RESAMPLING METHODOLOGIES AND RELIABLE TAIL ESTIMATION

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Abstract. Resampling methodologies, like the *generalised jackknife* and the *bootstrap* are important tools for a reliable semi-parametric estimation of parameters of extreme or even rare events. Among these parameters we mention the *extreme value index*, denoted ξ, the primary parameter in *statistics of extremes*, and the *extremal index*, denoted θ, a measure of clustering of extreme events. Most of the semi-parametric estimators of these parameters show the same type of behaviour: nice asymptotic properties, but a high variance for small k, the number of upper order statistics used in the estimation, a high bias for large k, and the need for an adequate choice of k. After a brief reference to some estimators of the aforementioned parameters and their asymptotic properties we present algorithms for an adaptive reliable estimation of ξ and θ.

Keywords and phrases. *Bootstrap and jackknife methodologies; semi-parametric estimation; statistics of extremes.*

1. Introduction

Resampling methodologies have recently revealed to be extremely fruitful in the field of *statistics of extremes*. Among others, we mention the importance of the *generalized jackknife* (GJ) (Gray and Schucany, 1972) and the *bootstrap* (Efron, 1979) for a reliable semi-parametric estimation of any parameter of extreme or even rare events, like a *high quantile*, the *expected shortfall*, the *return period* of a high level or the two primary parameters of extreme events, the *extreme value index* (EVI) and the *extremal index* (EI).

In order to illustrate such topics, we consider essentially a GJ *minimum-variance reduced-bias* (MVRB) class of estimators of a positive EVI. The MVRV EVI-estimators were introduced and studied in Caeiro *et al.* (2005). The GJ-MVRB EVI-estimators were studied in Gomes *et al.* (2013). We further consider a GJ Leadbetter-Nandagopalan EI-estimator, introduced and studied in Gomes *et al.* (2008c). In Section 2, we begin with a brief introduction to *extreme value theory* (EVT). Both the EVI and the EI are defined, and first, second and third-order conditions in EVT are made explicit. In Section 3, a set of classical and reduced-bias EVI and EI-estimators is presented. Section 4 is dedicated to a brief reference to resampling methodologies and its use in a reliable EVI and EI-estimation. In Sections 5 and 6 we respectively present a few illustrative case-studies and some overall conclusions.

2. EVT—A BRIEF INTRODUCTION

2.1. **The EVI.** We shall use the notation ξ for the EVI for maxima, the shape parameter in the *extreme value* (EV) cumulative distribution function (cdf),

$$\mathrm{EV}_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}),\ 1+\xi x > 0 & \text{if} \quad \xi \neq 0 \\ \exp(-\exp(-x),\ x \in \mathbb{R} & \text{if} \quad \xi = 0, \end{cases} \tag{1}$$

and we shall consider models with a heavy right-tail, i.e. an underlying right tail or survival function,

$$\overline{F} := 1 - F \in \mathcal{R}_{-1/\xi}, \quad \text{for some} \quad \xi > 0,$$

where the notation $\mathcal{R}_\alpha$ stands for the class of regularly-varying functions with an index of regular variation equal to α, i.e., positive measurable functions $g(\cdot)$ such that for all $x > 0$, $g(tx)/g(t) \to x^\alpha$, as $t \to \infty$ (see Bingham *et al.*, 1987).

2.2. **First, second and third-order frameworks.** Then (Gnedenko, 1943), F is in the domain of attraction for maxima of a Fréchet-type *extreme value* cdf, i.e. an EV_ξ cdf with $\xi > 0$, in the sense that given a sequence of random samples, $(X_1, \ldots, X_n)$, it is possible to linearly normalise the sequence of maximum values $\{X_{n:n} := \max(X_1, \ldots, X_n)\}_{n \geq 1}$ and get convergence to a non-degenerate random variable (rv), with cdf EV_ξ, defined in (1), with $\xi > 0$. We then write

$$F \in \mathcal{D}_{\mathrm{M}}(\mathrm{EV}_{\xi>0}) =: {\mathcal{D}_M}^+.$$

In this same context of heavy right-tails, and with the notation

$$U(t) = F^{\leftarrow}(1 - 1/t),\ t \geq 1,$$

$F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$ the *generalized inverse function* of the underlying model F, we can further say (de Haan, 1984) that

$$F \in {\mathcal{D}_{\mathrm{M}}}^+ \quad \Longleftrightarrow \quad \overline{F} \in \mathcal{R}_{-1/\xi} \quad \Longleftrightarrow \quad U \in \mathcal{R}_\xi, \tag{2}$$

the so-called *first-order conditions.*

For consistent semi-parametric EVI-estimation, in the whole $\mathcal{D}_M^+$, we merely need to assume the validity of one of the *first-order conditions*, like $U \in \mathcal{R}_\xi$, and to work with adequate functionals, dependent on an *intermediate tuning* parameter k, the number of top *order statistics* (os's) involved in the estimation. This means that k needs to be such that

$$k = k_n \to \infty \quad \text{and} \quad k_n = o(n), \text{ as } n \to \infty. \tag{3}$$

To obtain information on the non-degenerate asymptotic behaviour of semi-parametric EVI-estimators, we need further assuming a *second-order condition*, ruling the rate of convergence in any of the *first-order conditions* in (2). The *second-order parameter*, ρ (≤ 0), rules such a rate of convergence, and it is the parameter appearing in the limiting result,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \frac{x^\rho - 1}{\rho}, \tag{4}$$

where we are using the interpretation of the Box-Cox transformation as the logarithm when the power equals zero. We often assume that (4) holds for every $x > 0$. Then, $|A|$ must compulsory be in $\mathcal{R}_\rho$ (Geluk and de Haan, 1987). For technical simplicity, we usually further assume that $\rho < 0$, writing

$$A(t) =: \xi \beta t^\rho, \tag{5}$$

dependent on the vector (β, ρ) of second-order parameters.

To obtain full information on the asymptotic bias of any corrected-bias EVI-estimator, it is often necessary to further assume a *general third-order condition*, ruling now the rate of convergence in the second-order condition in (4), which guarantees that, for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} - \frac{x^\rho - 1}{\rho}}{B(t)} = \frac{x^{\rho+\rho'} - 1}{\rho + \rho'}, \tag{6}$$

where $|B|$ must then be in $\mathcal{R}_{\rho'}$.

More restrictively, and equivalently to the aforementioned condition in (6) with $\rho = \rho' < 0$, we often consider a *Pareto third-order condition*, i.e., a Pareto-type class of models, with a tail function

$$1 - F(x) = C x^{-1/\xi} \big(1 + D_1 x^{\rho/\xi} + D_2 x^{2\rho/\xi} + o\big(x^{2\rho/\xi}\big)\big), \tag{7}$$

as $x \to \infty$, with $C > 0$, $D_1,\ D_2 \neq 0$, $\rho < 0$.

Then we can choose in the aforementioned general *third-order condition*, in (6),

$$B(t) = \beta'\ t^{\rho} = \frac{\beta' A(t)}{\beta\xi} =: \frac{\zeta\ A(t)}{\xi}, \quad \beta,\ \beta' \neq 0, \quad \zeta = \frac{\beta'}{\beta}, \tag{8}$$

with β and β' 'scale' second and third-order parameters, respectively, and A the function in (5).

2.3. **The EI.** The EI is a parameter of extreme events related to the clustering of exceedances of high thresholds, a situation that occurs with stationary sequences (Leadbetter, 1973). We thus assume to be working with a strictly stationary sequence of rv's, $\{X_n\}_{n\geq 1}$, from F, under the long range dependence condition **D** (Leadbetter *et al.*, 1983) and the local dependence condition **D**" (Leadbetter and Nandagopalan, 1989).

Definition 1. The stationary sequence $\{X_n\}_{n\geq 1}$ from an underlying model F is said to have an extremal index θ $(0 < \theta \leq 1)$ if, for all $\tau > 0$, we can find a sequence of levels $u_n = u_n(\tau)$ such that, with $\{Y_n\}_{n\geq 1}$ the associated *independent, identically distributed* (iid) sequence (i.e., an iid sequence from the same F),

$$\mathbb{P}\left(Y_{n:n} \leq u_n\right) = F^n(u_n) \underset{n\to\infty}{\longrightarrow} e^{-\tau} \quad \text{and} \quad \mathbb{P}\left(X_{n:n} \leq u_n\right) \underset{n\to\infty}{\longrightarrow} e^{-\theta\tau}.$$

Remark 2. **D** and **D**" are straightforwardly valid for iid data, and $\theta = 1$.

For dependent sequences there can thus appear a 'shrinkage' of maximum values, but the limiting cdf of $X_{n:n}$, linearly normalized, is still an EV cdf, i.e. from the cdf EV_ξ family, in (1).

The *extremal index* can also in most cases be defined as:

$$\begin{aligned}\theta &= \frac{1}{\text{limiting mean size of clusters}} \\ &= \lim_{n\to\infty} \mathbb{P}(X_2 \leq u_n | X_1 > u_n) = \lim_{n\to\infty} \mathbb{P}(X_1 \leq u_n | X_2 > u_n),\end{aligned}$$

$$u_n:\ F(u_n) = 1 - \tau/n + o(1/n),\ \text{as } n \to \infty,\ \text{with } \tau > 0, \text{fixed.}$$

The ARMAX processes will be the ones used here for illustration. Such processes are based on an iid sequence of innovations $\{Z_i\}_{i\geq 1}$, with cdf H, and are defined through the relation,

$$X_i := \beta \max(X_{i-1}, Z_i), \quad i \geq 1, \quad 0 < \beta < 1.$$

The ARMAX sequences have a stationary distribution F, dependent on H through the stationarity equation $F(\beta x)/F(x) = H(x)$ (Alpuim, 1989). Conditions **D** and **D**" hold for these sequences and they can possess an extremal index $\theta < 1$.

For illustration, we shall consider ARMAX processes with Fréchet innovations. If $H(x) = \Phi_\xi^{\beta^{-1/\xi}-1}(x)$, then $F(x) = \Phi_\xi(x) = \exp\left(-x^{-1/\xi}\right)$, $x \geq 0$, and $\theta = 1 - \beta^{1/\xi}$.

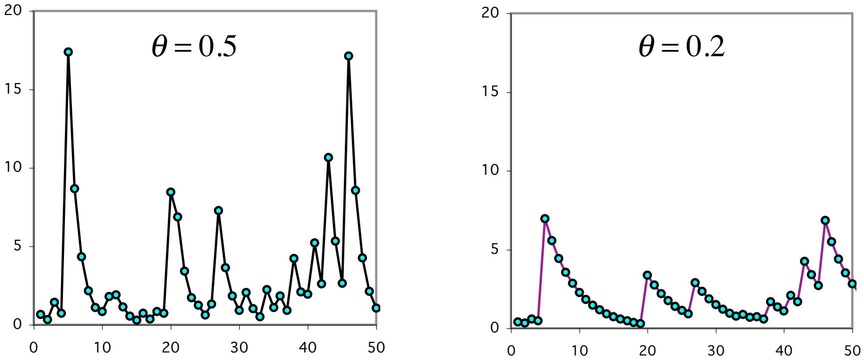


FIGURE 1. Sample paths of ARMAX processes with extremal index $\theta =$ 0.5 (*left*) and 0.2 (*right*)

Notice the richness of these processes regarding clustering of exceedances. Note also that for the same underlying model F there is a 'shrinkage of maximum values', together with the exhibition of larger and larger 'clusters of exceedances' of high values, as θ decreases.

3. EVI AND EI-ESTIMATORS

3.1. **Classical EVI-estimators.** For models in $\mathcal{D}_{\mathrm{M}}^+$, the classical EVI-estimators are the Hill estimators (Hill, 1975), averages of the *log-excesses*,

$$V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \qquad 1 \leq i \leq k < n,$$

i.e.,

$$\mathrm{H}_n(k) \equiv \mathrm{H}(k) := \tfrac{1}{k}\sum_{i=1}^{k} V_{ik}, \quad 1 \leq k < n. \tag{9}$$

But these EVI-estimators have often a strong asymptotic bias for moderate up to large values of k, of the order of $A(n/k)$, with A the function in (4), and the adequate accommodation of this bias has recently been extensively addressed in the literature.

3.2. **Second-order reduced-bias (SORB) EVI-estimators.** We mention the pioneering papers by Peng (1998), Beirlant *et al.* (1999), Feuerverger and Hall (1999) and Gomes *et al.* (2000; 2002), among others. In these papers, authors are led to SORB EVI-estimators, with asymptotic variances larger than or equal to $(\xi\ (1-\rho)/\rho)^2$, where $\rho(<0)$ is the aforementioned 'shape' second-order parameter, in (4). Note that $(\xi\ (1-\rho)/\rho)^2$ is the minimal asymptotic variance of an 'asymptotically unbiased' EVI-estimator in Drees' class of functionals (Drees, 1998).

3.3. **MVRB EVI-estimators.** Recently, Caeiro *et al.* (2005), Gomes *et al.* (2007) and Gomes *et al.* (2008*b*) considered, in different ways, the problem of *corrected-bias* EVI-estimation, being able to *reduce the bias without increasing the asymptotic variance*, which was shown to be kept at ξ^2, the asymptotic variance of Hill's estimator, the maximum likelihood (ML) estimator of ξ for an underlying Pareto cdf, $F_P(x) = 1 - (x/C)^{-1/\xi}$, $x \geq C$. Those estimators, called MVRB, from *minimum-variance reduced-bias*, are all based on an adequate 'external' consistent estimation of the pair of second-order parameters, $(\beta,\rho) \in (\mathbb{R},\mathbb{R}^-)$, in (5), done through estimators denoted $(\hat\beta,\hat\rho)$, and outperform the classical estimators for all k. We shall now consider the simplest class of MVRB EVI-estimators in Caeiro *et al.* (2005), a *corrected-Hill* (CH) EVI-estimator with the functional form

$$\overline{\mathrm{H}}(k) \equiv \overline{\mathrm{H}}_{\hat\beta,\hat\rho}(k) := \mathrm{H}(k)\Big(1 - \frac{\hat\beta}{1-\hat\rho}\left(\frac{n}{k}\right)^{\hat\rho}\Big). \tag{10}$$

For the estimation of (β,ρ), and following Gomes and Pestana (2007) (see also Gomes *et al.*, 2014, among others), we consider the following:

Algorithm 3.1 (Second-order parameters' estimation).
Given $\underline{\mathbf{x}}_n := (x_1,\ldots,x_n)$, *an observed value of the random sample* $\underline{\mathbf{X}}_n := (X_1,\ldots,X_n)$,

S1: *Compute, for the tuning parameters* $\tau = 0$ *and* $\tau = 1$ *the observed values of the simplest* ρ*-estimator in* Fraga Alves *et al.* (2003),

$$\hat\rho_\tau(k) \equiv \hat\rho_\tau(k;\underline{\mathbf{X}}_n) := -\left|\frac{3(V_\tau(k;\underline{\mathbf{X}}_n)-1)}{V_\tau(k;\underline{\mathbf{X}}_n)-3}\right|,$$

where

$$V_\tau(k;\underline{\mathbf{X}}_n) := \begin{cases} \dfrac{\left(M_n^{(1)}(k;\underline{\mathbf{X}}_n)\right)^\tau - \left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)^{\tau/2}}{\left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)^{\tau/2} - \left(M_n^{(3)}(k;\underline{\mathbf{X}}_n)/6\right)^{\tau/3}}, & \text{if} \quad \tau \neq 0, \\[2ex] \dfrac{\ln M_n^{(1)}(k;\underline{\mathbf{X}}_n) - \ln\left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)/2}{\ln\left(M_n^{(2)}(k;\underline{\mathbf{X}}_n)/2\right)/2 - \ln\left(M_n^{(3)}(k;\underline{\mathbf{X}}_n)/6\right)/3}, & \text{if} \quad \tau = 0, \end{cases}$$

is defined for any tuning parameter $\tau \in \mathbb{R}$, *with*

$$M_n^{(j)}(k;\underline{\mathbf{X}}_n) := \frac{1}{k}\sum_{i=1}^{k}\left(\ln X_{n-i+1:n} - \ln X_{n-k:n}\right)^j, \quad j = 1, 2, 3.$$

With $\lfloor x \rfloor$ *denoting the integer part of* x, *consider* $\{\hat\rho_\tau(k)\}_{k\in\mathcal{K}}$, *with* $\mathcal{K} = \left(\lfloor n^{0.995}\rfloor,\ \lfloor n^{0.999}\rfloor\right)$, *compute their median, denoted* χ_τ, *and further compute* $I_\tau := \sum_{k\in\mathcal{K}}\left(\hat\rho_\tau(k) - \chi_\tau\right)^2$, $\tau = 0, 1$. *Next choose the tuning parameter* $\tau^* = 0$ *if* $I_0 \leq I_1$; *otherwise, choose* $\tau^* = 1$.

S2: *Work with* $\hat\rho \equiv \hat\rho_{\tau^*}(k_1)$ *and* $\hat\beta = \hat\beta_{\hat\rho}(k_1)$, *where*

$$k_1 = \left\lfloor n^{1-\epsilon}\right\rfloor, \quad \epsilon = 0.001,$$

and with

$$d_\alpha(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha},\ D_\alpha(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i}{k}\right)^{-\alpha} i \ln\frac{X_{n-i+1:n}}{X_{n-i:n}},\ \alpha \in \mathbb{R},$$

$$\hat\beta_{\hat\rho}(k) \equiv \hat\beta_{\hat\rho}(k;\underline{\mathbf{X}}_n) := \left(\frac{k}{n}\right)^{\hat\rho}\frac{d_{\hat\rho}(k)\ D_0(k) - D_{\hat\rho}(k)}{d_{\hat\rho}(k)\ D_{\hat\rho}(k) - D_{2\hat\rho}(k)}$$

is the β*-estimator introduced and studied in* Gomes and Martins (2002).

For recent overviews on reduced-bias EVI-estimation see Reiss and Thomas (2007, Chapter 6), Gomes *et al.* (2008a), Beirlant *et al.* (2012) and Gomes and Guillou (2014).

3.4. Asymptotic comparison of classical and MVRB EVI-estimators.

The Hill estimator reveals usually a high asymptotic bias. Indeed, with $\mathcal{N}(\mu,\sigma^2)$ denoting a normal rv with mean value μ and variance σ^2, it follows from the results of de Haan and Peng (1998) that under the *general second-order condition*, in (4),

$$\sqrt{k}\,(\mathrm{H}(k) - \xi) \overset{d}{=} \mathcal{N}(0,\sigma_{_H}^2) + b_{_H}\sqrt{k}A(n/k) + o_p\big(\sqrt{k}A(n/k)\big),$$

where $\sigma_{_H}^2 = \xi^2$, and for $\rho < 0$ and $A(t) = \xi\beta t^\rho$, already defined in (5), the bias $b_{_H}\sqrt{k}A(n/k) = \xi\ \beta\ \sqrt{k}\ (n/k)^\rho/(1-\rho)$ can be very large (going to infinity), moderate (going to a constant) or small (going to zero) as $n \to \infty$, depending on the rate of increase of the sequence k_n with n.

This non-null asymptotic bias of the order of $A(n/k)$, together with a rate of convergence of the order of $1/\sqrt{k}$, leads to sample paths with a high variance for small k, a high bias for large k, and a very sharp mean square error (MSE) pattern, as a function of k.

Under the same conditions as before, $\sqrt{k}\left(\overline{\mathrm{H}}(k)-\xi\right)$ is asymptotically normal with variance also equal to ξ^2 but with a null mean value. Indeed, from the results in Caeiro *et al.* (2005), we know that it is possible to adequately estimate the second-order parameters β and ρ, through for instance *Algorithm* 3.1, so that we get

$$\sqrt{k}\left(\overline{\mathrm{H}}(k)-\xi\right) \stackrel{d}{=} \mathcal{N}(0,\xi^2)+o_p\big(\sqrt{k}A(n/k)\big).$$

Consequently, $\overline{\mathrm{H}}(k)$ outperforms $\mathrm{H}(k)$ for all k. Under the validity of the aforementioned third-order condition related to Pareto-type class of models, i.e. the condition in (7), and with ζ defined in (8), we can then adequately estimate the vector of second-order parameters, (β,ρ), and write (Caeiro *et al.*, 2009)

$$\sqrt{k}\left(\overline{\mathrm{H}}(k)-\xi\right) \stackrel{d}{=} \mathcal{N}(0,\xi^2)+b_{\overline{\mathrm{H}}}\sqrt{k}A^2(n/k)+o_p\big(\sqrt{k}A^2(n/k)\big),$$
$$b_{\overline{\mathrm{H}}}=\frac{1}{\xi}\left(\frac{\zeta}{1-2\rho}-\frac{1}{(1-2\rho)^2}\right),$$

i.e. the bias is now of the order of $A^2(n/k)$.

3.5. **Classical EI-estimators.** Given a sample $(X_1,\ldots,X_n)$ and chosen a suitable threshold u, with I_A the indicator function of A, a possible estimator of θ (Leadbetter and Nandagopalan, 1989) is given by

$$\hat\theta_n^N=\hat\theta_n^N(u):=\frac{\sum\limits_{j=1}^{n-1} I_{[X_j>u,X_{j+1}\le u]}}{\sum\limits_{j=1}^{n} I_{[X_j>u]}}=\frac{\sum\limits_{j=1}^{n-1} I_{[X_j\le u<X_{j+1}]}}{\sum\limits_{j=1}^{n} I_{[X_j>u]}}.$$

To have consistency, the high level u must be such that $n(1-F(u_n))=c_n\tau=\tau_n$, $\tau_n\to\infty$ and $\tau_n/n\to 0$ (Nandagopalan, 1990). Indeed, the intermediate sequence k_n, in (3), in an EVI-estimation is being replaced, in an EI-estimation, by the sequence $\tau_n=c_n\tau$ with $c_n\to\infty$ as $n\to\infty$.

To make the semi-parametric EI-estimation closer to the semi-parametric EVI-estimation, it is sensible to consider (see Gomes *et al.*, 2008c) a deterministic level

$u \in [X_{n-k:n}, X_{n-k+1:n})$ and the estimator

$$\hat\theta_n^{\mathrm{N}}(k) \equiv \hat\theta_n^{\mathrm{N}}(u) := \frac{1}{k}\sum_{j=1}^{n-1} I_{[X_j \le X_{n-k:n} < X_{j+1}]}. \tag{11}$$

Bias assumption on the data structures. For iid data ($\theta = 1$):

$$\mathbb{E}\left\{\hat\theta_n^{\mathrm{N}}(k)\right\} = 1 + \left(\frac{1}{2\,k} - \frac{k}{n}\right)(1+o(1)).$$

Moreover, for ARMAX processes, we get

$$\mathbb{E}\left\{\hat\theta_n^{\mathrm{N}}(k)\right\} = \theta - \left(\frac{\theta(\theta+1)}{2}\left(\frac{k}{n}\right) - \frac{3-2\,\theta}{2\,k}\right)(1+o(1)).$$

We shall thus consider the EI-estimator as a function of k, the number of os's higher than the chosen threshold, as given in (11). Moreover, we shall further assume a sensible structure for the asymptotic bias, given by

$$\mathrm{Bias}\left\{\hat\theta_n^{\mathrm{N}}(k)\right\} = \varphi_1(\theta)\left(\frac{k}{n}\right) + \varphi_2(\theta)\left(\frac{1}{k}\right) + o\left(\frac{1}{k}\right) + o\left(\frac{k}{n}\right), \tag{12}$$

as $n \to \infty$, and for any intermediate k (see Gomes *et al.*, 2008c).

In the semi-parametric EI-estimation we have thus to cope with problems similar to the ones appearing in the EVI-estimation: *increasing bias, as the threshold decreases and a high variance for high thresholds*, and it is sensible to ask whether it is possible to improve the performance of estimators through the use of resampling methods.

We are next interested in the use of the GJ methodology, in order to reduce the bias of the MVRB EVI-estimators, in (10) and the classical EI-estimators, in (11). In statistics we often put the question, *"May the combination of information improve the quality of estimators of a certain parameter or functional"*? The *jackknife* or GJ are resampling methodologies, which usually give a positive answer to such a question. Indeed, the main objectives of the *jackknife methodology* are:

1. Bias and variance estimation of a certain statistic, only through manipulation of observed data $\underline{x}$.
2. The building of estimators with bias and MSE smaller than those of an initial set of estimators.

4. Resampling methodologies

As mentioned in the very beginning of this article, the use of resampling methodologies has revealed to be promising in the estimation of the nuisance parameter k, and in the reduction of bias of any estimator of a parameter of extreme events. If we ask how to choose the tuning parameter k in the EVI-estimation, either through $\mathrm{H}(k)$ or through $\overline{\mathrm{H}}(k)$, given respectively in (9) and (10), we usually consider the estimation of $k_0^{\mathrm{H}} := \arg\min_k \mathrm{MSE}(\mathrm{H}(k))$ or $k_0^{\overline{\mathrm{H}}} = \arg\min_k \mathrm{MSE}(\overline{\mathrm{H}}(k))$. To obtain estimates of k_0^{H} and $k_0^{\overline{\mathrm{H}}}$ one can then use a *double-bootstrap* method applied to an adequate *auxiliary statistic* which tends to **zero** and has an asymptotic behaviour similar to the one of either $\mathrm{H}(k)$ (Draisma *et al.*, 1999, Gomes and Oliveira, 2001, Danielsson *et al.*, 2001, among others) or $\overline{\mathrm{H}}(k)$ (Gomes, *et al.*, 2011, 2012). See also, Gomes *et al.* (2014), for a short review on the role of bootstrap in statistics of extremes.

But at such optimal levels, we still have a non-null asymptotic bias even when we work with the CH EVI-estimator $\overline{\mathrm{H}}$, in (10). If we still want to remove such a bias, we can make use of the GJ methodology. It is then enough to consider an adequate pair of estimators of the parameter of extreme events under consideration, and to build a *reduced-bias affine combination* of them. In Gomes *et al.* (2000; 2002), also among others, we can find an application of this technique to the Hill estimator and in Gomes *et al.* (2013) an application to the CH EVI-estimators, in (10). To illustrate here the use of these methodologies in EVT, we again apply the GJ methodology to the aforementioned MVRB estimators $\overline{\mathrm{H}}(k)$ in Caeiro *et al.* (2005), just as performed in Gomes *et al.* (2013).

4.1. **The jackknife methodology and bias reduction.** The pioneering EVI reduced-bias estimators are, in a certain sense, GJ estimators, i.e., affine combinations of well-known estimators of ξ. The GJ statistic was introduced by Gray and Shucany (1972): Let $T_n^{(1)}$ and $T_n^{(2)}$ be two biased estimators of ξ, with similar bias properties, i.e.,

$$\mathrm{Bias}(T_n^{(i)}) = \xi + \phi(\xi) d_i(n), \quad i = 1, 2.$$

Then, if $q = q_n = d_1(n)/d_2(n) \neq 1$, the affine combination

$$T_n^{\mathrm{G}} := \left(T_n^{(1)} - qT_n^{(2)}\right)/(1-q)$$

is an unbiased estimator of ξ.

4.2. **A GJ corrected-bias EVI-estimator.** Given $\overline{\mathrm{H}}$, defined in (10), the most natural GJ rv is the one associated to the random pair $\left(\overline{\mathrm{H}}(k), \overline{\mathrm{H}}(\lfloor\theta k\rfloor)\right)$, $0 < \theta < 1$, i.e.

$$\overline{\mathrm{H}}^{\mathrm{GJ}(q,\theta)}(k) := \frac{\overline{\mathrm{H}}(k) - q\ \overline{\mathrm{H}}(\lfloor\theta k\rfloor)}{1-q}, \quad 0 < \theta < 1,$$

with

$$q = q_n = \frac{\mathrm{Bias}_\infty\left\{\overline{\mathrm{H}}(k)\right\}}{\mathrm{Bias}_\infty\left\{\overline{\mathrm{H}}(\lfloor\theta k\rfloor)\right\}} = \frac{A^2(n/k)}{A^2(n/\lfloor\theta k\rfloor)} \underset{n/k\to\infty}{\longrightarrow} \theta^{2\rho}.$$

It is thus sensible to consider $q = \theta^{2\rho}$, $\theta = 1/2$, and, with $\hat\rho$ a consistent estimator of ρ, the GJ EVI-estimator,

$$\overline{\mathrm{H}}^{\mathrm{GJ}}(k) := \frac{2^{2\hat\rho}\ \overline{\mathrm{H}}(k) - \overline{\mathrm{H}}(\lfloor k/2\rfloor)}{2^{2\hat\rho}-1}. \tag{13}$$

Then (Gomes *et al.*, 2013), and provided that $\hat\rho - \rho = o_p(1)$,

$$\sqrt{k}\left(\overline{\mathrm{H}}^{\mathrm{GJ}}(k) - \xi\right) \overset{d}{=} \mathcal{N}(0,\sigma^2_{_{GJ}}) + o_p\big(\sqrt{k}A^2(n/k)\big),$$

with

$$\sigma^2_{_{\mathrm{GJ}}} = \xi^2(1 + 1/(2^{-2\rho}-1)^2).$$

We have thus the ‘old’ trade-off between variance and bias, as happened with all the aforementioned SORB EVI-estimators associated with classical EVI-estimators. The bias decreases, but the variance increases. However, at optimal levels in the sense of minimal MSE, these third-order reduced-bias GJ EVI-estimators can often beat the MVRB EVI-estimators.

4.3. **Asymptotic bias and efficiency of an affine combination of corrected-bias EVI-estimators.** Just as mentioned before, the most obvious affine combination associated with the CH EVI-estimator $\overline{\mathrm{H}}(k)$, is

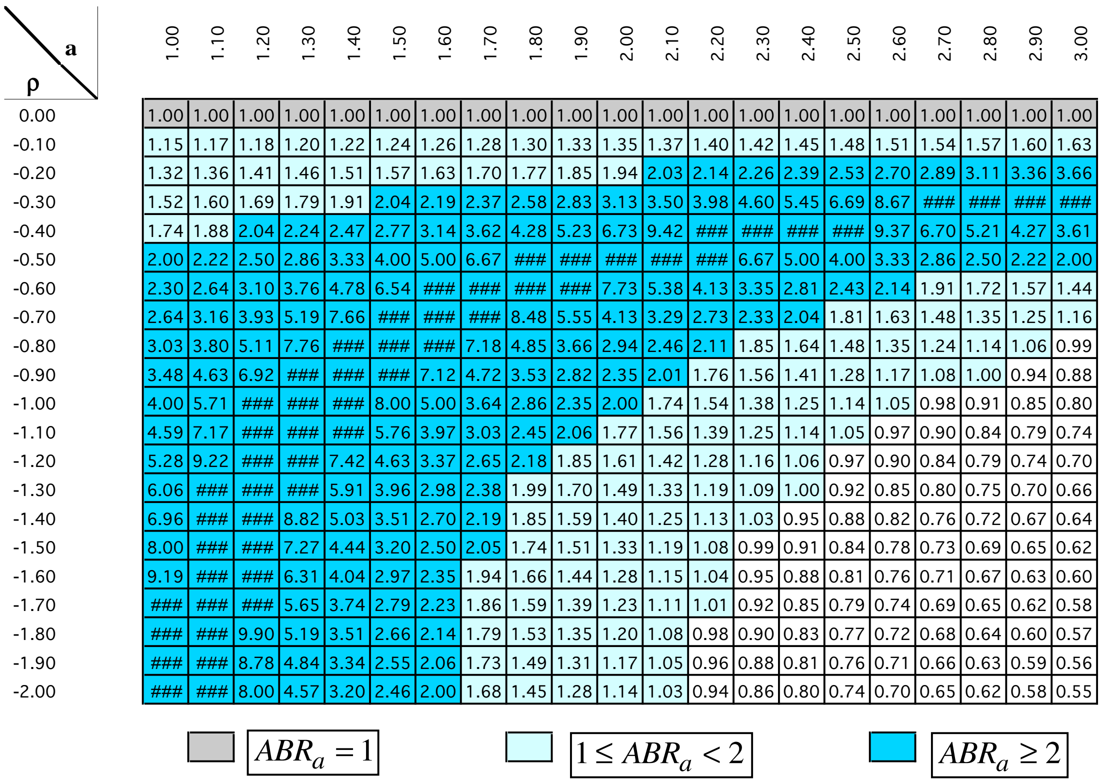
$$\overline{\mathrm{H}}^{\mathrm{GJ}(a)}(k) := a\overline{\mathrm{H}}(\lfloor k/2\rfloor) + (1-a)\overline{\mathrm{H}}(k). \tag{14}$$

We have thus a class of estimators parameterized in the *tuning* parameter a, to be chosen in the most adequate way.

We next refer the behaviour of the asymptotic bias and the efficiency of the affine combination in (14), referring first their asymptotic properties for a possibly non-optimal choice of $a > 1$. For a fixed level k, the reduction in the asymptotic bias of $\overline{\mathrm{H}}^{\mathrm{GJ}(a)}(k)$ comparatively with $\overline{\mathrm{H}}(k)$ is measured by the indicator:

$$\mathrm{ABR}_a := \lim_{n\to\infty}\left(\left|\frac{\mathrm{Bias}_\infty\left\{\overline{H}(k)\right\}}{\mathrm{Bias}_\infty\left\{\overline{\mathrm{H}}^{\mathrm{GJ}(a)}(k)\right\}}\right|\right) = \frac{1}{|1-a(1-2^{2\rho})|}. \tag{15}$$

In Figure 2 we present, in the (a,ρ)-plane, the values of the indicator ABR_a, in (15), independent of the tail index ξ.



ρ \ a	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00	2.10	2.20	2.30	2.40	2.50	2.60	2.70	2.80	2.90	3.00
0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.10	1.15	1.17	1.18	1.20	1.22	1.24	1.26	1.28	1.30	1.33	1.35	1.37	1.40	1.42	1.45	1.48	1.51	1.54	1.57	1.60	1.63
-0.20	1.32	1.36	1.41	1.46	1.51	1.57	1.63	1.70	1.77	1.85	1.94	2.03	2.14	2.26	2.39	2.53	2.70	2.89	3.11	3.36	3.66
-0.30	1.52	1.60	1.69	1.79	1.91	2.04	2.19	2.37	2.58	2.83	3.13	3.50	3.98	4.60	5.45	6.69	8.67	###	###	###	###
-0.40	1.74	1.88	2.04	2.24	2.47	2.77	3.14	3.62	4.28	5.23	6.73	9.42	###	###	###	###	9.37	6.70	5.21	4.27	3.61
-0.50	2.00	2.22	2.50	2.86	3.33	4.00	5.00	6.67	###	###	###	###	###	6.67	5.00	4.00	3.33	2.86	2.50	2.22	2.00
-0.60	2.30	2.64	3.10	3.76	4.78	6.54	###	###	###	###	7.73	5.38	4.13	3.35	2.81	2.43	2.14	1.91	1.72	1.57	1.44
-0.70	2.64	3.16	3.93	5.19	7.66	###	###	###	8.48	5.55	4.13	3.29	2.73	2.33	2.04	1.81	1.63	1.48	1.35	1.25	1.16
-0.80	3.03	3.80	5.11	7.76	###	###	###	7.18	4.85	3.66	2.94	2.46	2.11	1.85	1.64	1.48	1.35	1.24	1.14	1.06	0.99
-0.90	3.48	4.63	6.92	###	###	###	7.12	4.72	3.53	2.82	2.35	2.01	1.76	1.56	1.41	1.28	1.17	1.08	1.00	0.94	0.88
-1.00	4.00	5.71	###	###	###	8.00	5.00	3.64	2.86	2.35	2.00	1.74	1.54	1.38	1.25	1.14	1.05	0.98	0.91	0.85	0.80
-1.10	4.59	7.17	###	###	###	5.76	3.97	3.03	2.45	2.06	1.77	1.56	1.39	1.25	1.14	1.05	0.97	0.90	0.84	0.79	0.74
-1.20	5.28	9.22	###	###	7.42	4.63	3.37	2.65	2.18	1.85	1.61	1.42	1.28	1.16	1.06	0.97	0.90	0.84	0.79	0.74	0.70
-1.30	6.06	###	###	###	5.91	3.96	2.98	2.38	1.99	1.70	1.49	1.33	1.19	1.09	1.00	0.92	0.85	0.80	0.75	0.70	0.66
-1.40	6.96	###	###	8.82	5.03	3.51	2.70	2.19	1.85	1.59	1.40	1.25	1.13	1.03	0.95	0.88	0.82	0.76	0.72	0.67	0.64
-1.50	8.00	###	###	7.27	4.44	3.20	2.50	2.05	1.74	1.51	1.33	1.19	1.08	0.99	0.91	0.84	0.78	0.73	0.69	0.65	0.62
-1.60	9.19	###	###	6.31	4.04	2.97	2.35	1.94	1.66	1.44	1.28	1.15	1.04	0.95	0.88	0.81	0.76	0.71	0.67	0.63	0.60
-1.70	###	###	###	5.65	3.74	2.79	2.23	1.86	1.59	1.39	1.23	1.11	1.01	0.92	0.85	0.79	0.74	0.69	0.65	0.62	0.58
-1.80	###	###	9.90	5.19	3.51	2.66	2.14	1.79	1.53	1.35	1.20	1.08	0.98	0.90	0.83	0.77	0.72	0.68	0.64	0.60	0.57
-1.90	###	###	8.78	4.84	3.34	2.55	2.06	1.73	1.49	1.31	1.17	1.05	0.96	0.88	0.81	0.76	0.71	0.66	0.63	0.59	0.56
-2.00	###	###	8.00	4.57	3.20	2.46	2.00	1.68	1.45	1.28	1.14	1.03	0.94	0.86	0.80	0.74	0.70	0.65	0.62	0.58	0.55

FIGURE 2. Asymptotic bias reduction (ABR) indicator

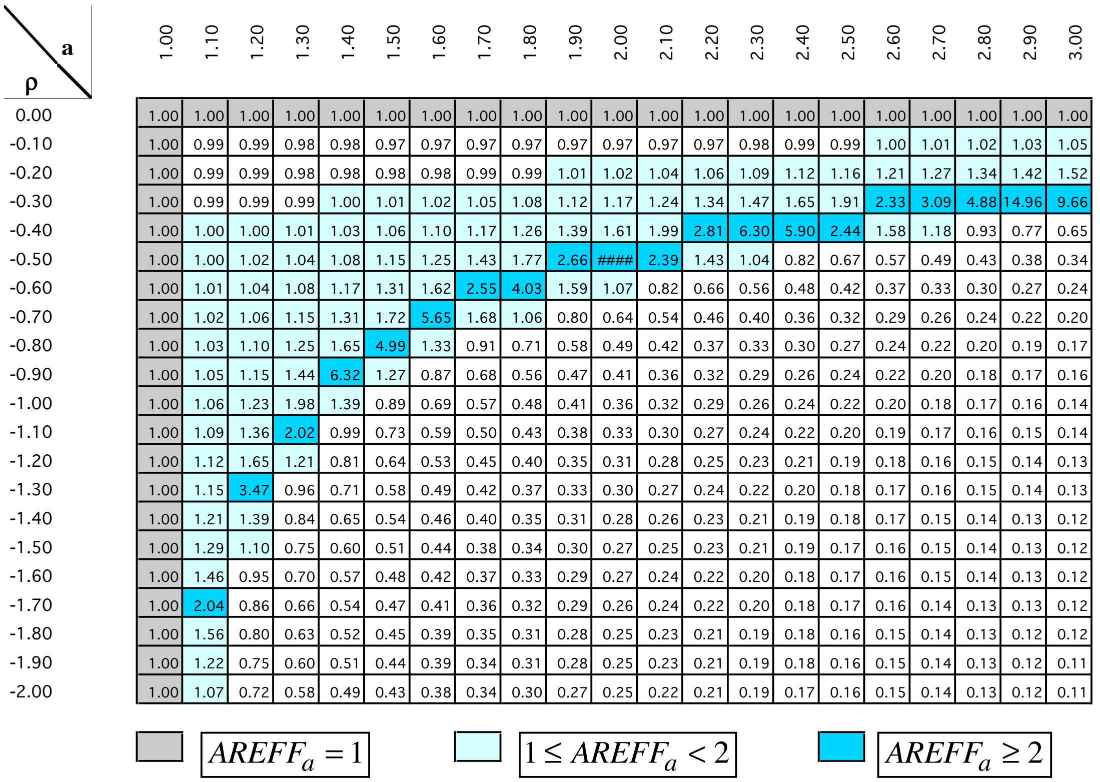
As usual, let us define the asymptotic efficiency of $\overline{\mathrm{H}}^{\mathrm{GJ}(a)}(k)$ relatively to $\overline{\mathrm{H}}(k)$ as the quotient between the two asymptotic MSEs, computed at optimal levels. Provided that $a \neq 1/(1-2^{2\rho})$, we have the indicator

$$\mathrm{AREF}_a := \frac{\mathrm{MSE}_\infty\left\{\overline{\mathrm{H}}(k_0^{\overline{\mathrm{H}}})\right\}}{\mathrm{MSE}_\infty\{\overline{\mathrm{H}}^{\mathrm{GJ}(a)}(k_0^{\overline{\mathrm{H}}^{\mathrm{GJ}(a)}})\}} = \left(\frac{(a^2+1)^{2\rho}}{1-a(1-2^{2\rho})}\right)^{\frac{2}{1-4\rho}}. \tag{16}$$

We next show, in Figure 3, and again in the (a,ρ)-plane, the values of the asymptotic relative efficiency indicator, in (16), which becomes infinity for $a = 1/(1-2^{2\rho})$, the value of a associated with the GJ rv.

Some general comments:

- The reduction in bias is achieved in a wide region of the (a,ρ)-plane, making almost irrelevant a choice of the *tuning parameter* a.
- However, it is clear that for a reduction in MSE we indeed need to work close to the line $a = 1/(1-2^{2\rho})$. This justifies the introduction of the GJ estimator $\overline{\mathrm{H}}^{\mathrm{GJ}}(k)$, in (13).



ρ \ a	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00	2.10	2.20	2.30	2.40	2.50	2.60	2.70	2.80	2.90	3.00
0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.10	1.00	0.99	0.99	0.98	0.98	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.97	0.98	0.99	0.99	1.00	1.01	1.02	1.03	1.05
-0.20	1.00	0.99	0.99	0.98	0.98	0.98	0.98	0.99	0.99	1.01	1.02	1.04	1.06	1.09	1.12	1.16	1.21	1.27	1.34	1.42	1.52
-0.30	1.00	0.99	0.99	0.99	1.00	1.01	1.02	1.05	1.08	1.12	1.17	1.24	1.34	1.47	1.65	1.91	2.33	3.09	4.88	14.96	9.66
-0.40	1.00	1.00	1.00	1.01	1.03	1.06	1.10	1.17	1.26	1.39	1.61	1.99	2.81	6.30	5.90	2.44	1.58	1.18	0.93	0.77	0.65
-0.50	1.00	1.00	1.02	1.04	1.08	1.15	1.25	1.43	1.77	2.66	####	2.39	1.43	1.04	0.82	0.67	0.57	0.49	0.43	0.38	0.34
-0.60	1.00	1.01	1.04	1.08	1.17	1.31	1.62	2.55	4.03	1.59	1.07	0.82	0.66	0.56	0.48	0.42	0.37	0.33	0.30	0.27	0.24
-0.70	1.00	1.02	1.06	1.15	1.31	1.72	5.65	1.68	1.06	0.80	0.64	0.54	0.46	0.40	0.36	0.32	0.29	0.26	0.24	0.22	0.20
-0.80	1.00	1.03	1.10	1.25	1.65	4.99	1.33	0.91	0.71	0.58	0.49	0.42	0.37	0.33	0.30	0.27	0.24	0.22	0.20	0.19	0.17
-0.90	1.00	1.05	1.15	1.44	6.32	1.27	0.87	0.68	0.56	0.47	0.41	0.36	0.32	0.29	0.26	0.24	0.22	0.20	0.18	0.17	0.16
-1.00	1.00	1.06	1.23	1.98	1.39	0.89	0.69	0.57	0.48	0.41	0.36	0.32	0.29	0.26	0.24	0.22	0.20	0.18	0.17	0.16	0.14
-1.10	1.00	1.09	1.36	2.02	0.99	0.73	0.59	0.50	0.43	0.38	0.33	0.30	0.27	0.24	0.22	0.20	0.19	0.17	0.16	0.15	0.14
-1.20	1.00	1.12	1.65	1.21	0.81	0.64	0.53	0.45	0.40	0.35	0.31	0.28	0.25	0.23	0.21	0.19	0.18	0.16	0.15	0.14	0.13
-1.30	1.00	1.15	3.47	0.96	0.71	0.58	0.49	0.42	0.37	0.33	0.30	0.27	0.24	0.22	0.20	0.18	0.17	0.16	0.15	0.14	0.13
-1.40	1.00	1.21	1.39	0.84	0.65	0.54	0.46	0.40	0.35	0.31	0.28	0.26	0.23	0.21	0.19	0.18	0.17	0.15	0.14	0.13	0.12
-1.50	1.00	1.29	1.10	0.75	0.60	0.51	0.44	0.38	0.34	0.30	0.27	0.25	0.23	0.21	0.19	0.17	0.16	0.15	0.14	0.13	0.12
-1.60	1.00	1.46	0.95	0.70	0.57	0.48	0.42	0.37	0.33	0.29	0.27	0.24	0.22	0.20	0.18	0.17	0.16	0.15	0.14	0.13	0.12
-1.70	1.00	2.04	0.86	0.66	0.54	0.47	0.41	0.36	0.32	0.29	0.26	0.24	0.22	0.20	0.18	0.17	0.16	0.14	0.13	0.13	0.12
-1.80	1.00	1.56	0.80	0.63	0.52	0.45	0.39	0.35	0.31	0.28	0.25	0.23	0.21	0.19	0.18	0.16	0.15	0.14	0.13	0.12	0.12
-1.90	1.00	1.22	0.75	0.60	0.51	0.44	0.39	0.34	0.31	0.28	0.25	0.23	0.21	0.19	0.18	0.16	0.15	0.14	0.13	0.12	0.11
-2.00	1.00	1.07	0.72	0.58	0.49	0.43	0.38	0.34	0.30	0.27	0.25	0.22	0.21	0.19	0.17	0.16	0.15	0.14	0.13	0.12	0.11

▨ $AREFF_a = 1$

□ $1 \leq AREFF_a < 2$

🟦 $AREFF_a \geq 2$

FIGURE 3. Asymptotic relative efficiency (AREFF) indicator

- There is then a high reduction in the MSE of the GJ $\overline{\text{H}}$-estimators, at optimal level, in the sense of minimal MSE as a function of k, comparatively with the original $\overline{\text{H}}$-estimators, also at optimal level.
- The sample paths of these corrected-bias estimators are usually quite stable. The choice of the optimal level is thus of a smaller importance.
- But, even so, we can use the *bootstrap methodology* for the choice of such an optimal level, as already mentioned in this article.

4.4. **A GJ corrected-bias EI-estimator.** Since the bias term of the aforementioned classical EI-estimator reveals two main components of different orders, as can be seen in (12), we need to use an affine combination of three EI-estimators, i.e. an order-2 GJ-statistic.

Let $\underline{X} = (X_1, \ldots, X_n)$ be a sample from F, and let $T_n = T_n(\underline{X}, F)$ be an estimator of a functional $\theta(F)$, or of a parameter θ. If the bias of our estimator reveals two main terms that we would like to remove, the GJ methodology advises us to deal with three estimators with the same type of bias:

Definition 3. Given three estimators $T_n^{(1)}$, $T_n^{(2)}$ and $T_n^{(3)}$ of θ, such that

$$E\left\{T_n^{(i)} - \theta\right\} = d_1(\theta)\ \varphi_1^{(i)}(n) + d_2(\theta)\ \varphi_2^{(i)}(n),\ \ i = 1, 2, 3,$$

the GJ-statistic (of order 2) is given by

$$T_n^{GJ} := \left\| \begin{matrix} T_n^{(1)} & T_n^{(2)} & T_n^{(3)} \\ \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} \end{matrix} \right\| \Bigg/ \left\| \begin{matrix} 1 & 1 & 1 \\ \varphi_1^{(1)} & \varphi_1^{(2)} & \varphi_1^{(3)} \\ \varphi_2^{(1)} & \varphi_2^{(2)} & \varphi_2^{(3)} \end{matrix} \right\|,$$

with $||A||$ denoting, as usual, the determinant of the matrix A.

Straightforwardly, one may state:

Proposition 1. *T_n^{GJ} is unbiased for the estimation of θ.*

Moreover, although the variance of T_n^{GJ} is always larger than the variance of the original estimators, the MSE of T_n^{GJ} is often smaller than that of any of the statistics $T_n^{(i)}$, $i = 1, 2, 3$.

Given the information on the bias of the extremal index estimator $\hat\theta_n^{\mathrm{N}}(k)$, in (11), as stated in (12), let us consider, just as in Gomes *et al.* (2008c), the levels k, $\lfloor \delta k \rfloor + 1$ and $\lfloor \delta^2 k \rfloor + 1$, dependent of a *tuning parameter* δ, $0 < \delta < 1$, and the class of estimators,

$$\hat\theta_n^{\mathrm{GJ}(\delta)}(k) := \frac{(\delta^2+1)\ \hat\theta_n^{\mathrm{N}}\left([\delta k]+1\right) - \delta\left(\hat\theta_n^{\mathrm{N}}\left([\delta^2 k]+1\right) + \hat\theta_n^{\mathrm{N}}(k)\right)}{(1-\delta)^2}. \tag{17}$$

Among the members of this class, the aforementioned authors have been heuristically led to the choice $\delta = 1/4$. Distributional properties of

$$\hat\theta_n^{\mathrm{GJ}}(k) := \hat\theta_n^{\mathrm{GJ}(1/4)}(k) \tag{18}$$

have so far been obtained *only* through simulation techniques, and are next briefly presented.

In Figure 4, we picture the sample paths of $\hat\theta_n^{\mathrm{N}}(k)$, in (11), and $\hat\theta_n^{\mathrm{GJ}}(k) \equiv \hat\theta_n^{\mathrm{GJ}(1/4)}(k)$, in (18), with $\hat\theta_n^{\mathrm{GJ}(\delta)}(k)$ generally given in (17), for a stationary Fréchet(1) ARMAX sample of size $n = 5000$, with $\theta = 0.5$. Note the reasonably high stability around the target value $\theta = 0.5$, of the sample path of the GJ EI-estimator for a wide range of k-values, comparatively to that of Nandagopalan's estimator.

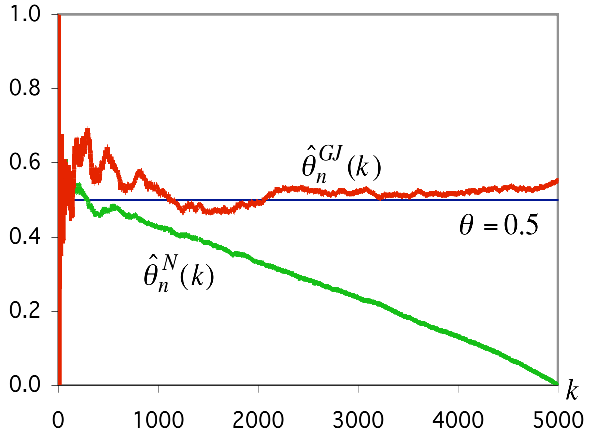


FIGURE 4. Sample paths of $\hat{\theta}_n^{\mathrm{N}}(k)$ and $\hat{\theta}_n^{\mathrm{GJ}}(k) \equiv \hat{\theta}_n^{\mathrm{GJ}(1/4)}(k)$, for a stationary Fréchet(1) ARMAX sample of size $n = 5000$, with $\theta = 0.5$

In Figure 5, to exhibit the influence of the *tuning parameter* δ in the GJ EI-estimator, we present the expected values and MSEs of such an estimator, associated with $\delta = 0.1,\ 0.4$ and 0.5, for the same structure as before.

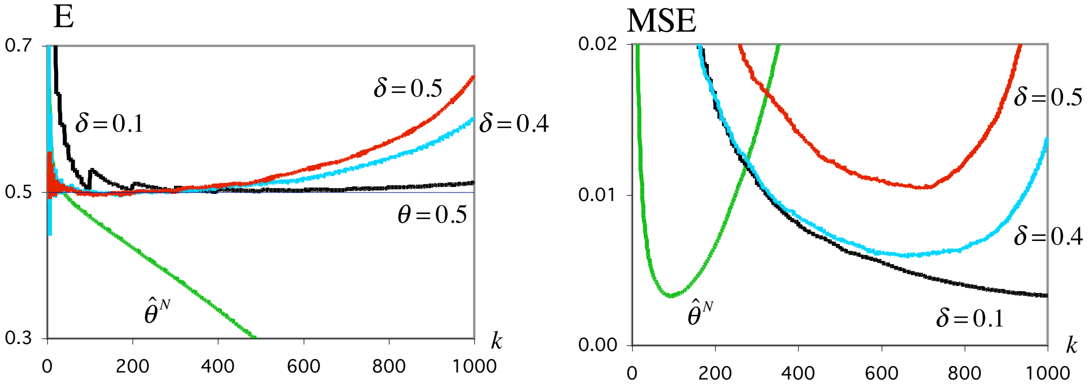


FIGURE 5. Expected value (E) and MSE of $\hat{\theta}_n^{\mathrm{GJ}(\delta)}(k)$, in (17), associated with $\delta = 0.1,\ 0.4$ and 0.5, for a stationary Fréchet(1) ARMAX sample of size $n = 5000$, with $\theta = 0.5$

Remark 4. The mean value stability around the target value θ, for a wide range of k-levels, is true for all θ and for all models simulated in Gomes *et al.* (2008c). But the

GJ EI-estimator, $\hat{\theta}_n^{\mathrm{GJ}}$, in (18), may be not able to overpass for small θ, the original EI-estimator, $\hat{\theta}_n^{\mathrm{N}}$, in (11), regarding MSE at optimal levels. Extra investment is thus needed on the 'optimal choice' of the three levels to be used in the building of a GJ EI-estimator or on the use of extra resampling or sub-sampling techniques, as initially performed in Gomes *et al.* (2008c). These authors have used simple subsampling techniques, briefly sketched in the following, in order to attain a smaller MSE at optimal levels.

4.5. **Effect of sampling frequency on the EI of an ARMAX process.** From the articles of Robison and Tawn (2000), Scotto *et al.* (2003) and Martins and Ferreira (2004), among others, we get the following result for stationary sequences under **D** and **D"** conditions: If we consider the sub-sample $\mathbf{V} = \left\{X_{(n-1)T}\right\}_{n\geq 1}$ we have,

$$\theta_V = \theta_X + \lim_{n\to\infty} \frac{\sum_{i=0}^{T-2} P(X_0 \leq u < X_1, X_{T-i} > u)}{\tau/n}.$$

For ARMAX sequences,

$$\theta_{\mathbf{V}} = 1 - (1-\theta_{\mathbf{X}})^T \iff \theta_{\mathbf{X}} = 1 - (1-\theta_{\mathbf{V}})^{1/T}.$$

Sub-sampling may thus improve the performance of an EI-estimator. After the implementation of different subsampling algorithms, we here advance with the following simple algorithm.

Algorithm 4.1.

Fix T (possibly $T=2$), and compute $r = \lfloor n/T \rfloor$;

S1: *Consider, for $i = 1,\ldots,T$, $\mathbf{V}_i = \left(X_i, X_{T+i}, \cdots, X_{(r-1)T+i}\right)$, the T subsamples of size r, and compute the estimates $\hat{\theta}_{\mathbf{V}_i}^{\mathrm{GJ}}(j)$, $j = 1,2,\ldots,r-1$;*

S2: *Compute*

$$\hat{\theta}_{sub|T}^{\mathrm{GJ}}(k) = 1 - \frac{1}{T}\sum_{i=1}^{T}\left(1-\hat{\theta}_{\mathbf{V}_i}^{\mathrm{GJ}}(j)\right)^{1/T},$$

for thresholds $k = (j-1)T+1,\ldots,j\ T$, $j = 1,2,\ldots,r-1$.

The use of the previous algorithm in $\hat{\theta}^{\mathrm{GJ}}$, in (17), with $\delta = 1/4$, enable us to achieve, at optimal levels, a MSE smaller than that of $\hat{\theta}^{\mathrm{GJ}}$, even for small values of θ, the most problematic ones, i.e., the ones for which the GJ EI-estimator had not been able to overpass the original estimator, regarding MSE at optimal levels. For small θ (here illustrated with $\theta = 0.2$), we are able to overpass the original estimator at optimal levels, when we consider the GJ statistic with $\delta = 1/4$, defined in (18), together with the use of subsampling techniques with $T = 2$ or 3.

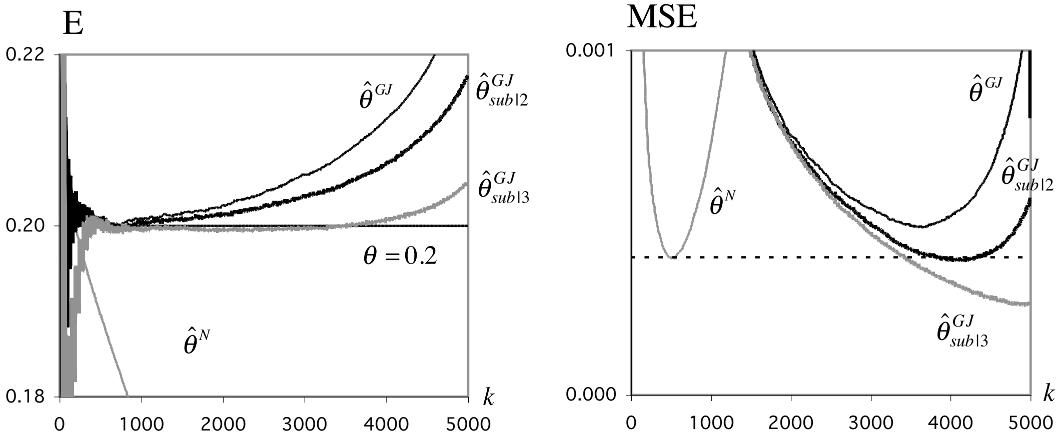


Figure 6. Behaviour of the GJ statistic with $\delta = 1/4$, together with the use of subsampling techniques with $T = 2$ or 3

5. Case studies

5.1. **The GJ EVI-estimation applied to insurance data.** We next consider an illustration of the performance of the estimators under study, through the analysis of automobile claim amounts exceeding 1,200,000 Euro over the period 1988-2001, gathered from several European insurance companies co-operating with the same re-insurer (Secura Belgian Re), with a size $n = 371$. This data set was already studied in Beirlant *et al.* (2004; 2008) and Vandewalle and Beirlant (2006), as an example to excess-of-loss reinsurance rating and heavy-tailed distributions in car insurance. It is clear from the box-and-whiskers plot in Figure 7 that data have been left-censored and that the right tail of the underlying model is quite heavy.

Regarding the EVI-estimation, note that whereas the Hill EVI-estimator is unbiased for the estimation of ξ when the underlying model is a strict Pareto model, it always exhibits a relevant bias when we have only Pareto-like tails, as happens here and can be seen in Figure 8.

The corrected-bias estimators, which are 'asymptotically unbiased', have a smaller bias, exhibit more stable sample paths as functions of k, and enable us to take a decision upon the estimate of ξ to be used, even with the help of any heuristic stability criterion, like the 'largest run' method suggested in Gomes and Figueiredo (2006). A bootstrap algorithm, not detailed here, but fully sketched in Gomes *et al.* (2012), helps us to provide an adaptive choice for corrected-bias EVI-estimators. We have got $\hat{k}_{0|\mathrm{H}} = 56$, $\hat{k}_{0|\overline{H}} = 158$,

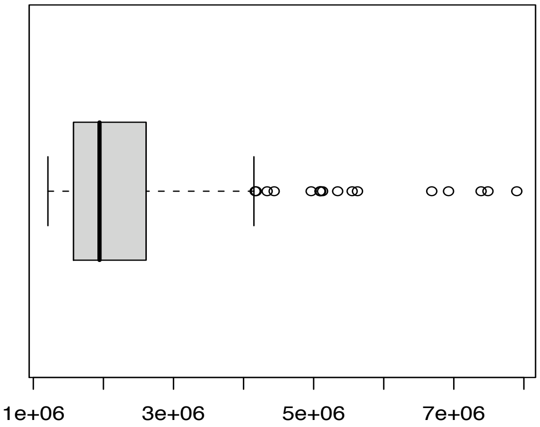


FIGURE 7. Box-and-whiskers plot associated with Secura data

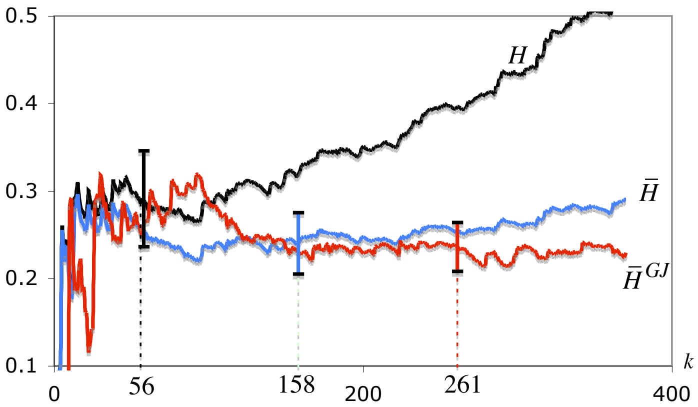


FIGURE 8. Estimates and 95% bootstrap confidence intervals of the extreme value index ξ for the Secura Belgian Re data

$\hat{k}_{0|\overline{\mathrm{H}}^{\mathrm{GJ}}} = 261$, and the EVI-estimates $\mathrm{H}^* = 0.286$, $\overline{\mathrm{H}}^* = 0.240$ and $\overline{\mathrm{H}}^{\mathrm{GJ}^*} = 0.236$, the values pictured in Figure 8. The associated bootstrap 95% confidence intervals were $(0.236, 0.346)$, $(0.205, 0, 275)$ and $(0.208, 0.264)$, with sizes 0.110, 0.070 and 0.056, respectively for the Hill, the corrected-Hill and the generalized jackknife. Indeed, both bootstrap confidence intervals and asymptotic confidence intervals are easily associated with the estimates presented, the smallest size (with a high coverage probability) being related to $\overline{\mathrm{H}}^{\mathrm{GJ}^*}$.

5.2. **The GJ EI-estimation applied to financial data.** We now consider the performance of the above mentioned estimators in the analysis of Euro-UK Pound daily exchange rates from January 4, 1999 until December 14, 2004. Working with the $n_0 = 725$

positive log-returns, we picture as an illutration, and as already done in Gomes *et al.* (2008c), the sample paths of $\hat{\theta}^{\mathrm{N}}(k)$ and $\hat{\theta}^{\mathrm{GJ}}(k)$, as functions of k.

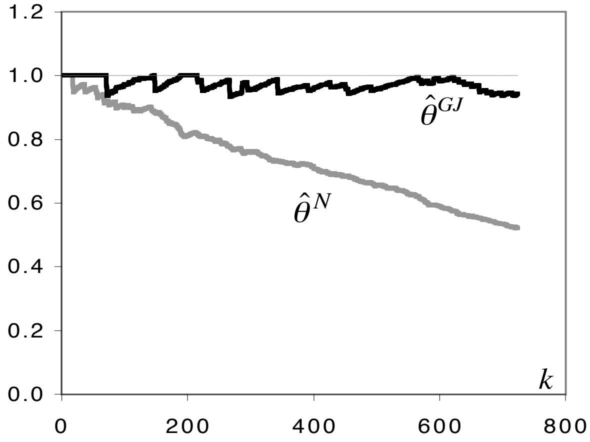


FIGURE 9. Sample paths of $\hat{\theta}^{\mathrm{N}}(k)$ and $\hat{\theta}^{\mathrm{GJ}}(k)$, as functions of k for the Euro-UK Pound log-returns under study

6. SOME OVERALL CONCLUSIONS

(1) The most attractive features of the GJ estimators are their stable sample paths (for a wide region of k values), close to the target value, and the 'bath-tube' MSE patterns.

(2) Regarding the EI-estimators, the choice $\delta = 1/4$ (heuristically based) in $\hat{\theta}_n^{\mathrm{GJ}(\delta)}(k)$, defined in (17), provides sample paths with a high stability. However reduction of MSE at optimal levels, relative to the original $\hat{\theta}_n^{\mathrm{N}}$ is not always achieved. Such an objective can be attained only with the extra use of a subsampling algorithm. Further investment is thus welcome.

(3) Again: the insensitivity of the mean value and sample path to changes in k is indeed the nicest feature of these GJ-estimators.

(4) And the tail bootstrap has revealed to be of high importance in the choice of the optimal threshold, in the sense of minimal MSE.

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