon) \qquad \forall\, \varepsilon > 0$$

provided both emergences are computed using the same W_p or LP metric on $\mathcal{M}_1(X)$.

1·3. *Main results*

Let X be either $[0,1]$ or $\mathbb{S}^1$, endowed with the Euclidean metric. Denote by $\mathrm{Homeo}_+(X,d)$ the set of order preserving homeomorphisms of X with the uniform metric D_{C^0} given by

$$D_{C^0}(f,g) = \sup_{x \in X} \big\{d(f(x),g(x)),\, d(f^{-1}(x),g^{-1}(x))\big\}.$$

The set $\mathrm{Homeo}_+(X,d)$ with this distance is a Baire space. Our starting point is the following property of the topological emergence of these homeomorphisms.

Theorem 1. *If* $X = [0,1]$ *or* $X = \mathbb{S}^1$ *endowed with the Euclidean metric, then every map in* $\mathrm{Homeo}_+(X,d)$ *has zero topological emergence.*

Now let (X,d) be a compact connected smooth manifold X (with or without boundary) of dimension at least two. We will consider both the space $\mathrm{Homeo}(X,d)$ of homeomorphisms on X with the uniform metric D_{C^0} and its subset $\mathrm{Homeo}_\mu(X,d)$ of those homeomorphisms which preserve a Borel probability measure μ on X. For reasons we will explain later, we are mainly interested in $\mathbb{OU}-$probability measures (so named after the work [**27**] of Oxtoby and Ulam; see also [**1**]), which comply with the following conditions:

(C_1) [Non-atomic] For every $x \in X$ one has $\mu(\{x\}) = 0$.

(C_2) [Full support] For every nonempty open set $U \subset X$ one has $\mu(U) > 0$.

(C_3) [Boundary with zero measure] $\mu(\partial X) = 0$.

It is known that the set of $\mathbb{OU}-$probability measures is generic in $\mathcal{M}_1(X)$ (see [**11**]).

The next result shows that, contrary to Theorem 1, in a higher dimensional setting the topological emergence of C^0-generic conservative homeomorphisms attains its maximum possible value.

Theorem 2. *Let X be a compact smooth manifold with dimension* $\dim X \geqslant 2$*, d be a metric compatible with the smooth structure of X and μ be a $\mathbb{OU}-$probability measure on X. There are C^0-Baire generic subsets $\mathfrak{R} \subset \mathrm{Homeo}(X, d)$ and $\mathfrak{R}_\mu \subset \mathrm{Homeo}_\mu(X, d)$ such that*

$$\overline{\mathrm{mo}}\,(\mathcal{M}_f^{\mathrm{erg}}(X),\, \mathrm{W}_p) \,=\, \dim X \qquad \forall\, f \in \mathfrak{R}$$

and

$$\mathrm{mo}\,(\mathcal{M}_f^{\mathrm{erg}}(X),\, \mathrm{W}_p) \,=\, \dim X \qquad \forall\, f \in \mathfrak{R}_\mu$$

In the measure preserving setting, Theorem 2 is a consequence of the fact that, if μ is a $\mathbb{OU}-$probability measure on X, then a C^0-generic element f in $\mathrm{Homeo}_\mu(X)$ is ergodic (cf. [**27**]), has a dense set of periodic points (cf. [**10**]) and the shadowing property [**16**], and therefore satisfies the specification property (cf. [**11**]). This implies that the set of ergodic probability measures is dense in $\mathcal{M}_f(X)$, so we are left to show that the metric order of $\mathcal{M}_f(X)$ is equal to $\dim X$. This is easier to prove since $\mathcal{M}_f(X)$ is convex. The argument ultimately depends on the fact that the existence of pseudo-horseshoes is C^0-generic in the conservative context (see [**10**] and Section 4 for more details).

The proof of Theorem 2 for dissipative (that is, non-conservative) homeomorphisms also builds on the construction of pseudo-horseshoes, which were introduced in [**36**] and redesigned in [**7**] to satisfy two conditions: to exist in all sufficiently small scales and to exhibit an adequate separation of sufficiently large sets of points in all steps of their construction. However, as the denseness of ergodic probability measures on the set of the invariant ones is not C^0-generic within $\mathrm{Homeo}(X)$ (cf. [**17**, **11**, **21**]), the argument in the conservative case does not extend to the non-conservative context. To prove that the topological emergence in this setting is C^0-generically maximal we will carry out another upgrade on the construction of the pseudo-horseshoes to guarantee that we can find sufficiently many ergodic probability measures adequately separated with respect to a Wasserstein metric (see Section 5 for more details).

Given a homeomorphism $f\colon X \to X$, the map which assigns to each nonempty compact $f-$invariant subset Z of X the topological entropy of the restriction of f to Z fails to satisfy the intermediate value property. See, for instance, the minimal homeomorphism on the $2-$torus with positive entropy presented in [**31**]. A. Katok asked whether the metric entropy map satisfies the intermediate value property. More precisely, Katok conjectured that, for every C^2 diffeomorphism $f\colon X \to X$, acting on a compact connected manifold X with finite topological entropy, and for every $c \in [0, h_{\mathrm{top}}(f)]$, there is a Borel $f-$invariant and ergodic probability measure μ such that the metric entropy $h_\mu(f)$ is equal to c. This conjecture has been positively answered in a number of contexts (cf. [**33**] and references therein). After Theorem 2, one may likewise ask if the image of the metric emergence map of a C^0-generic $f \in \mathrm{Homeo}_\mu(X)$ is $[0, \overline{\dim}_{\mathrm{B}} X]$. Supporting this question is the fact that for every continuous self-map f of a compact metric space there exists a Borel $f-$invariant probability measure μ such that $\mathcal{E}_\mu(f) = \mathcal{E}_{\mathrm{top}}(f)$ (cf. [**3**]). Our next result generalizes this assertion, providing a proof of the counterpart of Katok's conjecture for the metric emergence.

Theorem 3. *Let $f\colon X \to X$ be a continuous map on a compact metric space (X, d). Then:*

(a) *The set*

$$\mathcal{B}_f(X) \,=\, \big\{\mu \in \mathcal{M}_f(X)\colon\, \sup_{\varepsilon>0} \mathcal{E}_\mu(f)(\varepsilon) > 1\big\}$$

is convex.

(b) *The restriction to $\mathcal{B}_f(X)$ of the metric emergence is quasiconvex since*

$$\mathcal{E}_{t\,\mu\,+\,(1-t)\,\nu}(f) \,=\, \max\big\{\mathcal{E}_\mu(f), \mathcal{E}_\nu(f)\big\} \qquad \forall\,\mu,\,\nu \in \mathcal{B}_f(X), \quad \forall\, t \in]0,1[.$$

(c) *For every $0 \leqslant \beta \leqslant \mathcal{E}_{\mathrm{top}}(f)$ there is $\mu \in \mathcal{M}_f(X)$ such that $\mathcal{E}_\mu(f) = \beta$.*

It is known that for C^0-generic volume preserving homeomorphisms the Lebesgue measure is ergodic (cf [**27**]), so its metric emergence map is constant and equal to one. On the other hand, by Theorem 2, for C^0-generic volume preserving homeomorphisms on a compact manifold with dimension at least two, one has $\mathcal{E}_{\mathrm{top}}(f) = \mathrm{dim}X$. This indicates that, C^0-generically in the space of volume preserving homeomorphisms, the probability measure whose metric emergence attains the maximal value $\mathcal{E}_{\mathrm{top}}(f)$ is not the Lebesgue measure. Yet, in the space of C^r diffeomorphisms, $r \geqslant 1$, in any surface, there exists a C^r-open subset for whose generic maps the Lebesgue measure has metric emergence equal to two (cf. [**3**, Theorem D]).

The proof of Theorem 3 relies on the following intermediate value property for the upper metric order map, which is of independent interest.

Theorem 4. *Let (Z,d) be a compact metric space. The upper metric order function defined on the space of subsets of Z has the intermediate value property. More precisely, if $0 \leqslant \beta \leqslant \overline{\mathrm{mo}}\,(Z)$, then there exists a subset $Y_\beta \subset Z$ such that $\overline{\mathrm{mo}}\,(Y_\beta) = \beta$.*

2. *Preliminary information*

For future use, in this section we will recall some definitions and previous results.

2·1. *Metrics on $\mathcal{M}_1(X)$*

Given a compact metric space (X,d) it is known that the space $\mathcal{M}_1(X)$ of the Borel probability measures on X is compact if endowed with the weak$^*-$topology. Moreover, there are metrics on $\mathcal{M}_1(X)$ inducing this topology, the classic ones being the *Wasserstein distances* and the *Lévy-Prokhorov distance.* The former are defined by

$$\mathrm{W_p}(\mu,\nu) = \inf_{\pi\,\in\,\Pi(\mu,\nu)} \left(\int_{X\times X} [d(x,y)]^p \; d\pi(x,y)\right)^{1/p}$$

where $p \in [1,+\infty[$ and $\Pi(\mu,\nu)$ denotes the set of probability measures on the product space $X \times X$ with marginals μ and ν (see [**34**] and references therein for more details). The latter is defined by

$$\mathrm{LP}(\mu,\nu) = \inf\Big\{\varepsilon > 0\colon\; \forall\, E \in \mathfrak{B} \quad \forall\;\varepsilon\text{−neighborhood } V_\varepsilon(E) \text{ of } E \text{ one has}$$
$$\nu(E) \leqslant \mu(V_\varepsilon(E)) + \varepsilon \quad \text{ and } \quad \mu(E) \leqslant \nu(V_\varepsilon(E)) + \varepsilon\Big\}.$$

The reader may find more information about this distance in [**4**].

Throughout the text we will say that two probability measures on a compact metric space X *are* $\varepsilon-$*apart* if their supports are at a distance at least ε in the Hausdorff metric, which we denote by dist_H; more precisely,

$$\text{dist}_H\big(\text{supp}(\mu),\, \text{supp}(\nu)\big) \,\geqslant\, \varepsilon$$

where

$$\begin{aligned}&\text{dist}_H\big(\text{supp}(\mu),\, \text{supp}(\nu)\big)\\ &\qquad = \max\Big\{ \sup_{x\,\in\,\text{supp}(\mu)} \text{dist}\big(x, \text{supp}(\nu)\big),\ \sup_{y\,\in\,\text{supp}(\nu)} \text{dist}\big(y, \text{supp}(\mu)\big)\Big\}\end{aligned} \tag{2·1}$$

and $\text{dist}\big(a,\, A\big) = \inf\, \big\{d(a,x)\,|\, x\in A\big\}$.

For example, if $N \in \mathbb{N}$ and $\{x_1,\, x_2,\, \cdots, x_N\}$ is an $\varepsilon-$separated subset of X, then the Dirac measures $\delta_{x_1},\, \delta_{x_2}, \cdots, \delta_{x_N}$ are pairwise $\varepsilon-$apart.

Remark 1. *In* [**3**, Theorem 1.6]*, the authors proved that, if* $\mathcal{C} \subset \mathcal{M}_1(X)$ *is a convex subset and we denote by* $\mathcal{A}(\mathcal{C},\varepsilon)$ *the maximal cardinality of pairwise* $\varepsilon-$*apart probability measures in* $\mathcal{C}$*, then*

$$\min\Big\{\inf_{p\,\in\,[1,+\infty[}\ \underline{\text{mo}}\,(\mathcal{C}, \text{W}_p),\ \ \underline{\text{mo}}\,(\mathcal{C}, \text{LP})\Big\} \,\geqslant\, \liminf_{\varepsilon\,\to\,0^+} \frac{\log \mathcal{A}(\mathcal{C},\varepsilon)}{-\log\varepsilon}.$$

2·2. *Pseudo-horseshoes*

The main tool to prove our first theorem is a class of compact invariant sets, called pseudo-horseshoes. Such structures were used in [**36**] to prove that C^0-generic homeomorphisms, acting on compact manifolds (X, d) with dimension greater than one, have infinite topological entropy; and later in [**7**] to show the existence of a C^0-Baire generic subset $\mathfrak{R}_0 \subset \text{Homeo}(X, d)$ where the metric mean dimension is maximal, equal to $\dim X$. In what follows we recall the main definitions and properties of pseudo-horseshoes on manifolds. We refer the reader to [**17**, **18**], where one finds other relevant properties of the atractors and pseudo-horseshoes of generic homeomorphisms.

2·2·1. *Pseudo-horseshoes in* $\mathbb{R}^k$

Consider in $\mathbb{R}^k$ the norm

$$\|(x_1, \cdots, x_k)\| \,=\, \max_{1\,\leqslant\, i\,\leqslant k}\ |x_i|.$$

Given $r > 0$ and $x \in \mathbb{R}^k$, denote $D^k_r(x) = \big\{y \in \mathbb{R}^k\colon \|x-y\| \leqslant r\big\}$ and $D^k_r = D^k_r\big((0,\ldots,0)\big)$. For $1 \leqslant j \leqslant k$, let $\pi_j\colon \mathbb{R}^k \to \mathbb{R}^j$ be the projection on the first j coordinates. Let us now define pseudo-horseshoes with N legs ($N \geqslant 2$).

Definition 5. *Fix* $r > 0$, $x = (x_1, \cdots, x_k)$ *and* $y = (y_1, \cdots, y_k)$ *in* $\mathbb{R}^k$*, and take an open set* $U \subset \mathbb{R}^k$ *containing* $D^k_r(x)$*. Having fixed a positive integer* N*, we say that a homeomorphism* $\varphi\colon U \to \mathbb{R}^k$ *has a* pseudo-horseshoe of type N at scale r connecting x to y *if the following conditions are satisfied:*

(i) $\varphi(x) = y$.

(ii) $\varphi\Big(D^k_r(x)\Big) \subset \text{int}\Big(D^{k-1}_r(\pi_{k-1}(y))\Big)\, \times\, \mathbb{R}$.

(iii) *For* $i = 0, 1, \ldots, \big[\frac{N}{2}\big]$,

$$\varphi\Big(D^{k-1}_r(\pi_{k-1}(x))\, \times\, \Big\{x_k - r + \frac{4ir}{N}\Big\}\Big) \subset \text{int}\Big(D^{k-1}_r(\pi_{k-1}(y))\Big)\, \times\, (-\infty,\, y_k - r).$$

(iv) *For* $i = 0, 1, \ldots, \left[\frac{N-1}{2}\right]$,

$$\varphi\left(D_r^{k-1}(\pi_{k-1}(x)) \times \left\{x_k - r + \frac{(4i+2)r}{N}\right\}\right) \subset \operatorname{int}\left(D_r^{k-1}(\pi_{k-1}(y))\right) \times (y_k+r,\, +\infty).$$

(v) *For each* $i \in \{0, \ldots, N-1\}$, *the intersection*

$$V_i = D_r^k(y) \cap \varphi\left(D_r^{k-1}(x) \times \left[x_k - r + \frac{2ir}{N},\, x_k - r + \frac{(2i+2)r}{N}\right]\right)$$

is connected and satisfies:

(a) $V_i \cap (D_r^{k-1}(y) \times \{-r\}) \neq \emptyset$;

(b) $V_i \cap (D_r^{k-1}(y) \times \{r\}) \neq \emptyset$;

(c) *each connected component of* $V_i \cup \partial D_r^k(y)$ *is simply connected.*

Each V_i *is called a* vertical strip *of the pseudo-horseshoe.*

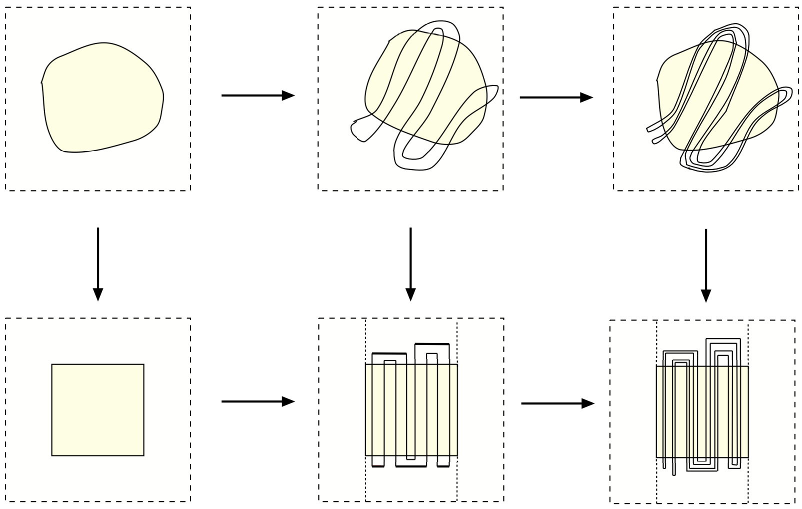

Fig. 1. Positive iterates of a pseudo-horseshoe on a compact manifold (top) and their geometric representation on $\mathbb{R}^k$ (bottom) using local charts which are signaled by downward arrows.

2·2·2. *Pseudo-horseshoes in manifolds*

So far, pseudo-horseshoes were defined in open sets of $\mathbb{R}^k$. Now we convey this notion to manifolds, where the number of legs $N \geqslant 2$ is determined by $k = \dim X$ and the scale $\varepsilon > 0$.

Definition 6. *Let* (X, d) *be a compact smooth manifold of dimension* $\dim X$. *Given* $f \in \mathrm{Homeo}(X, d)$ *and constants* $0 < \alpha < 1$, $\delta > 0$, $0 < \varepsilon < \delta$ *and* $q \in \mathbb{N}$, *we say that* f *has a* coherent $(\delta,\, \varepsilon,\, q,\, \alpha)-$pseudo-horseshoe *if we may find a pairwise disjoint family of open subsets* $(\mathcal{U}_i)_{0 \leqslant i \leqslant q-1}$ *of* X *such that*

$$f(\mathcal{U}_i) \cap \mathcal{U}_{(i+1) \bmod q} \neq \emptyset \qquad \forall\ i$$

and a collection $(\phi_i)_{0 \leqslant i \leqslant q-1}$ *of homeomorphisms*

$$\phi_i \colon D_\delta^{\dim X} \subset \mathbb{R}^{\dim X} \quad \to \quad \mathcal{U}_i \subset X$$

satisfying, for every $0 \leqslant i \leqslant q-1$*:*

(i) $(f \circ \phi_i)\,(D_\delta^{\dim X}) \subset \mathcal{U}_{(i+1) \bmod q}$.

(ii) *The map*

$$\psi_i = \phi^{-1}_{(i+1) \bmod q} \circ f \circ \phi_i \colon \quad D_\delta^{\dim X} \to \mathbb{R}^{\dim X}$$

has a pseudo-horseshoe of type $\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor$ *at scale* δ *connecting* $x = 0$ *to itself with vertical strips* $\{V_{i,j}\}_j$ *with* $j \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$.

(iii) *Denoting* $H_{i,j} = \psi_i^{-1}(V_{i,j})$ *for every* $j \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$*, which we refer to as horizontal strips, one has for every* $j_1 \neq j_2 \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$

$$\min\Big\{ \inf\{\|a - b\| \colon \, a \in V_{i,j_1},\, b \in V_{i,j_2}\},\, \inf\{\|z - w\| \colon \, z \in H_{i,j_1},\, w \in H_{i,j_2}\}\Big\} > \varepsilon.$$

(iv) *For every* $0 \leqslant i \leqslant q-1$ *and every* $j_1 \neq j_2 \in \{1, 2, \ldots, \lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor\}$*, the horizontal strip* H_{i,j_1} *crosses the vertical strip* $V_{(i+1) \bmod q, j_2}$*, where by crossing we mean that there exists a foliation of each horizontal strip* $H_{i,j} \subset D_\delta^{\dim X}$ *by a family* $\mathcal{C}_{i,j}$ *of continuous curves* $c \colon [0,1] \to H_{i,j}$ *such that* $\psi_i(c(0)) \in D_\delta^{\dim X - 1} \times \{-\delta\}$ *and* $\psi_i(c(1)) \in D_\delta^{\dim X - 1} \times \{\delta\}$.

Regarding the parameters $(\delta, \varepsilon, q, \alpha)$ that identify the pseudo-horseshoe, we remark that δ is a small scale determined by the size of the q domains and the charts so that item (i) of Definition 6 holds; ε is the scale at which a large number (which is inversely proportional to ε and involves α) of finite orbits is separated, to comply with the demand (ii)-(iii) of Definition 6; and α is conditioned by the room in the manifold needed to build the convenient amount of $\varepsilon-$separated points. In [**36**], Yano constructed pseudo-horseshoes of type N for every $N \geqslant 2$, while coherent pseudo-horseshoes were introduced and constructed in [**7**].

Coherent $(\delta, \varepsilon, q, \alpha)-$pseudo-horseshoes have three important features. (We refer the reader to [**7**] for more details.) Firstly, these pseudo-horseshoes persist under C^0-small perturbations. Secondly, every homeomorphism which has a coherent $(\delta, \varepsilon, q, \alpha)-$pseudo-horseshoe also has a $(q, \varepsilon)-$separated set with at least $\lfloor \left(\frac{1}{\varepsilon}\right)^{\alpha \dim X} \rfloor$ elements. The third main property of coherent pseudo-horseshoes is the following proposition.

Fix a strictly decreasing sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ in the interval $]0,1[$ converging to zero and let $L > 0$ be a bi-Lipschitz constant for the charts of a finite atlas of X. Denote by $\mathcal{O}(\varepsilon_k, \alpha)$ the set of homeomorphisms $g \in \mathrm{Homeo}(X, d)$ such that g has a coherent $(\delta, L\varepsilon_k, q, \alpha)-$pseudo-horseshoe, for some $\delta > 0$, $q \in \mathbb{N}$ and $L > 0$.

Proposition 1 ([**7**])**.** *For every* $\alpha \in\,]0,1[$ *and* $k \in \mathbb{N}$*, the set* $\mathcal{O}(\varepsilon_k, \alpha)$ *is* C^0-*open. Moreover, given* $K \in \mathbb{N}$*, the union*

$$\mathcal{O}_K(\alpha) \;=\; \bigcup_{\substack{k \,\in\, \mathbb{N} \\ k \,\geqslant\, K}} \mathcal{O}(\varepsilon_k, \alpha)$$

is C^0-dense in Homeo(X,d). *In particular,*

$$\mathfrak{R}_0 = \bigcap_{\alpha \in]0,1[\cap \mathbb{Q}} \; \bigcap_{K \in \mathbb{N}} \; \bigcup_{\substack{k \in \mathbb{N} \\ k \geqslant K}} \mathcal{O}(\varepsilon_k, \alpha)$$

is a C^0-Baire generic subset of Homeo(X,d).

Regarding the conservative setting, given a $\mathbb{OU}-$probability measure μ on X, the perturbation technics in [**15**] allow us to make C^0-small perturbations of any $\mu-$preserving homeomorphism in order to create coherent pseudo-horseshoes. In particular, with this strategy, one ensures that the space $\mathcal{O}_\mu(\varepsilon_k, \alpha)$ of homeomorphisms in Homeo$_\mu(X,d)$ exhibiting a coherent $(\delta, L\varepsilon_k, q, \alpha)-$pseudo-horseshoe is C^0-open and dense in Homeo$_\mu(X,d)$. A detailed construction of these C^0-open dense subsets $\mathcal{O}_\mu(\varepsilon_k, \alpha)$ was carried in [**23**, Theorem A], leading to a result similar to Proposition 1 in the setting of volume preserving homeomorphisms: the set

$$\mathfrak{R}_1 = \bigcap_{\alpha \in]0,1[\cap \mathbb{Q}} \; \bigcap_{K \in \mathbb{N}} \; \bigcup_{\substack{k \in \mathbb{N} \\ k \geqslant K}} \mathcal{O}_\mu(\varepsilon_k, \alpha)$$

is C^0-Baire generic in Homeo$_\mu(X,d)$.

2·3. *Specification property*

According to Bowen [**6**], a continuous map $f\colon X \to X$ on a compact metric space (X,d) satisfies the *specification property* if for any $\delta > 0$ there exists $T(\delta) \in \mathbb{N}$ such that any finite block of iterates by f can be $\delta-$shadowed by an individual orbit provided that the time lag of each block is larger than the prefixed time $T(\delta)$. More precisely, f satisfies the specification property if for any $\delta > 0$ there exists an integer $T(\delta) \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, any points $x_1, \ldots, x_k$ in X, any sequence of positive integers $n_1, \ldots, n_k$ and every choice of integers $T_1, \ldots, T_k$ with $T_i \geqslant T(\delta)$, there exists a point x_0 in X such that

$$d\big(f^j(x_0),\, f^j(x_1)\big) \leqslant \delta \qquad \forall\, 0 \leqslant j \leqslant n_1$$

and

$$d\Big(f^{j+n_1+T_1+\cdots+n_{i-1}+T_{i-1}}(x_0)\;,\; f^j(x_i)\Big) \leqslant \delta \qquad \forall\, 2 \leqslant i \leqslant k \quad \forall\, 0 \leqslant j \leqslant n_i.$$

It is known that full shifts on finitely many symbols satisfy the specification property; besides, factors of maps with the specification property also enjoy this property (cf. [**11**]). Moreover, if μ is a $\mathbb{OU}-$probability measure, the specification property is C^0-Baire generic in Homeo$_\mu(X,d)$ (cf. [**16**]).

The importance of the specification property in the study of the topological emergence is illustrated by the fact that it guarantees the denseness of the set of periodic measures in the space of invariant probability measures (cf. [**11**]), together with the following result, essentially stated by Bochi in [**5**].

Lemma 1. *Let (X,d) be a compact metric space and $f\colon X \to X$ be a continuous map such that $\overline{\mathcal{M}_f^{\text{erg}}(X)} = \mathcal{M}_f(X)$. Take a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers satisfying $\lim_{n \to +\infty} \varepsilon_n = 0$. Assume that there exist constants C, $\gamma > 0$ such that, for every $n \in \mathbb{N}$, there is an $f-$invariant finite subset $F_n \subset X$ containing only periodic orbits and satisfying the conditions:*

(i) *any two distinct orbits in F_n are uniformly ε_n-separated (in the Hausdorff distance) from each other;*

(ii) *the number of periodic orbits of F_n is bounded from below by $C\,(1/\varepsilon_n)^\gamma$.*
Then

$$\limsup_{\varepsilon \to 0^+} \frac{\log\, \log \mathcal{E}_{\rm top}(f)(\varepsilon)}{-\log \varepsilon} \geqslant \gamma.$$

In particular, if $\gamma = \overline{\dim}_{\rm B}\, X$, then $\mathcal{E}_{\rm top}(f) = \overline{\dim}_{\rm B}\, X$.

Proof. Fix $n \in \mathbb{N}$ and denote by $N_{\rm per}(F_n)$ the the number of periodic orbits of F_n. Consider the set of ergodic probability measures supported on each orbit in F_n, whose distinct elements are ε_n-apart due to condition (i). Given $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $\varepsilon_n \leqslant \varepsilon$ for every $n \geqslant N$. Then

$$\mathcal{A}(\mathcal{M}_f(X), \varepsilon) \geqslant \mathcal{A}(\mathcal{M}_f(X), \varepsilon_n) \geqslant N_{\rm per}(F_n) \qquad \forall\, n \geqslant N$$

where $\mathcal{A}(\mathcal{M}_f(X), \varepsilon)$ is the maximal cardinality of pairwise $\varepsilon-$apart probability measures in $\mathcal{M}_f(X)$. According to Remark 1, these inequalities together with the condition (ii) imply that

$$\underline{\rm mo}\,(\mathcal{M}_f(X), {\rm W}_p) \geqslant \gamma.$$

Moreover, by assumption, the closure of $\mathcal{M}_f^{\rm erg}(X)$ is $\mathcal{M}_f(X)$, so

$$\underline{\rm mo}\,(\mathcal{M}_f^{\rm erg}(X), {\rm W}_p) = \underline{\rm mo}\,(\mathcal{M}_f(X), {\rm W}_p) \geqslant \gamma$$

as claimed. This proves the first statement of the lemma.

In the particular case of $\gamma = \overline{\dim}_{\rm B}\, X$, we conclude more, since, as a consequence of [**3**, Equation 2.2, Theorem 1.3], we know that

$$\overline{\rm mo}\,(\mathcal{M}_f^{\rm erg}(X), {\rm W}_p) \leqslant \overline{\dim}_{\rm B}\, X.$$

Thus, ${\rm mo}\,(\mathcal{M}_f^{\rm erg}(X), {\rm W}_p) = \overline{\dim}_{\rm B}\, X$.

3. *Proof of Theorem* 1

Assume that $X = [0,1]$. Given $f \in {\rm Homeo}_+([0,1])$, it is immediate to conclude that the non-wandering set of f, say $\Omega(f)$, coincides with the set of fixed points (we denote by ${\rm Fix}(f)$). Indeed, each orbit by f is a monotonic bounded sequence, so it converges and, by the continuity of f, the limit is a fixed point. In particular, one has

$$\mathcal{M}_f^{\rm erg}([0,1]) = \Big\{\delta_x\colon\; f(x) = x\Big\}$$

hence $(\mathcal{M}_f^{\rm erg}(X), W_p)$ is isometric to a subset of (X, d). This implies that $\mathcal{E}_{\rm top}(f)(\varepsilon) = \mathcal{O}(\varepsilon^{-1})$ for every sufficiently small $\varepsilon > 0$, and so $\mathcal{E}_{\rm top}(f) = 0$. In particular, for any $f \in {\rm Homeo}_+([0,1])$

$$0 = \mathcal{E}_{\rm top}(f) = \sup\, \big\{\dim_B(\mu)\colon\; \mu \in \mathcal{M}_f^{\rm erg}([0,1])\big\}$$

where

$$\dim_B(\mu) = \limsup_{\varepsilon \to 0^+} \frac{\log\, \mu(B(x,\varepsilon))}{\log\, \varepsilon}.$$

If $f \in {\rm Homeo}_+(\mathbb{S}^1)$ has rational rotation number $\rho(f)$, then there is a conjugation between the restriction of f to its non-wandering set $\Omega(f)$ and the restriction of the rotation $R_{\rho(f)}$ to a closed subset of $\mathbb{S}^1$. Thus, every non-wandering point of f is periodic and all the periodic points have the same period (say m). Moreover, $\mathbb{S}^1 \setminus \Omega(f)$ is a union of open intervals and each of these intervals is mapped onto itself by the iterate f^m in a fixed-point free manner. In particular, in each of these intervals one has either $f^m(x) < x$ for every x or $f^m(x) > x$ for every x. So, the orbit by f^m of any point of each of these open intervals converges to a periodic point of f with period m. (The proofs of the previous assertions may be found in [**26**].) Thus,

$$\mathcal{M}_f^{\mathrm{erg}}(\mathbb{S}^1) \,=\, \Big\{\frac{1}{m}\sum_{j=0}^{m-1}\delta_{f^j(x)}\colon\; f^m(x)=x\Big\}$$

and, similarly to the context of the interval, $\mathcal{E}_{\mathrm{top}}(f)(\varepsilon) \,=\, \mathcal{O}(\varepsilon^{-1})$ for every small $\varepsilon > 0$. Thus the topological emergence of f is zero.

Finally, if $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ has irrational rotation number $\rho(f)$ then f is uniquely ergodic (cf. [**35**]), so it has zero topological emergence. This ends the proof of the theorem.

Remark 2. *It is known ([**30**]) that there exists a C^0-open and dense set of homeomorphisms $\mathfrak{O} \subset \mathrm{Homeo}_+(\mathbb{S}^1)$ such that every $f \in \mathfrak{O}$ has rational rotation number. The proof of Theorem 1 ensures that, for every $f \in \mathfrak{O}$,*

$$0 \,=\, \mathcal{E}_{\mathrm{top}}(f) \,=\, \sup\,\big\{\dim_B(\mu)\colon\; \mu \in \mathcal{M}_f^{\mathrm{erg}}(\mathbb{S}^1)\big\}.$$

*In case $f \in \mathrm{Homeo}_+(\mathbb{S}^1)$ has irrational rotation number then its non-wandering set is either the whole circle or a minimal Cantor set $\Omega(f)$. Moreover, given $0 < \tau < 1$, there are examples of orientation preserving $C^{1+\tau}-$diffeomorphisms of the circle with irrational rotation number and whose non-wandering set is a Cantor set of positive box dimension equal to τ (cf. [**22**, Theorem 4.2]). For such a diffeomorphism f one has*

$$0 \,=\, \mathcal{E}_{\mathrm{top}}(f) \,<\, \sup\,\big\{\dim_B(\mu)\colon\; \mu \in \mathcal{M}_f^{\mathrm{erg}}(\mathbb{S}^1)\big\} < 1.$$

4. *Proof of Theorem 2: conservative setting*

Let X be a compact smooth manifold with dimension at least two, d be a metric compatible with the smooth structure of X and μ be a $\mathbb{OU}-$probability measure on X. Denote by $\mathfrak{R}_s$ the C^0-Baire generic subset of $\mathrm{Homeo}_\mu(X,d)$ formed by homeomorphisms which satisfy the specification property (cf. [**16**]).

Recall from Subsection 2·2 that, given $\alpha \in \,]0,1[$, a strictly decreasing sequence $(\varepsilon_k)_{k\,\in\,\mathbb{N}}$ in the interval $]0,1[$ converging to zero, a bi-Lipschitz constant $L > 0$ for the charts of a finite atlas of M and $k \in \mathbb{N}$, we denote by $\mathcal{O}_\mu(\varepsilon_k,\alpha)$ the set of homeomorphisms $g \in \mathrm{Homeo}_\mu(X,d)$ such that g has a coherent $(\delta, L\varepsilon_k, q, \alpha)-$pseudo-horseshoe, for some $\delta > 0$, $q \in \mathbb{N}$ and $L > 0$.

(The constant L depends only on the fixed atlas and will be fixed throughout.)

For every $K \in \mathbb{N}$, define

$$\mathcal{O}_{\mu,\,K}(\alpha) \,=\, \bigcup_{\substack{k\,\in\,\mathbb{N}\\ k\,\geqslant\,K}} \mathcal{O}_\mu(\varepsilon_k,\alpha).$$

The set $\mathcal{O}_{\mu,\,K}(\alpha)$ is C^0-open and dense in $\mathrm{Homeo}_\mu(X,d)$ (cf. [**15**]). Thus the intersection

$$\mathfrak{R}_\mu \,=\, \mathfrak{R}_s \,\cap\, \left(\bigcap_{\alpha\,\in\,]0,1[\,\cap\,\mathbb{Q}}\;\bigcap_{K\,\in\,\mathbb{N}} \mathcal{O}_{\mu,\,K}(\alpha)\right)$$

is C^0-Baire generic in $\mathrm{Homeo}_\mu(X,d)$.

Given $\alpha \in \,]0,1[\,\cap \mathbb{Q}$ and $K \in \mathbb{N}$, any homeomorphism $g \in \mathfrak{R}_\mu$ has a coherent $(\delta, L\varepsilon_k, q, \alpha)-$pseudo-horseshoe Λ_k, for some $\delta > 0$, $q \in \mathbb{N}$, $L > 0$ and $k \geqslant K$. Therefore (cf. [**7**, Proposition 6.1]), there exists a finite subset $F_K \subset \mathcal{M}_g(X)$ formed by probability measures supported on $g-$periodic orbits of period q which are ε_k-apart from each other, and whose cardinality satisfies

$$\# F_K \geqslant \left\lfloor \Big(\frac{1}{L\varepsilon_k}\Big)^{\alpha \dim X} \right\rfloor.$$

Therefore, by Lemma 1, the upper metric order of $\mathcal{M}_g(X)$ is bigger or equal to $\dim X$, since

$$\mathrm{mo}\,(\mathcal{M}_g(X),\, \mathrm{W}_p) \geqslant \alpha \dim X$$

and $\alpha \in \,]0,1[\,\cap \mathbb{Q}$ is arbitrary. Moreover, the converse inequality

$$\mathrm{mo}\,(\mathcal{M}_g(X),\, \mathrm{W}_p) \leqslant \dim X$$

always holds (see (1·2)). So, $\mathrm{mo}\,(\mathcal{M}_g(X),\, \mathrm{W}_p) = \dim X$.

We are left to deduce from the previous equality that the topological emergence is maximal. As every $g \in \mathfrak{R}_\mu$ satisfies the specification property, the closure of the space $\mathcal{M}_g^{\mathrm{erg}}(X)$ is equal to $\mathcal{M}_g(X)$ (cf. Subsection 2·3). Thus,

$$\mathrm{mo}\,(\mathcal{M}_g^{\mathrm{erg}}(X),\, \mathrm{W}_p) = \mathrm{mo}\,(\mathcal{M}_g(X),\, \mathrm{W}_p) = \dim X$$

and similar equalities hold regarding the metric LP. This confirms that

$$\limsup_{\varepsilon \to 0^+} \frac{\log\, \log \mathcal{E}_{\mathrm{top}}(g)(\varepsilon)}{-\log \varepsilon} = \dim X \qquad \forall\, g \in \mathfrak{R}_\mu$$

and the proof of Theorem 2 for $\mathrm{Homeo}_\mu(X, d)$ is complete.

Denote by $\mathrm{Per}\, f$ the set of periodic points of $f\colon X \to X$. The previous proof has the following consequence:

Corollary 1. *Under the assumptions of Theorem* 2*, if* $f \in \mathfrak{R}_\mu$ *then*

$$\mathcal{E}_{\mathrm{top}}(f_{|\overline{\mathrm{Per}\, f}}) = \dim X.$$

5. *Proof of Theorem* 2*: non-conservative setting*

The argument in the previous section also shows that

$$\mathrm{mo}(\mathcal{M}_f(X),\, \mathrm{W}_p) = \dim X = \mathrm{mo}(\mathcal{M}_f(X),\, \mathrm{LP}) \qquad \forall\, f \in \mathfrak{R}_0$$

where $\mathfrak{R}_0$ is the C^0-generic subset of $\mathrm{Homeo}(X, d)$ defined in Proposition 1. However, the proof we presented for the conservative case does not entirely apply to $\mathrm{Homeo}(X, d)$. Indeed, whereas a C^0-generic homeomorphism in $\mathrm{Homeo}_\mu(X, d)$ is ergodic [**27**], hence transitive, there exists a C^0-open and dense set of homeomorphisms in $\mathrm{Homeo}(X, d)$ which display absorbing regions (cf. [**28**, Lemma 3.1] or [**17**]), and so those maps are not transitive. As transitivity is a necessary condition for the denseness of the ergodic probability measures in the space of invariant ones (cf. [**11**, **21**]), a typical homeomorphism in $\mathrm{Homeo}(X, d)$ does not satisfy the requirements needed to apply Lemma 1. Actually, such a strategy cannot even be pursued within a coherent pseudo-horseshoe, since an arbitrarily C^0-small perturbation of these structures also allows us to create open trapping regions. Therefore we need to refine the construction of the set $\mathfrak{R}_0$ in order to ensure the existence of an adequate amount of ergodic probability measures at appropriate scales.

5·1. *Topological horseshoes*

We start by establishing a strengthened version of Proposition 1.

Proposition 2. *Fix a strictly decreasing sequence* $(\varepsilon_k)_{k \in \mathbb{N}}$ *in the interval* $]0,1[$ *such that* $\lim_{k \to +\infty} \varepsilon_k = 0$. *For every* $\alpha \in]0,1[$ *and* $k \in \mathbb{N}$, *there exists a* C^0-*open subset* $\widehat{\mathcal{O}}(\varepsilon_k, \alpha) \subset \mathcal{O}(\varepsilon_k, \alpha)$ *such that:*

(i) *Given* $K \in \mathbb{N}$, *the union*

$$\widehat{\mathcal{O}}_K(\alpha) = \bigcup_{\substack{k \in \mathbb{N} \\ k \geqslant K}} \widehat{\mathcal{O}}(\varepsilon_k, \alpha)$$

is C^0-*dense in* $\mathrm{Homeo}(X, d)$.

(ii) *There exists a constant* $C > 0$ *such that, if* $h \in \widehat{\mathcal{O}}(\varepsilon_k, \alpha)$, *then:*

- *The map* h *has a coherent* $(\delta,\, L\varepsilon_k,\, q,\, \alpha)-$*pseudo-horseshoe, for some* $\delta > 0$, $q \in \mathbb{N}$ *and* $L > 0$.
- *The map* h *has a collection of* $\lfloor \left(\frac{1}{\varepsilon_k}\right)^{\alpha \dim X} \rfloor^q$ *periodic orbits of period* q *whose supports are* ε_k^q-*apart in the Hausdorff metric.*
- *There exists a subset* $E_h(X) \subset \mathcal{M}_h^{\mathrm{erg}}(X)$ *whose cardinality is larger than*

$$C \, \exp\left(\frac{1}{C} \Big\lfloor \Big(\frac{1}{\varepsilon_k}\Big)^{\alpha \dim X} \Big\rfloor^q\right)$$

and such that any two distinct elements in $E_h(X)$ *are* $8^{-\frac{1}{p}} \varepsilon_k^q-$*separated in the* W_p *distance.*

Proof. We start with the construction of the C^0-generic set $\mathfrak{R}_0$ described in Proposition 1. Then, given a homeomorphism $f \in \mathfrak{R}_0$, to overcome the lack of specification within f we will make a small local C^0-perturbation of f to obtain a homeomorphism g whose restriction to a fixed arbitrarily small open subset U of X is a C^1-diffeomorphism and exhibits in U a horseshoe (that is, a closed invariant set restricted to which the dynamics is conjugate to a full shift on a finite alphabet), where the periodic specification property is valid. Clearly we cannot expect that this horseshoe persists under small C^0-perturbations; but a well chosen finite number of its periodic points and the periodic orbits that shadow them may be turned permanent by a C^0-small perturbation.

The first main difficulty of this argument is to adjust the size of the needed C^0 perturbations with the separation rates of the strips in the horseshoe, in order to be able to apply the combinatorial approach of [**3**, Theorem 1.6]. The second difficulty is to ensure that the ergodic probability measures supported on all of these orbits are distinct and sufficiently separated in the W_p metric.

Let us briefly recall the reasoning to prove [**7**, Proposition 7.1]. Given $\delta > 0$, a homeomorphism of X can be arbitrarily C^0-approximated by another homeomorphism, say f, which has both a $q-$absorbing disk B with diameter smaller than δ, for some $q \in \mathbb{N}$, and a C^0-open neighborhood $\mathcal{W}_f$ in $\mathrm{Homeo}(X, d)$ such that, for every $g \in \mathcal{W}_f$, the disk B is still $q-$absorbing for g. Then, by extra q arbitrarily small C^0-perturbations, we get a homeomorphism $g \in \mathcal{W}_f$ exhibiting a coherent $(\delta,\, \varepsilon_k,\, q,\, \alpha)-$pseudo-horseshoe in the finite union $\hat{B}$ of the domains $\big(f^j(B)\big)_{0 \leqslant j < q}$.

The previous construction is performed by an isotopy in $\hat{B}$, so we may assume that the homeomorphism g is C^1-smooth on the open domain $\hat{B}$ and that there exists an open subset

$$Q \subset \hat{B} \tag{5·1}$$

such that the maximal invariant set

$$\Lambda = \bigcap_{n \in \mathbb{Z}} g^n(Q)$$

is a horseshoe, g^q has a horseshoe with

$$N = \Big\lfloor \Big(\frac{1}{\varepsilon_k}\Big)^{\alpha \dim X} \Big\rfloor^q$$

strips and the Hausdorff distance between any two such strips is bounded from below by ε_k^q. Let $T = T(\varepsilon_k^q/4) \in \mathbb{N}$ given by the specification property (cf. Subsection 2·3) for the map g restricted to Λ. Take an even positive integer ℓ (depending on g and ε_k) satisfying

$$q\ell \geqslant T \tag{5·2}$$

and such that the diameter of each connected component of $\bigcap_{|n| \leqslant \ell/2} g^n(Q)$ is strictly smaller than ε_k^q.

Using the methods of [**10**], one can perform a finite number of arbitrarily small C^0-perturbations so that there is a C^0-open neighborhood of g where a fixed finite number of periodic orbits become permanent, that is, persist under small C^0-perturbations of the dynamics. Therefore:

Lemma 2. *There exists a C^0-open neighborhood $\widehat{\mathcal{W}_g} \subset \mathcal{W}_g$ of g in* Homeo(X, d) *such that every $h \in \widehat{\mathcal{W}_g}$ satisfies the following conditions:*

(a) *The homeomorphism h has a coherent $(\delta,\, L\varepsilon_k,\, q,\, \alpha)-$pseudo-horseshoe, for some $\delta > 0$, $q \in \mathbb{N}$ and $L > 0$.*

(b) *The homeomorphism h has a collection $\mathcal{F}_h$ of $N = \lfloor \big(\frac{1}{\varepsilon_k}\big)^{\alpha \dim X} \rfloor^q$ permanent periodic orbits of period q, and the Hausdorff distance between these orbits is bounded from below by ε_k^q.*

(c) *The intersection $\bigcap_{n \in \mathbb{Z}} h^n(Q)$ is a pseudo-horseshoe, where Q is the open set satisfying the conditions that follow* (5·1). *Moreover, the diameter of each connected component of $\bigcap_{|n| \leqslant \ell/2} h^n(Q)$ is strictly smaller than ε_k^q.*

(d) *Assume that N is an even integer (otherwise, replace N by $2\lfloor N/2 \rfloor$). For any collection*

$$\underline{P} = (P_1,\, P_2,\, \cdots,\, P_{\frac{N}{2}})$$

of $N/2$ periodic orbits in $\mathcal{F}_h$ there is a periodic orbit $\wp = \wp(\underline{P})$ with period $q\ell N/2 + TN/2$ which $\varepsilon_k^q/4-$shadows the pseudo-orbit

$$\Big(\underbrace{(P_1,\, P_1,\, \cdots,\, P_1)}_{\ell},\, \underbrace{(P_2,\, P_2,\, \ldots,\, P_2)}_{\ell},\, \cdots,\, \underbrace{(P_{\frac{N}{2}},\, P_{\frac{N}{2}},\, \cdots,\, P_{\frac{N}{2}})}_{\ell} \Big)$$

with a time lag of T iterates in between, that is,

$$dist_H\big(h^{\ell+(s-1)q}(P),\, h^{\ell+(s-1)q}(\wp(P))\big) \; < \; \varepsilon_k^q/4$$

for every $0 \leqslant \ell \leqslant q$, $1 \leqslant s \leqslant k$ *and* $1 \leqslant t \leqslant L$.

We define the subset $\widehat{\mathcal{O}}_K(\varepsilon_k, \alpha)$ as the union of the previously obtained open domains $\widehat{\mathcal{W}_g}$. By construction, this is a C^0-open subset of Homeo(X, d) and, given $K \in \mathbb{N}$, the union $\widehat{\mathcal{O}}_K(\alpha) = \bigcup_{k \in \mathbb{N}, k \geqslant K} \widehat{\mathcal{O}}(\varepsilon_k, \alpha)$ is C^0-dense in Homeo(X, d). We are left to prove that, if $h \in \widehat{\mathcal{O}}(\varepsilon_k, \alpha)$, then there exists a subset $E_h(X) \subset \mathcal{M}_h^{\mathrm{erg}}(X)$ with the properties listed in Proposition 2. For that we will apply the combinatorial estimates used in the proof of [**3**, Theorem 1.6].

According to [**3**], every maximal $\frac{N}{4}-$separated set in the space

$$F = \Big\{\beta\colon \{1, 2, \ldots, N\} \to \{0, 1\} \text{ such that } \sum_{i=1}^{N} \beta(i) = \frac{N}{2}\Big\}$$

endowed with the Hamming metric, has cardinality bounded from below by $D_1 e^{C_1 N}$, for some uniform constants $D_1, C_1 > 0$. Given $h \in \widehat{\mathcal{O}}(\varepsilon_k, \alpha)$, fix a $N/4-$maximal separated set $F' \subset F$ and consider the space $E_h(X)$ of ergodic probability measures defined by

$$\mu_\beta = \frac{1}{q\ell N/2 + TN/2} \sum_{j=0}^{q\ell N/2 + TN/2 - 1} \beta(i_j)\, \delta_{h^j(\wp(\underline{P}_\beta))}$$

where $\beta \in F'$, $\underline{P}_\beta = (P_{i_1}, P_{i_2}, \ldots, P_{i_{\frac{N}{2}}})$ and $\beta(i_j) = 1$ for every $1 \leqslant j \leqslant N/2$. Note that the cardinality of $E_h(X)$ coincides with the one of F'.

We claim that any two probability measures in $E_h(X)$ are $8^{-\frac{1}{p}} \varepsilon_k^q-$separated in the metric W_p. Firstly, observe that, if $\beta_1, \beta_2 \in F'$ are distinct, then

$$\underline{P}_{\beta_1} = (P_{i_1}, P_{i_2}, \cdots, P_{i_j}, \cdots, P_{i_{\frac{N}{2}}}) \neq \underline{P}_{\beta_2} = (P_{k_1}, P_{k_2}, \cdots, P_{k_j}, \cdots P_{k_{\frac{N}{2}}})$$

and these two vectors differ in at least $N/4$ entries (that is, there are at least $N/4$ values of $1 \leqslant j \leqslant N/2$ such that $P_{i_j} \neq P_{k_j}$). Moreover, using item (d) of Lemma 2, we conclude that, for any such values of j, one has

$$\mathrm{dist}_H\Big(h^{t+(j-1)q\ell}(\wp(\underline{P}_{\beta_1})),\ h^{t+(j-1)q\ell}(\wp(\underline{P}_{\beta_2}))\Big) > \varepsilon_k^q/2 \qquad \forall\, 0 \leqslant t \leqslant q\ell$$

where dist_H stands for the Hausdorff distance defined in (2·1). Due to the choice of ℓ

(see (5·2)), given $\pi \in \Pi(\mu_{\beta_1}, \mu_{\beta_2})$ one has

$$\begin{aligned}\int_{X\times X} [d(x,y)]^p \, d\pi(x,y) &= \int_{\mathrm{supp}(\mu_{\beta_1})\times \mathrm{supp}(\mu_{\beta_2})} [d(x,y)]^p \, d\pi(x,y) \\ &\geqslant \frac{\varepsilon_k^{pq}}{2}\, \pi\Big(\big\{(x,y)\in X\times X\colon d(x,y) > \frac{\varepsilon_k^q}{2}\big\}\Big) \\ &\geqslant \frac{\varepsilon_k^{pq}}{2}\, \frac{q\ell N/4}{q\ell N/2 + TN/2} \\ &= \frac{\varepsilon_k^{pq}}{4}\, \frac{q\ell}{q\ell + T} \\ &\geqslant \frac{\varepsilon_k^{pq}}{8}.\end{aligned}$$

Thus

$$W_p(\mu_{\beta_1}, \mu_{\beta_2}) = \inf_{\pi \in \Pi(\mu_{\beta_1}, \mu_{\beta_2})} \Big(\int_{X\times X} [d(x,y)]^p \, d\pi(x,y)\Big)^{1/p} \geqslant 8^{-\frac{1}{p}} \varepsilon_k^q.$$

This ends the proof of Proposition 2.

5·2. *Estimate of the topological emergence*

It is immediate to deduce from Proposition 2 that the set

$$\mathfrak{R} = \bigcap_{\alpha \in \,]0,1[\,\cap \mathbb{Q}} \; \bigcap_{K \in \mathbb{N}} \; \bigcup_{\substack{k \in \mathbb{N} \\ k \geqslant K}} \widehat{\mathcal{O}}(\varepsilon_k, \alpha) \tag{5·3}$$

is C^0-Baire generic in $\mathrm{Homeo}(X,d)$. We will show that

$$\overline{\mathrm{mo}}\,(\mathcal{M}_f^{\mathrm{erg}}(X), \mathrm{W}_p) = \dim X \qquad \forall\, f \in \mathfrak{R}.$$

The estimates in the previous subsection show that, given $K \in \mathbb{N}$, an integer $k \geqslant K$ and a rational number $\alpha \in]0,1[$, there are $q_k \in \mathbb{N}$

and a set of cardinality

$$D_1 \exp\Big(\big\lfloor\Big(\frac{1}{\varepsilon_k}\Big)^{\alpha \dim X}\big\rfloor^{q_k}\Big)$$

formed by ergodic probability measures which are $8^{-\frac{1}{p}} \varepsilon_k^{q_k}-$separated in the Wasserstein metric W_p. This implies that, for each small $\delta > 0$, there exists $k_\delta \geqslant 1$ such that

$$S_{\mathcal{M}_f^{\mathrm{erg}}(X)}(8^{-\frac{1}{p}} \varepsilon_k^{q_k}) \geqslant \exp(\exp(-q_k \log(\varepsilon_k)\,(\alpha \dim X - \delta)))$$

for every $k \geqslant k_\delta$. Therefore,

$$\frac{\log\log S_{\mathcal{M}_f^{\mathrm{erg}}(X)}(8^{-\frac{1}{p}} \varepsilon_k^{q_k})}{-\log(8^{-\frac{1}{p}} \varepsilon_k^{q_k})} \geqslant \frac{-q_k \log(\varepsilon_k)\,(\alpha \dim X - \delta)}{-q_k \log(\varepsilon_k) - \log(8^{-\frac{1}{p}})}.$$

Consequently, taking lim sup as k goes to $+\infty$, we get

$$\overline{\mathrm{mo}}\,(\mathcal{M}_f^{\mathrm{erg}}(X),\,\mathrm{W}_p) \;\geqslant\; \alpha \dim X - \delta.$$

As δ and α can be chosen arbitrarily close to 0 and 1, respectively, we conclude that

$$\overline{\mathrm{mo}}\,(\mathcal{M}_f^{\mathrm{erg}}(X),\,\mathrm{W}_p) \;=\; \dim X$$

as claimed. The proof of Theorem 2 is complete.

5·3. *Pseudo-physical measures*

Assume that the manifold X is endowed with a volume reference measure, which we call Lebesgue measure. Given $\mu \in \mathcal{M}_f(X)$, denote by $\mathcal{L}_\omega(x,f)$ the set of accumulation points in the weak$^*-$topology of the sequence $\big(\mathrm{e}_n^f(x)\big)_{n\,\in\,\mathbb{N}}$ of $n^{\mathrm{th}}-$empirical measures associated to x by f. The measure μ is called *physical* if the set of those $x \in X$ for which $\mathcal{L}_\omega(x,f) = \{\mu\}$ has positive Lebesgue measure in X. Recall from [**8**] that $\mu \in \mathcal{M}_f(X)$ is said to be *pseudo-physical* if, for every $\varepsilon > 0$, the set

$$A_\varepsilon(\mu) \;=\; \big\{x \in X\colon\; \mathrm{dist}(\mu,\nu) \;<\; \varepsilon \quad \forall\,\nu \in\, \mathcal{L}_\omega(x,f)\big\}$$

has positive Lebesgue measure, where dist stands for any distance inducing in $\mathcal{M}(X)$ the weak$^*-$topology.

Remark 3. *It was proved in* [**19**, Proposition 1.2] *that, if X is an infinite compact metric space and $f\colon X \to X$ is a continuous map with the specification property, then there is a residual subset Y of X such that*

$$\overline{\dim}_{\mathrm{B}}\,\big(\mathcal{L}_\omega(x,f)\big) = +\infty \qquad \forall\, x \in Y.$$

When μ is a $\mathbb{OU}-$probability measure, as the specification property is C^0-generic in $\mathrm{Homeo}_\mu(X,d)$ *so is the previous equality.*

Let $\mathcal{O}_f(X)$ be the set of pseudo-physical measures of f and $\mathcal{M}_f^{\mathrm{per}}(X)$ be the set of periodic Dirac measures of f. It is known (cf. [**9**, Theorem 1]) that, for a C^0-generic f in $\mathrm{Homeo}(X,d)$, one has

$$\overline{\mathcal{M}_f^{\mathrm{erg}}}(X) \;=\; \overline{\mathcal{M}_f^{\mathrm{per}}}(X) \;=\; \mathcal{O}_f(X)$$

where the closures are taken in the weak$^*-$topology. Moreover, for a C^0-generic f in $\mathrm{Homeo}(X,d)$, the set $\mathcal{O}_f(X)$ has empty interior in $\mathcal{M}_f(X)$, so $\mathcal{M}_f(X) \setminus \mathcal{O}_f(X)$ is an open dense subset of $\mathcal{M}_f(X)$ which does not intersect $\overline{\mathcal{M}_f^{\mathrm{erg}}}(X)$ (cf. [**9**, Theorem 2]). Therefore, in spite of $\mathcal{O}_f(X)$ being meager,

$$\overline{\mathrm{mo}}\,(\mathcal{M}_f^{\mathrm{erg}}(X),\,\mathrm{W}_p) \;=\; \overline{\mathrm{mo}}\,(\mathcal{O}_f(X),\,\mathrm{W}_p)$$

and similarly regarding the metric LP. Hence, from Theorem 2 we conclude that:

Corollary 2. *For a C^0-generic f in* $\mathrm{Homeo}(X,d)$ *one has*

$$\overline{\mathrm{mo}}\,(\mathcal{O}_f(X),\,\mathrm{W}_p) \;=\; \dim X.$$

6. *Proof of Theorem* 4

The content of this section is inspired by the intermediate value property of the upper box dimension of bounded subsets of the Euclidean space $\mathbb{R}^\ell$, $\ell \in \mathbb{N}$, proved in [**13**, Theorem 2].

Let (Z, d) be a compact metric space and fix an arbitrary $0 \leqslant \beta \leqslant \overline{\mathrm{mo}}\,(Z)$.

If $\beta = 0$, we take $Y_\beta = \{z\}$ for any $z \in Z$; if, otherwise, $\beta = \overline{\mathrm{mo}}\,(Z)$, we just consider $Y_\beta = Z$.

Now we assume that $\beta \in \,]\,0, \overline{\mathrm{mo}}\,(Z)\,[$. We start by showing that, in order to evaluate the upper metric order of Y, which is given by the limit

$$\overline{\mathrm{mo}}\,(Y) \;=\; \limsup_{\varepsilon \to 0^+} \frac{\log\log S_Y(\varepsilon)}{-\log \varepsilon}$$

we may use balls of radius λ^j, for $j \in \mathbb{N}$ and any choice of $0 < \lambda < 1$. More precisely:

Lemma 3. *Given $\lambda \in \,]0, 1[$, for every subset Y of Z one has*

$$\overline{\mathrm{mo}}\,(Y) \;=\; \limsup_{j \to +\infty} \frac{\log\log S_Y(\lambda^j)}{-\log \lambda^j}.$$

Proof. Given $\lambda \in \,]0, 1[$ and $\varepsilon > 0$, there is a positive integer j such that $\lambda^{j+1} < \varepsilon \leqslant \lambda^j$. Then, as $S_Y(\lambda^j) \leqslant S_Y(\varepsilon) \leqslant S_Y(\lambda^{j+1})$ we conclude that

$$\frac{\log\log S_Y(\lambda^{j+1})}{-\log \lambda^{j+1}} \geqslant \frac{\log\log S_Y(\varepsilon)}{-\log \varepsilon} \geqslant \frac{\log\log S_Y(\lambda^j)}{-\log \lambda^j}.$$

Consequently,

$$\limsup_{\varepsilon \to 0^+} \frac{\log\log S_Y(\varepsilon)}{-\log \varepsilon} = \limsup_{j \to +\infty} \frac{\log\log S_Y(\lambda^j)}{-\log \lambda^j}.$$

This completes the proof of Lemma 3.

Let us resume the proof of the theorem when $0 < \beta < \overline{\mathrm{mo}}\,(Z)$. We start by choosing $\lambda \in \,]0, \frac{1}{2}[$. By compactness of Z, for each $k \in \mathbb{N}$ there is a finite open covering $\mathcal{U}_k$ of Z by balls of radius λ^k whose corresponding balls of radius λ^{k+1} are pairwise disjoint. In particular, there exists a partition $\mathcal{P}_k$ of Z made up of elements whose diameter is bounded by λ^k and whose inner diameter is bounded from below by λ^{k+1}.

As $\overline{\mathrm{mo}}\,(Z) > \beta$, there are infinitely many positive integers k such that $S_Z(\lambda^{k-1})$ is bigger than $\lfloor \exp(\lambda^{-\beta k}) \rfloor$. Let $k_1 \in \mathbb{N}$ be the smallest of them, which satisfies

$$S_Z(\lambda^{k_1 - 1}) > \lfloor \exp(\lambda^{-\beta k_1}) \rfloor. \tag{6·1}$$

As the diameter of the elements of the partition $\mathcal{P}_{k_1}$ is smaller than λ^{k_1} and $0 < \lambda < \frac{1}{2}$, we are sure that any two $\lambda^{k_1 - 1}$–separated points belong to different elements of the partition $\mathcal{P}_{k_1}$. Therefore, there exist at least $\lfloor \exp(\lambda^{-\beta k_1}) \rfloor$ elements of the partition $\mathcal{P}_{k_1}$ which intersect Z. Moreover, since the upper metric order is finitely stable (cf. [**3**]), that is, for any collection $\{B_j\}_{1 \leqslant j \leqslant n}$ of subsets of Z one has

$$\overline{\mathrm{mo}}\,(\bigcup_{1 \leqslant j \leqslant n} B_j) \;=\; \max_{1 \leqslant j \leqslant n} \overline{\mathrm{mo}}\,(B_j)$$

then there exists a partition element $E_{k_1} \in \mathcal{P}_{k_1}$ such that

$$\overline{\mathrm{mo}}\,(E_{k_1}) \;=\; \overline{\mathrm{mo}}\,(Z). \tag{6·2}$$

Select a finite sample of points

$$\widehat{Y}_{k_1} \;=\; \Big\{ x_{1,i} \colon \; 1 \leqslant i \leqslant \lfloor \exp(\lambda^{-\beta k_1}) \rfloor \Big\} \subset Z$$

which belong to different elements of the partition $\mathcal{P}_{k_1} \setminus \{E_{k_1}\}$. Afterwards, take the set

$$Y_1 \,=\, \widehat{Y}_{k_1} \,\cup\, E_{k_1}.$$

By construction, the equality (6·2) and the finite stability of the upper metric order, one has

(i) $\overline{\text{mo}}\,(Y_1) \,=\, \overline{\text{mo}}\,(Z)$;

(ii) $\#\{E \in \mathcal{P}_{k_1} \colon\, E \cap Y_1 \neq \emptyset\} \,=\, \lfloor \exp(\lambda^{-\beta k_1}) \rfloor + 1$;

(iii) $\#\{E \in \mathcal{P}_{k} \colon\, E \cap Y_1 \neq \emptyset\} \,\leqslant\, \lfloor \exp(\lambda^{-\beta k}) \rfloor$ for every $1 \leqslant k < k_1$.

The last item (iii) is due to the choice of k_1 as the smallest value of all positive integers k satisfying the inequality (6·1).

By item (i), one can take the smallest integer $k_2 > k_1$ such that

$$S_{Y_1}(\lambda^{k_2-1}) > \lfloor \exp(\lambda^{-\beta k_2} \rfloor$$

and so there are at least $\lfloor \exp(\lambda^{-\beta k_2}) \rfloor$ elements of the partition $\mathcal{P}_{k_2}$ which intersect Y_1. Thus, there exists $E_{k_2} \in \mathcal{P}_{k_2}$ such that

$$\overline{\text{mo}}\,(E_{k_2}) \,=\, \overline{\text{mo}}\,(Z).$$

Again, take a finite collection of points

$$\widehat{Y}_{k_2} \,=\, \Big\{x_{2,i} \colon\, 1 \leqslant i \leqslant \lfloor \exp(\lambda^{-\beta k_2}) \rfloor \Big\} \subset Y_1$$

belonging to different elements of the partition $\mathcal{P}_{k_2} \setminus \{E_{k_2}\}$. Afterwards, consider the set

$$Y_2 \,=\, \widehat{Y}_{k_1} \,\cup\, E_{k_2}$$

which satisfies

(iv) $\overline{\text{mo}}\,(Y_2) \,=\, \overline{\text{mo}}\,(Z)$;

(v) $\#\{E \in \mathcal{P}_{k_2} \colon\, E \cap Y_2 \neq \emptyset\} \,=\, \lfloor \exp(\lambda^{-\beta k_2}) \rfloor + 1$;

(vi) $\#\{E \in \mathcal{P}_{k} \colon\, E \cap Y_2 \neq \emptyset\} \,\leqslant\, \lfloor \exp(\lambda^{-\beta k}) \rfloor$ for every $k_1 < k < k_2$.

Proceeding recursively, one constructs a nested sequence of sets

$$Y_{n+1} \,\subset\, Y_n \,\subset\, \ldots \,\subset\, Y_2 \,\subset\, Y_1 \,\subset Z$$

whose upper metric orders coincide with $\overline{\text{mo}}\,(Z)$ and, moreover, such that

$$\#\{E \in \mathcal{P}_{k_n} \colon\, E \cap Y_n \neq \emptyset\} \,=\, \lfloor \exp(\lambda^{-\beta k_n}) \rfloor + 1 \tag{6·3}$$

and

$$\#\{E \in \mathcal{P}_{k} \colon\, E \cap Y_n \neq \emptyset\} \,\leqslant\, \lfloor \exp(\lambda^{-\beta k}) \rfloor \quad \forall\, k_{n-1} < k < k_n. \tag{6·4}$$

In particular, bringing together equations (6·3) and (6·4), and the fact that the inner diameter of $\mathcal{P}_k$ is bounded from below by λ^{k+1}, we conclude that the subset of Z defined by

$$Y_\beta \,=\, \bigcap_{n \,\in\, \mathbb{N}} Y_n$$

has upper metric order $\overline{\text{mo}}\,(Y_\beta) = \beta$. The proof of the theorem is complete.

7. *Proof of Theorem* 3

Let (X, d) be a compact metric space and denote by $\mathcal{M}_1(\mathcal{M}_1(X))$ the space of probability measures, defined on the Borel subsets of the space $\mathcal{M}(X)$, endowed with a metric D which induces the weak$^*-$topology. Given $\varepsilon > 0$ and a probability measure $\eta \in \mathcal{M}_1(\mathcal{M}_1(X))$, the *quantization number of* η *at scale* $\varepsilon > 0$, denoted by $Q_\eta(\varepsilon)$, is the least integer $N \in \mathbb{N}$ such that there exists a probability measure $\zeta \in \mathcal{M}_1(\mathcal{M}_1(X))$ supported on a set of cardinality N and satisfying $D(\eta, \zeta) \leqslant \varepsilon$. By [**3**, Proposition 3.2], *the quantization number* $Q_\eta(\varepsilon)$ *for the* $1-$*Wasserstein metric* W_1 is the minimal cardinality N of any set

$$F \,=\, \big\{\theta_1, \ldots, \theta_N\big\} \,\subset\, \mathcal{M}_1(X)$$

such that

$$\int_{\mathcal{M}_1(X)} W_1(\theta,\, F)\, d\eta(\theta) \,\leqslant\, \varepsilon.$$

We refer the reader to [**14**] for more details regarding this notion which aims at evaluating how close, in the Wasserstein or LP metric, is each $\eta-$almost every $\theta \in \mathcal{M}_1(X)$ to measures with finite support.

Given a continuous map $f\colon X \to X$, $\varepsilon > 0$ and $\mu \in \mathcal{M}_f(X)$, one has (cf. [**3**, Proposition 3.12])

$$\mathcal{E}_\mu(f)(\varepsilon) \;=\; Q_{\widehat{\mu}}(\varepsilon) \tag{7·1}$$

where $\widehat{\mu} \in \mathcal{M}_1(\mathcal{M}_f(X))$ is the ergodic decomposition of μ and $Q_{\widehat{\mu}}(\varepsilon)$ is the quantization number of $\widehat{\mu}$ for the metric W_1 on $\mathcal{M}_1(X)$. This characterization of the metric emergence map will be a crucial ingredient in the proof of Theorem 3.

We start by establishing a connection between the metric emergence maps of two $f-$invariant probability measures with the corresponding emergence map of a convex combination of them.

Lemma 4. *Given* $\mu, \nu \in \mathcal{M}_f(X)$, *let* $\tau_t = t\mu + (1-t)\nu$ *be a convex combination of* μ *and* ν *for some* $t \in\,]0, 1[$. *Then,*

$$\max\Big\{Q_{\widehat{\mu}}\left(\frac{\varepsilon}{t}\right),\, Q_{\widehat{\nu}}\left(\frac{\varepsilon}{1-t}\right)\Big\} \,\leqslant\, Q_{\widehat{\tau_t}}(\varepsilon) \,\leqslant\, 2\max\Big\{Q_{\widehat{\mu}}(\varepsilon),\, Q_{\widehat{\nu}}(\varepsilon)\Big\}.$$

Proof. Consider $\mu, \nu \in \mathcal{M}_f(X)$ and $\tau_t = t\mu + (1-t)\nu$ for some $t \in\,]0, 1[$. Fix $\varepsilon > 0$ and let $F \subset \mathcal{M}_1(X)$ be a finite subset such that

$$\int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\tau_t}(\theta) \,\leqslant\, \varepsilon. \tag{7·2}$$

By the ergodic decomposition theorem, the probability measures $\widehat{\mu}$ and $\widehat{\nu}$ in $\mathcal{M}_1(\mathcal{M}_f(X))$ satisfy

$$\mu \,=\, \int_{\mathcal{M}_f^{\text{erg}}(X)} \delta_\theta\, d\widehat{\mu}(\theta) \qquad \text{and} \qquad \nu \,=\, \int_{\mathcal{M}_f^{\text{erg}}(X)} \delta_\theta\, d\widehat{\nu}(\theta)$$

where $\mathcal{M}_f^{\text{erg}}(X)$ stands for the space of extremes of the convex set $\mathcal{M}_f(X)$. Or, equivalently,

$$\int_X \varphi\, d\mu \,=\, \int_{\mathcal{M}_f^{\text{erg}}(X)} \Big(\int_X \varphi\, d\theta\Big)\, d\widehat{\mu}(\theta) \quad \text{and} \quad \int_X \varphi\, d\nu \,=\, \int_{\mathcal{M}_f^{\text{erg}}(X)} \Big(\int_X \varphi\, d\theta\Big)\, d\widehat{\nu}(\theta)$$

for every continuous function $\varphi\colon X \to \mathbb{R}$. Therefore, the probability measure $\widehat{\tau_t}$ on

$\mathcal{M}_1(\mathcal{M}_f(X))$ satisfies

$$\widehat{\tau}_t \,=\, t\widehat{\mu} \,+\, (1-t)\widehat{\nu}$$

and so

$$\int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\tau}_t(\theta) \\ = t \int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\mu}(\theta) \,+\, (1-t) \int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\nu}(\theta). \tag{7·3}$$

This equality together with (7·2) and the fact the three integrands above are non-negative imply that

$$\int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\mu}(\theta) \,\leqslant\, \frac{\varepsilon}{t} \qquad \text{and} \qquad \int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\nu}(\theta) \,\leqslant\, \frac{\varepsilon}{1-t}$$

which ultimately yields to

$$\max\left\{ Q_{\widehat{\mu}}\left(\frac{\varepsilon}{t}\right),\, Q_{\widehat{\nu}}\left(\frac{\varepsilon}{1-t}\right)\right\} \,\leqslant\, Q_{\widehat{\tau}_t}(\varepsilon). \tag{7·4}$$

Regarding the second inequality in the statement of Lemma 4, we notice that if, for $\varepsilon > 0$, the sets $F_1 \subset \mathcal{M}_1(X)$ and $F_2 \subset \mathcal{M}_1(X)$ are finite with minimal cardinality such that

$$\int_{\mathcal{M}_1(X)} W_1(\theta, F_1)\, d\widehat{\mu}(\theta) \,\leqslant\, \varepsilon \qquad \text{and} \qquad \int_{\mathcal{M}_1(X)} W_1(\theta, F_2)\, d\widehat{\nu}(\theta) \,\leqslant\, \varepsilon$$

then, by (7·3), the union $F = F_1 \cup F_2$ satisfies

$$\int_{\mathcal{M}_1(X)} W_1(\theta, F)\, d\widehat{\tau}_t(\theta) \,\leqslant\, \varepsilon.$$

Since $\#F \leqslant 2\, \max\{\#F_1, \#F_2\}$, we deduce that

$$Q_{\widehat{\tau}_t}(\varepsilon) \,\leqslant\, 2\, \max\left\{Q_{\widehat{\mu}}(\varepsilon),\, Q_{\widehat{\nu}}(\varepsilon)\right\}.$$

We now resume the proof of Theorem 3.

(a) Recall that

$$\mathcal{B}_f(X) \,=\, \left\{\mu \in \mathcal{M}_f(X)\colon\; \sup_{\varepsilon>0} \mathcal{E}_\mu(f)(\varepsilon) > 1\right\}$$

and note that, in $\mathcal{B}_f(X)$, the limit (1·4) that estimates the metric emergence is well defined. Moreover, since the metric emergence map

$$\varepsilon > 0 \qquad \mapsto \qquad \mathcal{E}_\mu(f)(\varepsilon)$$

is decreasing, one has

$$\sup_{\varepsilon>0} \mathcal{E}_\mu(f)(\varepsilon) > 1 \quad \Leftrightarrow \quad \exists\, \varepsilon_\mu > 0\colon \quad \mathcal{E}_\mu(f)(\varepsilon_\mu) > 1 \\ \Leftrightarrow \quad \exists\, \varepsilon_\mu > 0\colon \quad \mathcal{E}_\mu(f)(\varepsilon) > 1 \quad \forall\, 0 < \varepsilon \leqslant \varepsilon_\mu. \tag{7·5}$$

Consider μ, $\nu \in \mathcal{B}_f(X)$, that is, $f-$invariant probability measures such that

$$\sup_{\varepsilon>0} \mathcal{E}_\mu(f)(\varepsilon) > 1 \qquad \text{and} \qquad \sup_{\varepsilon>0} \mathcal{E}_\nu(f)(\varepsilon) > 1.$$

Let $\varepsilon_\mu > 0$ and $\varepsilon_\nu > 0$ as in (7·5). Given $0 < t < 1$, take

$$\varepsilon_0 \;=\; \min\left\{t\varepsilon_\mu,\, (1-t)\varepsilon_\nu\right\} > 0.$$

Thus, $\frac{\varepsilon_0}{t} \leqslant \varepsilon_\mu$ and $\frac{\varepsilon_0}{1-t} \leqslant \varepsilon_\nu$. By Lemma 4, if $\tau_t = t\mu + (1-t)\nu$ then

$$\begin{aligned} Q_{\widehat{\tau}_t}(\varepsilon_0) \;&\geqslant\; \max\left\{Q_{\widehat{\mu}}\left(\frac{\varepsilon_0}{t}\right),\, Q_{\widehat{\nu}}\left(\frac{\varepsilon_0}{1-t}\right)\right\} \\ &\geqslant\; \max\left\{Q_{\widehat{\mu}}(\varepsilon_\mu),\, Q_{\widehat{\nu}}(\varepsilon_\nu)\right\} \;>\; 1. \end{aligned} \tag{7·6}$$

That is, $\mathcal{E}_{\tau_t}(\varepsilon_0) > 1$, hence τ_t belongs to $\mathcal{B}_f(X)$.

(b) Now we will show that, if $\mathcal{B}_f(X)$ is nonempty, then the metric emergence map is quasi-convex on $\mathcal{B}_f(X)$. Given $t \in \,]0,1[$ and $\mu, \nu \in \mathcal{B}_f(X)$, consider the convex combination $\tau_t = t\mu + (1-t)\nu$. We claim that

$$\mathcal{E}_{t\mu\,+\,(1-t)\nu}(f) \;=\; \max\left\{\mathcal{E}_\mu(f),\, \mathcal{E}_\nu(f)\right\}. \tag{7·7}$$

Indeed, as a consequence of item (a), (7·4) and (7·1), we obtain

$$\begin{aligned} \mathcal{E}_{t\mu\,+\,(1-t)\nu}(f) \;&=\; \limsup_{\varepsilon\,\to\,0^+} \frac{\log\log Q_{\widehat{\tau}_t}(\varepsilon)}{-\log\varepsilon} \\ &\geqslant\; \limsup_{\varepsilon\,\to\,0^+} \frac{\log\log Q_{\widehat{\mu}}\left(\frac{\varepsilon}{t}\right)}{-\log\frac{\varepsilon}{t} - \log t} \\ &=\; \mathcal{E}_\mu(f). \end{aligned}$$

A similar estimate yields $\mathcal{E}_{t\mu\,+\,(1-t)\nu}(f) \geqslant \mathcal{E}_\nu(f)$. Hence,

$$\mathcal{E}_{t\mu\,+\,(1-t)\nu}(f) \;\geqslant\; \max\{\mathcal{E}_\mu(f),\, \mathcal{E}_\nu(f)\}. \tag{7·8}$$

Conversely, by Lemma 4, one has

$$Q_{\widehat{\tau}_t}(\varepsilon) \;\leqslant\; 2\max\left\{Q_{\widehat{\mu}}(\varepsilon),\, Q_{\widehat{\nu}}(\varepsilon)\right\}$$

which, by (7·1), implies

$$\mathcal{E}_{t\mu\,+\,(1-t)\nu}(f) \;\leqslant\; \max\{\mathcal{E}_\mu(f),\, \mathcal{E}_\nu(f)\}. \tag{7·9}$$

Bringing together (7·8) and (7·9), we get

$$\mathcal{E}_{t\mu\,+\,(1-t)\nu}(f) \;=\; \max\{\mathcal{E}_\mu(f), \mathcal{E}_\nu(f)\}.$$

We observe that we have also shown that, in general, the metric emergence is not affine.

(c) Take $\beta \in [0,\, \mathcal{E}_{\text{top}}(f)]$. The following argument is inspired by the proof of [**3**, Theorem E], where the case $\beta = \mathcal{E}_{\text{top}}(f)$ was addressed.

Assume that $\mathcal{E}_{\text{top}}(f) > 0$ and fix $\beta \in [0,\, \mathcal{E}_{\text{top}}(f)[$. By Theorem 4 applied to $Z = \mathcal{M}_f^{\text{erg}}(X)$, whose upper metric order $\overline{\text{mo}}\,(Z)$ is precisely $\mathcal{E}_{\text{top}}(f)$, there exists a subset $Y_\beta \subset Z$ such that $\overline{\text{mo}}\,(Y_\beta) = \beta$. Therefore, by [**3**, Theorem 3.9] we may find a probability measure $\nu \in \mathcal{M}_1(\mathcal{M}_f^{\text{erg}}(X))$ such that $\overline{q_0}(\nu) = \overline{\text{mo}}\,(Y_\beta)$, where $\overline{q_0}$ stands for the quantization of ν. Then the probability measure $\mu = \int_{\mathcal{M}_1(X)} \eta\, d\nu(\eta)$ is $f-$invariant, so we may apply [**3**, Proposition 3.12] to μ and thus conclude that

$$\mathcal{E}_\mu(f) \;=\; \limsup_{\varepsilon\,\to\,0^+} \frac{\log\log \mathcal{E}_\mu(f)(\varepsilon)}{-\log\varepsilon} \;=\; \overline{q_0}(\nu) \;=\; \overline{\text{mo}}\,(Y_\beta) \;=\; \beta.$$

This proves item (c) and ends the proof of Theorem 3.

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