Dynamics and Geometry on Homogeneous Spaces
An Introduction

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Contents

we start by defining Riemann surfaces, holomorphic and meromorphic functions on them and also holomorphic maps between two Riemann surfaces.

Riemann surface :
- two dimensional real manifolds
- additional structure that we will define next

n-dimensional manifold = Hausdorff topological space X such that every point $a \in X$ has an open neighbourhood which is homeomorphic to an open neighbourhood of $\mathbb{R}^n$

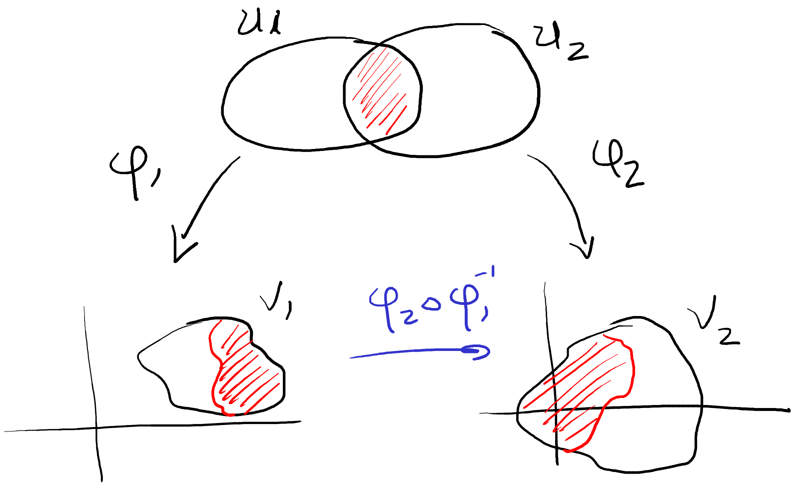
Def:
Let X be a two-dimensional real manifold. A complex chart on X is a homeomorphism

$$\varphi : U \to V \qquad \left(\begin{array}{c} U \subseteq X \\ V \subseteq \mathbb{C} \end{array} \right)$$

of an open subset U of X onto an open subset of $\mathbb{C}$.
Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$, are said to be holomorphically compatible if the map

$$\varphi_2 \circ \varphi_1^{-1} : \underset{\substack{\cap \\ \mathbb{C}}}{\varphi_1(U_1 \cap U_2)} \to \underset{\substack{\cap \\ \mathbb{C}}}{\varphi_2(U_1 \cap U_2)}$$

is biholomorphic.



Def:
Let X be a two-dimensional real manifold. A complex atlas on X is a sytem

$$\mathcal{A} = \{ (U_i, \varphi_i : U_i \to V_i) : i \in I \}$$

of charts which are holomorphically compatible and which cover X, i.e $\bigcup_{i \in I} U_i = X$

Two complex atlases $\mathcal{A}$ and $\mathcal{A}'$ on X are called <u>analytically equivalent</u> if every chart of $\mathcal{A}$ is holomorphically compatible with every chart of $\mathcal{A}'$.

<u>Rem</u>:

1. $\left.\begin{array}{l} \varphi: U \to V \text{ complex chart on } X \\ U_1 \subseteq U, \quad U_i \text{ open} \end{array}\right\} \Rightarrow \varphi|_{U_1}$ is a complex chart on X that is holomorphically compatible with φ

2. Since the compostion of biholomorphic maps is again biholomorphic, the notion of analytic equivalence of complex atlases is an equivalence relation

$\rightsquigarrow$

<u>Def</u>:

A complex structure on a two-dimensional real manifold X is an equivalence class of analytically equivalent atlases on X.

Every complex struture Σ on X contains a unique maximal atlas $\mathcal{A}^*$

$\rightsquigarrow$

If $\mathcal{A}$ is an arbitrary atlas in Σ, then $\mathcal{A}^*$ consits of all complex charts on X which are holomorphically compatible with every chart of $\mathcal{A}$.

Def:

A Riemann surface is a pair (X, Σ), where X is a connected two-dimensional real manifold and Σ is a complex structure on X.

usually we will write X instead of (X, Σ)
(if it is clear which is Σ)

Rmk:

Locally, a Riemann surface X is nothing but an open subset of the complex plane $\mathbb{C}$

$$\varphi : \mathcal{U} (\subseteq X) \longrightarrow V (\subseteq \mathbb{C}) \quad \text{bijection}$$

$\Rightarrow$ we may carry over Riemann surfaces the notions from complex analysis in the plane that remain invariant under biholomorphic mappings

i.e. notions not depending on the choice of a particular chart.

Examples:

$\mathbb{R}^2 \simeq \mathbb{C}$
$(x, y) \quad z = x + iy$

1. The complex plane $\mathbb{C}$

we have that $\mathbb{C}$ is a two-dimensional real manifold and the complex structure on $\mathbb{C}$ is the one whose atlases contain a unique chart given by the identity map

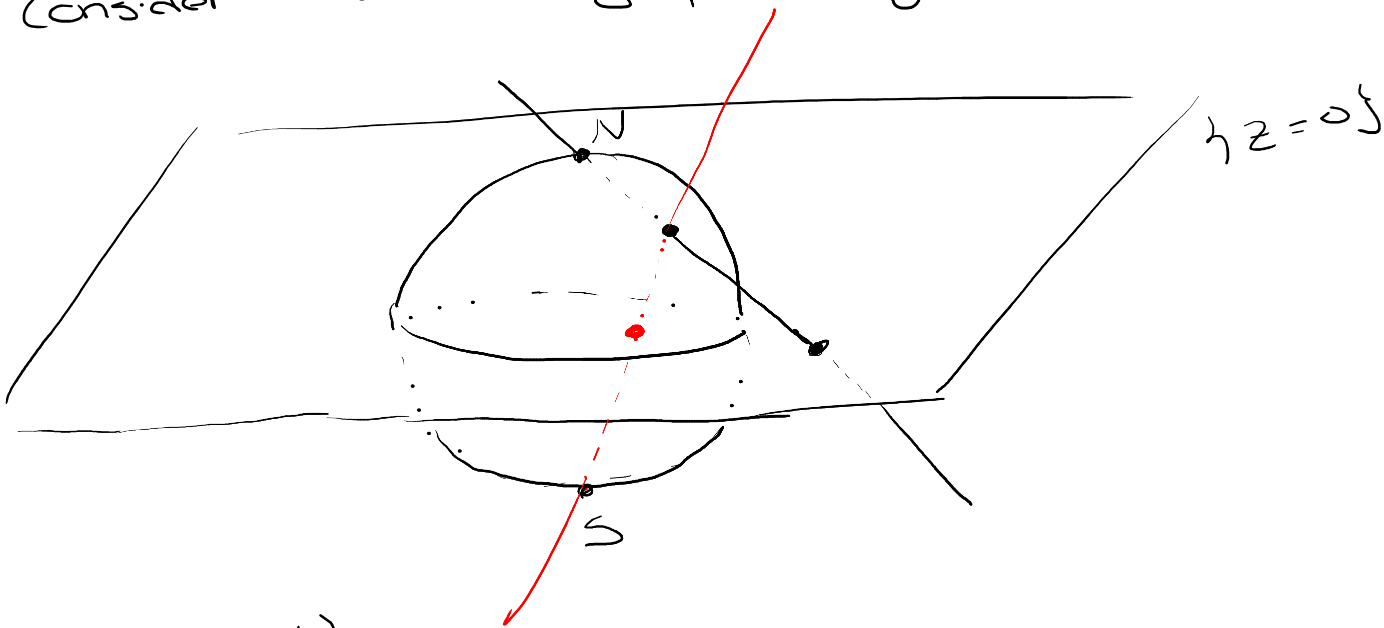
$$\mathcal{A} = \{ (\mathbb{C}, \varphi : \mathbb{C} \longrightarrow \mathbb{C}) \}$$
$$z \longmapsto z$$

2. <u>Open subsets of $\mathbb{C}$</u>

3. <u>The 2-sphere S^2</u>

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$$

Consider the stereographic projection



$N = (0,0,1)$

$S = (0,0,-1)$

we will cover S^2 by the following two open sets

$\mathcal{U}_1 = S^2 \setminus \{(0,0,1)\}$ and $\mathcal{U}_2 = S^2 \setminus \{(0,0,-1)\}$

$(\mathcal{U}_1 \cup \mathcal{U}_2 = S^2)$

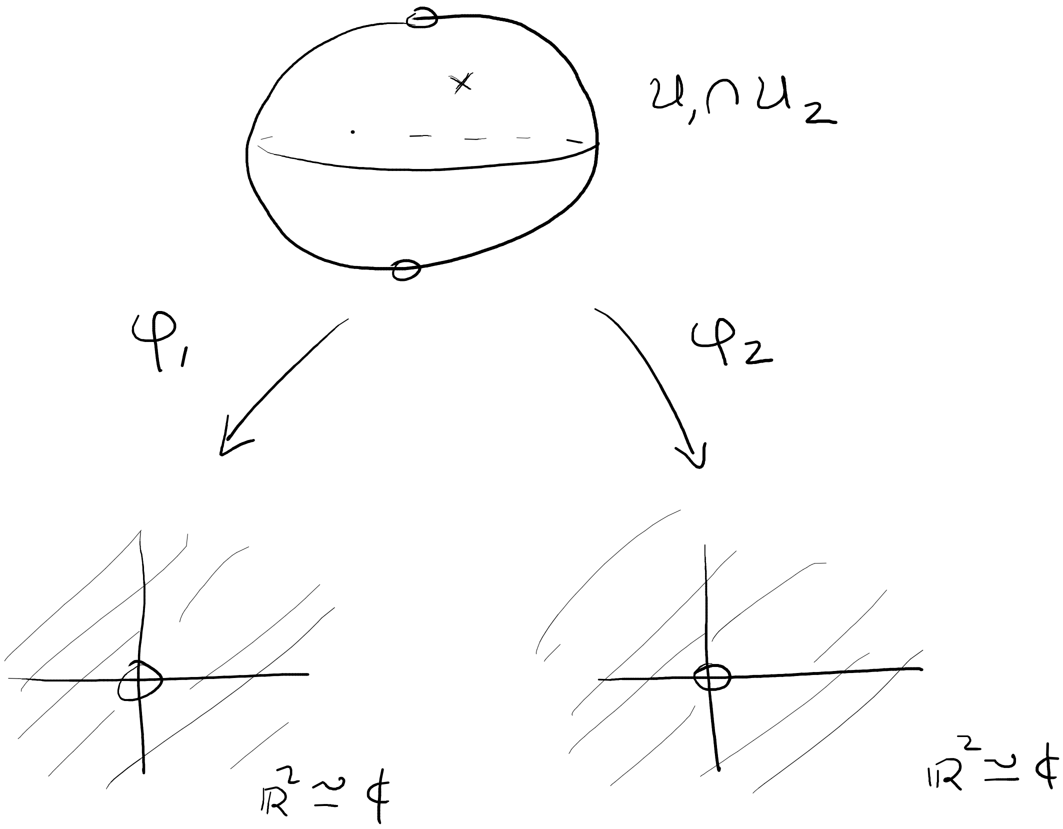
and we will consider the following two charts on these open sets:

$$\varphi_1 : \mathcal{U}_1 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

$$\varphi_2 : \mathcal{U}_2 \longrightarrow \mathbb{R}^2$$
$$(x,y,z) \longmapsto \left(\frac{x}{1+z}, -\frac{y}{1+z} \right)$$

ξ

change of sign $\rightsquigarrow$ related with the conjugation on the complex plane
$\mathbb{C} \simeq \mathbb{R}^2$



$z \simeq (X, Y)$
$z = X + iY$

$\tilde{z} \simeq (\tilde{X}, \tilde{Y})$
$\tilde{z} = \tilde{X} + i\tilde{Y}$

$$(X+iY).(\tilde{X}+i\tilde{Y}) = \left(\frac{x}{1-z} + i\frac{y}{1-z}\right)\cdot\left(\frac{x}{1+z} - i\frac{y}{1+z}\right)$$

$$= \frac{x^2+y^2}{1-z^2}$$

$$= 1$$

$x^2+y^2+z^2=1$
$\Rightarrow x^2+y^2 = 1-z^2$

$$z.\tilde{z} = 1 \implies \tilde{z} = \frac{1}{z}$$

Then we have

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$z \longmapsto \frac{1}{z}$$

Since $\varphi_2 \circ \varphi_1^{-1}$ is (clearly) biholomorphic, we have presented a complex structure on S^2. So S^2 is a Riemann surface.

4. <u>The projective line $\mathbb{C}\mathbb{P}(1)$</u>

$\mathbb{C}\mathbb{P}(1) = \mathbb{C} \cup \{\infty\}$

Let us equipp $\mathbb{C}\mathbb{P}(1)$ with the following topology:
the open sets are

- the usual open sets $U \subseteq \mathbb{C}$ <u>and</u>
- $V \cup \{\infty\}$ where $V \subseteq \mathbb{C}$ is the complement of a compact set $K \subseteq \mathbb{C}$

with this topology, $\mathbb{C}\mathbb{P}(1)$ is a compact Hausdorff topological space homeomorphic to the 2-sphere S^2.

Set

$$\mathcal{U}_1 = \mathbb{C}\mathbb{P}(1) \setminus \{\infty\} = \mathbb{C}$$
$$\mathcal{U}_2 = \mathbb{C}\mathbb{P}(1) \setminus \{0\} = \mathbb{C}^* \cup \{\infty\}$$

on these sets we define the following charts

$$\varphi_1 : \mathcal{U}_1 \longrightarrow \mathbb{C}$$
$$z \longmapsto z$$

$$\varphi_2 : \mathcal{U}_2 \longrightarrow \mathbb{C}$$
$$z \longmapsto \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C}^* \\ 0 & \text{if } z = \infty \end{cases}$$

Since

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$z \longmapsto \frac{1}{z}$$

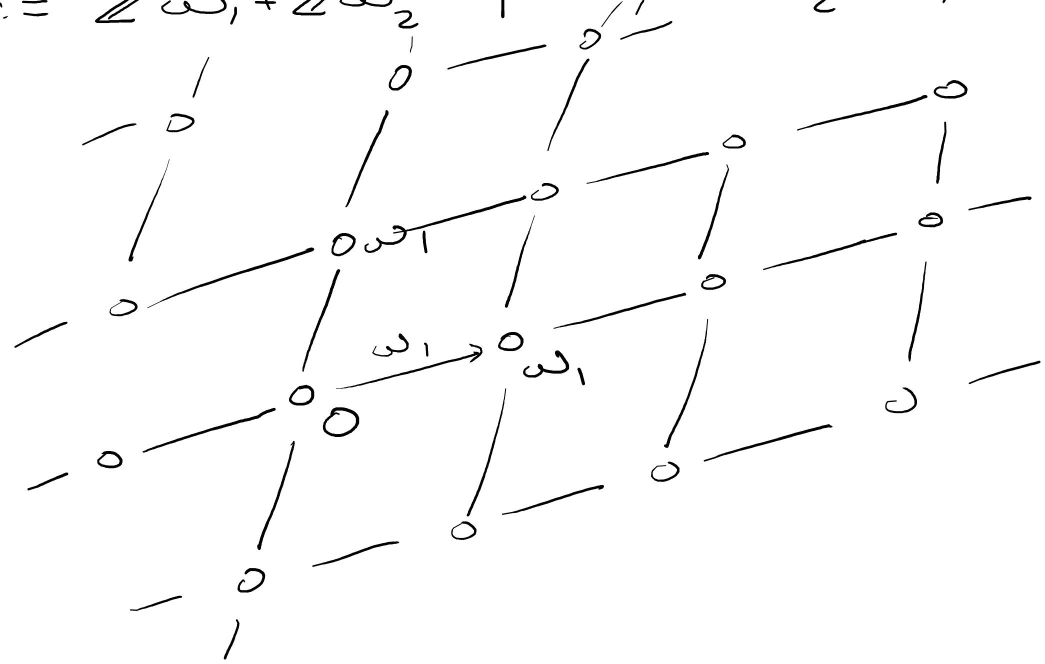
This is biholomorphic. Thus $\mathbb{C}P(1)$ is a Riemann surface.

5. The tori

$\mathbb{R}^2$ $\quad \mu_1 = (\omega_1, y_1)$ $\quad \mu_2 = (\omega_2, y_2)$ $\quad$ L.I

Consider $\omega_1, \omega_2 \in \mathbb{C}$ that are linearly independent over $\mathbb{R}$. Let the

$$\Pi := \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \{ n\omega_1 + m\omega_2 : n, m \in \mathbb{Z} \}$$



Π is called the lattice spanned by ω_1 and ω_2

$z, z' \in \mathbb{C}$

$z \sim z' \iff z - z' \in \Pi$

$\sim$ is an equivalence relation on $\mathbb{C}$
their equivalence classes are denoted by $\mathbb{C}/\Pi$

canonical projection $\longrightarrow$ $\pi : \mathbb{C} \longrightarrow \mathbb{C}/_{\Pi}$

$z \longmapsto [z]$ $\leftarrow$ equivalence class of z

Let us introduce on $\mathbb{C}/_{\Pi}$ the following topology

$$\mathcal{U} \subseteq \mathbb{C}/_{\Pi} \text{ open } \iff \pi^{-1}(\mathcal{U}) \subseteq \mathbb{C} \text{ is open}$$

with this topology, $\mathbb{C}/_{\Pi}$ is a Hausdorff topological space (and the quotient map $\pi : \mathbb{C} \to \mathbb{C}/_{\Pi}$ is continuous)

<u>Rmk</u>:

- $\mathbb{C}$ connected $\Rightarrow$ $\mathbb{C}/_{\Pi}$ connected

- $\mathbb{C}/_{\Pi}$ compact : it is covered by the image under π of the following compact subset of $\mathbb{C}$

  $K = \{ \lambda\omega_1 + \beta\omega_2 \, , \, \lambda, \beta \in [0,1] \}$

- π is an open map

  $\Updownarrow$

  the image of every open set $V \subseteq \mathbb{C}$ is open.

  $\Updownarrow$

  we have to check that $\hat{V} = \pi^{-1}(\pi(V))$ is open whenever V is open. We have

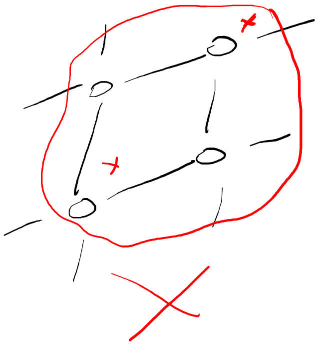
$$\hat{V} = \bigcup_{\omega \in \Pi} (\omega + V)$$

V open $\Longrightarrow$ $\omega + V$ open $\forall \omega \in \Pi$

$\Longrightarrow$ $\hat{V}$ open

The complex structure on $\mathbb{C}/\Pi$ is then defined as follows:

- Let V be an open set of $\mathbb{C}$ admitting no points equivalent under Π

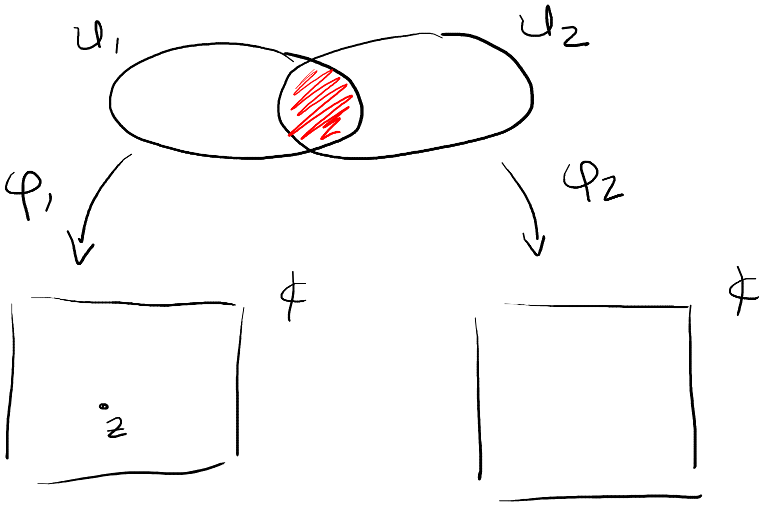


$\Longrightarrow$ $\mathcal{U} := \pi(V)$ is open

$\pi|_V$ homeomorphism

$\Longrightarrow$ $\varphi = \pi^{-1} : \mathcal{U} \to V$ is a complex chart on $\mathbb{C}/\Pi$

- Consider then the set of all charts obtained as before. We just to show that every two charts φ_1, φ_2 as above are holomorphically compatible.



$$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(\mathcal{U}_1 \cap \mathcal{U}_2) \to \varphi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$$

Recall : $\varphi_i = \pi^{-1}$

$\Longrightarrow \quad \forall z \in \varphi_1(U_1 \cap U_2)$, $\pi(\varphi_2 \circ \varphi_1^{-1}(z)) = \varphi_1^{-1}(z) = \pi(z)$

$\Longrightarrow \quad \varphi_2 \circ \varphi_1^{-1}(z) - z \in \Gamma$

$\Longrightarrow \quad \varphi_2 \circ \varphi_1^{-1}(z) - z \equiv cte$ on every connected component of $\varphi_1(U_1 \cap U_2)$

Γ is discrete
$\varphi_2 \circ \varphi_1^{-1}$ is continuous

$\Longrightarrow \quad \varphi_2 \circ \varphi_1^{-1}$ is holomorphic

The same argument allows us to say that $(\varphi_2 \circ \varphi_1^{-1})^{-1}$ is holomorphic, and so $\varphi_2 \circ \varphi_1^{-1}$ is biholomorphic so that φ_1 and φ_2 are holomorphically compatible.

<u>Remark</u>:

$$S^1 = \{ (x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \}$$

we have that $\mathbb{C}/\Gamma$ is homeomorphic to $S^1 \times S^1$

$$S^1 \simeq \{ z \in \mathbb{C} : |z| = 1 \}$$

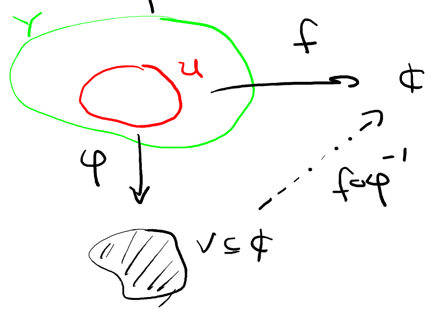
$$\lambda \omega_1 + \beta \omega_2 \in \mathbb{C}/\Gamma \longmapsto (e^{2\pi i \lambda}, e^{2\pi i \beta})$$

Definition:

Let X be a Riemann surface and $Y \subseteq X$ an open subset of X. A function $f: Y \to \mathbb{C}$ is said to be <u>holomorphic</u> if for every chart $\varphi: U \to V$ (where $U \subseteq Y$, $V \subseteq \mathbb{C}$) on Y, the function

$$f \circ \varphi^{-1} : V \to \mathbb{C}$$

is holomorphic in the usual sense.



The set of all holomorphic functions on Y will be denoted by $\mathcal{O}(Y)$.

Every chart $\varphi: U(\subseteq X) \to V(\subseteq X)$ is clearly holomorphic. we call φ a <u>local coordinate</u> on (U, φ). Usually it used z instead of φ.

$\mathcal{O}(Y)$ is $\mathbb{C}$-algebra : the sum and the product of holomorphic functions of Y is also holomorphic
the constant function is holomorphic.

An immediate consequence of the Riemann Removable Singularities Theorem on the complex plane is :

Thm:

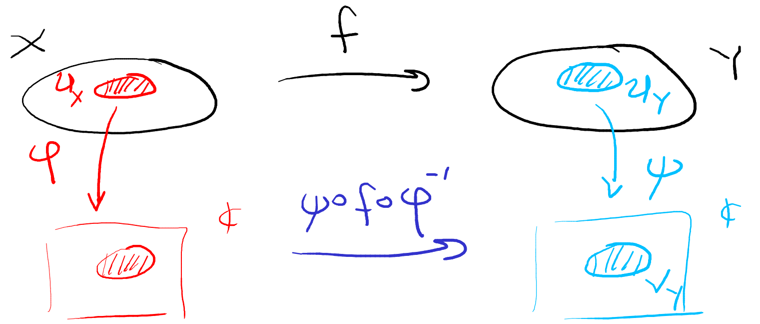
Let U be an open subset of a Riemann surface X and $p \in U$. Suppose that $f \in \mathcal{O}(U \setminus \{p\})$ is bounded in a neighbourhood of p. Then f can be extended to a <u>unique</u> function $\tilde{f} \in \mathcal{O}(U)$.

Def:

Let X and Y be two Riemann surfaces. A continuous mapping $f: X \to Y$ is said <u>holomorphic</u> if for every pair of charts in X on Y $(\varphi: U_X \to V_X$ and $\psi: U_Y \to V_Y)$ satisfying $f(U_X) \subseteq U_Y$, the mapping

$$\psi \circ f \circ \varphi^{-1} : V_X \to V_Y$$

is holomorphic in the usual sense $(V_X, V_Y \subseteq \mathbb{C})$
$(U_X \subseteq X, U_Y \subseteq Y)$



A map $f : X \to Y$ is called <u>biholomorphic</u> if it is injective and surjective and both $f : X \to Y$ and $f^{-1} : Y \to X$ are holomorphic.

Two Riemann surfaces X, Y are called isomorphic if there is a biholomorphism map $f : X \to Y$.

A biholomorphic map from a Riemann surface X to itself, $f : X \to X$, is said an <u>automorphism</u> of X.

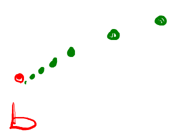
<u>Theorem</u>: (Identity Theorem)

Let X, Y be two (connected) Riemann surfaces and $f_1 : X \to Y$, $f_2 : X \to Y$ two holomorphic maps. Assume that f_1 and f_2 coincide on a set $\mathcal{U} \subseteq X$ having a limit point $q \in X$.
Then f_1 and f_2 coincide on all X.

<u>Proof</u>: (consequence of the identity theorem for holomorphic functions on $\mathbb{C}$)

Let
$$G = \{ p \in X : \exists\, \mathcal{U} \text{ neighbourhood of } : f_1|_{\mathcal{U}} \equiv f_2|_{\mathcal{U}} \}$$

G is clearly an open set. Let us check that G is also closed.. So, let $b \in X$ be a point in ∂G.



Since f_1 and f_2 are continuous, the follows that $f_1(b) = f_2(b)$.
Consider then charts

$$\varphi : \mathcal{U} \longrightarrow V \qquad b \in \mathcal{U} \subseteq X, \ V \subseteq \mathbb{C}$$
$$\psi : \mathcal{U}' \longrightarrow V' \qquad f(b) \in \mathcal{U}' \subseteq Y, \ V' \subseteq \mathbb{C}$$

such that $b \in \mathcal{U}$, $f_1(\mathcal{U}) \subseteq \mathcal{U}'$ and $\mathcal{U}$ is connected. The maps

$$\overline{f_i} := \psi \circ f_i \circ \varphi^{-1} : V \longrightarrow V' \qquad V, V' \subseteq \mathbb{C}$$

is (by definition) holomorphic and since $\varphi^{-1}(b)$ is an accumulation point of V of points where $\overline{f_1}$ and $\overline{f_2}$ coincide, the Identity theorem for holomorphic functions on domains in $\mathbb{C}$ implies that $\overline{f_1}$ and $\overline{f_2}$ coincide on V.
There follows that

$$f_1|_{\mathcal{U}} \equiv f_2|_{\mathcal{U}}$$

So G is closed.
Being X connected and G simultaneously open and closed, we have

$$G = \emptyset \quad \text{or} \quad G = X$$

However $q \in G$, so that $G = X$ and, consequently, $f_1 \equiv f_2$.

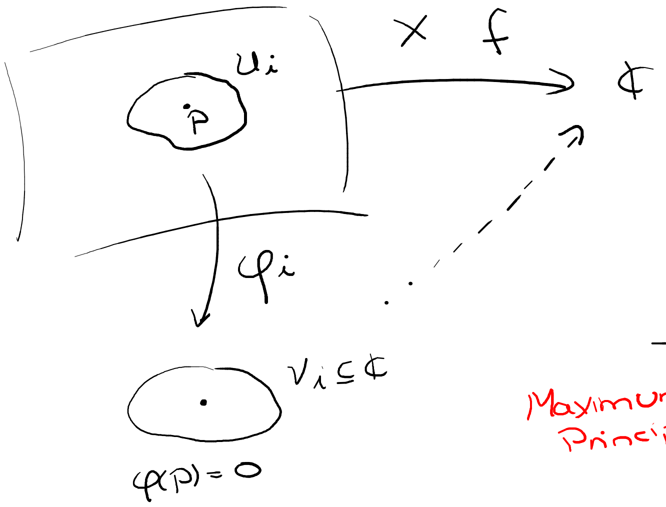
Consequence:

Thm:
Let X be a compact and connected Riemann surface. Then every holomorphic function $f : X \to \mathbb{C}$ must be constant.

Proof:
Consider the function $|f| : X \longrightarrow \mathbb{R}$. Since S is
$$z \longmapsto |f(z)|$$
compact

$$\exists\, p \in X : |f(p)| \text{ is maximum on } X$$

Consider then an atlas for X, $\mathcal{A} = \{(\mathcal{U}_i, \varphi_i)\}$ and let $i : p \in \mathcal{U}_i$



Consider the map

$f \circ \varphi^{-1} : V_i \subseteq \mathbb{C} \longrightarrow \mathbb{C}$

which is, by definition, holomorphic

$\overset{\text{Maximum Modulus Principle}}{\Longrightarrow}$ $f \circ \varphi^{-1}$ is constant on V_i

$\Longrightarrow$ f is constant on U_i

The previous Theorem ensures that f is constant on X.

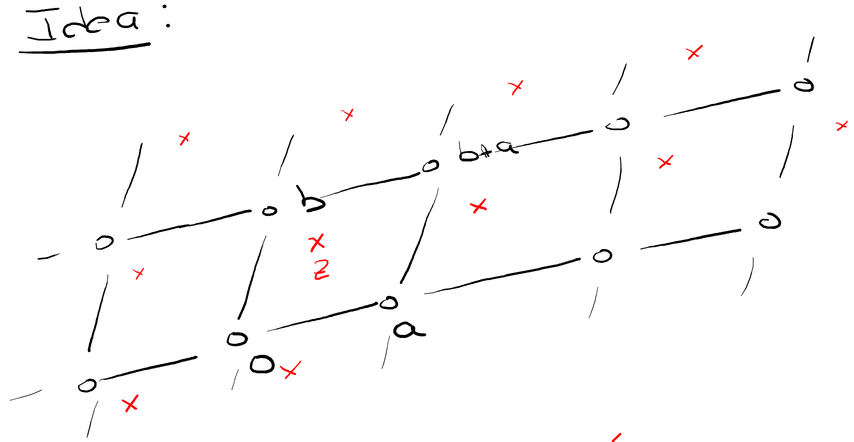
Example:

we have that $\mathbb{C}P(1)$ admit no non-constant holomorphic functions.

Exercise:

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function that is 2-periodic in the following sense. There exist $a, b \in \mathbb{C}^*$ ($b/a \notin \mathbb{R}$) such $f(z) = f(z+a) = f(z+b)$ for every $z \in \mathbb{C}$. Then f must be constant.

Idea:



$\Pi = \{ \lambda a + \beta b : \lambda, \beta \in \mathbb{Z} \}$

f defined on $\mathbb{C}$ has natural projection $\bar{f}$ on $\mathbb{C}/\Pi$

$\bar{f}$ is then a holomorphic function on $\mathbb{C}/\Pi$

$\Longrightarrow$ $\bar{f}$ is constant

$\mathbb{C}/\Pi$ is compact

$\Longrightarrow$ f is constant

Def:

Let X be a Riemann surface. By a meromorphic function on X we mean a holomorphic function $f : X' \to \mathbb{C}$, where X' is an open subset of X satisfying the following conditions

(i) $X \setminus X'$ constains only isolated points

(ii) $\forall p \in X \setminus X'$ we have $\lim\limits_{\omega \to p} |f(\omega)| = \infty$

The points of $X \setminus X'$ are called the poles of f and the set of meromorphic functions on X is denoted by $\mathcal{M}(X)$.

Def:

Let X be a Riemann surface and $p \in X$. Let $f : X \setminus \{p\} \to \mathbb{C}$ be a holomorphic function.. We say that p is an essential singularity if p is neither a pole nor a removable singularity (i.e. if f does not admit a holomorphic extension to p)

Consider a polynomial function on $\mathbb{C}$

$$f : \mathbb{C} \to \mathbb{C}$$
$$z \mapsto z^n + a_1 z^{n-1} + \dots + a_n \quad , \; a_n \in \mathbb{C}$$

$$z^n\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_n}{z^n}\right)$$

- $f \in \mathcal{O}(\mathbb{C})$
- $\lim\limits_{z \to \infty} |f(z)| = \infty \quad \Longrightarrow \quad f \in \mathcal{M}(\mathbb{C}\mathbb{P}(1))$
- $\bar{f}$ is a holomorphic function from $\mathbb{C}\mathbb{P}(1)$ to $\mathbb{C}\mathbb{P}(1)$

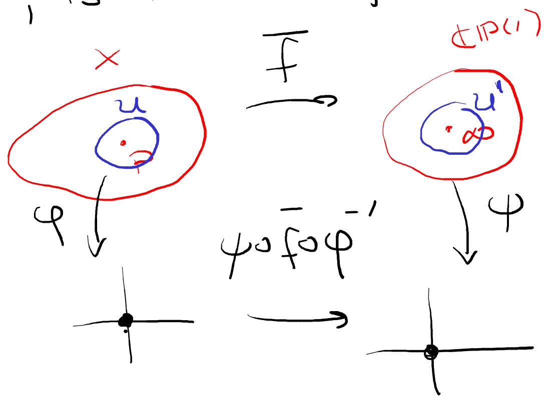
$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f"] \arrow[d, phantom, "\cap" rotate=-90] & \mathbb{C} \arrow[d, phantom, "\cap" rotate=-90] \\
\mathbb{C}\mathbb{P}(1) \arrow[r, "\bar{f}"] & \mathbb{C}\mathbb{P}(1)
\end{tikzcd}$$

Thm:

Let X be a Riemann surface and $f \in M(X)$. Denote by $\mathcal{P}$ the set of poles of f. Consider then the func. $\bar{f} : X \to \mathbb{C}P(1)$ defined as

$$\bar{f}(p) = \begin{cases} f(p) & , \text{ if } p \in X \setminus \mathcal{P} \\ \infty & , \text{ otherwise} \end{cases}$$

Then $\bar{f} : X \to \mathbb{C}P(1)$ is a holomorphic map. Conversely, if $\bar{f} : X \to \mathbb{C}P(1)$ is holomorphic map, then $\bar{f}$ is either identically equal to ∞ or $\bar{f}^{-1}(\infty)$ consists of isolated points and $f : X \setminus \bar{f}^{-1}(\infty) \to \mathbb{C}$ defined as $f(p) = \bar{f}(p)$ $\forall p \in X \setminus \bar{f}^{-1}(\infty)$, is a meromorphic function on X.



Proof:

$f \in M(X)$

$\mathcal{P} (\subseteq X)$ set of poles of f

$\bar{f}$ as in the statment of the theorem

- $\bar{f}$ is clearly continuos

Fix $p \in \mathcal{P}$, consider a chart $\varphi : \mathcal{U} \to V$, where $p \in \mathcal{U} \subseteq X$, $V \subseteq \mathbb{C}$. Consider also a chart $\psi : \mathcal{U}' \to V'$ $\boxed{\text{for } \mathbb{C}P(1)}$ where $\infty \in \mathcal{U}' \subseteq \mathbb{C}P(1)$, $V' \subseteq \mathbb{C}$. Assume also $\bar{f}(\mathcal{U}) \subseteq \mathcal{U}'$. For simplification, assume

$$\varphi(p) = 0 \quad \text{and} \quad \psi(\infty) = 0$$

we have to show that

$$g : \psi \circ \bar{f} \circ \varphi^{-1} : V \to V'$$

is holomorphic (for every charts as above)

- $g(\mathbb{D}\setminus\{0\})$ is bounded
- $\lim_{z \to 0} g(z) = 0$

$\Rightarrow \bar{f}$ is holomorphic by the Riemann Removable Singularity theorem.

The converse follows from the identity theorem.

<u>Examples</u>:

1. $P(z) = z^3 - 3z^2 + 3$

$P : \mathbb{C} \to \mathbb{C}$ holomorphic

2. $F : \mathbb{C}\setminus\{1, i, -i\} \to \mathbb{C}$

$$F(z) = \frac{z^2+2}{z^3-1} \qquad (z-1)(z^2+1)$$

F is holomorphic on $\mathbb{C}\setminus\{1, i, -i\}$

F admits a holomorphic extention viewed as a holomorphic function from $\mathbb{C}$ to $\mathbb{C}P(1)$

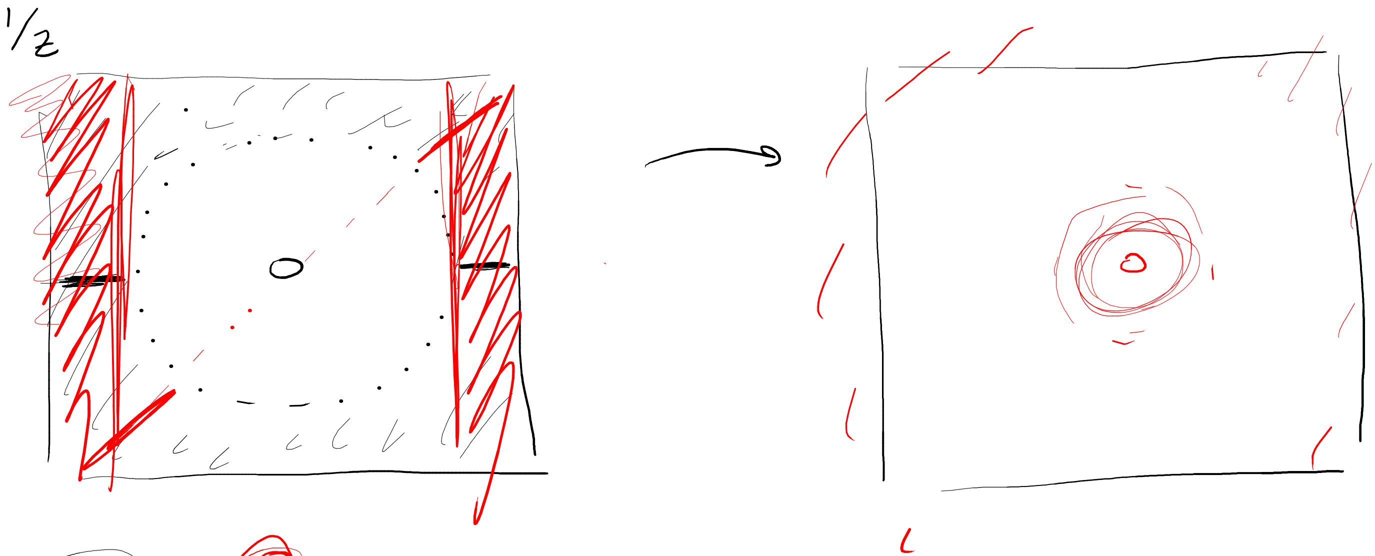
$$\lim_{\substack{z \to 1 \\ i \\ -i}} |F(z)| = \infty \qquad \text{poles : } 1, i, -i$$

3. $F : \mathbb{C}^* \longrightarrow \mathbb{C}$

$F(z) = e^{1/z}$

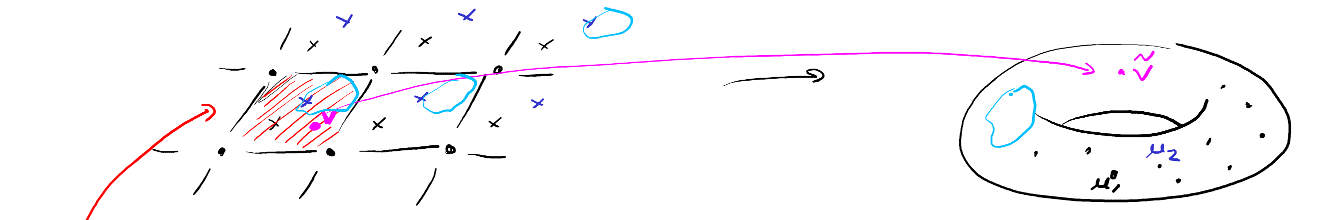
F is holomorphic on $\mathbb{C}^*$

0 is an essential singularity for F



$e^{a+ib} = e^a(\cos b + i \operatorname{sen} b)$

$z \to 0 \implies \frac{1}{z} \to \infty$



$\{\mu_n\}$

$z \sim z' \iff z' - z \in \Gamma$

$\exists\, k_n : \mu_{k_n} \longrightarrow \tilde{v}$

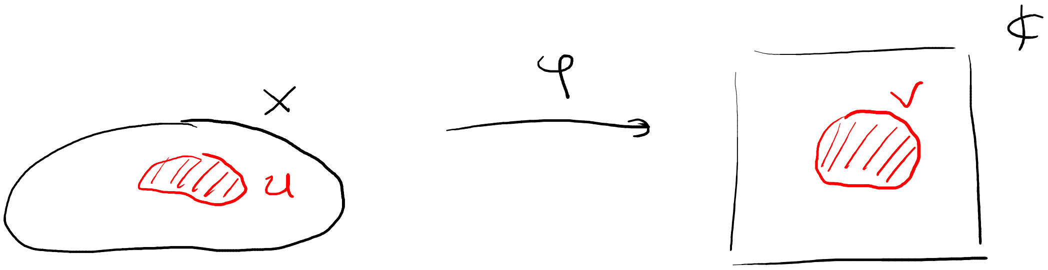
$\Lambda = \underbrace{\{\lambda\omega_1 + \beta\omega_2 : \lambda, \beta \in [0,1]\}}_{\text{compact on } \mathbb{C}}$

$\left.\begin{array}{l}\{v_n\} \subseteq \Lambda \\ \Lambda \text{ compact}\end{array}\right\} \Longrightarrow \{v_n\}$ has a convergente subsequence

v ← limite

φ complex chart $\Longrightarrow$ φ is holomorphic

$\varphi : \mathcal{U}(\subseteq X) \longrightarrow V(\subseteq \mathbb{C})$



$S = \{\psi : \mathcal{U}_i \to V_i\}$ complex charts such that $\mathcal{U}_i \cap \mathcal{U} \neq \emptyset$

φ is holomorphic $\overset{\text{def}}{\iff}$ $\varphi \circ \psi^{-1}$ is holomorphic in the usual sense

φ is a chart : $\varphi \circ \varphi^{-1} = \mathrm{id}$ that is clearly holomorphic

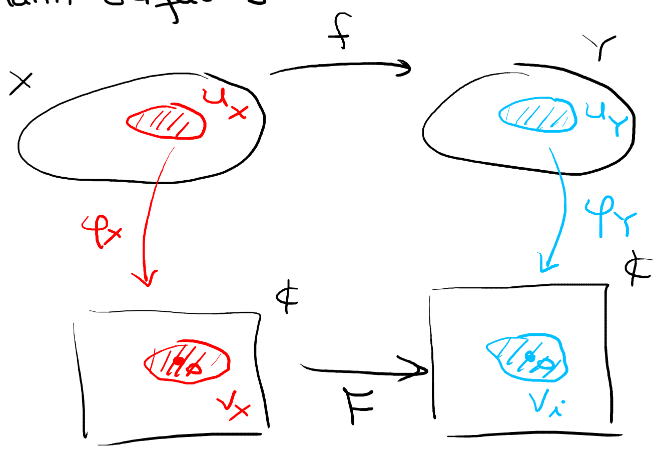
$\underbrace{\varphi \circ \psi^{-1}}$ with $\psi \in S$

it is holomorphic by the compatibility condition of a complex atlas

$f : X \longrightarrow Y$ $\qquad$ X, Y Riemann surfaces

$\varphi_X : \mathcal{U}_X \longrightarrow V_X \qquad \mathcal{A}_X = \{(\mathcal{U}_X, \varphi_X)\}$

$\varphi_Y : \mathcal{U}_Y \longrightarrow V_Y \qquad \mathcal{A}_Y = \{(\mathcal{U}_Y, \varphi_Y)\}$



$F = \varphi_Y \circ f \circ \varphi_X^{-1}$

$F(0) = 0$

- F holomorphic at 0

  $\Longrightarrow$ Laurent series : $\quad F(z) = \sum_{j \geq k} a_j z^j \quad , \quad k \in \mathbb{N}$

- F meromorphic at 0

  $\Longrightarrow$ Laurent series : $\quad F(z) = \sum_{j \geq -k} a_j z^j \quad k \in \mathbb{N}$

  $a_{-k} \neq 0 \Longrightarrow k$ is called <u>the order of the pole</u>

- F has an essential singularity at 0

  $\Longrightarrow$ Laurent series : $\quad F(z) = \sum_{-\infty}^{+\infty} a_j z^j$

  <u>and</u> $\forall k \in \mathbb{Z} \quad \exists j < k : a_j \neq 0$

<u>In the case of the previous examples</u>

1. $P(z) = 3 - 3z^2 + z^3$ holomorphic ($a_j = 0, \forall j<0$)

2. $F(z) = \dfrac{z^2+1}{z^3-1}$ Singular at $z = 1$

$\omega = z - 1$

$$\tilde{F}(\omega) = \frac{(\omega+1)^2+1}{(\omega+1)^3-1} = \frac{\omega^2+2\omega+2}{\omega^3+3\omega^2+3\omega}$$

$$= \frac{\omega^2+2\omega+2}{\omega(3+3\omega+\omega^2)}$$

$$= \frac{1}{\omega}(2+2\omega+\omega^2)(3+3\omega+\omega^2)^{-1}$$

$$= \frac{1}{\omega}(2+2\omega+\omega^2)\left(\frac{1}{3} - \frac{1}{3}\omega + \frac{2}{9}\omega^2 - \frac{1}{9}\omega^3 + \dots\right)$$

Pole of order 1

3. $G(z) = e^z$ holomorphic on $\mathbb{C}$

L.S.E at $0 \in \mathbb{C}$

$$G(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$F(z) = e^{1/z} = G\left(\frac{1}{z}\right)$$

$$= 1 + \frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \frac{1}{3!}\cdot\frac{1}{z^3} + \frac{1}{4!}\cdot\frac{1}{z^4} + \dots$$

$\forall N \in \mathbb{N} \ \exists k < N : a_k \neq 0$

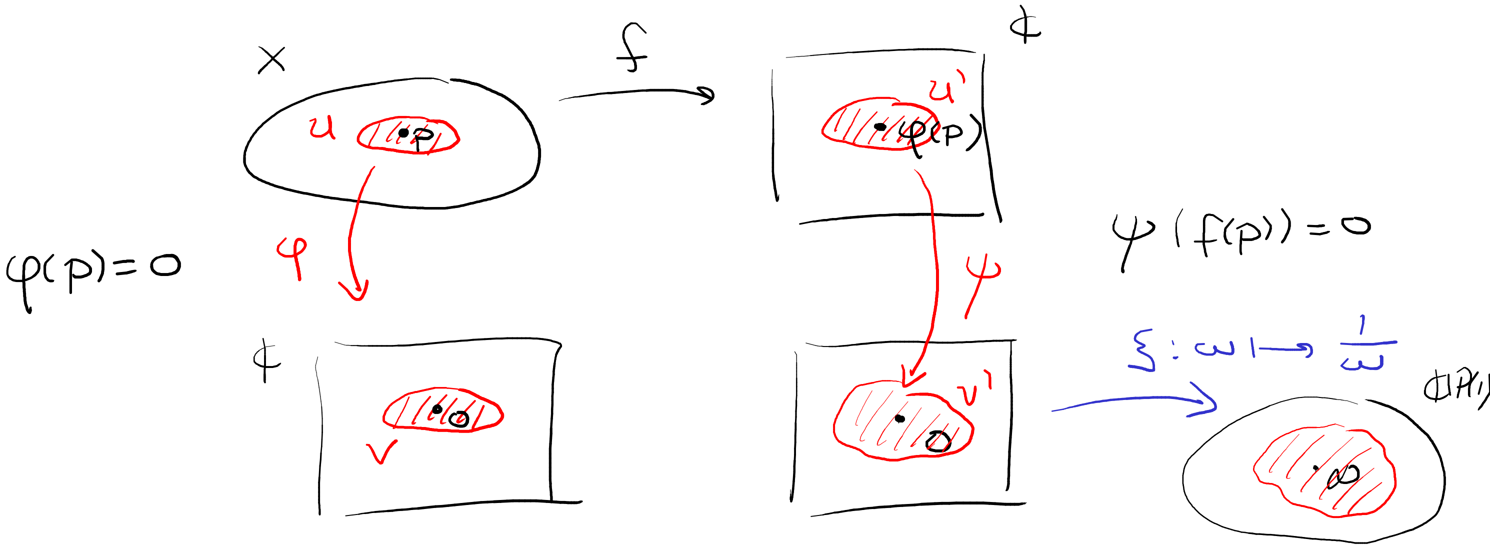
$\Rightarrow 0$ is an essential singularity of F.

Rmk (with respect to the previous Thm)

f meromorphic with a pole at $\boxed{p \in X}$ $(f: X \to \mathbb{C})$

Consider charts $\varphi: U \to V$ $\quad U \subseteq X$, $V \subseteq \mathbb{C}$

$\psi: U' \to V'$ $\quad U' \subseteq \mathbb{C}$, $V' \subseteq \mathbb{C}$



$$\Rightarrow \exists k \in \mathbb{N} : (\psi \circ f \circ \varphi^{-1})(z) = \frac{a_0}{z^k} + \frac{a_1}{z^{k-1}} + \frac{a_2}{z^{k-2}} + \dots$$

with $a_0 \neq 0$

$$= \frac{1}{z^k}\left(a_0 + a_1 z + a_2 z^2 + \dots\right)$$

pole of order k at 0

Consider on V' the map $\xi : \omega \mapsto \frac{1}{\omega}$.

$$(\xi \circ \psi \circ f \circ \varphi^{-1})(z) = \xi\left(\frac{1}{z^k}(a_0 + a_1 z + a_2 z^2 + \dots)\right)$$

$$= \underbrace{z^k \left(a_0 + a_1 z + a_2 z^2 + \dots\right)^{-1}}_{\text{holomorphic nearby } z=0 \text{ since } a_0 \neq 0}$$

Exercise:

Show that meromorphic functions on $\mathbb{CP}(1)$ are rational functions (i.e. quotient of polynomial function).

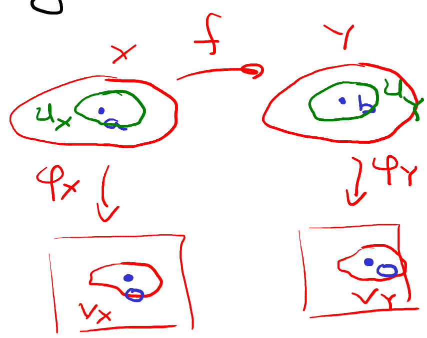
Local behaviour of holomorphic mappings

X, Y Riemann surfaces

$f : X \to Y$ non-constant holomorphic map

Thm:

Let $a \in X$ and $b = f(a)$. Then, there exist $k \geq 1$ $(k \in \mathbb{N})$ and charts $\varphi_X : \mathcal{U}_X \to V_X$ and $\varphi_Y : \mathcal{U}_Y \to V_X$ (with $\mathcal{U}_X \subseteq X$, $\mathcal{U}_Y \subseteq Y$, $V_X, V_Y \subseteq \mathbb{C}$) satisfying the following:



i) $a \in \mathcal{U}_X$, $\varphi_X(a) = 0$

$b \in \mathcal{U}_Y$, $\varphi_Y(b) = 0$

ii) $f(\mathcal{U}_X) \subseteq \mathcal{U}_Y$

iii) $F = \varphi_Y \circ f \circ \varphi_X^{-1} : V_X \to V_Y$ takes on the form

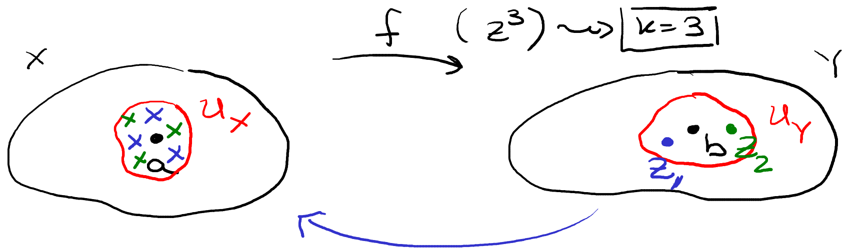
$$F(z) = z^k$$

Interpretation:

The value of k corresponds to the following: for every neighbourhood $\mathcal{U}$ of a that we fix, there exist neighbourhoods $\mathcal{U}_X$ of a and neighbourhoods $\mathcal{U}_Y$ of $b = f(a)$ such that

$$\# f^{-1}(z) \cap \mathcal{U}_X = k$$

for every $z \in \mathcal{U}_Y$, $z \neq b$.



k is called the <u>multiplicity</u> of f at a.

$(k = 1 \rightsquigarrow f$ locally injective$)$

<u>Picard Theorem</u> $\Rightarrow$ f has an essential singularity at a

$f : X \longrightarrow \mathbb{C}$

Then, for all $w \in \mathbb{C}$ except perhaps one value, the equation $f(z) = w$ has <u>infinitly many solutions</u> in $U \subseteq X$

<u>Calculation of the automorphism groups of some Riemann surfaces</u>

1. $\quad \mathrm{Aut}(\mathbb{C}) = \{ z \longmapsto az + b \ : \ a, b \in \mathbb{C},\ a \neq 0 \}$

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an automorphism of $\mathbb{C}$. Let then $f^* : \mathbb{C}^* \longrightarrow \mathbb{C}$ be the function defined as

$$f^*(z) = f\left(\frac{1}{z}\right)$$

1) f^* is also injective

In fact:
$$f^*(z_1) = f^*(z_2)$$
$$\Rightarrow f\left(\frac{1}{z_1}\right) = f\left(\frac{1}{z_2}\right)$$
$$\overset{f \text{ injective}}{\Longrightarrow} \frac{1}{z_1} = \frac{1}{z_2}$$
$$\Rightarrow z_1 = z_2$$

(2') f^* is clearly holomorphic on $\mathbb{C}^*$

It then follows that 0 is not an essential singularity of f^* (recall the interpretation of essential singularity). More precisely, one of the following holds

- f^* has a pole at $0 \in \mathbb{C}$ (and the order of the pole must be 1)
- f^* admits a holomorphic and injective extension to $\mathbb{C}$

f holomorphic and then

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

$$\Rightarrow f^*(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

we know that 0 is not an essential singularity for f^* so that there exist $N \in \mathbb{N}$: $a_n = 0$ $\forall n \geq N$

$\Longrightarrow$ f is polynomial

$\underset{f \text{ is injective}}{\Longrightarrow}$ f is polynomial of degree 1.

$\boxtimes$

2. $$\mathrm{Aut}(\mathbb{C}P(1)) = \Big\{ \underbrace{z \longmapsto \frac{az+b}{cz+d}}_{\text{Möbius transformations}} : a,b,c,d \in \mathbb{C}, \ ad-bc \neq 0 \Big\}$$

Let us first note that all Möbius transformations are clearly automorphisms of $\mathbb{C}P(1)$. So, let

$$f(z) = \frac{az+b}{cz+d}$$

for some constants $a,b,c,d \in \mathbb{C}$ with $ad-bc \neq 0$

- Injectivity

$$f(z_1) = f(z_2)$$

$$\Rightarrow \quad \frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$

$$\Rightarrow \quad (az_1+b)(cz_2+d) = (az_2+b)(cz_1+d)$$

$$\Rightarrow \quad (z_1-z_2)(ad-bc) = 0$$

$$\overset{ad-bc\neq 0}{\Longrightarrow} \quad z_1 - z_2 = 0$$

$$\Rightarrow \quad z_1 = z_2$$

- surjectivity

$$w = f(z)$$

$$\Leftrightarrow \quad w = \frac{az+b}{cz+d}$$

$$\Leftrightarrow \quad w(cz+d) = az+b$$

$$\Leftrightarrow \quad (cw-a)z = b - wd$$

$$\Leftrightarrow \quad z = \frac{-dw+b}{cw+a}$$

$\forall w \in \mathbb{C}\setminus\{\frac{a}{c}\}$ we have $w = f\left(\frac{-dw+b}{cw+a}\right)$

In turn, $\frac{a}{c} = f(\infty)$

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{\cancel{z}\left(a+\frac{b}{z}\right)}{\cancel{z}\left(c+\frac{d}{z}\right)} = \frac{a}{c}$$

- f is holomorphic and the inverse is also holomorphic. The inverse takes on the form

$$w \mapsto \frac{-dw+b}{cw-a}$$

which is clearly a möbius transformation as well. So the möbius transformations are automorphisms of $\mathbb{C}\mathbb{P}(1)$.

Rmk: The set of möbius transformations is closed under composition (exercise) $\rightarrow$ it is a group

Let us then prove that the group of möbius transformations is, in fact, the automorphism group of $\mathbb{C}P(1)$. We are thus going to prove the following:

Proposition:

Let $f: \mathbb{C}P(1) \rightarrow \mathbb{C}P(1)$ be holomorphic. Then f is a rational functio, i.e.

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomial (with complex coefficients) and q is not identically zero.

The fact that the automorphisms are reduced to the Möbius transformations immediately follows. In fact, they are the unique bijective functions among the rational functions.

$$\omega = \frac{p(z)}{q(z)} \iff p(z) - \omega q(z) = 0$$

So it remains to prove the proposition.

Proof of the Proposition:

Since the Riemann Sphere is compact, we have that f can have only finitely many poles. Furthermore, unless f is constant (case that can be excluded from our discussion since in that case f is clearly rational), f possesses at least one pole. Indeed, assume without loss of generality that $\infty \in \mathbb{C}P(1)$ is not a pole for f. Consider then $\bar{f}$ the restriction of f to $\mathbb{C} \subseteq \mathbb{C}P(1)$.

$$\bar{f}: \mathbb{C} \longrightarrow \mathbb{C}P(1)$$

Liouville's theorem guarantes that there exists a sequence $\{z_n\} \subseteq \mathbb{C}$ such that $|f(z)| \rightarrow \infty$. Since $f(\infty) \neq \infty$ and $\mathbb{C}P(1)$ is compact, the sequence $\{z_n\}$ contains a convergent subsequence $\{z_{k_n}\} \longrightarrow z^*$. Clearly

$f(z^*) = \infty$, i.e f has a pole at z^* (and $z^* \neq \infty$).

Let $z_1, z_2, \dots, z_n \in \mathbb{C}$ stands for the poles of f and let $k_1, k_2, \dots, k_n$ be their associated multiplicities.
order of the pole

Next, conside the polynomial function

$$q(z) = \prod_{j=1}^{n} (z - z_j)^{k_j} \quad , \quad z \in \mathbb{C}$$

(that can also be seen as a function $q : \mathbb{C}\mathbb{P}(1) \to \mathbb{C}\mathbb{P}(1)$) and also the function $p : \mathbb{C}\mathbb{P}(1) \longrightarrow \mathbb{C}\mathbb{P}(1)$ defined by

$$p(z) = f(z) \cdot q(z)$$

we intend to prove that p is a polynomial function.

- p is holomorphic on $\mathbb{C}$ $\rightsquigarrow$ each z_i is a removable singularity for p (the zeros of q are the poles of f and they have the same multiplicity)

- p is meromorphic at ∞ since so are f and q.

$\Longrightarrow$ p is polynomial ☒

Rmk:

we have

$$\mathrm{Aut}(\mathbb{C}\mathbb{P}(1)) = \left\{ z \longmapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} \, , \, ad-bc \neq 0 \right\}$$

$$= \left\{ z \longmapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C} : ad-bc = 1 \right\}$$

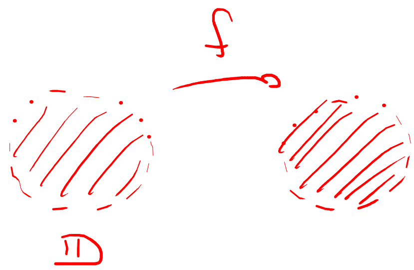
$$\simeq PGL_2(\mathbb{C})$$

$$\simeq PSL_2(\mathbb{C})$$

3. $\operatorname{Aut}(\mathbb{D}) = \left\{ z \longmapsto \dfrac{az+b}{\bar{b}z+\bar{a}} \;:\; a,b\in\mathbb{C},\ |a|^2-|b|^2=1 \right\}$

$\operatorname{Aut}(\mathbb{D})$ is a subgroup of Möbiustransformations.

Let then f be a Möbius transformation taking on the form
$f(z) = \dfrac{az+b}{\bar{b}z+\bar{a}}$ for some constants $a,b\in\mathbb{C}$ satisfying
$|a|^2-|b|^2=1$.



Fix then $z\in\mathbb{C}$ with $|z|=1$. We have

$$
\begin{aligned}
|f(z)|^2 &= f(z)\cdot\overline{f(z)} \\
&= \frac{az+b}{\bar{b}z+\bar{a}}\cdot\frac{\bar{a}\bar{z}+\bar{b}}{b\bar{z}+a} \\
&= \frac{az+b}{\bar{b}z+\bar{a}}\cdot\frac{\bar{a}z^{-1}+\bar{b}}{bz^{-1}+a} \\
&= \frac{\cancel{az+b}}{\cancel{\bar{b}z+\bar{a}}}\cdot\frac{\cancel{\bar{a}+\bar{b}z}}{\cancel{b+az}} \\
&= 1
\end{aligned}
$$

$|z|=1$
$\Rightarrow z\cdot\bar{z}=1$
$\Rightarrow \bar{z}=z^{-1}$

(I am sending $\partial\mathbb{D}$ into $\partial\mathbb{D}$)

We have that f takes the unit to itself. The interior of the unit disc, $\mathbb{D}$, is then taken to itself or the exterior of the unit disc. Since

$$|f(0)|^2 = f(0)\cdot\overline{f(0)} = \frac{b}{\bar{a}}\cdot\frac{\bar{b}}{a} = \frac{|b|^2}{|a|^2} < 1$$

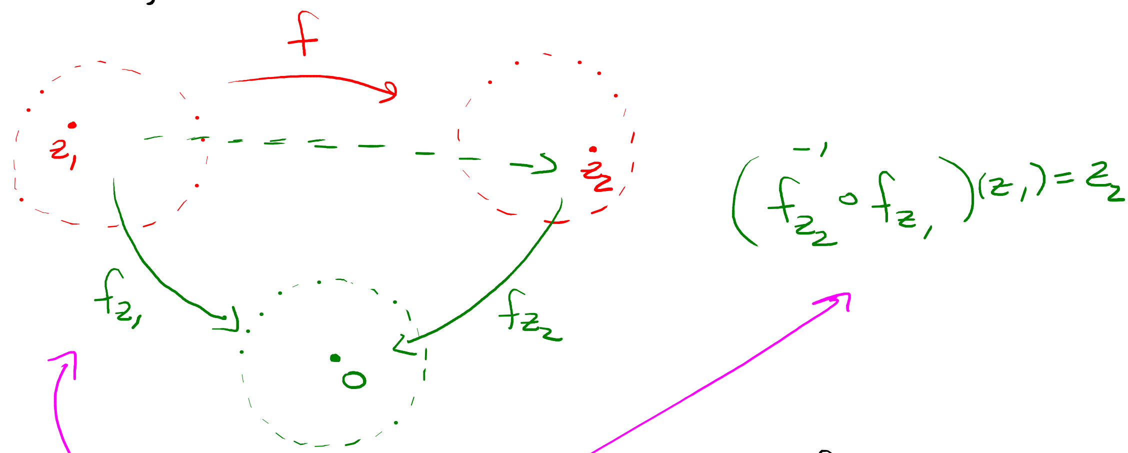
because $|a|^2-|b|^2=1$

We have then that $f(\mathbb{D}) \subseteq \mathbb{D}$.
$=$

f is an automorphism of $\mathbb{D}$ and we have to check that all automorphisms of $\mathbb{D}$ take on this form.

This is based on the following two facts:

(1) The subgroup of Möbius transformation in question acts transitively on $\mathbb{D}$. In other words, for every pair of points $z_1, z_2 \in \mathbb{D}$ that we fix, there exists an element f as above such that $f(z_1) = z_2$.



To prove this fact, it suffices to show that for every $z_0 \in \mathbb{D}$ that we fix, there exists an element f_{z_0} in the element in question group taking z_0 to the origin. The element in question is clearly

$$f_{z_0}(z) = \frac{z - z_0}{\bar{z}_0 z + 1}$$

$$\frac{az+b}{\bar{b}z+\bar{a}}$$

Rmk: Since $f_{z_0}(0) = -z_0$, we have that

$$f_{z_0}^{-1} = f_{-z_0}$$

(2) Thm:

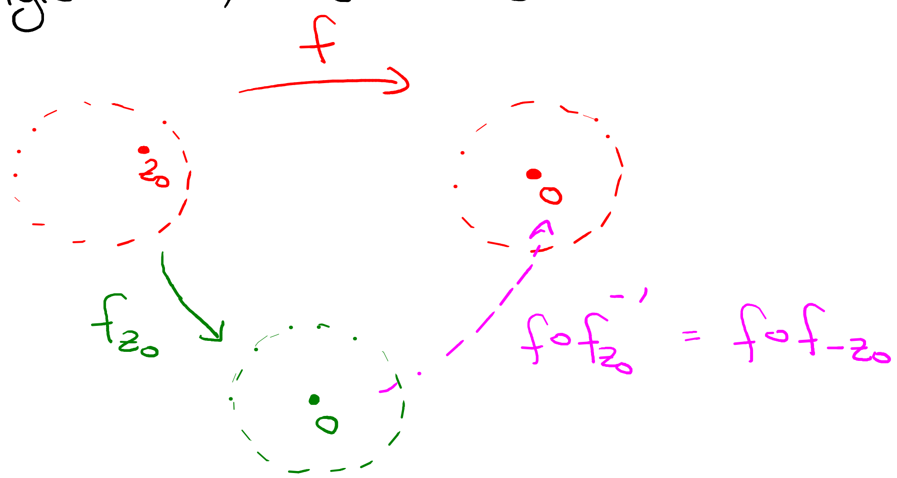
Let f be an endomorphism of $\mathbb{D}$ fixing the origin. Then f is an automorphism if and only if f is a rotation, i.e. f takes on the form

$$f(z) = e^{i\theta} z$$

for some $\theta \in [0, 2\pi[$.

So, let then f be an arbitrary automorphism of $\mathbb{D}$ and let $z_0 \in \mathbb{D}$ be the element such that $f(z_0) = 0$. We have that

$f \circ f_{-z_0}$ is an automorphism of $\mathbb{D}$ fixing the origin. Thus, according to the above theorem, $f \circ f_{-z_0}$ is a rotation of some angle θ ; $R_\theta(z) = e^{i\theta} z$



$$\Longrightarrow \quad f(z) = \left(R_\theta \circ f_{z_0}\right)(z)$$

Since both R_θ and f_{z_0} are elements of the group in question, so is f

$$R_\theta(z) = e^{i\theta} z = \frac{e^{i\frac{\theta}{2}} z}{e^{-i\frac{\theta}{2}}} = \frac{e^{i\frac{\theta}{2}} z}{\overline{e^{i\frac{\theta}{2}}}}$$

$(b=0)$

$\left(a = e^{i\frac{\theta}{2}}\right)$

It then follows that $Aut(\mathbb{D})$ can also be written in the form

$$Aut(\mathbb{D}) = \left\{ z \longmapsto e^{i\theta} \cdot \frac{z-a}{-\bar{a}z+1} : \begin{array}{l} a \in \mathbb{D} \\ \theta \in [0, 2\pi[\end{array} \right\}$$

Proof of the Theorem:

It is clear that a rotation is an automorphism of the disc fixing the origin. We have to prove that all automorphisms with this property are rotations.

Fix $f : \mathbb{D} \to \mathbb{D}$ an automorphism fixing the origin. Let

$$g : \mathbb{D}^* \longrightarrow \mathbb{C}$$

be the function on the puncture disc defined by

$$g(z) = \frac{f(z)}{z}$$

we have that g is holomorphic in $\mathbb{D}^*$. Furthermore, we have that 0 is a removable singularity of g. In fact

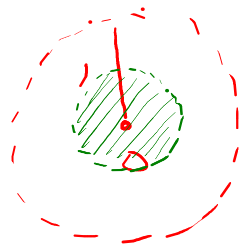
$$\lim_{z \to 0} g(z) = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0)$$

continuous and, to be checked analytic

Then g admits an analytic extension to the entire disc, namely

$$g(z) = \begin{cases} \dfrac{f(z)}{z} & , \text{ if } z \neq 0 \\ f'(0) & , \text{ if } z = 0 \end{cases}$$

For every radius $0 < r < 1$ that we fix, we have



$$\sup_{|z| \leq r} |g(z)| = \sup_{|z| = r} |g(z)| = \sup_{|z| = r} \frac{|f(z)|}{|z|}$$

$$= \sup_{|z| = r} \frac{|f(z)|}{r} \leq \frac{1}{r}$$

Maximum Principle

and, by letting $r \to 1^-$, we get that

$$\sup_{z \in \mathbb{D}} |g(z)| \leq 1$$

and, consequently, $|g(z)| \leq 1$ for every $z \in \mathbb{D}$.

Rmk: If $|g(z)| = 1$ for some $z \in \mathbb{D}$, then g is constant. More precisely, $g(z) \equiv e^{i\theta}$ for some $\theta \in [0, 2\pi[$.

$\Longrightarrow$ f is a rotation in this case

$$g(z) = e^{i\theta} \iff \frac{f(z)}{z} = e^{i\theta}$$
$$\iff f(z) = e^{i\theta} z$$

It is then sufficient to prove that $|f'(0)| = 1$.

Let us prove that this occurs.

f automorphism $\Longrightarrow$ f^{-1} automorphism

we can apply the argument above to both f and f^{-1}

$$\Longrightarrow \quad |f'(0)| \leq 1 \quad \wedge \quad |(f^{-1})'(0)| \leq 1$$

However

$$|(f^{-1})'(0)| = \frac{1}{|f'(0)|}$$

$(f^{-1} \circ f)(0) = 0$

$(f^{-1})'(f(0)) \cdot f'(0) = 1$

$$\Longrightarrow \quad |(f^{-1})'(0)| = |f'(0)| = 1$$

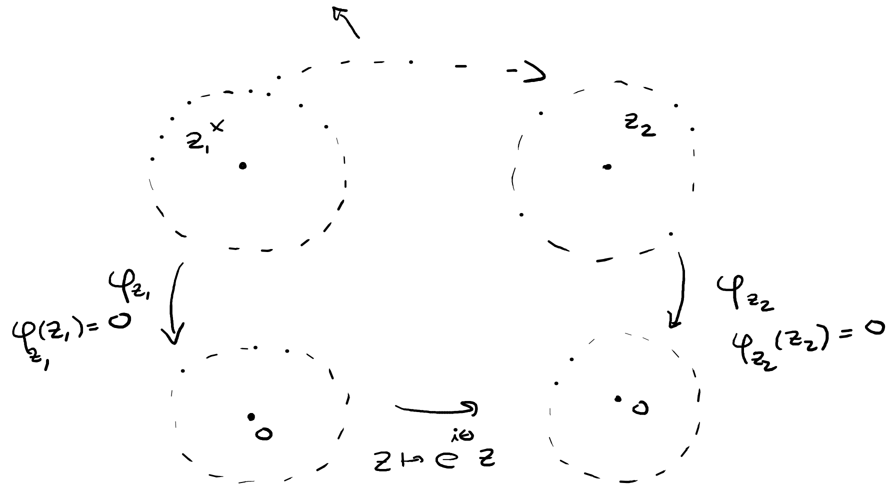
$\Longrightarrow$ f is a rotation from the remark

$\boxed{}$

$$\mathrm{Aut}(\mathbb{D}) \simeq \left\{ \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}$$

$$\mathrm{Aut}(\mathbb{D}) = \left\{ \frac{az+b}{\bar{b}z+\bar{a}} : a,b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}$$

$$= \left\{ e^{i\theta} \frac{z-b}{-\bar{b}z+1} : \theta \in [0, 2\pi[,\ b \in \mathbb{D} \right\}$$



<u>Def</u>:

Let A and B be two domains of $\mathbb{C}$ (open and connected subsets of $\mathbb{C}$). We say that A and B are biholomorphically equivalent if there is a bihomorphic map $\phi : A \to B$

<u>Example</u>:

The unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ is biholomorphically equivalent to the upper half plane $\mathbb{H} := \{ z \in \mathbb{C} : \mathrm{Im}(z) > 0 \}$. In fact, there is a biholomorphic function $\varphi : \mathbb{H} \to \mathbb{D}$ between these two domamains, namely

$$\varphi(z) = \frac{z-i}{z+i}$$

- φ is clearly injective (it is a Möbius transformation)

$$\frac{z-i}{z+i} = \frac{\omega - i}{\omega + i}$$

$$\Rightarrow z = \omega$$

- we need to show that it maps $\mathbb{H}$ onto $\mathbb{D}$

we have $\quad (\varphi(z) \in \mathbb{D})$

$$|\varphi(z)| < 1 \iff \left| \frac{z-i}{z+i} \right| < 1$$

$$\iff |z-i| < |z+i|$$

$$\iff |z-i|^2 < |z+i|^2$$

$$\iff \cancel{|z|^2} + iz - i\bar{z} + \cancel{1} < \cancel{|z|^2} - iz + i\bar{z} + \cancel{1}$$

$$\iff -2\,\mathrm{Im}(z) < 2\,\mathrm{Im}(z)$$

$$\iff \mathrm{Im}(z) > 0$$

$$\iff z \in \mathbb{H}$$

The result follows. $\boxtimes$

Thm:

Let A and B be two biholomorphically equivalent domains. Denote by f a biholomorphic map between A and B. Then $\phi : \mathrm{Aut}(A) \to \mathrm{Aut}(B)$ defined by

$$\phi(h) = f \circ h \circ f^{-1}$$

is an isomorphism between the automorphisms groups.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, dashed, "\phi(h)"'] & B \end{tikzcd}$$

$$\phi(h) = f \circ h \circ f^{-1}$$

Corollary:

The group of automorphisms of the upper half plane is isomorphic the group of automorphisms of the disc. More precisely, we have

$$\mathrm{Aut}(\mathbb{H}) = \left\{ \varphi^{-1} \circ h \circ \varphi \ : \ h \in \mathrm{Aut}(\mathbb{D}), \ \varphi(z) = \frac{z-i}{z+i} \right\}$$

<u>Proof of the thm</u>:

- ϕ is a homomorphism: $\quad (h_1, h_2 \in \mathrm{Aut}(A))$

$$\begin{aligned} \phi(h_1 \circ h_2) &= f \circ (h_1 \circ h_2) \circ f^{-1} \\ &= (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) \\ &= \phi(h_1) \circ \phi(h_2) \end{aligned}$$

$$\forall h_1, h_2 \in \mathrm{Aut}(A)$$

- ϕ is injective

$$\begin{aligned} & \phi(h_1) = \phi(h_2) \\ \iff & f \circ h_1 \circ f^{-1} = f \circ h_2 \circ f^{-1} \\ \iff & f^{-1} \circ f \circ h_1 \circ f^{-1} \circ f = f^{-1} \circ f \circ h_2 \circ f^{-1} \circ f \\ \iff & h_1 = h_2 \end{aligned}$$

- ϕ is surjective

<u>Rmk</u>: given $h \in \mathrm{Aut}(A)$, since f, f^{-1} and h are all biholomorphic maps, we have that $\phi(h)$ is biholomorphic as well, so $\phi(h) \in \mathrm{Aut}(B)$.

Fix $g \in Aut(B)$. From the argument above we have that

$$\begin{tikzcd} A \arrow[r, dashed, "h"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ B \arrow[r, "g"'] & B \end{tikzcd}$$

$$h = f^{-1} \circ g \circ f \in Aut(A)$$

and, clearly,

$$\phi(h) = g$$

Hence, ϕ is an isomorphism. $\boxed{//}$

Uniformization theorem

Let us start to recall the following:

- $\mathbb{C}\mathbb{P}(1)$ is not biholomorphically equivalent to $\mathbb{C}$ or $\mathbb{D}$. In fact, they are not even topologically equivalent.

- Also, $\mathbb{C}$ and $\mathbb{D}$ are not biholomorphically equivalent by Liouville's theorem.

- The 3 above mentioned Riemann surfaces are simply connected. We also talked about another simply connected Riemann surface, namely $\mathbb{H}$, and we have proved that it is biholomorphically equivalent to $\mathbb{D}$.

In fact, up to biholomorphically equivalence, $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{CP}(1)$ are the unique simply connected Riemann surfaces.

<u>Thm:</u>

Let S be a simply connected Riemann surface. Then S is bihomorphically equivalent to $\mathbb{C}$, $\mathbb{CP}(1)$ or $\mathbb{D}$.

- S bihomorphically equivalent to $\mathbb{C}$ $\rightsquigarrow$ parabolic
- " " " " $\mathbb{CP}(1)$ $\rightsquigarrow$ elliptic
- " " " " $\mathbb{D}$ $\rightsquigarrow$ hyperbolic

<u>Thm:</u>

Every Riemann surface S is biholomorphically equivalent to D/G, where D is $\mathbb{C}$, $\mathbb{CP}(1)$ or $\mathbb{D}$ and G is a freely action discontinuous group of Möbius transformation that preserves D.

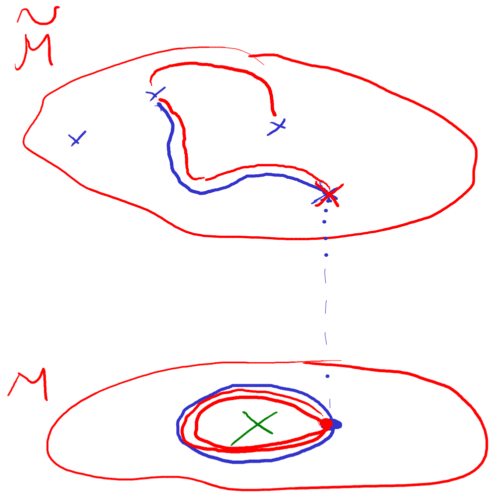
Given a topological manifold M (in our case we are interested in Riemann surfaces), a new manifold $\tilde{M}$ can be constructed so that the following properties are satisfied:

(1) There exists a surjective local homeomorphism

$$\pi : \tilde{M} \to M$$

(2) $\tilde{M}$ is simply connected

(3) every closed curve on M that is not homotopically trivial can be lifted to an open curve on $\tilde{M}$ that is uniquely determined by the closed curved on M and the initial point on $\tilde{M}$.



$\tilde{M}$ is said the universal covering of M

Rmk :

M Riemann surface $\Rightarrow$ it can be introduced a Riemann surface structure on $\tilde{M}$ in such a way that

$$\pi : \tilde{M} \to M$$

is holomorphic. Furthermore there is a group G of automorphisms of $\tilde{M}$ such that

$$M \simeq \tilde{M}/G$$

Let us precise some notions.

Homotopy of curves and Fundamental Group

A <u>curve</u> α on a topological space X is a continuous map $\alpha : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$

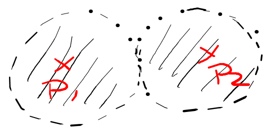
$\alpha(0) \mapsto$ initial point of α

$\alpha(1) \mapsto$ end point of α

Def:
A topological space is called pathwise connected if any two points $p_1, p_2 \in X$ can be joined by a curve, i.e. there exists a curve $\alpha : I \to X$ such that $\alpha(0) = p_1$ and $\alpha(1) = p_2$.

pathwise connected $\Longrightarrow$ connected

$$\left.\begin{array}{l} X = U_1 \cup U_2 \\ U_1, U_2 \text{ open sets} \end{array}\right\} \Longrightarrow U_1 = \emptyset \text{ or } U_2 = \emptyset$$



we say that X is locally patwise connected if for every point $p \in X$, there exists a neighbourhood of p that is pathwise connected

$$\left.\begin{array}{l} \text{connected} \\ \text{locally pathwise connected} \end{array}\right\} \Longrightarrow \text{pathwise connected}$$

Def:

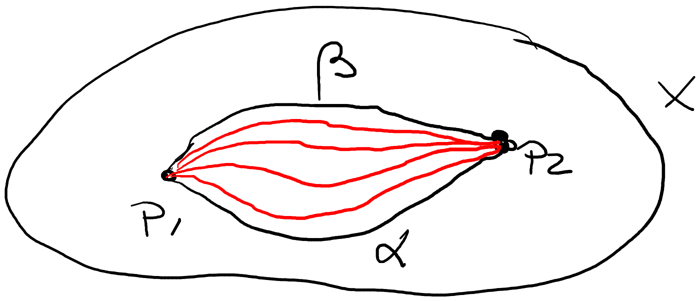
Let X be a topological space and consider $P_1, P_2 \in X$. Let $\alpha, \beta : I \to X$ be two curves such that $\alpha(0) = \beta(0) = P_1$ and $\alpha(1) = \beta(1) = P_2$. We say that α and β are homotopic, and we write $\alpha \sim \beta$ if there exists a continuous map

$$F : \underset{t}{I} \times \underset{s}{I} \longrightarrow X$$

such that

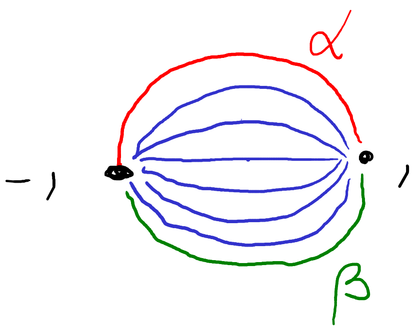
(1) $F(t,0) = \alpha(t)$ for every $t \in I$
$F(t,1) = \beta(t)$

(2) $F(0,s) = P_1$ for every $s \in I$
$F(1,s) = P_2$

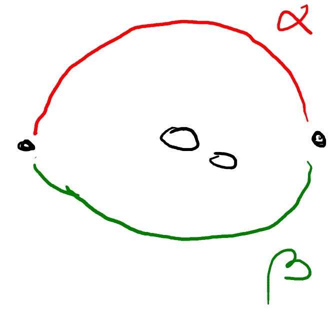


Example:

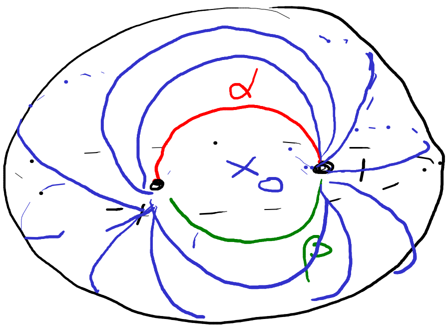
On $\mathbb{C}$, the curve $\alpha(t) = e^{\pi i t}$ is homotopic to the curve $\beta(t) = e^{-\pi i t}$.



However the two curves are not homotopic on $\mathbb{C}^*$.



The curves α and β are homotopic on $\mathbb{C}\mathbb{P}(1)\setminus\{0\}$



The notion of homotopy between two curves with p_1 and p_2 as initial and final poins, respectively, defines an equivalence relation in the space of curves with the mentioned property. In this case

$$\gamma_1 \text{ homotopic to } \gamma_2 \;:\; \gamma_1 \sim \gamma_2$$

So

- $\gamma_1 \sim \gamma_1$
- $\gamma_1 \sim \gamma_2 \iff \gamma_2 \sim \gamma_1$
- $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3 \implies \gamma_1 \sim \gamma_3$

<u>In fact</u> :

- $\gamma_1 \sim \gamma_1$: it suffices to take $F(t,s) = \gamma_1(t) \quad \forall s \in I$

- $\gamma_1 \sim \gamma_2 \iff \gamma_2 \sim \gamma_1$

$$\gamma_1 \sim \gamma_2 \longmapsto \begin{array}{l} F : I \times I \to X \\ F(t,0) = \gamma_1(t) \\ F(t,1) = \gamma_2(t) \end{array}$$

The homotopy map from γ_2 to γ_1 is

$$G : I \times I \to X$$

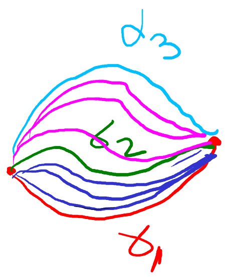
$$G(t,s) = F(t, 1-s) \quad \forall t, s \in I$$

$$\implies \gamma_2 \sim \gamma_1$$

- $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$

$F : I \times I \to X$ homotopy map from γ_1 to γ_2

$G : I \times I \to X$ homotopy map from γ_2 to γ_3



Define $H : I \times I \to X$ in the following way

$$H(t,s) = \begin{cases} F(t, 2s) & , \text{ if } s \in [0, \tfrac{1}{2}] \\ G(t, 2s-1) & , \text{ if } s \in]\tfrac{1}{2}, 1] \end{cases}$$

The map H is clearly continuous since

$$F(t,1) = \gamma_2(t) = G(t,0) \quad \forall t \in I$$

H is a homotopy map from γ_1 to γ_3

$$\implies \gamma_1 \sim \gamma_3$$

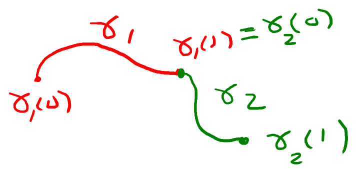
The equivalence classes of the homotopy equivalence relation are called homotopy classes and the homotopy class of $\gamma : I \to X$ will be denoted by $[\gamma]$.

Def:

The product of two curves $\gamma_1 : [0,1] \to X$ and $\gamma_2 : [0,1] \to X$ such that $\gamma_1(1) = \gamma_2(0)$ (i.e. final point of γ_1 = initial point of γ_2), commonly denoted by $\gamma_1 * \gamma_2$, is the curve defined as follows:

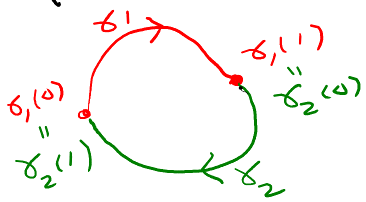
$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & , t \in [0, \tfrac{1}{2}] \\ \gamma_2(2t-1) & , t \in]\tfrac{1}{2}, 1] \end{cases}$$

Rmk: The product of curves is not a commutative opperator

- 

  In this case $\gamma_2 * \gamma_1$ is not even defined since $\gamma_2(1) \neq \gamma_1(0)$.

- A necessary condition for $\gamma_2 * \gamma_1$ to be defined (assuming that $\gamma_1 * \gamma_2$ is defined) is that $\gamma_1 * \gamma_2$ is a closed curve (to be defined)



Def:
A curve $\gamma : [0,1] \to X$ is said closed if $\gamma(0) = \gamma(1) = x_0$.
The point x_0 is called the base point of the closed curve.

Prop:
The set of equivalence classes for the closed curves on to a topological space X equipped with base point $x_0 \in X$ with the product

$$[\gamma_1].[\gamma_2] = [\gamma_1 * \gamma_2]$$

forms a group.

Proof:
To begin with, let us check that the product in question is well defined:

γ_1, γ_2 closed curves with base point ω_0

$$\left| \begin{array}{l} \gamma_1(0) = \gamma_1(1) = \omega_0 \\ \gamma_2(0) = \gamma_2(1) = \omega_0 \end{array} \right.$$

In particular, $\gamma_1(0) = \gamma_2(0) = \omega_0$ so that $\gamma_1 * \gamma_2$ is well-defined

Furthermore, $(\gamma_1 * \gamma_2)(0) = \gamma_1(0) = \omega_0$

$(\gamma_1 * \gamma_2)(1) = \gamma_2(1) = \omega_0$

$\Rightarrow \gamma_1 * \gamma_2$ is a closed curve with base point ω_0

Now:

(a) associativity: <u>exercise</u>

(b) identity element:

Consider the "constant curve" α given by

$$\alpha(t) = \omega_0 \quad \forall t \in [0,1]$$

(<u>Notation</u>: for simplicity, we will denote it by ω_0)

For every closed curve α with base point ω_0, we clearly have $\gamma * \omega_0 \sim \gamma$. In fact the homotopy map is:

$$\boxed{\gamma * \omega_0(t) = \begin{cases} \gamma(2t) & , t \in [0, \tfrac{1}{2}[\\ \omega_0 & , t \in [\tfrac{1}{2}, 1] \end{cases}}$$

$$F : [0,1] \times [0,1] \to X$$

$$F(t,s) = \begin{cases} \gamma\left(\dfrac{2t}{1+s}\right) & , t \in [0, \tfrac{s+1}{2}] \\ \omega_0 & , t \in]\tfrac{s+1}{2}, 1] \end{cases}$$

$y = a\omega + b$

$\omega = 0 \mapsto y = \frac{1}{2}$

$\omega = 1 \mapsto y = 1$

$y = \frac{1}{2}\omega + \frac{1}{2}$

$\omega = 0 \mapsto y = 1$

$\omega = 1 \mapsto y = 2$

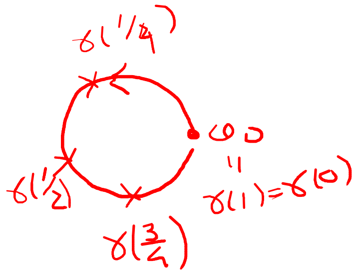
$y = \omega + 1$

$F(t,0) = (\gamma * \omega_0)(t)$

$F(t,1) = \gamma(t)$

$F(0,s) = F(1,s) = \omega_0 \qquad \forall s \in [0,1]$

(c) existence of an inverse for every element



$\gamma : [0,1] \to X$: $\gamma(0) = \gamma(1) = \omega_0$

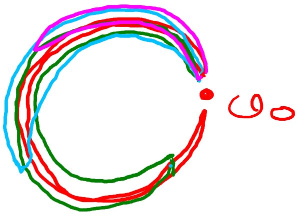
Consider then the curve $\bar{\gamma} : [0,1] \to X$

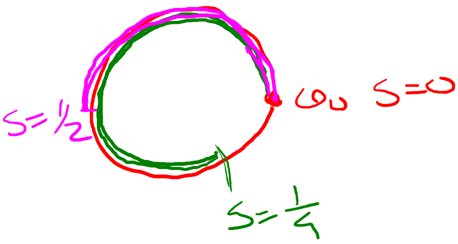
defined as: $\bar{\gamma}(t) = \gamma(1-t)$ $\forall t \in [0,1]$

Exercise: construct the homotopy map of

$$[\gamma * \bar{\gamma}] = [\omega_0]$$

$$\gamma * \bar{\gamma} \sim \omega_0$$





Def:

The set of equivalence classes of closed curves on a topological space X with base point $\omega_0 \in X$, equipped with the product $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2]$ is called the and it is fundamental group of X with base point ω_0 denoted by $\pi_1(X, \omega_0)$.

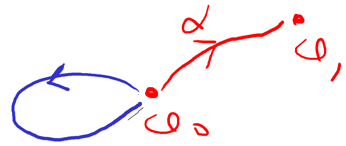
Proposition:

Let X be a topological space and assume that X is pathwise connected. For every two points $\omega_0, \omega_1 \in X$, there exists an isomorphism between $\pi_1(X, \omega_0)$ and $\pi_1(X, \omega_1)$.

Proof:

By assumption X is pathwise connected. So, by fixing two points $\omega_0, \omega_1 \in X$ let $\alpha : [0,1] \to X$ stands for the (continuous) curve joining ω_0 to ω_1. Consider then

$$\begin{aligned} \gamma_\alpha : \pi_1(X,\omega_0) &\longrightarrow \pi_1(X,\omega_1) \\ [\beta] &\longmapsto [\alpha^{-1} * \beta * \alpha] \end{aligned}$$



- γ_α is homomorphism

$$\begin{aligned} \gamma_\alpha[\beta_1 * \beta_2] &= [\alpha^{-1} * \beta_1 * \beta_2 * \alpha] \\ &= [(\alpha^{-1} * \beta_1 * \alpha) * (\alpha^{-1} * \beta_2 * \alpha)] \\ &= [\alpha^{-1} * \beta_1 * \alpha] \cdot [\alpha^{-1} * \beta_2 * \alpha] \\ &= \gamma_\alpha[\beta_1] \cdot \gamma_\alpha[\beta_2] \end{aligned}$$

- The inverse is given by

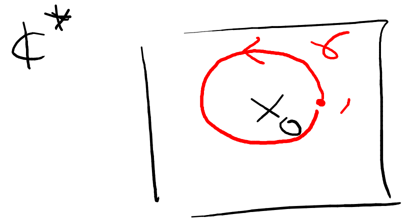
$$\begin{aligned} \Delta_\alpha : \pi_1(X,\omega_1) &\longrightarrow \pi_1(X,\omega_0) \\ [\beta] &\longmapsto [\alpha * \beta * \alpha^{-1}] \end{aligned}$$

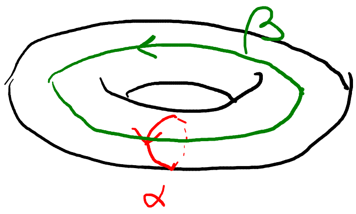
<u>Def:</u>

A topological space X is called <u>simply connected</u> if it is pathwise connected and its fundamental group is trivial, i.e. every closed curve is homotopic to a constant curve (e.g. to the base point).

<u>Examples:</u>

- $\mathbb{C}$, $\mathbb{D}$ and $\mathbb{C}P(1)$ are simply connected
- $\mathbb{C}P(1)\setminus\{p\}$ is simply connected $\mathbb{C}P(1)\setminus\{p\} \cong \mathbb{C}$
- $\mathbb{C}^*$, $\mathbb{C}/\Gamma$ (tori) are not simply connected

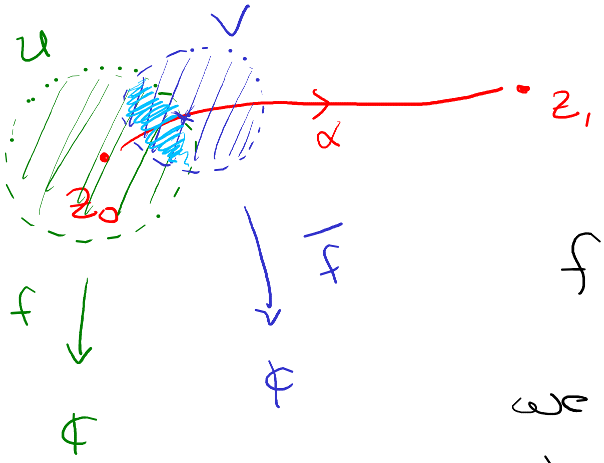




γ is not homotopic to a constant curve

Let us prove that $\mathbb{C}^*$ is not simply connected

<u>Proposition</u>: (Monodromy principle)
Let $A (\subseteq \mathbb{C})$ be <u>simply connected</u> and let $z_0 \in A$. Let f be an analytic function in a neighbourhood of z_0. Suppose that f can be analytically continued along any path joining z_0 to another point $z_1 \in A$. Then this continuation defines a (single-valued) analytic continuation of f on A.



$$f(U \cap V) = \bar{f}(U \cap V)$$

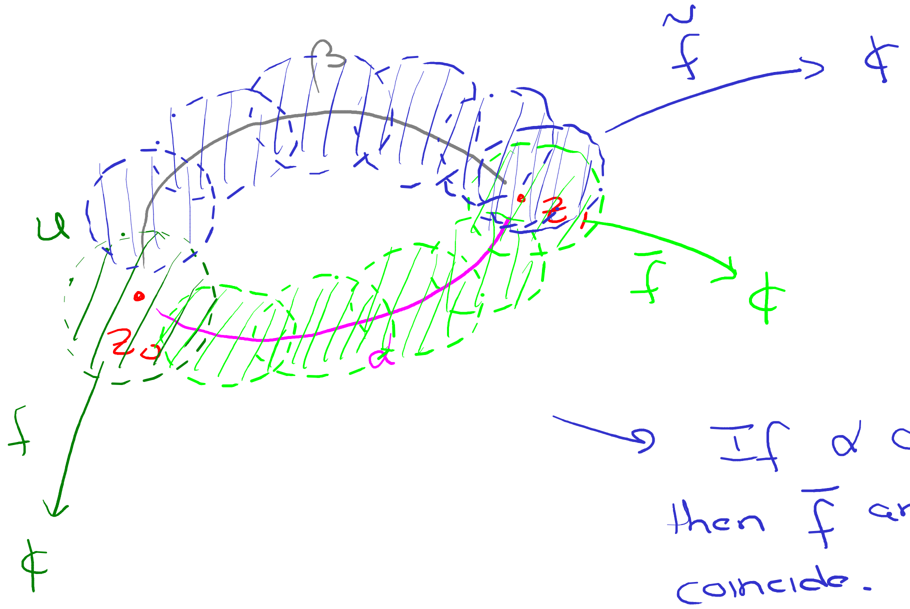
$$\Downarrow$$

we say that $\bar{f}$ is an analytic continuation of f

Suppose there exists $g: V \to \mathbb{C}$ that is

- analytic on V
- $g(U \cap V) = f(U \cap V) = \bar{f}(U \cap V)$

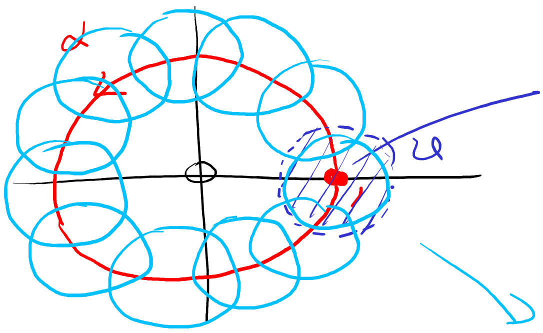
Since $U \cap V$ is an non-empty open set, from the Identity Theorem we know that g and $\bar{f}$ coincide in the entire V.



If α and β are not homotopic, then $\bar{f}$ and $\tilde{f}$ may not coincide.

Example :

$\mathbb{C}^*$, $\alpha(t) = e^{2\pi i t}$, $t \in [0,1]$



$\int_\alpha \frac{1}{z} \neq 0$

$$f : \mathcal{U} \longrightarrow \mathbb{C}$$
$$z \longmapsto \ln z \quad (= \ln|z| + i \arg z)$$

when I go back the argument differs by 2π

If I denote by $\bar{f}$ the analytic continuation along α at a small neighbourhood of $1 \in \mathbb{C}$ we have

$$\bar{f}(z) - f(z) = 2\pi i$$

This means that the analytic continuation of f along α is not single-valued, it is multivalued.

$\xRightarrow{\text{Monodromy Principle}}$ $\mathbb{C}^*$ is not simply connected

Exercises :

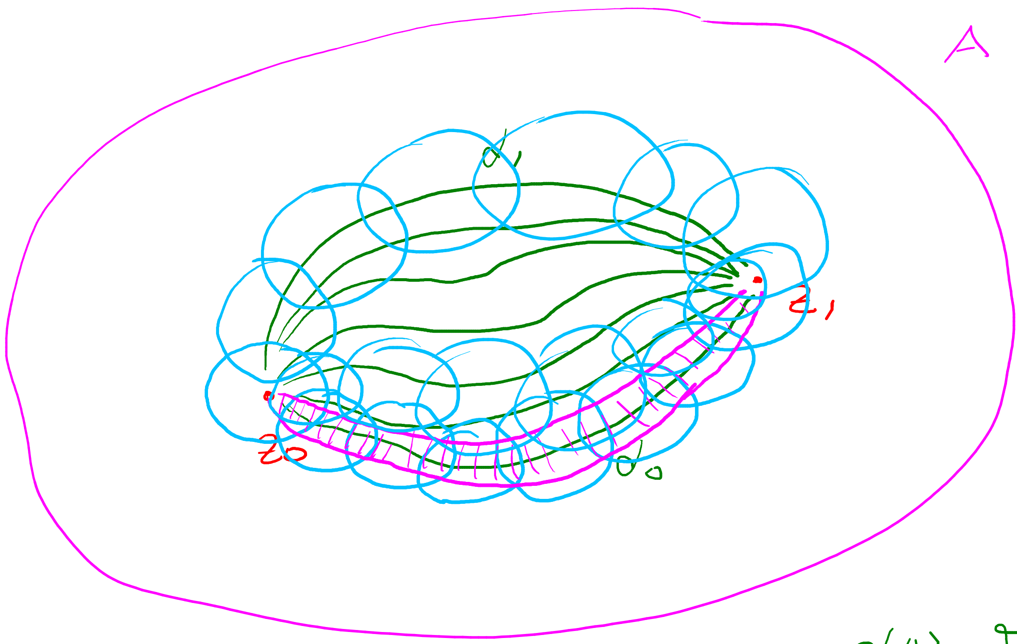
1. Prove that $\pi_1(S^1) \simeq \mathbb{Z}$

2. Prove that $\mathbb{C}P(1)$ is simply connected

3. $\pi_1(S^n)$ is trivial , $\forall\, n \geq 2$

4. $\mathbb{C}/\Pi$ (tori) is not simply connected

Idea of the proof:



$\alpha_s(t) = F(t,s)$

$\Downarrow$

homotopy map between α_0 and α_1

$s_0, s_{k_1}, s_{k_2}, \ldots, s_{k_n}$ $= 1$

Induced Homomorphisms

Let X, Y be two topological spaces and $f: X \to Y$ a continuous mapping between X and Y. There is an induced homomorphism

$$f_* : \pi_1(X, \omega_0) \longrightarrow \pi_1(Y, y_0)$$
$$[\gamma] \longmapsto [f\circ\gamma]$$

where $f(\omega_0) = y_0$

- Note that f_* is well-defined. In fact, if $\alpha_1, \alpha_2 : I \to X$ are two closed curves with base point ω_0 that are homotopic, then $f\circ\alpha_1, f\circ\alpha_2 : I \to Y$ are two homotopic closed curves on Y with base point y_0

$\alpha_s(t) = F(t,s)$

$$\Updownarrow$$

Let $F : I \times I \to X$ be the homotopy mapping between α_1, α_2 i.e.

$$F(t,0) = \alpha_1(t) \quad , \forall t \in I$$
$$F(t,1) = \alpha_2(t) \quad , \forall t \in I$$
$$F(0,s) = F(1,s) = \omega_0 \quad , \forall s \in I$$

F is, in particular continuous. Since f is continuous, the composition

$$f\circ F : I \times I \to Y$$

is an homotopy mapping between $f\circ\alpha_1$ and $f\circ\alpha_2$

$$(f\circ F)(t,0) = (f\circ\alpha_1)(t) \quad \forall t \in I$$
$$(f\circ F)(t,1) = (f\circ\alpha_2)(t) \quad \forall t \in I$$
$$(f\circ F)(0,s) = (f\circ F)(1,s) = f(\omega_0) = y_0 \quad \forall s \in I$$

$$\Rightarrow [f\circ\alpha_1] = [f\circ\alpha_2] \in \pi_1(Y, y_0)$$

- f_* is a group homomorphism since $f\circ(\alpha*\beta) = (f\circ\alpha)*(f\circ\beta)$

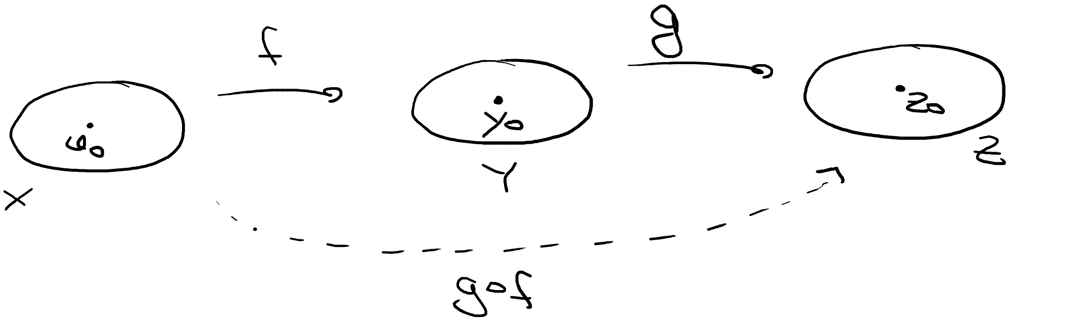
$$f_*([\alpha].[\beta]) = f_*([\alpha*\beta])$$
$$= [f(\alpha*\beta)]$$
$$= [(f\circ\alpha)*(f\circ\beta)]$$
$$= [f\circ\alpha].[f\circ\beta]$$
$$= f_*[\alpha] . f_*[\beta]$$

$$f(\alpha*\beta) = \begin{cases} f(\alpha(2s)) & s\in[0,\tfrac{1}{2}] \\ f(\beta(2s-1)) & s\in]\tfrac{1}{2},1] \end{cases}$$
$$= (f\circ\alpha)*(f\circ\beta)$$

Properties:

i) $f : X \longrightarrow Y$ $\qquad$ f, g continuous

$g : Y \longrightarrow Z$ $\qquad$ X, Y, Z topological spaces

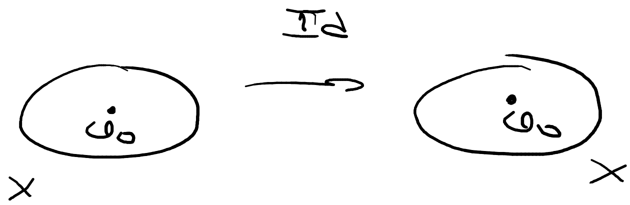


$$(g \circ f)_* = g_* \circ f_*$$

$$(g\circ f)_*([\alpha]) = [(g \circ f)\circ \alpha]$$
$$= [g \circ (f \circ \alpha)]$$
$$= g_*[f \circ \alpha]$$
$$= (g_* \circ f_*)[\alpha]$$

$[\alpha] \in \pi_1(X, x_0)$

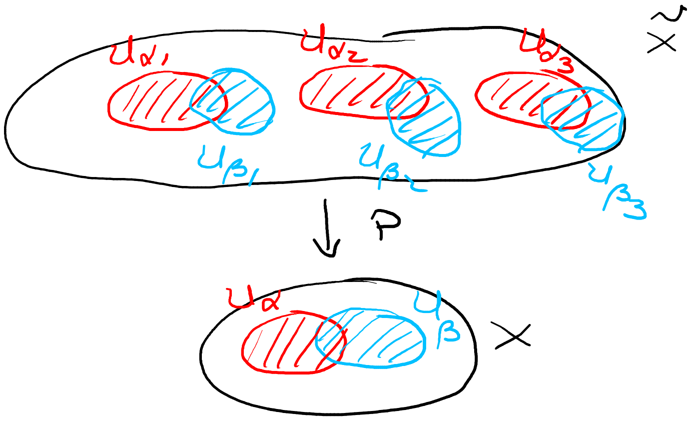
ii) $Id_* = Id$



$$Id_*[\alpha] = [Id \circ \alpha]$$
$$= [\alpha]$$

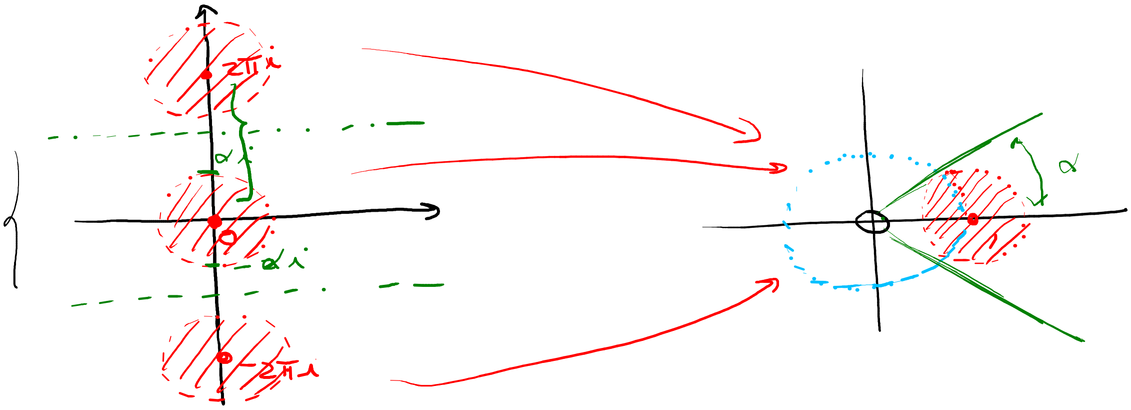
Covering spaces:

Def: Let $X, \tilde{X}$ be two topological spaces. We say that $p : \tilde{X} \to X$ is a covering map if there exists an opening covering of X by open sets $\{U_\alpha\}$ such that for every α, $p^{-1}(U_\alpha)$ is a disjoint union of open sets $\{\tilde{U}_{\alpha_j}\}$ in $\tilde{X}$ and $p|_{\tilde{U}_{\alpha_j}} : \tilde{U}_{\alpha_j} \to U_\alpha$ is a homeomorphism for every j.



<u>Examples:</u>

1. $$p : \mathbb{C} \longrightarrow \mathbb{C}^* \qquad \text{is a covering map}$$
$$z \longmapsto e^z$$



Fix $a \in \mathbb{C}^*$ and let $b \in \mathbb{C}$ be such that $p(b) = a$, i.e. $e^{2\pi i b} = a$. Since f is a local homeomorphism, there exists open neighbourhoods V_0 of b and $\mathcal{U}$ of a such that $p|_V : V \to \mathcal{U}$ is a homeomorphism. Then

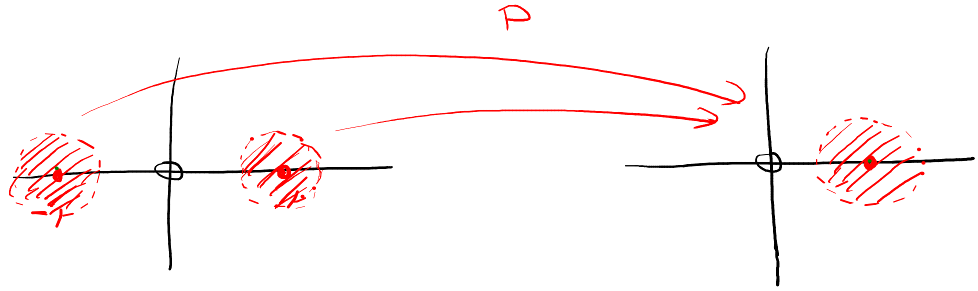
$$p^{-1}(\mathcal{U}) = \bigcup_{n \in \mathbb{Z}} \Big(\underbrace{V_0 + 2\pi i n}_{V_n} \Big)$$

The open sets V_n are clearly disjoint in the case where V is contained in a ball of radius smaller than π.

2. Fix $k \in \mathbb{N}$, $k \geq 2$ and let

$$p : \mathbb{C}^* \longrightarrow \mathbb{C}^* \qquad p_k \text{ is a covering map}$$
$$z \longmapsto z^k$$

$\boxed{k=2}$

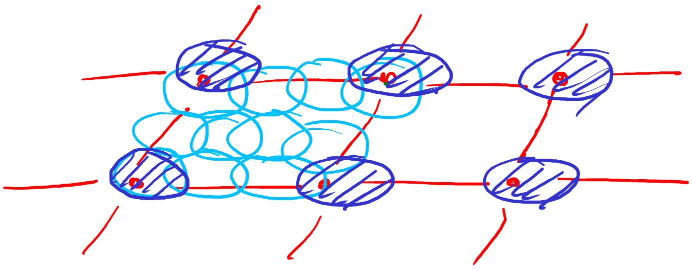


Proof : Exercise

3. Let $\Gamma \subseteq \mathbb{C}$ be a lattice and let

$$\pi : \mathbb{C} \longrightarrow \mathbb{C}/\Gamma$$

be canonical quotient map. Prove that π is a covering map



4. Let $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ and let

$$p : \mathbb{D} \longrightarrow \mathbb{C}$$
$$z \longmapsto z$$

be the canonical injection. Although p is a local homeomorphism, it is not a covering map. In fact there is no neighbourhood of the point $1 \; (\in \mathbb{C})$ satisfying the required properties

5. what about $p : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto z^k$, $k \in \mathbb{N}$, $k \geq 2$?

It is not a covering

$\boxed{k=2}$

$p(\sqrt{\varepsilon}) = \varepsilon$

$p(-\sqrt{\varepsilon}) = \varepsilon$ $\longrightarrow$ $p|_U$ is not injective !!!

Thm:

Suppose that X is a Riemann surface, $\tilde{X}$ is a Hausdorff topological space and $p: \tilde{X} \to X$ is a local homeomorphism. Then there is a unique complex structure on $\tilde{X}$ such that p is holomorphic.

Def:

Let $p: \tilde{X} \to X$ be a covering map for the topological spaces / manifolds X and $\tilde{X}$. Let Y be another topopological space / manifold and let $f: Y \to X$ a continuous map. A lift of f is a continuous map $\tilde{f}: Y \to \tilde{X}$ such that $f = p \circ \tilde{f}$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] \arrow[ur, phantom, shift right=3ex, "\circlearrowleft"] & X \end{tikzcd}$$

Thm (Uniqueness of Lifting)

Let X, Y be (connected) topological spaces and $p: Y \to X$ a local homeomorphism. Let Z be another (connected) topological space and $f: Z \to X$ a continuous map. If $\tilde{f}_1$ and $\tilde{f}_2$ $(\tilde{f}_1, \tilde{f}_2 : Z \to Y)$ are two liftings of f such that $\tilde{f}_1(z_0) = \tilde{f}_2(z_0)$ for some $z_0 \in Z$, then $\tilde{f}_1$ and $\tilde{f}_2$ coincide on the entire Z.

Proof:

Let $\quad A = \{ z \in Z : \tilde{f}_1(z) = \tilde{f}_2(z) \}$

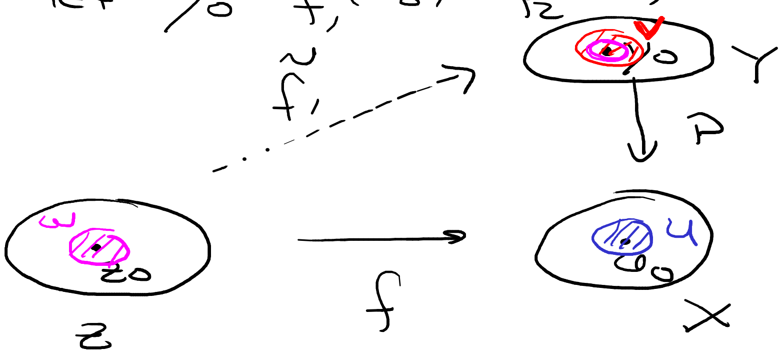
$\boxed{1^{st} : A \text{ is closed}}$

$\hookrightarrow$ In fact, A is the pre-image of the diagonal $\Delta = Y \times Y$ under the map

$$\tilde{F} : Z \longrightarrow Y \times Y$$
$$z \longmapsto (\tilde{f}_1(z), \tilde{f}_2(z))$$

$\boxed{2^{nd} : A \text{ is open}}$

Fix $z_0 \in A$ and let $y_0 = \tilde{f}_1(z_0) = \tilde{f}_2(z_0)$



Let $\omega_0 = p(y_0)$. Recalling that p is a local homeomorphism there exist V (neighbourhood of y_0) and $\mathcal{U}$ (neighb. of ω_0) such that $p(V) = \mathcal{U}$

Consider next ω, neighbourhood of z_0, such that

$$\tilde{f}_i(\omega) \subseteq V \qquad i = 1, 2$$

we have that $\tilde{f}_1$ and $\tilde{f}_2$ coincide on ω. In fact, denoting by φ the inverse of the restriction of p to V

$$\varphi = (p|_V)^{-1} : \mathcal{U} \longrightarrow V$$

we get that

$$\tilde{f}_i = \varphi \circ (f|_\omega) \qquad \forall i = 1, 2$$

Thus $\omega \subseteq A$ and, hence, A is open.

Finally, since A is non-empty ($z_0 \in A$)

in the statement of the result

- open
- closed
- Z connected

$\Longrightarrow$ $A = Z$

Prop:

X, Y, Z Riemann surfaces, $p: Y \to X$ holomorphic map. If p is a local homeomorphism then every lifting $g: Z \to Y$ of a holomorphic function $f: Z \to X$ is also holomorphic.

Proof: Follows from an argument of the proof of the previous theorem.

Particular case of lifting that will be considered:

lifting of curves $\alpha: [0,1] \to X$

$\leadsto$

By the uniqueness lifting thm, if a lifting $\tilde{\alpha}: [0,1] \to Y$ exists, that it is uniquely determined by the initial point of the lifting that we choose.

Def:

A continuous map $p: Y \to X$ is said to have the curve lifting property if for every curve $\mu : [0,1] \to X$ and every point $y_0 \in Y$ such that $p(y_0) = \mu(0)$, there exists a curve $\tilde{\mu} : [0,1] \to Y$ such that

$$\tilde{\mu}(0) = y_0$$

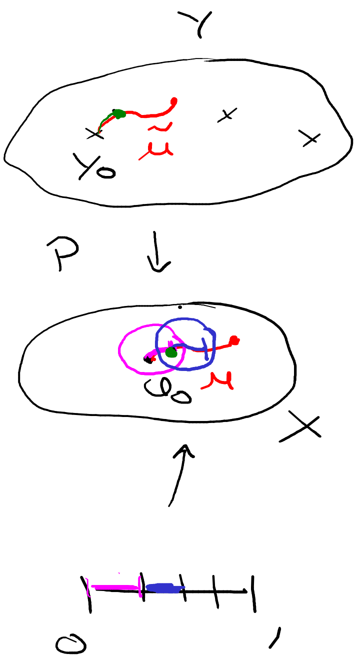
$$p \circ \tilde{\mu} = \mu$$

Thm:

Every covering map $p : Y \to X$ of (connected) topological spaces X, Y has the curve lifting property.

Proof:

Let $\alpha : [0,1] \to X$ be a curve and let $y_0 \in Y$ such that $p(y_0) = \alpha(\omega_0)$



Since $[0,1]$ is compact, there exists a partition of it

$$0 = t_0 < t_1 < t_2 < \dots < t_n = 1$$

and open sets $U_k \subseteq X$, $k = 1, \dots, n$ satisfying the following properties

(i) $\mu([t_{k-1}, t_k]) \subseteq U_k$

(ii) $p^{-1}(U_k) = \bigcup_j V_{k,j}$

$V_{k,j} \subseteq Y$ open sets such that $p|_{V_{j,j}}$ are homeomorphisms

Induction on k will be used to prove the existence of the lifting

- $k=0 \longrightarrow$ it suffices to take the pre-image of $u|_{[0,t_1]}$ through $p|_{V_{1,j}}$, where $V_{1,j}$ is the open set of Y on (ii) containing the point y_0.

- $k \geq 1$ and let $\tilde{u} : [0, t_{k-1}] \to Y$ be the lift of $u|_{[0,t_{k-1}]}$ already constructed. Let $y_{k-1} = \tilde{u}(t_{k-1})$. There exists j such that $y_{k-1} \in V_{k,j}$. we take $p|_{V_{k,j}}$, which is a local homeomorphism, and

$$\tilde{u}|_{[t_{k-1}, t_k]} = \varphi \circ u|_{[t_{k-1}, t_k]}$$

where $\varphi = \left(p|_{V_{k,j}}\right)^{-1}$.

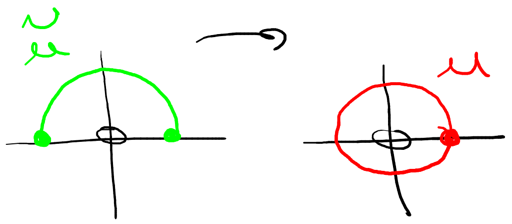
$\boxtimes$

<u>Rmk:</u>

The lift of closed curves can be an open curve.

<u>Example</u>:

$$p : \mathbb{C}^* \longrightarrow \mathbb{C}^*$$
$$z \longmapsto z^2$$



$u(t) = e^{2\pi i t}$, $t \in [0,1]$

(closed curve)

$\tilde{u}(t) = e^{\pi i t}$, $t \in [0,1]$

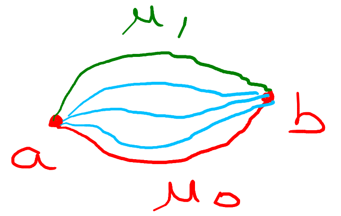
↳ open curve

Lifting of homotopies of curves

Thm:

X, Y Hausdorff top. spaces and $p : Y \to X$ local homeomorphism. Fix $a, b \in X$ and $\tilde{a} \in Y$ such that $p(\tilde{a}) = a$. Let

$$\mu_s : [0,1] \to X$$



a family of homotopic curves where

$$\mu_s(t) = F(t,s) \qquad ; \quad F : I \times I \to X$$

F is the homotopy map (in particular $\mu_s(0) = a$, $\mu_s(1) = b$, $\forall s \in I$). If every curve μ_s can be lifted to a curve $\tilde{\mu}_s$ with $\tilde{\mu}_s(0) = \tilde{a}$, then $\tilde{\mu}_s(1)$ coincide for every $s \in [0,1]$ and the curves $\tilde{\mu}_s$ are all homotopy.

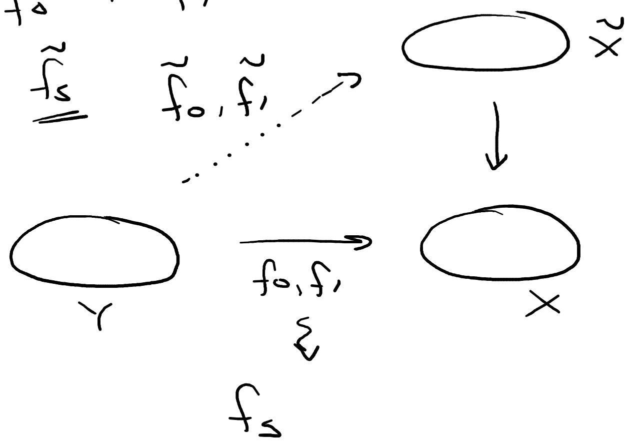
if p is a covering map we can always lift the curves,

Rmk: p covering

$\Rightarrow$ the lift of homotopic curves are always homotopic !!!

More generally:

Thm:
Suppose $X, \tilde{X}$ are two manifolds, $p: \tilde{X} \to X$ local homeomorphism.
If we have a homotopy between two maps $f_0, f_1 : Y \to X$, where Y is another manifold, and if these maps can be lifted to maps $\tilde{f}_0, \tilde{f}_1 : Y \to \tilde{X}$, then $\tilde{f}_0$ and $\tilde{f}_1$ are homotopic as well.



Rmk: curves is a particular case of functions
we saw that lift of curves is always possible if p is a covering map
what about lift of functions? when can we lift them?

Thm (lifting of maps)
Let $p: \tilde{X} \to X$ be a covering where $X, \tilde{X}$ are two manifolds.
Let $f: Y \to X$ be a continuous map, where Y is a manifold.
Then, for every choice of points $y_0 \in Y$ and $\tilde{\omega}_0 \in \tilde{X}$ such that $f(y_0) = p(\tilde{\omega}_0) = \omega_0$, there exists a lifting $\tilde{f}: Y \to \tilde{X}$ such that $\tilde{f}(y_0) = \tilde{\omega}_0$ if and only if

$$\boxed{f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{\omega}_0))}$$

The lift is unique.

Corollary 1:
Every covering map $p: \tilde{X} \to X$, where $X, \tilde{X}$ are manifolds, has the curve lifting property.

<u>Proof</u> : curves : $\alpha : I \longrightarrow X$

In that case $Y = I = [0,1]$ simply connected

$\Downarrow$

$\pi_1(I,0) = id$

(trivial)

$\Rightarrow \alpha_*(\pi_1(I,0)) = id \subseteq p_*(\pi(\tilde{X},\tilde{\omega}_0))$

<u>Corollary 2</u> :

$X, \tilde{X}, Y$ manifolds

$p : \tilde{X} \to X$ covering map

$f : Y \to X$

If we assume Y to be simply connected, then there exists a (unique) lifting $\tilde{f} : Y \to \tilde{X}$.

<u>Rmks</u> :

1. $X, \tilde{X}$ manifolds

$p : \tilde{X} \to X$ covering map

$p_* : \pi_1(\tilde{X}, \tilde{\omega}_0) \longrightarrow \pi_1(X, \omega_0)$ $\qquad p(\tilde{\omega}_0) = \omega_0$

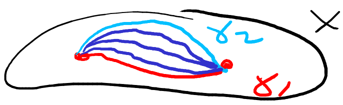
<u>claim</u> : p_* is injective

$[\gamma_1], [\gamma_2] \in \pi_1(\tilde{X}, \tilde{\omega}_0)$

$\alpha_1 = p(\gamma_1) \qquad \alpha_2 = p(\gamma_2)$

If $p_*([\gamma_1]) = p_*([\gamma_2]) \Rightarrow \alpha_1, \alpha_2$ are homotopic loops (closed curves)

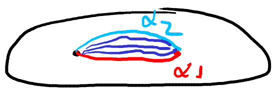
$\tilde{X}$



$\underset{\text{Lift homotopy thm}}{\Longrightarrow}$ γ_1, γ_2 are homotopic loops

(curves have to be closed ...)

$\Rightarrow [\gamma_1] = [\gamma_2]$



X

$\blacksquare$

2. $p : \tilde{X} \to X$ covering map ($X, \tilde{X}$ manifolds)

Consider then the map

$$\# : X \to \mathbb{N} \cup \{\infty\}$$
$$\omega \mapsto \# p^{-1}(\omega)$$

Claim : $\#$ is locally constant

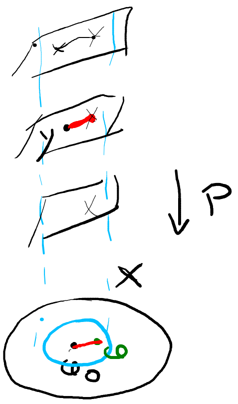
Proof

X manifold $\Rightarrow$ $\underline{\# p^{-1}(\omega)}$ is constant on X

$\downarrow$

The cardinality of $p^{-1}(\omega)$ is called the number of sheets of the covering

ex : $p : \mathbb{C}^* \to \mathbb{C}^*$ is a 2-sheet covering
$z \mapsto z^2$



choose a curve $u : [0,1] \to X$ joining ω_0 to ω.
If $y \in p^{-1}(\omega_0)$ is an arbitrary point, there exists a unique lifting $\tilde{u} : [0,1] \to \tilde{X}$ of u such that $\tilde{u}(0) = y$.
By taking the map

$$\varphi : p^{-1}(\omega_0) \to p^{-1}(\omega)$$
$$y \mapsto \tilde{u}(1)$$

This map is bijective by the uniqueness of lifts.

4. The number of sheets of a covering map
$p : \tilde{X} \to X$ $X, \tilde{X}$ manifolds
is the index of $P_*(\pi_1(\tilde{X}, \tilde{\omega}_0))$ over $\pi_1(X, \omega_0)$ ($p(\tilde{\omega}_0) = \omega_0$)

$\Updownarrow$

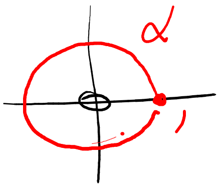
Recall that $P_*(\pi_1(\tilde{X}, \tilde{q}))$ is a subgroup of $\pi_1(X, \omega_0)$ so that the index is well defined.

<u>Exercise</u>: To prove that there exists a bijective map between $p^{-1}(\omega_0)$ and the equivalence classes of g, where g is a closed curve with base point at ω_0.

<u>Example</u>: $p: \mathbb{C}^* \longrightarrow \mathbb{C}^*$ 2-sheet covering
$z \longmapsto z^2$

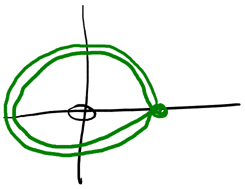
Fix $1 \in \mathbb{C}^*$. We have $\pi_1(\mathbb{C}^*, 1) \simeq \mathbb{Z}$

In fact, $\pi_1(\mathbb{C}^*, 0) = \langle \alpha \rangle$, $\alpha(t) = e^{2\pi i t}$



$$P_*(\pi_1(\mathbb{C}^*, 0)) = \langle \alpha * \alpha \rangle$$

2-turns around the origin



Thus $[\alpha] \in \pi_1(X, \omega_0)$
$[\alpha] \notin P_*(\pi_1(\tilde{X}, \tilde{\omega}_0))$

5. X manifold
Y Hausdorff topological space
$p: Y \to X$ local homeomorphism with the curve lifting property

$\Rightarrow p$ is a covering

<u>Classification of covering spaces</u>

From now on, let X be a manifold (or a Riemann surfaces)

Question : How many covering spaces of X do we have ?

(we will see that we will have a "largest" one, called the universal covering of X, and the other covering spaces can be obtained from the universal covering as quotients.)

$\rightsquigarrow$ Let us precise.

Let X be a manifold and $p : \tilde{X} \to X$ a covering map. Fix a point $\omega_0 \in X$ and $\tilde{\omega}_0 \in p^{-1}(\omega_0)$

$\Rightarrow$ $p_*(\pi_1(\tilde{X}, \tilde{\omega}_0))$ is a subgroup of $\pi_1(X, \omega_0)$

we have then a map

$$\varphi : \tilde{X} \longrightarrow \text{subgroups of } \pi_1(X, \omega_0)$$

Question : Is that map surjective ?

I.e. for every subgroup $G < \pi_1(X, \omega_0)$ that we fix, does there exist $\tilde{X}$ and a covering map $p : \tilde{X} \to X$ such that

$$p_*(\pi_1(\tilde{X}, \tilde{\omega}_0)) = G$$

$\rightsquigarrow$ Particular case : Does there exist $\tilde{X}$ and $p : \tilde{X} \to X$ covering map such that $p_*(\pi_1(\tilde{X}, \tilde{\omega}_0))$ is trivial?

p_* injective $\Rightarrow$ if such $\tilde{X}$ exists, then $\tilde{X}$ is simply connected.

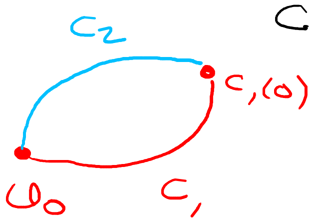
Thm:

Let X be a manifold and fix a point $\omega_0 \in X$. Fixed a subgroup $H < \pi_1(X, \omega_0)$, then there exists a covering $p : \tilde{X} \to X$ such that $p_*(\pi_1(\tilde{X}, \tilde{\omega}_0)) = H < \pi_1(X, \omega_0)$ with $p(\tilde{\omega}_0) = \omega_0$.

Proof:

We begin by considering the set of all paths $c : [0,1] \to X$ with $c(0) = \omega_0$. On this set, we introduce the following equivalence relation

$$\begin{aligned} c_1 &: [0,1] \to X \\ c_2 &: [0,1] \to X \end{aligned}, \quad c_1(0) = c_2(0) = \omega_0$$



$$c_1 \sim c_2 \iff \begin{cases} ① & c_1(1) = c_2(1) \\ ② & [c_2^{-1} * c_1] \in H < \pi_1(X, \omega_0) \end{cases}$$

We call $\tilde{X}$ the set formed by the equivalence classes described above. A point of $\tilde{X}$ is an equivalence class of a path $c \in X$. In particular, there is a natural map

$$\begin{aligned} p : \tilde{X} &\longrightarrow X \\ c &\longmapsto c(1) \end{aligned}$$

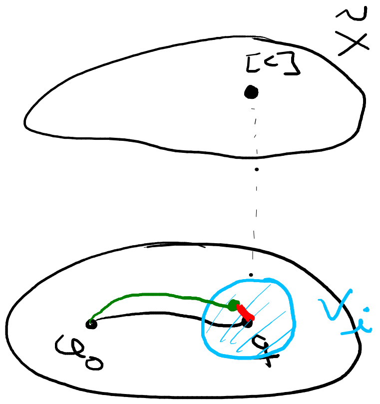
well defined by ① !!

[To have structure of a manifold, we use p to define the charts of the structure]

The remainder of the proof consists of endowing $\tilde{X}$ with a structure of differential manifold for which $p : \tilde{X} \to X$ is the desidered covering map.

(1) Topology on $\tilde{X}$

Let us fix a point $[c] \in \tilde{X}$ and consider the point $q = c(1) \in X$.

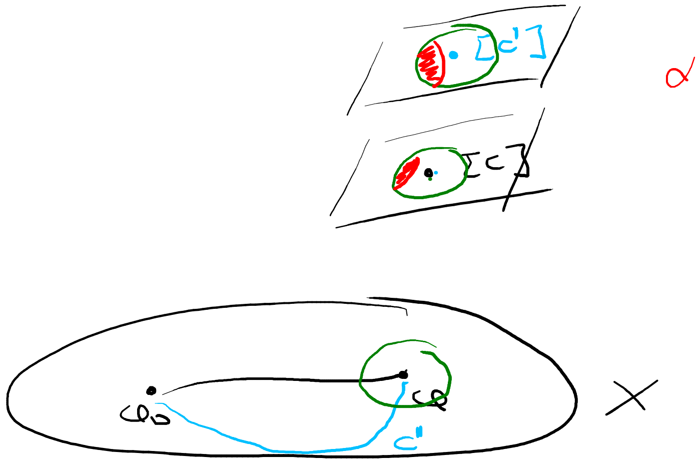


A base of neighbourhoods $\{\mathcal{U}_i\}$ for $[c]$ is defined as follows. Denote by $\{V_i\}$ a basis of (open) neighbourhoods for the point $q \in X$. The sets $\mathcal{U}_i \subseteq \tilde{X}$ are defined by setting

$$[\alpha] \in \mathcal{U}_i \iff \begin{cases} \alpha(1) \in V_i \\ \text{if } \beta \text{ is a path (on } V_i\text{) joining} \\ \alpha(1) \text{ to } q \text{, then } [\bar{c}^{-1} * \beta * \alpha] \in H \end{cases}$$

[Rmk: union of open sets are open
finite intersection of open sets is open]

By constructions, p yields a homeomorphism from $\mathcal{U}_i$ to V_i. In particular p is continuous (the pre-image of open sets is open). Moreover, the V_i can be used to have "$\mathcal{U}_i$" around each point lying in the fiber $p^{-1}(q) \subseteq \tilde{X}$. and the map p satisfies the conditions to be a covering map



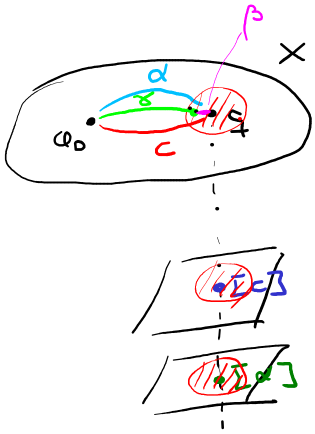
$[(c')^{-1} * c] \notin H$

Let us then prove that for a neighbourhood arbitrarily small of $\omega \in X$, the pre-image are pairwise disjoint.

Fix c : $c(1) = q$, $[c] \in \mathcal{U}_j$ for some j

Fix then α : $\alpha(1) = q$, $[\alpha] \notin \mathcal{U}_j$ (this means that $[\alpha^{-1} * c] \notin H$)

$([\alpha] \in \mathcal{U}_k$ for some $k \neq j)$



Assume now that $\mathcal{U}_j \cap \mathcal{U}_k \neq \emptyset$. Let then $[\gamma] \in \mathcal{U}_j \cap \mathcal{U}_k$.
we have that

- $\gamma(1) \in V_i$
- if β joins in V_i, $\alpha(1)$ to q, then the loops

$$[c^{-1} * \beta * \gamma] \in H$$
$$[\alpha^{-1} * \beta * \gamma] \in H$$

Since H is a subgroup, we must have that

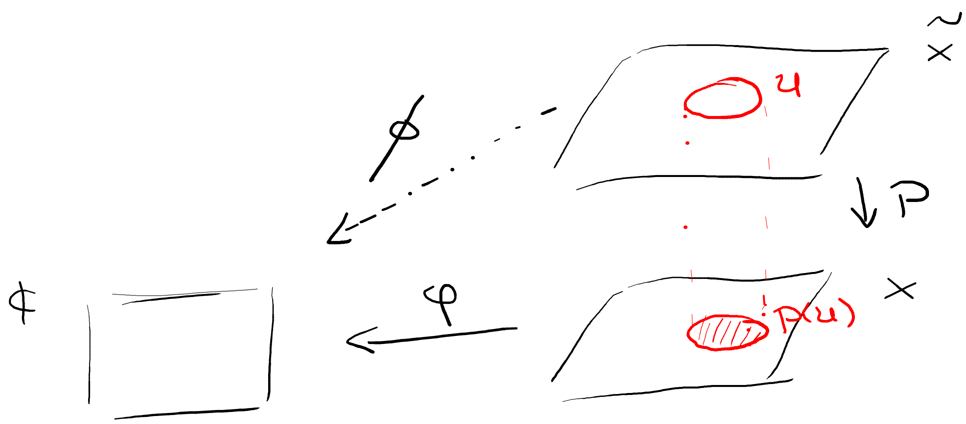
$$[(\alpha^{-1} * \beta * \gamma) * (c^{-1} * \beta * \gamma)^{-1}] \in H$$

i.e.
$$[\alpha^{-1} * c] \in H$$

contradicting our assumption. The result follows.

To have the structure of a manifold on $\tilde{X}$, we use p to define the charts of the structure.

Let $\mathcal{U} \subseteq \tilde{X}$ be an open set such that $p : \mathcal{U} \to p(\mathcal{U}) \subseteq X$ is a homeomorphisms. Up to reducing $\mathcal{U}$, we can assume that $p(\mathcal{U})$ is contained in the domain of definition of a chart $\{\varphi\}$ of X



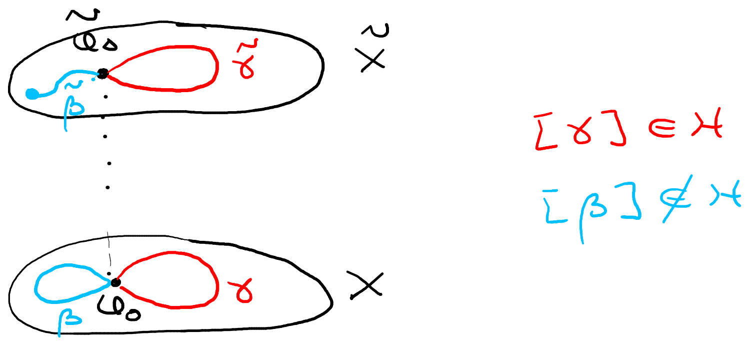
we then define charts $\{\phi\}$ on $\mathcal{U}$ by letting

$$\boxed{\phi = \varphi \circ p}$$

In particular $\tilde{X}$ has the same structure of X and p is compatible with this structure.

- if X is differentiable manifold, so is $\tilde{X}$ and $p:\tilde{X}\to X$ is differentiable
- if X is a Riemann surface, so is $\tilde{X}$ and $p:\tilde{X}\to X$ is holomorphic.

It only remains to show that $p_*(\pi_1(\tilde{X},\tilde{\omega}_0)) = H$. Fix a point $\tilde{\omega}_0 \in p^{-1}(\omega_0)$. Since p_* is injective, it suffices to show that a loop $c:[0,1]\to X$ $(c(0)=c(1)=\omega_0)$ lifts in a loop at $\tilde{X}$ if and only if $[c]\in H$



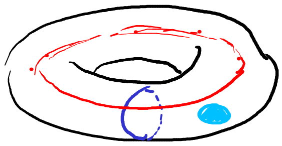
Fix a loop $c:[0,1]\to X$. Let $\tilde{c}$, $\tilde{c}(0)=\tilde{\omega}_0$, a lift of the loop c. If $\tilde{c}(1)\neq\tilde{\omega}_0$ (i.e. if $\tilde{c}$ is not a loop) then $\tilde{c}$ represents a point in $\tilde{X}$ different from the $[\tilde{\omega}_0]$ ($\tilde{\omega}_0$ identified with the constant curve). In particular

$$[c] = [\omega_0^{-1} * c] \notin H.$$

Conversely, if $\tilde{c}(1)=\tilde{\omega}_0$ (i.e. if $\tilde{c}$ is a loop) then $\tilde{c}$ defines the same point in $\tilde{X}$ than $\tilde{\omega}_0$ (the constant curve) and this means, by definition, that $[c]\in H$. This completes the proof of the theorem. ∎

Examples:

$\mathbb{C}/\Pi$ tori

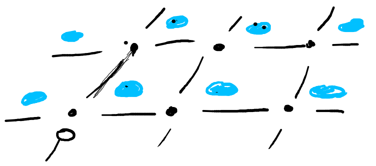


$\pi_1(\mathbb{C}/\Pi) \simeq \mathbb{Z} \oplus \mathbb{Z}$

Subgroups of $\pi_1(\mathbb{C}/\Pi)$ $\begin{cases} \{0\} \\ \mathbb{Z} \oplus \{0\} \end{cases}$

- $p_1 : \mathbb{R}^2 \longrightarrow \mathbb{C}/\Pi$

  $p_1(s,t) = \left(e^{2\pi i s}, e^{2\pi i t}\right)$

  the natural projection is a covering such that

  $$p_*(\underbrace{\pi_1(\mathbb{R}^2, 0)}_{\mathbb{R}^2 \text{ is simply connected}}) = \{0\}$$

- $p_2 : \underset{\text{(cylinder)}}{S^1 \times \mathbb{R}} \longrightarrow \mathbb{C}/\Pi$

  $p_2(z,t) = \left(z, e^{2\pi i t}\right)$

  p_2 is a covering map such that

  $$p_*(\pi_1(S^1 \times \mathbb{R})) = \mathbb{Z} \oplus \{0\}$$

Corollary:

Suppose that X is a manifold. Then there is a simply connected manifold $\tilde{X}$ and a covering map $p : \tilde{X} \to X$.

Rmk:

Given X manifold, $\omega_0 \in X$, $H < \pi_1(X, \omega_0)$. Let X_1, X_2 be two manifolds and $p_i : X_i \longrightarrow X$ two coverings such that

$$p_*(\pi_1(\tilde{X}_i, \tilde{\omega}_0)) = H$$

Then there is an isomorphism between X_1 and X_2.

Def

A simply connected manifold covering X is called the **universal covering** of X.

$$\begin{tikzcd} & X_2 \arrow[ld, dashed, bend left] \arrow[d, "p_2"] \\ X_1 \arrow[ru, dashed, "f"] \arrow[r, "p_1"'] & X \end{tikzcd}$$

$f : X_1 \to X_2$

$g : X_2 \to X_1$

$f \rightsquigarrow$ lifting of p_1 through p_2 : $\underbrace{p_1 = p_2 \circ f}$

$g \rightsquigarrow$ lifting of p_2 through p_1 : $p_2 = p_1 \circ g$

$p_1 = p_1 \circ g \circ f$

(locally) $\Rightarrow$ $g \circ f = id$

$\Rightarrow$ $g \circ f = id$ everywhere

Identity thm.

Deck Transformations

Let the X and $H < \pi_1(X, \omega_0)$ be as before. We know that

$$\boxed{p^{-1}(\omega_0) \text{ is in bijection with the equivalence classes } \pi_1(X,\omega_0)/H}$$

Let $p : \tilde{X} \to X$ be a covering

Def :

A deck transformation $\sigma : \tilde{X} \to \tilde{X}$ is a homeomorphism (diffeo, hol) such that the diagram below commutes.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\sigma"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowright"] & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

$$\boxed{p \circ \sigma = p}$$

we have that σ preserves the fibers of p ($p^{-1}(y), y \in X$)

Examples:

1. $p : \mathbb{R}^2 \to \mathbb{R}^2/\Pi$ canonical projection

$\Pi = \{ \alpha.(1,0) + \beta(0,1) : \alpha, \beta \in \mathbb{Z} \}$

$\sigma : \mathbb{R}^2 \to \mathbb{R}^2$
$\sigma(x,y) = (x+m, y+n)$ $m, n \in \mathbb{Z}$

2. $p : \mathbb{C}^* \to \mathbb{C}^*$, $k \in \mathbb{N}$, $k \geq 2$
$z \mapsto z^k$

ω a k-root of the unit

$\sigma : \mathbb{C}^* \to \mathbb{C}^*$ $\boxed{p \circ \sigma = p}$
$z \mapsto \omega z$

$(\omega z)^k = \omega^k z^k = z^k$

$$(p \circ \sigma)(z) = p(\sigma(z)) = p(\omega z)$$
$$= (\omega z)^k = \omega^k z^k = z^k = p(z)$$

The interest of this definition arises from the following proposition.

Prop:

Let $H < \pi_1(X, x_0)$ and denote by $p : \tilde{X} \to X$ the covering map associated to H. Then:

(1) $\pi_1(X, x_0)$ acts on $\tilde{X}$ by deck transformations

(2) An element $[c] \in \pi_1(X, x_0)$ acts trivially on $\tilde{X}$ if and only if $[c] \in H$.

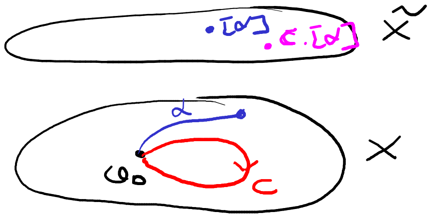
Proof:

It amounts to properly define the action, which is done as follows. Recall that $\tilde{X} = \{ [\alpha] : \alpha : [0,1] \to X, \alpha(0) = x_0 \}$ with

$$[\alpha] = [\beta] \iff \begin{cases} \alpha(1) = \beta(1) \\ [\beta^{-1} * \alpha] \in H \end{cases}$$

Given $[c] \in \pi_1(X, x_0)$ we define $c : \tilde{X} \to \tilde{X}$ by letting

$$c.[\alpha] = [\alpha * c^{-1}]$$



$[\alpha]$, $c.[\alpha]$ projects on the same point of X since the end point of the curves α and $\alpha * c^{-1}$ are the same

$\Rightarrow$ c sends fibers to fibers. It is also straightforward to check that c is a homeomorphism of $\tilde{X}$ to $\tilde{X}$.

$\Rightarrow$ $c : \tilde{X} \to \tilde{X}$ is a deck transformation.

we have $\boxed{\rho : \pi_1(X, \omega_0) \to \text{Deck transformations}}$

Note that $c : \tilde{X} \to \tilde{X}$ coincides with the identity if and only if $[c] \in H$. Indeed

$$c.[\alpha] = [\alpha * c^{-1}]$$

and

$$\begin{aligned} c.[\alpha] = [\alpha] &\iff [\alpha * c^{-1}] = [\alpha] \\ &\iff \alpha^{-1} \circ \alpha \circ c^{-1} \in H \\ &\iff c^{-1} \in H \\ &\iff c \in H \end{aligned}$$

Finally, we have that the assignment

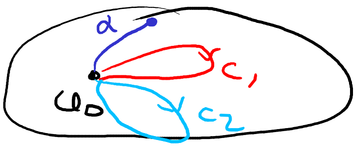
$$c \in \pi_1(X, \omega_0) \rightsquigarrow c : \tilde{X} \to \tilde{X}$$

is indeed an action, i.e it satisfies

$$c_2 * c_1 \in \pi_1(X, \omega_0) \rightsquigarrow c_2 \circ c_1 : \tilde{X} \to \tilde{X}$$

$$\Updownarrow$$

$$\begin{aligned} (c_2 * c_1)[\alpha] &= [\alpha * (c_2 * c_1)^{-1}] = [\alpha * c_1^{-1} * c_2^{-1}] \\ &= c_2 [\alpha * c_1^{-1}] = c_2(c_1, [\alpha]) \\ &= (c_2 \circ c_1)[\alpha] \end{aligned}$$



Let $\Gamma \subseteq \mathrm{Aut}(\tilde{X})$ denote the set of deck transformations arising from $\pi_1(X, \omega_0)$. Clearly each element in Γ $\mathrm{Aut}(\tilde{X})$ corresponds to an equivalence class in $\pi_1(X, \omega_0)/H$ since H is the kernel of the action.

In other words, there is

$$\rho : \pi_1(X, \omega_0) \longrightarrow \mathrm{Aut}(\tilde{X}, \omega_0)$$

whose kernel is presicely H.

Quotients of Group actions

* Free action: we say that Γ acts freely on $\tilde{X}$ if the following is satisfied: if there is $\tilde{\omega} \in \tilde{X}$ and $\gamma \in \Gamma$ such that $\gamma(\tilde{\omega}) = \tilde{\omega}$, then $\gamma = id$.

  (i.e. there is no element of Γ with fixed points bar the identity)

* Properly discontinuous: we say that the action of Γ is properly discontinuous if for any compact set $K \subseteq \tilde{X}$ and for every point $\tilde{\omega} \in \tilde{X}$, the cardinality of the set

$$\{ \gamma \in \Gamma : \gamma(\tilde{\omega}) \in K \}$$

is finite.

Assume that Γ acts freely and properly discontinuously on a manifold $\tilde{X}$.

$\Updownarrow$ Exercise

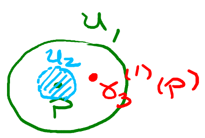
every point $\tilde{\omega} \in \tilde{X}$ admits a neighbourhood $U \subseteq \tilde{X}$ such that $\{\gamma(U)\}_{\gamma \in \Gamma}$ are pairwise disjoint, i.e. $\gamma_1(U) \cap \gamma_2(U) = \emptyset \quad \forall \gamma_1 \neq \gamma_2$

$\Updownarrow$

every point $\tilde{\omega} \in \tilde{X}$ admits a neighbourhood $U \subseteq \tilde{X}$ such that

$\forall \gamma \in \Gamma, \gamma \neq id$ we have $\gamma(U) \cap U = \emptyset$

Idea: Assume for a contradiction that this is not true



$\exists\, \gamma_1^{(1)}, \gamma_2^{(1)} : \gamma_1^{(1)}(U_1) \cap \gamma_2^{(1)}(U_1) \neq \emptyset$

$\Updownarrow$

$\underbrace{\gamma_1^{(1)} \circ (\gamma_2^{(1)})^{-1}}_{\gamma_3^{(1)}}(U_1) \cap U_1 \neq \emptyset$

$\gamma_3^{(1)} \in \Gamma : \gamma_3^{(1)}(U_1) \cap U_1 \neq \emptyset$

action is free $\Rightarrow \gamma_3^{(1)}(p) \neq p$

I repeat the process for U_2

$\exists\, \gamma_1^{(2)}, \gamma_2^{(2)} : \gamma_1^{(2)}(U_2) \cap \gamma_2^{(2)}(U_2) \neq \emptyset$

$\gamma_3^{(2)} \in \Gamma : \gamma_3^{(2)}(U_2) \cap U_2 \neq \emptyset$

$\underbrace{\gamma_1^{(2)} \circ (\gamma_2^{(2)})^{-1}}_{\gamma_3^{(2)}}(U_2) \cap U_2 \neq \emptyset$

action is free $\Rightarrow \gamma_3^{(2)}(p) \neq p$

Exercise : Continue the argument and arrive to a contradiction.

Γ acting freely and properly discontinuously

$\Downarrow$

$\Gamma.\omega = \{ \gamma\omega : \gamma \in \Gamma \}$ is a closed subset of $\tilde{X}$

Rmk :

$\tilde{X}$ compact manifold
Γ acts properly discontinuously and freely on $\tilde{X}$ $\Big\}$ $\Rightarrow$ Γ is finite

$\Downarrow$ in fact

there is a bijection between the orbit of ω ($\Gamma.\omega$) and Γ and $\underline{\Gamma.\omega}$ $\underline{\text{is closed}}$.

$y \notin \Gamma.\omega \Rightarrow \exists\, U$ neighbourhood of y such that $\Gamma.\omega \cap U = \emptyset$

$\circledast \Downarrow$

Suppose that $\forall\, U$ neighbourhood of y we have $\Gamma.\omega \cap U \neq \emptyset$

$\Rightarrow \gamma_n \in \Gamma : \gamma_n.\omega \longrightarrow y$

Let $y_n = \gamma_n.\omega$ $(\Leftrightarrow \omega = \gamma_n^{-1}.y_n)$. we must have

$\#\{y_n\} < \infty$ since the action is properly discontinuous

$\Rightarrow \exists\, n_0 : \forall n \geq n_0 ,\ \gamma_n.\omega = y$

$\Rightarrow y \in \Gamma.\omega$

contradiction !!

$\circledast \Rightarrow \tilde{X} \setminus (\Gamma.\omega)$ is open

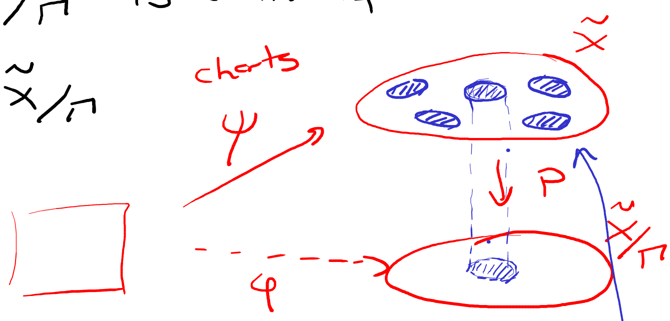
$\Rightarrow \Gamma.\omega$ is closed

Prop:

Assume that Γ acts freely and properly discontinuously on a manifold $\tilde{X}$. Then the quotient $\tilde{X}/\Gamma$ is a manifold and the canonical projection

$$p : \tilde{X} \longrightarrow \tilde{X}/\Gamma$$

is a covering map.



Proof:

Fix $y \in p(\omega)$, $\omega \in \tilde{X}$, $y \in \tilde{X}/\Gamma$ and let $\mathcal{U}$ be an arbitrarily small neighbourhood of ω (in the sense that $\mathcal{U}$ does not contain two points of any orbit of Γ)

disjoint

Since p is an open map, $V = p(\mathcal{U})$ is an open set and, consequently, an open neighbourhood of y. We have that

$$p^{-1}(V) = \underbrace{\bigcup_{g \in \Gamma} g.\mathcal{U}}_{\text{union of disjoint open sets}}$$

$\Rightarrow$ $p|_{g.\mathcal{U}}$ is an homeomorphism onto V $\blacksquare$

Examples:

For a given covering $p : \tilde{X} \longrightarrow X$, $p(\tilde{\omega}_0) = \omega_0$, the action of $\Gamma = \rho(\pi_1(X, \omega_0)) \subseteq \mathrm{Aut}(\tilde{X}, \tilde{\omega}_0)$ is free and properly discontinuous

Corollary: $X = \tilde{X}/\Gamma$

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{\omega}_0) \arrow[dl, "p"'] \arrow[dr, "\pi"] \arrow[d, phantom, "\circlearrowright"] & \\
X \arrow[rr, dashed, bend left=10, "f"] & & \tilde{X}/\Gamma \arrow[ll, dashed, bend left=10, "g"]
\end{tikzcd}$$

locally p, π are homeomorphisms

- $\pi \circ p^{-1} = f$
- $p \circ \pi^{-1} = g$

Corollary:

Assume that Π acts freely and properly discontinuously on a simply connected manifold $\tilde{X}$. Then the quotient $\tilde{X}/\Pi$ is a manifold, the canonical projection

$$p : \tilde{X} \longrightarrow \tilde{X}/\Pi$$

is a covering map and the fundamental group of $\tilde{X}/\Pi$ identifies with Π.

$$\rho : \pi_1(\tilde{X}/\Pi, \omega_0) \to Aut(\tilde{X}, \tilde{\omega}_0)$$

injective when $\tilde{X}$ is simply connected

$H = id$, $H = \ker \rho$

Recall that:

Thm: Every simply connected Riemann surface is isomorphic to one of the following Riemann surfaces: $\mathbb{C}$, $\mathbb{CP}(1)$, $\mathbb{D}$.

Thm: Every Riemann surface S is biholomorphically equivalent to D/Π, where D is $\mathbb{C}$, $\mathbb{CP}(1)$ or $\mathbb{D}$ and Π is a group acting freely and properly discontinuously on D.

To understand all Riemann surfaces, we have to understand all subgroups acting freely and properly discontinuously on the simply connected Riemann surfaces $\mathbb{C}$, $\mathbb{CP}(1)$ and $\mathbb{D}$.

$\boxed{\mathbb{CP}(1)}$

The only Riemann surface covered by $\mathbb{CP}(1)$ is $\mathbb{CP}(1)$ itself.

Let S be a Riemann surface and assume that S is covered by $\mathbb{C}\mathbb{P}(1)$

$$\begin{tikzcd} \mathbb{C}\mathbb{P}(1) \arrow[rr, dashed, "\sigma"] \arrow[dr, "p"'] & & \mathbb{C}\mathbb{P}(1) \arrow[dl, "p"] \\ & S & \end{tikzcd}$$

σ is a deck-transformation

$\sigma \in \mathrm{Aut}(\mathbb{C}\mathbb{P}(1))$

$\Rightarrow$ σ takes on the form $\sigma(z) = \dfrac{az+b}{cz+d}$: $ad-bc=1$

Every element $\sigma \in \mathrm{Aut}(\mathbb{C}\mathbb{P}(1))$ has a fixed point

$\Longrightarrow$ $\Gamma = \{ id \}$

Γ acts freely $\Longrightarrow$ $S = \mathbb{C}\mathbb{P}(1)/_{\Gamma} = \mathbb{C}\mathbb{P}(1)$

$\boxed{\mathbb{C}}$

$\mathrm{Aut}(\mathbb{C}) = \{ z \mapsto az+b \ : \ a,b \in \mathbb{C}, \ a \neq 0 \}$

$\Gamma \subseteq \mathrm{Aut}(\mathbb{C})$ acts freely $\Rightarrow$ $\sigma \in \Gamma$, $\sigma \neq id$
σ cannot have fixed points

$\Downarrow$

$\boxed{a = 1}$

assume $a = 2$

$\sigma(z) = z$

$\Leftrightarrow 2z + b = z$

$\Leftrightarrow \underline{\underline{z = -b}}$

So, we restrict ourselves to the subgroup of $\mathrm{Aut}(\mathbb{C})$ given by the translations.

$$\Gamma < \{ z \mapsto z+b \ : \ b \in \mathbb{C} \}$$

However, Γ acts properly discontinuously $\Rightarrow$ Γ is discrete

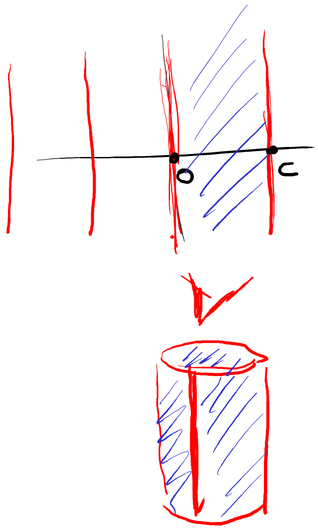
<u>Discret subgroups</u> :

1. $\Gamma = \{ id \}$
2. $\Gamma = \langle z \mapsto z + c \rangle = \{ z \mapsto z + nc : n \in \mathbb{Z} \}$ $\quad \underline{c \neq 0}$
3. $\Gamma = \langle z \mapsto z+\alpha, \ z \mapsto z+\beta \rangle$ $\quad \alpha/\beta \notin \mathbb{R}$
$= \{ z \mapsto z + n\alpha + m\beta : n,m \in \mathbb{Z} \}$

① $\longmapsto$ $\mathbb{C}$

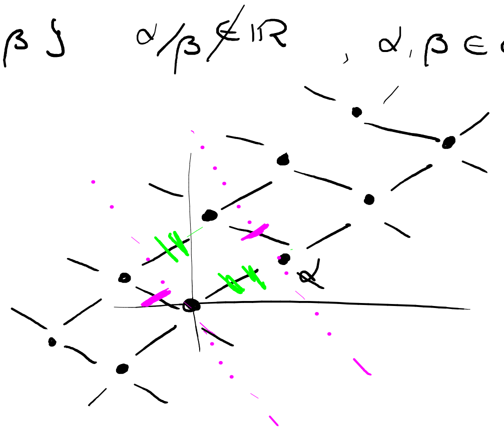
② $\longmapsto$ $\Gamma = \langle z \mapsto z + c \rangle$

$\Rightarrow S = \mathbb{C}/\Gamma$ cylinder



③ $\longmapsto$ $\Gamma = \langle z \mapsto z+\alpha,\ z \mapsto z+\beta \rangle$ $\alpha/\beta \notin \mathbb{R}$, $\alpha, \beta \in \mathbb{C}^*$

$\Rightarrow S = \mathbb{C}/\Gamma$ tori



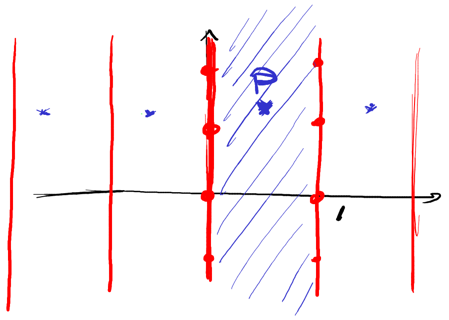
Questions :

Topologically, the toris are all the same and the cylinders are all the same as well. what about the analytic point of view? Are they the same?

$\boxed{\text{Cylinders}}$: analytically, they are all the same :

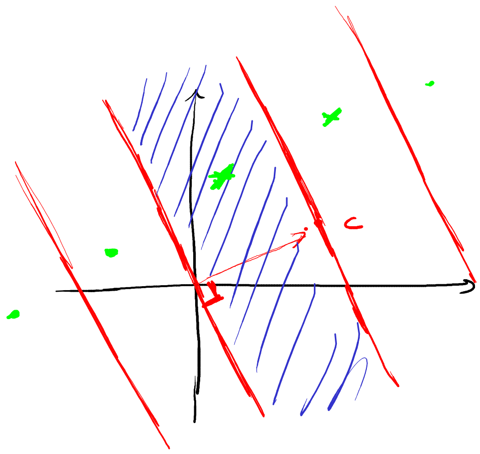
$C_1 = \mathbb{C}/\Gamma_1$, $\Gamma_1 = \langle z \mapsto z+1 \rangle$

$C_2 = \mathbb{C}/\Gamma_2$, $\Gamma_2 = \langle z \mapsto z+c \rangle$



$f(z) = z+1$

$\boxed{T(z) = cz}$



$g(z) = z+c$

we can check that

$$g \circ T = T \circ f$$

$$\Longrightarrow \quad g^n \circ T = T \circ f^n$$

This passes to the quotient

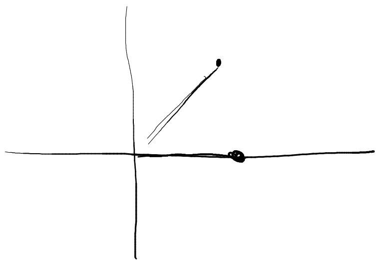
p on the fundamental domain $\rightsquigarrow$ $q = f^k(p)$

$$T(q) = T \circ f^k(q)$$
$$= g^k \circ T(q)$$
$$= g^k(T(q))$$

$\Rightarrow$ T induces a (holomorphic) map on the cylinders.

$\Rightarrow$ the cylinders are analytically the same.

<u>Exercise</u>: To look for the case of tori.



$z \mapsto z+1$

$z \mapsto z+\tau$

$\tau \in \mathbb{C} \setminus \mathbb{R}$

$\mathrm{Im}(\tau) > 0$

It remains to decide when $\tau_1 \neq \tau_2$ define the same torus

<u>Answer</u>: They define the same tori

$$\Updownarrow$$

$$\exists\, M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \; : \; \frac{a\tau_1 + b}{c\tau_1 + d} = \tau_2$$

Let us then check that the discret subgroups of Γ of the group of translations $\{ z \mapsto z+b : b \in \mathbb{C} \}$ are

① $\Gamma = \{ id \}$

② $\Gamma = \langle z \mapsto z+a \rangle = \{ z \mapsto z+na : n \in \mathbb{Z} \}$, $a \in \mathbb{C}^*$

③ $\Gamma = \langle z \mapsto z+a , z \mapsto z+b \rangle$ $\quad a, b \in \mathbb{C}^*$

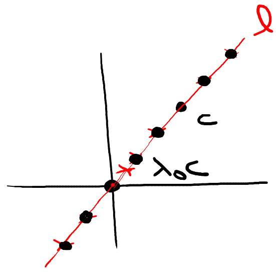
$= \{ z \mapsto z+na+mb : n, m \in \mathbb{Z} \}$ $\quad a/b \notin \mathbb{R}$

To begin with, assume that $\Gamma \neq \{ id \}$. Let then φ be an element of Γ, $\varphi \neq id$. We have that φ takes on the form

$$\varphi : z \longmapsto z+c$$

for some $c \in \mathbb{C}^*$. Consider now the elements of Γ (1-parameter family) taking on the form

$$\varphi_\lambda(z) = z + \lambda c \qquad \text{where } \underline{\underline{\lambda \in \mathbb{R}}}$$



Since Γ is discrete, there exists $0 < \lambda_0 \leq 1$ such that $\varphi_{\lambda_0} \in \Gamma$ and for all $\varphi_\lambda \notin \Gamma$ for all $0 < \lambda < \lambda_0$.

In that case

$$\langle z \mapsto z + \lambda_0 c \rangle \subseteq \Gamma$$

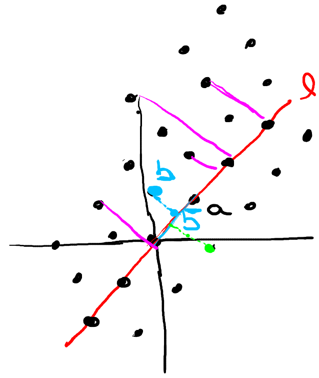
If $\Gamma \subseteq \langle z \mapsto z + \lambda_0 c \rangle$ as well, then we are in case ②. Let us then prove that if we are not in case ② (i.e $\Gamma \not\subseteq \langle z \mapsto z+\lambda_0 c \rangle$) then we are in case ③. To simplify notation, let $a = \lambda_0 c$ and denote by l the straight line passing through the origin and a.

Consider then the subset of $\mathbb{C}$ defined as

$$A = \{ d \in \mathbb{C} : \{ z \mapsto z+d \} \in \Gamma \}$$

This set has to be a discrete subset of $\mathbb{C}$ from the fact that Γ is a discrete subset of the translation group. Let

$$\bar{A} = \{ d \in A : d \text{ has minimum distance to } l \}$$



(the mentioned distance above is strictly positive since A is discrete). Let b be the element of $\bar{A}$ whose projection on l takes on the form

$$\bar{b} = \lambda a \qquad \text{for some } 0 \leq \lambda < 1$$

(angle between a and b is positive)

Then

$$\Pi = \langle z \mapsto z+a,\ z \mapsto z+b \rangle$$

otherwise, there would exist an element $z \mapsto z+\alpha \in \Pi$ whose α has distance to ℓ less than the distance of b, contradicting our assumption.

$\boxtimes$

All the other Riemann surfaces are covered by the disc.

Questions:

- How many Riemann surfaces covered by the disc do we have?
- How many groups acting freely and properly discontinuously on the disc do we have?

Recall that

$$\mathrm{Aut}(\mathbb{D}) = \left\{ \frac{az+b}{\bar{b}z+\bar{a}} : a,b \in \mathbb{C},\ |a|^2 - |b|^2 = 1 \right\}$$

$$= \left\{ e^{i\theta} \frac{z-a}{-\bar{a}z+1} : a \in \mathbb{D},\ \theta \in [0, 2\pi[\right\}$$

Classification of the elements of $\mathrm{Aut}(\mathbb{D})$

To begin with, recall the following theorem:

Brouwer's Fixed point theorem:

Let D be the closed unit ball in $\mathbb{R}^n$ and $f: D \to D$ a continuous map. Then there exists $\omega \in D$ such that

$$f(\omega) = \omega$$

Since the möbius transformations on $\mathrm{Aut}(\mathbb{D})$ can be (continuously) extended to $\overline{\mathbb{D}}$, we have that their extensions

$$\bar{f}: \overline{\mathbb{D}} \to \overline{\mathbb{D}}$$

possesses at least one fixed point on $\overline{\mathbb{D}}$.

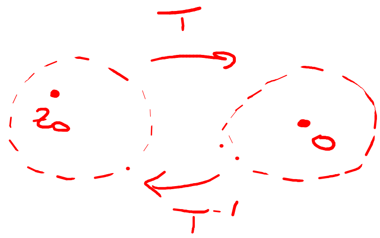
$\boxed{1^{st} \text{ case}}$

If f has a fixed point in $\mathbb{D}$, then f is called <u>elliptic</u>.

$\Updownarrow$

Let $z_0 \in \mathbb{D}$ stands for the fixed point of f. Let T be the element of $Aut(\mathbb{D})$ taking z_0 to the origin. Recall that T is given by

$$T(z) = \frac{z - z_0}{\overline{z_0} z + 1}$$



Consider then

$$R = T \circ f \circ T^{-1}$$

The map R has the origin as a fixed point. Furthermore, R is the composition of elements in $Aut(\mathbb{D})$, so that $R \in Aut(\mathbb{D})$. We have already checked that R is a rotation, i.e

$$R(z) = e^{i\theta} z$$

for some $\theta \in [0, 2\pi[$.

Being f conjugated to a rotation, z_0 is the <u>unique fixed point of f</u>.

möbius transformations

$$\frac{az+b}{cz+d} = z$$

$$\Leftrightarrow az+b = z(cz+d)$$

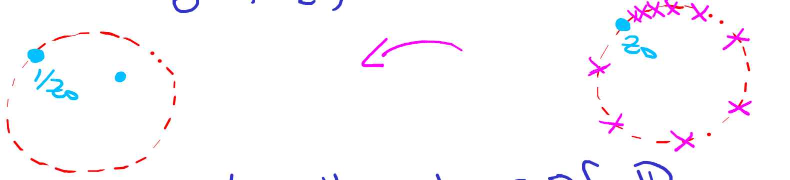
$\boxed{2^{nd} \text{ case}}$

Assume that f has no fixed points in $\mathbb{D}$. The fixed points are in the boundary of $\mathbb{D}$, $\overline{\mathbb{D}}$. Since f is a möbius transformation, it may have at most 2 fixed points in the boundary of $\mathbb{D}$.

If f has two distinct fixed points in the boundary of $\mathbb{D}$, then f is called <u>hyperbolic</u> or <u>loxodromic</u>.

$g = f(\frac{1}{z})$

$z \mapsto \frac{1}{z} = w$



g fixis one point in the interior of $\mathbb{D}$ $\Rightarrow$ we are in the 1st case

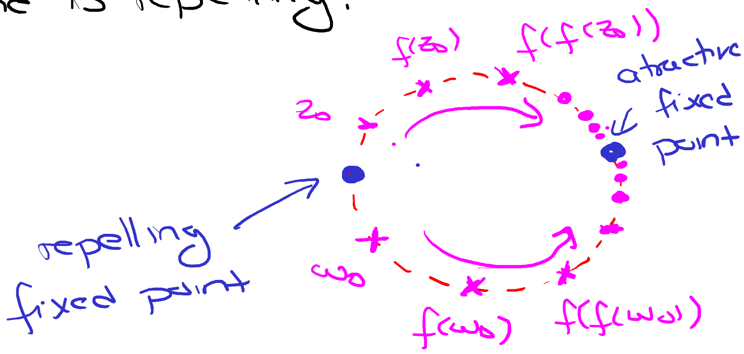
$\Rightarrow$ we cannot have fixed points in the boundary of $\mathbb{D}$

Conclusion: If we have a fixed point in the boundary of $\mathbb{D}$, then all fixed points will be in the boundary of $\mathbb{D}$.

f hyperbolic

$\Updownarrow$

In terms of the dynamics of f, one of the fixed points is attractive and the other one is repelling.



Idea of the proof:

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}} \qquad a, b \in \mathbb{C}, \quad |a|^2 - |b|^2 = 1$$

$$\boxed{SL_2(\mathbb{R})}$$

$\Updownarrow$

f is identified with the matrix given by

$$A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$$

Rmk: $|a|^2 - |b|^2 = 1$

$\Updownarrow$

$|A| = 1$

$z = \alpha + i\beta$ is a fixed point iff

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

the fixed points are then the eigenvectors of A

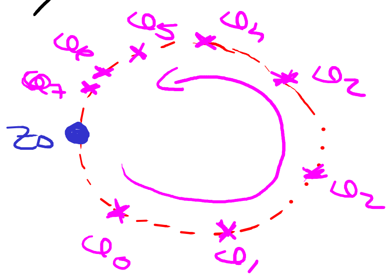
fixed points: z_1, z_2

$f'(z_1)$, $f'(z_2)$

$$|f'(z_1)| \cdot |f'(z_2)| = 1$$

we can also check that $|f'(z_i)| \neq 1$ so that one of them will be >1 and the other one <1.

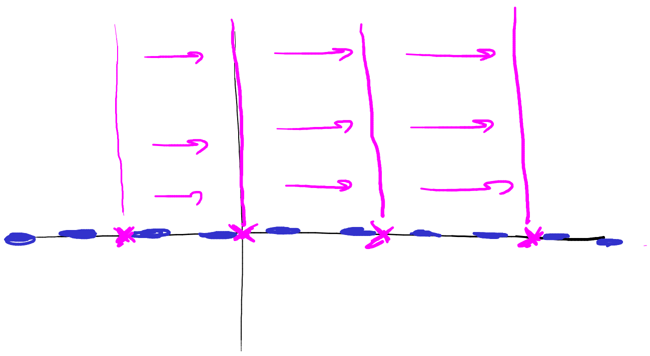
If f has a unique fixed point in the boundary of $\mathbb{D}$, then f is called parabolic



$\omega_i = (\underbrace{f \circ \dots \circ f}_{i \text{ times}})(\omega_0)$

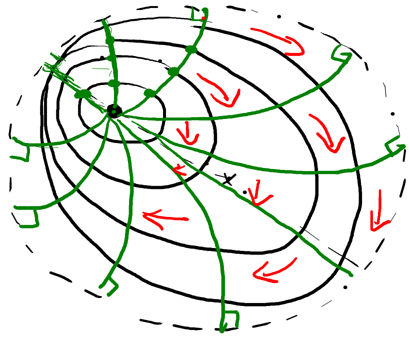
Recall that $\mathbb{D}$ is isomorphic to $\mathbb{H}$, the upper half-plane. The map f viewed in $\mathbb{H}$ is conjugated to the translation

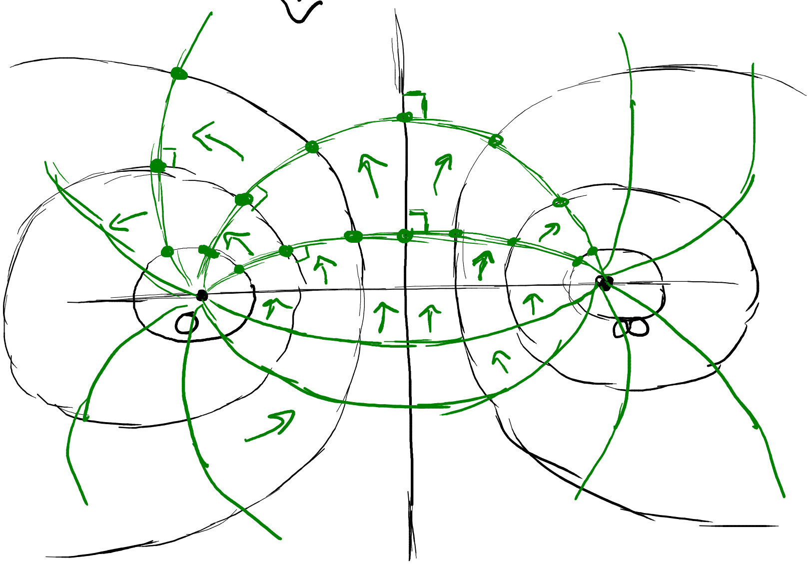
$$z \longmapsto z+1$$



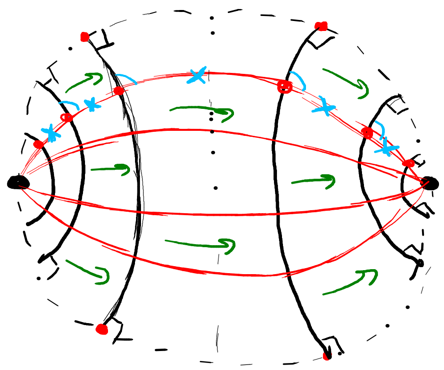
Pictures

Elliptic case :

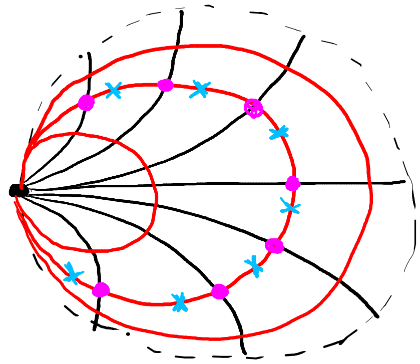




Hyperbolic case



Parabolic case



Thm:

Consider the transformations

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$$

$A = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$

$\text{trace}(A) = 2\,\text{Re}(a)$

where $a, b \in \mathbb{C}^*$, $|a|^2 - |b|^2 = 1$, we have that

- f is hyperbolic iff $|\text{Re}(a)| > 1$
- f is elliptic iff $|\text{Re}(a)| < 1$
- f is parabolic iff $|\text{Re}(a)| = 1$

Remarks:

- Recall that a cylinder may be covered by $\mathbb{C}$. More precisely a cylinder may be given as

$$\mathbb{C}/\Gamma \quad , \text{ where } \quad \Gamma = \langle z \mapsto z + c \rangle \qquad c \in \mathbb{C}^*$$

In particular we have proved that any two cylinders with the complex structure obtained from this quotient are isomorphic.

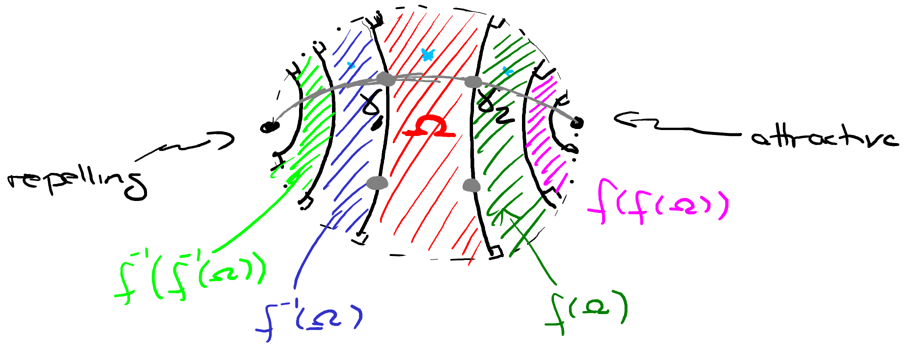
However, a cylinder can also be covered by the disc.

Consider a hyperbolic element

$$f(z) = \frac{az+b}{\bar{b}z+\bar{a}} \quad , \; a,b \in \mathbb{C} \; , \; |a|^2 - |b|^2 = 1$$

$$|Re(a)| > 1$$

Being f an hyperbolic elemnt, f has two fixed points (distinct fixed points) on the boundary of $\mathbb{D}$, one being attractive and the other one repelling.



Let then $\underbrace{\Omega \text{ be a fundamental domain for } f}$

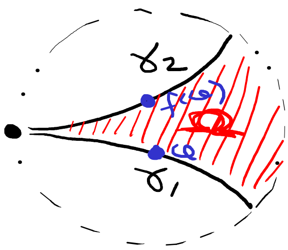
$$\bigcup_{k \in \mathbb{Z}} f^k(\Omega) = \mathbb{D} \qquad \text{and} \qquad int(f^k(\Omega)) \cap int(f^j(\Omega)) = \emptyset$$

If we identify the two boundaries γ_1 and γ_2 of the fundamental domain ($\gamma_2 = f(\gamma_1)$) by associating one element of γ_1 to its image through f, we obtain a manifold that is, topologically a cylinder C. A complex structure can naturally be defined on C by means of the quotient $\mathbb{D}/\langle f \rangle$. This complex structure is, however, completely different from the one obtained by taking the quotients of $\mathbb{C}$.

More than that, if we take different hyperbolic maps f_1, f_2, the complex structure arising from $\mathbb{D}/\langle f_1 \rangle$ and $\mathbb{D}/\langle f_2 \rangle$ may be different.

This will be seen later.

- Furthermore, a cylinder can also be obtained as the quotient of the disc by the group generated by a parabolic element. The idea is the same as in the case of hyperbolic elements.



The complex structure obtained as the quotient of $\mathbb{D}$ by the action of $\langle f \rangle$, f is parabolic, is distinct from the previous complex structures.

- The cylinder is the unique surface admitting complex structures of different nature, in the sense that it is covered by distinct surfaces.

to be seen later

Classification of compact (orientable) surfaces

Let S_1 and S_2 be two surfaces. The connected sum of S_1 and S_2, $S_1 \# S_2$, is obtained as follows. We "delete" an open ball from each surface S_1 and S_2, B_1 and B_2. The surfaces $S_1 \setminus B_1$ and $S_2 \setminus B_2$ have boundaries, denoted by ∂B_1 and ∂B_2, respectively, that are diffeomorphic to S^1.

Define then an orientation preserving diffeo

$$g : \partial B_1 \longrightarrow \partial B_2$$

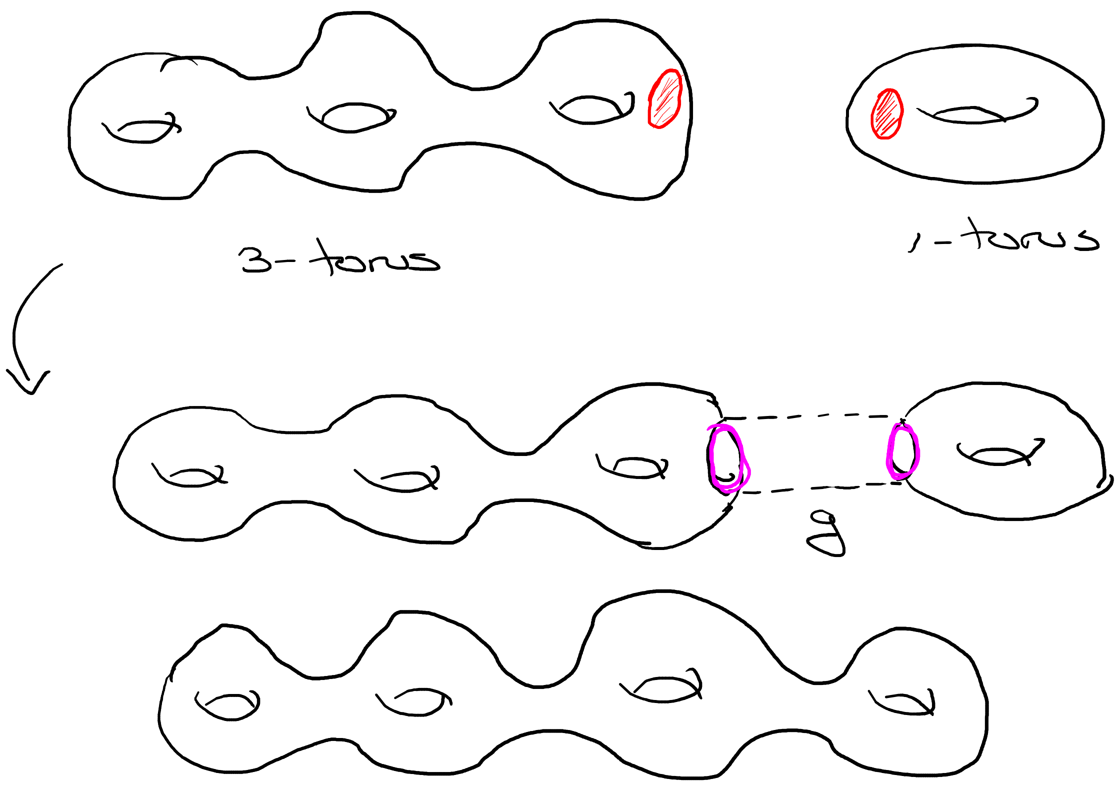
The sum is obtained by taking the disjoint union $(S_1 \setminus B_1) \cup_g (S_2 \setminus B_2)$, where we identify the points in ∂B_1 and ∂B_2 by g.

genus-0 torus $\rightsquigarrow$ surfaces homeomorphic to a sphere

genus-1 torus $\rightsquigarrow$ surfaces homeomorphic to a torus $(\mathbb{C}/\Lambda)$

⋮

genus-k torus $(k \geq 2)$ $\rightsquigarrow$ a surface obtained as the sum of a genus-$(k-1)$ torus with a genus-1 torus.



Classification theorem:

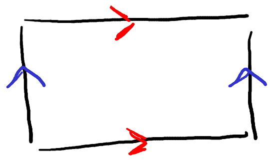
Let S be a compact surface. then S is diffeomorphic to a g-torus, for some $g \in \mathbb{N}_0$.

Question:

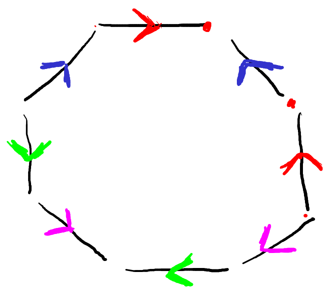
Does exist a Riemann surface of genus ? In other words, there exists a genus-2 torus admitting a complex struture ?

Topologically:

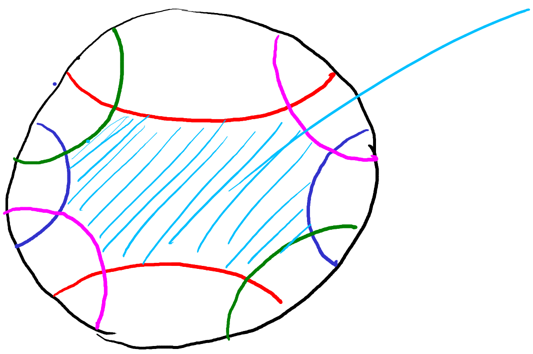
genus 1



genus 2



fundamental domain



f_1 (hyperbolic)

f_2 (hyperbolic)

f_3 "

f_4 "

$\mathbb{D}/\langle f_1, f_2, f_3, f_4 \rangle \rightsquigarrow$ $\boxed{\text{genus 2 riemann surface}}$

Riemannian Metrics

M = differentiable manifold

↳ we want to be able to measure the lengths of and angles between vector tangent to M

TM = total space of the tangent bundle

$p \in M$, T_pM carries the structure of a vector space.

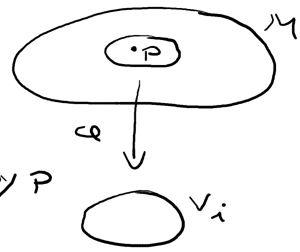
Def:

A Riemannian metric on a differentiable manifold M is a correspondence that associates to every point $p \in M$ a scalar product on the tangent space T_pM (i.e. a positive definite and symmetric bilinear form)

$$p \longmapsto \langle \cdot , \cdot \rangle_p$$

which depends smoothly on the base point p.

A Riemannian manifold is a differentiable manifold, equipped with a Riemannian metric.



If $x = \varphi(x_1, \dots, x_n)$ are local coordinates nearby p

$$q = \varphi(x_1, \dots, x_n)$$

position i

$$\frac{\partial}{\partial x_i}(q) = d\varphi(0, \dots, 0, 1, 0 \dots, 0)$$

$$\left\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \right\rangle = g_{ij}(x_1, \dots, x_n)$$

is a differentiable function

$$G = [\, g_{ij}(x) \,]$$

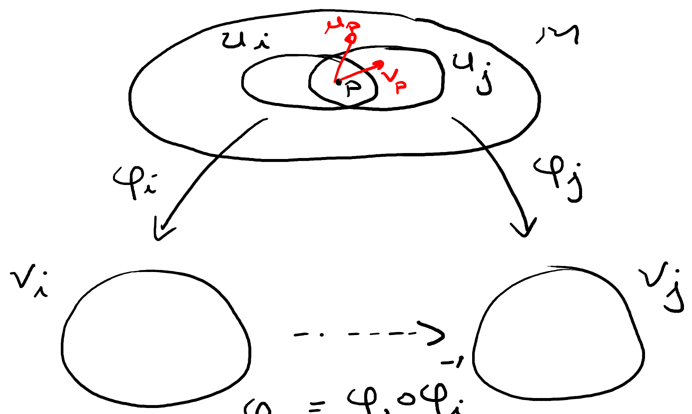
definite positive and symmetric matrices whose coefficients depend smoothly on φ

$$v = \sum v_i \frac{\partial}{\partial x_i}$$

$$u = \sum u_i \frac{\partial}{\partial x_i}$$

$$\langle u_p, v_p \rangle_p = u^T G v$$

Compatibility condition:



$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1}$$

$u_p, v_p \in T_pM$

$$u_p, v_p \;\overset{\varphi_i}{\rightsquigarrow}\; u, v \in \mathbb{R}^n \qquad u_p, v_p \;\overset{\varphi_j}{\rightsquigarrow}\; \bar{u}, \bar{v} \in \mathbb{R}^n$$

and

$$\bar{u} = D_{\varphi_i(p)} \varphi_{ji}\, u$$

$$\bar{v} = D_{\varphi_i(p)} \varphi_{ji}\, v$$

$$\langle u_p, v_p \rangle_p = u^T G_{\varphi_i(p)} \cdot v = (u, G_{\varphi_i(p)} \cdot v)$$

$$\langle u_p, v_p \rangle_p = \bar{u}^T \cdot H_{\varphi_j(p)} \cdot \bar{v} = (\bar{u}, H_{\varphi_j(p)} \bar{v})$$

$$= (D_{\varphi_i(p)} \varphi_{ji}\, u , H_{\varphi_j(p)} \cdot D_{\varphi_i(p)} \varphi_{ji}\, v)$$

$$= (u, (D_{\varphi_i(p)} \varphi_{ji})^T H_{\varphi_j(p)} \cdot D_{\varphi_i(p)} \varphi_{ji}\, v)$$

$$\boxed{(Au, v) = (u, A^T v)}$$

$$\Longrightarrow \quad G_{\varphi_i(p)} = (D_{\varphi_i(p)} \varphi_{ji})^T H_{\varphi_j(p)} \cdot D_{\varphi_i(p)} \varphi_{ji}$$

$$\boxed{G_i(p) = ((D\varphi)^T H_j \cdot D\varphi)(p)} \qquad \varphi = \varphi_{ji}$$

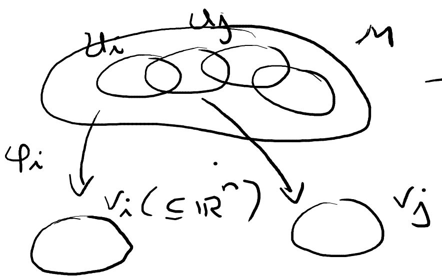
Given $v \in T_pM$, the length of v is given by

$$\|v\| := \langle v, v \rangle_p^{1/2} = \sqrt{\langle v, v \rangle_p}$$

Thm:

Every differentiable manifold may be equipped with a Riemannian metric.

Idea:



→ finite covering of M by open sets

we can equip V_i with a Riemannian metric by taking a Riemannian metric on $\mathbb{R}^n$ and by multiplying it by a bump function → $\sum$ Riemannian metrics

finite sum

Let now $\gamma : [a,b] \to M$ be a smooth curve on M ($[a,b] \subseteq \mathbb{R}$).

The length of γ is defined as follows:

$$L(\gamma) := \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt$$

(depends only on the "curve" $\gamma([a,b]) \subseteq M$ and not on the choice of the parametrization)

In local coordinates:

Assume that, in local coordinates, $\gamma(t)$ is given by

$$\gamma(t) = (\omega_1(t), \dots, \omega_n(t))$$

then

$$\frac{d\gamma}{dt}(t) = (\dot{\omega}_1(t), \dots, \dot{\omega}_n(t))$$

so that

$$L(\gamma) = \int_a^b \sqrt{\sum_{ij} g_{ij}(\omega_1(t), \dots, \omega_n(t)) \cdot \dot{\omega}_i(t) \cdot \dot{\omega}_j(t)} \, dt$$

Def:

Let M be a Riemannian manifold and let p, q be two points on M. The distance between p and q is defined as

$$d(p,q) := \inf \{ L(\gamma) \; : \; \gamma : [a,b] \to M \text{ smooth}, \gamma(a) = p, \gamma(b) = q \}$$

$\Downarrow$

Note that since M is connected, it is also path-connected (connected open sets of $\mathbb{R}^n$ are path-connected). Thus, for every pair of points $p, q \in M$, there exists a path $c : [a,b] \to M$ joining p to q.

$\Downarrow$

we can thus define a metric on M by letting

$$d(p,q) = \inf \{ L(\gamma) : \gamma \text{ paths joining } p \text{ to } q \}$$

It is straightforward to check that (M, d) is a metric space (i.e. d is a metric). Namely, we clearly have

$\forall p, q \in M$

* $d(p,q) \geq 0$
* $d(p,q) = 0 \iff p = q$
* $d(p,q) = d(q,p)$
* $d(p,q) \leq d(p,w) + d(w,q)$

Purpose: study / investigate the geometry of (M, d)

3 fundamental notions $\left\{ \begin{array}{l} * \text{ geodesics} \\ * \text{ Levi-Civita connection} \\ * \text{ curvature} \end{array} \right\}$ to be seen later as they become necessary

Geodesics:

Def:
A parametrized curve $\gamma : [a,b] \to M$ is said to be a geodesic if and only if it satisfies

(1) $\| \gamma'(t) \| = 1$

(2) for every $t \in [a,b]$

$$\underbrace{d(\gamma(t), \gamma(a))}_{\text{going along the curve}} = L(\gamma|_{[a,t]}) \quad (= t)$$

there is a partition $[a,b] = \cup [t_i, t_{i+1}]$ so that $d(\gamma(t_i), \gamma(t_{i+1})) = L(\gamma|_{[t_i, t_{i+1}]})$

A natural way to think about geodesics joining p to q is to think on a curve having minimum distance among the curves joining p to q.

Rmk: γ curve, $\gamma : [a,b] \to M$. There exists parametrizations of γ satisfying condition (1). The length of γ and of its reparametrization is the same.

Def / Proposition:

Let $(M, \langle , \rangle)$ be a Riemannian manifold. A diffeomorphism $f : M \to M$ is called an isometry if it verifies one (and hence both) of the following equivalent conditions

(1) For every pair of points $p, q \in M$ we have
$$d(f(p), f(q)) = d(p, q)$$

(2) For every point $p \in M$ and tangent vectors $u, v \in T_pM$ we have
$$\langle u, v \rangle_p = \langle D_pf \cdot u, D_pf \cdot v \rangle_{f(p)}$$

The isometry group of M will be denoted by $Iso(M)$.

Example:

$$\mathcal{M} = \mathbb{R}^2 \quad (\simeq \mathbb{C})$$

$\langle \, , \, \rangle$ = Euclidean metric

we have that a rotation is an isometry.

We have mentioned that every differentiable manifold can be equipped with a Riemaniann metric. For the case of the (open) disc $\mathbb{D}$ a little more can be said.

Prop:

There exist a Riemaniann metric on $\mathbb{D}$ such that $\boxed{\mathrm{Iso}(\mathbb{D}) = \mathrm{Aut}(\mathbb{D})}$

Proof:

Recall that

$$\mathrm{Aut}(\mathbb{D}) = \left\{ \underbrace{e^{i\theta} \frac{z - z_0}{-\bar{z}_0 z + 1}}_{\substack{\text{composition of the rotation (at } 0 \in \mathbb{C}\text{) of} \\ \text{angle } \theta \text{ with the möbius transformation} \\ \text{preserving the disc and sending } z_0 \text{ to } 0 \in \mathbb{C}}} \; : \; z_0 \in \mathbb{D} \, , \, \theta \in [0, 2\pi[\right\}$$

So, we have:

$$\begin{cases} * \text{ given } z_0 \in \mathbb{D} \, , \; \exists f \in \mathrm{Aut}(\mathbb{D}) : f(z_0) = 0 \\ * \text{ if } g \in \mathrm{Aut}(\mathbb{D}) \, , \; g(0) = 0 \implies g(z) = e^{i\theta} z \, , \; \theta \in [0, 2\pi[\end{cases}$$

$$\Updownarrow$$

$$\mathrm{Stab}(0) = \{ g \in \mathrm{Aut}(\mathbb{D}) : g(0) = 0 \}$$

Up to a constant, let us define the scalar product on the tangent space $T_0\mathbb{D}$ as the Euclidean one. In other words, given $u, v \in T_0\mathbb{D}$

$$\langle u, v \rangle_0 := (u, v)$$

where (u, v) stands for the Euclidean scalar product on $\mathbb{R}^2$.

Let us then see how to define a Riemaniann metric on $\mathbb{D}$ so that $\mathrm{Aut}(\mathbb{D}) \subseteq \mathrm{Iso}(\mathbb{D})$.

$\Rightarrow$

$\Downarrow$

Fix $z_0 \in \mathbb{D}$. We have $T_{z_0}\mathbb{D} \simeq \mathbb{C} \simeq \mathbb{R}^2$

Fixed $u, v \in T_{z_0}\mathbb{D}$ we define

$$\langle u, v \rangle_{z_0} := \langle D_{z_0}f \cdot u \, , \, D_{z_0}f \, v \rangle_0$$

$$= (D_{z_0}f \cdot u \, , \, D_{z_0}f \cdot v)$$

$f : f(z_0) = 0$

$f \in Aut(\mathbb{D})$

Note that this metric is well-defined. In fact, let $f_1, f_2 \in Aut(\mathbb{D})$ such that $f_1(z_0) = f_2(z_0) = 0$. We know that

$$f_1 \circ f_2^{-1} = Rot \qquad \text{(rotation)}$$

$$\Leftrightarrow f_1 = Rot \circ f_2$$

Since

$$\langle D_{z_0}f_1 \cdot u \, , \, D_{z_0}f_1 \cdot v \rangle_0 = (D_{z_0}f_1 \cdot u \, , \, D_{z_0}f_1 \cdot v)$$

$$= (D_{z_0}(Rot \circ f_2) \cdot u \, , \, D_{z_0}(Rot \circ f_2) \cdot v)$$

$$= (Rot' \cdot D_{z_0}f_2 \cdot u \, , \, Rot' \cdot D_{z_0}f_2 \cdot v)$$

Rotations are isometries $\rightarrow$

$$= (D_{z_0}f_2 \cdot u \, , \, D_{z_0}f_2 \cdot v)$$

$$= \langle D_{z_0}f_2 \cdot u \, , \, D_{z_0}f_2 \cdot v \rangle_0$$

The metric in question is unique up to a multiplicative constant, and it is called the hyperbolic metric. Let us present its expression.

$\Downarrow$

Recall that an element $f \in Aut(\mathbb{D})$ sending z_0 to 0 is given by

$$f(z) = \frac{z - z_0}{-\bar{z}_0 z + 1}$$

Let $u, v \in T_{z_0}\mathbb{D}$. We have then

$$\langle u, v \rangle_{z_0} = \langle D_{z_0}f \cdot u \, , \, D_{z_0}f \cdot v \rangle_0$$

$$= (D_{z_0}f \cdot u \, , \, D_{z_0}f \cdot v)$$

up to a ~~constant~~ and the constant can be chosen so that $\displaystyle = \frac{4}{(1-|z_0|^2)} \cdot (u, v)$

which means that the curvature of the metric is constant equal to -1.

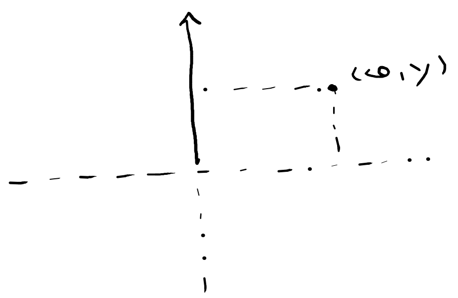
$\Downarrow$

The hyperbolic metric in the upper half-plane can be deduced by using the isomorphism between the two Riemann surfaces by pulling it back.

$\Downarrow$

$$\mathrm{Iso}(\mathbb{H}) = \mathrm{Aut}(\mathbb{H})$$

we have then



$z = x + iy \qquad (\simeq (x, y))$

$u, v \in T_z \mathbb{H}$

$\Downarrow$

$$\langle u, v \rangle_z = \frac{1}{y^2} (u, v)$$

Let us now try to understand the geodesics of these models. We will do it first on the upper half-plane model.

Rmk:

$\mathrm{Aut}(\mathbb{H})$ is $\underbrace{\text{simply transitive}}$ in $T_1 \mathbb{H}$ (where $T_1 \mathbb{H}$ stands for the unit tangent bundle)

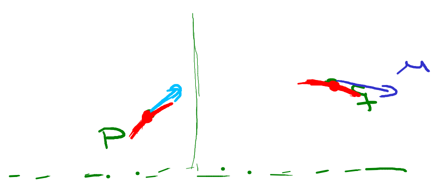
$\Downarrow$

Fixed (p, v) and (q, u)

$p, q \in \mathbb{H}$

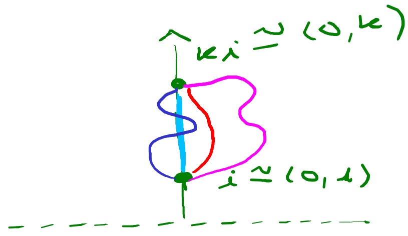
$v \in T_1 \mathbb{H} \quad (\|v\| = 1)$

$u \in T_1 \mathbb{H} \quad (\|u\| = 1)$



Then, $\exists !\ f \in \mathrm{Aut}(\mathbb{H})$ such that $\begin{cases} f(p) = q \\ D_p f \cdot v = u \end{cases}$

This remark allows us to say that it is sufficient to us to describe the geodesics through a sole point of $\mathbb{H}$ (or, more precisely, a geodesic through a sole point of $\mathbb{H}$)



Understanding geodesics

Fix the point $i \in \mathbb{H}$. Let c be a curve joining i and ki for some $k \in \mathbb{R}^+ \setminus \{1\}$ such $\|c'(t)\| = 1$.
Suppose that $c(t) = (\omega(t), y(t)) \in \mathbb{H}$ $\quad (\forall t \in [a, b])$

$\rightarrow c'(t) = (\omega'(t), y'(t))$

we intend to deduce the expression (or the image) of the curve c minimizing the distance between i and ki for metric in question (hyperbolic metric)

Recall that

$$L(c) = \int_a^b \|c'(t)\| \, dt$$

Furthermore, the norm $\|c'(t)\|$ with respect to the hyperbolic metric is given by

$$\|c'(t)\|^2 = \frac{1}{y^2(t)} \cdot \left(\overset{\geq 0}{\dot{\omega}^2(t)} + \dot{y}^2(t) \right) \geqslant \frac{\dot{y}^2(t)}{y^2(t)}$$

for every $\omega = \omega(t)$ that we consider. We set then

$$\int_a^b \|c'(t)\| \, dt \geqslant \int_a^b \frac{|\dot{y}(t)|}{y(t)} \, dt = L(\gamma)$$

recall that $y(t) > 0$

where γ is the curve contained in the imaginary axis of $\mathbb{H}$ joining i and ki.

$$\gamma(t) = i + t(k-1)i \simeq (0, 1 + t(k-1))$$

Furthermore, recall that γ does not necessarily satisfy $\|\gamma'(t)\| = 1$, so that γ is not necessarily the geodesic joining the two points. Let us deduce the expression of the geodesic (by reparametrizing γ). We must have

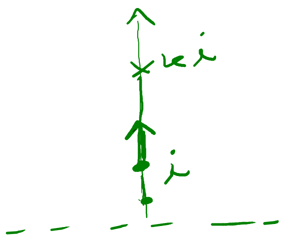
$$\alpha(t) = (0, y(t))$$

$$\alpha(a) = (0, 1)$$

$$\alpha(b) = (0, k) \qquad \boxed{k > 1}$$

$$\|\alpha'(t)\| = 1$$

$$\Longrightarrow \quad \|\alpha'(t)\|^2 = 1 \iff \frac{(y'(t))^2}{(y(t))^2} = 1$$



$$\Longrightarrow \quad y'(t) = y(t)$$

$$\Longrightarrow \quad y(t) = ke^t$$

assuming, for simplicity, $a = 0$ $\rightsquigarrow$

$$\Longrightarrow \quad y(t) = e^t$$

$\Rightarrow y(0) = 1$

$\Rightarrow k = 1$

$$\alpha(t) = (0, e^t) \simeq ie^t$$

Geodesics passing through i (for $t = 0$) and satisfying $\alpha'(0) = i$
($\alpha'(0) \simeq (0,1)$)

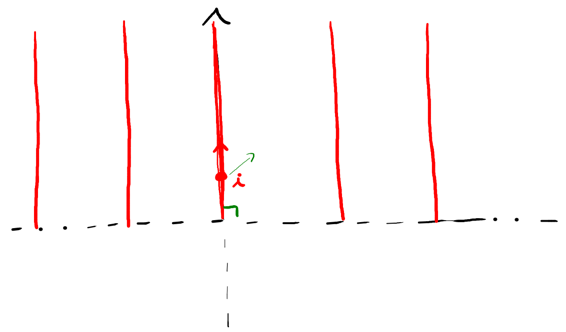
$$\Downarrow$$

$$\boxed{\alpha(t) = ie^t}$$

This geodesic parametriz all the other geodesics by using the action of the isometry group ($= \mathrm{Aut}(\mathbb{H})$)

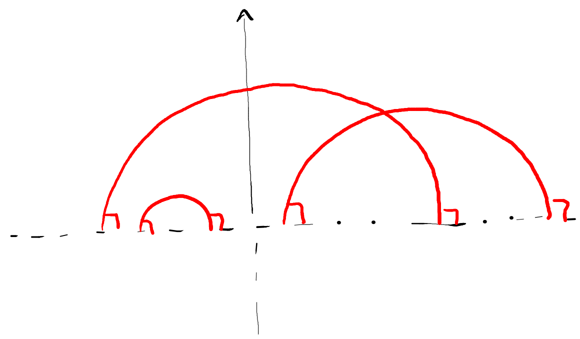
Pictures :

Geodesics (like unparametrized curves) on $\mathbb{H}$



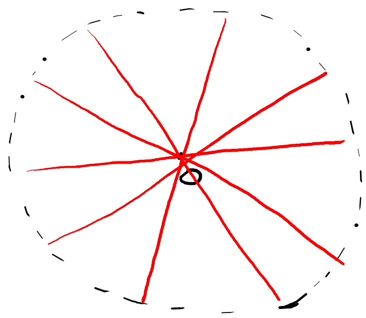
vertical lines (obtained from $\alpha(t) = ie^t$ by translations $z \mapsto z + \alpha$, $\alpha \in \mathbb{R}$)

or



they are half-circles orthogonal to the $\mathbb{R}$-axis.

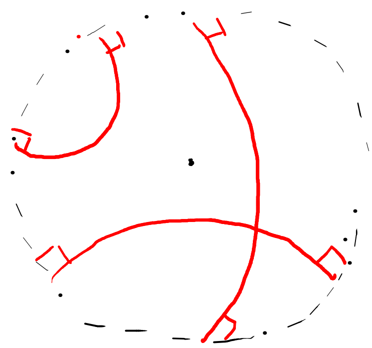
Geodesics on $\mathbb{D}$



"straight lines" through the origin

$\downarrow$

- all lines through $0 \in \mathbb{D}$



- Images of these lines by möbius transformations (preserving the disk)

$\rightsquigarrow$

preserve the class of euclidean circles (including lines in this class)

Recall that $T_1 \mathbb{H} \simeq \mathrm{Aut}(\mathbb{H})$ (bijection, diffeo, ...)

unit tangent space

$$T^1\mathbb{H} \longrightarrow \mathrm{Aut}(\mathbb{H})$$
$$(p, v) \longmapsto g \in \mathrm{Aut}(\mathbb{H}) : \begin{array}{l} g(p) = i \\ D_p g \cdot v = i \end{array}$$

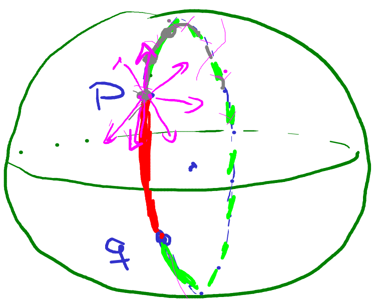
Let γ be the geodesic satisfying $\left\| \begin{array}{l} \gamma(0) = i \\ \gamma'(0) = i \end{array} \right.$. We checked that $\gamma(t) = ie^t$. This geodesic is globally minimizing

in the sense that for every points p, q on the image of γ, γ is the curve minimizing the distance between the two points

$p = \gamma(a)$ $\quad \gamma([a,b])$

$q = \gamma(b)$

Example:



- the red line is globally minimizing
- the green line is locally minimizing

Distance between two points

Choose two points on the image of the geodesic $\gamma(t) = ie^t$: i and yi for some $y \in \mathbb{R}^+$. We have

$$\gamma(0) = i$$
$$\gamma(t) = yi$$

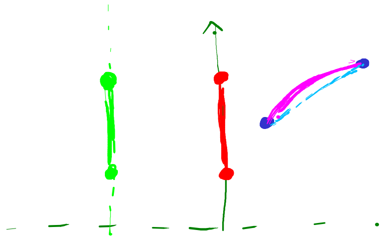
Since $\gamma(t) = yi$, we have $ie^{t} = yi \iff e^{t} = y \iff t = \ln y$.

Since γ is a geodesic

$$\mathrm{dist}_{\mathbb{H}}(\gamma(t), \gamma(0)) = L(\gamma|_{[0,t]}) = t$$

So,

$$\boxed{\mathrm{dist}_{\mathbb{H}}(i, yi) = \ln y}$$

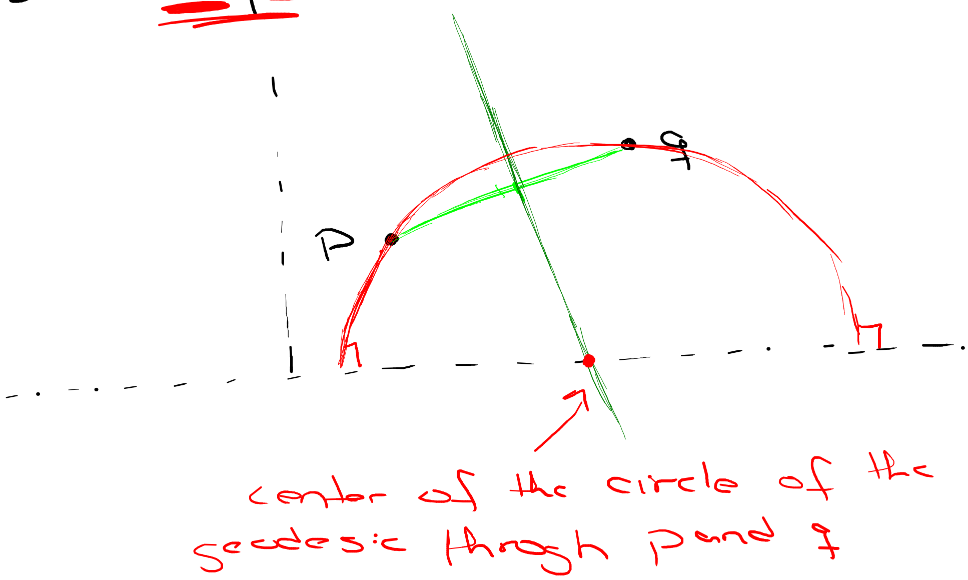


Rmk:

$\mathrm{Aut}(\mathbb{H})$ is simply transitive in the space of geodesics, the latter being identified to $T_1\mathbb{H}$

$\implies$ all geodesics on $\mathbb{H}$ are globally minimizing since they are obtained from the previous one by isometries.

Besides: given two points $p, q \in \mathbb{H}$, $p \neq q$, there is a unique geodesic through p and q



Given now $p, q \in \mathbb{H}$, $\mathrm{dist}_{\mathbb{H}}(p,q) = ?$

$$p = x_1 + i y_1 \qquad x_1, x_2 \in \mathbb{R}, \quad y_1, y_2 \in \mathbb{R}^+$$
$$q = x_2 + i y_2$$

$$\mathrm{dist}_{\mathbb{H}}(p,q) = 2\ln\left(\frac{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} + \sqrt{(x_2-x_1)^2+(y_2+y_1)^2}}{2\sqrt{y_1 y_2}}\right)$$

In the particular case where $x_2 = x_1$ we get:

$$\mathrm{dist}_{\mathbb{H}}(p,q) = 2\ln\left(\frac{\sqrt{(y_2-y_1)^2} + \sqrt{(y_2+y_1)^2}}{2\sqrt{y_1 y_2}}\right)$$

assumption
$y_2 > y_1$

$$= 2\ln\left(\frac{y_2 - \cancel{y_1} + y_2 + \cancel{y_1}}{2\sqrt{y_1 y_2}}\right)$$

$$= 2\ln\left(\sqrt{\frac{y_2}{y_1}}\right)$$

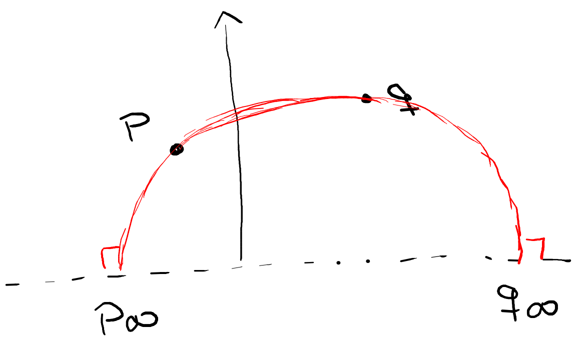
$$= 2\ln\left(\frac{y_2}{y_1}\right)^{1/2}$$

$$= \ln\left(\frac{y_2}{y_1}\right)$$

$$= \ln(y_2) - \ln(y_1)$$

In general: $\mathrm{dist}_{\mathbb{H}}(p,q) = |\ln(y_2) - \ln(y_1)|$

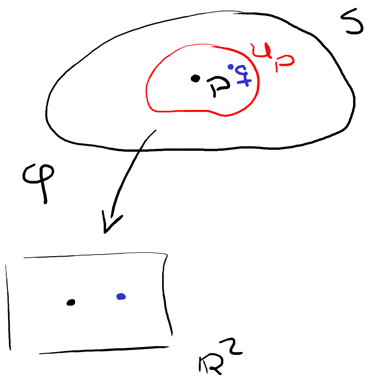
<u>Furthermore</u>:



$$\mathrm{dist}_{\mathbb{H}}(p,q) = \left|\ln\left(\frac{|pq_\infty| \cdot |q p_\infty|}{|p p_\infty| \cdot |q q_\infty|}\right)\right|$$

where $|pq|$ stands for the Euclidean distance between p and q.

More examples (many examples) of Riemann surfaces :



S orientable surface

$\langle\, , \,\rangle_R$ Riemaniann metric

$\Downarrow$

<u>Gauss</u> : $\forall p \in S$

$\exists \varphi : U_p \to V \subseteq \mathbb{R}^2$ (injective) such that

$$\langle\, , \,\rangle_R = f \cdot \varphi^* [(\, , \,)_{Eucl.}]$$

function (diff. ⇝ with the same regularity as the metric)

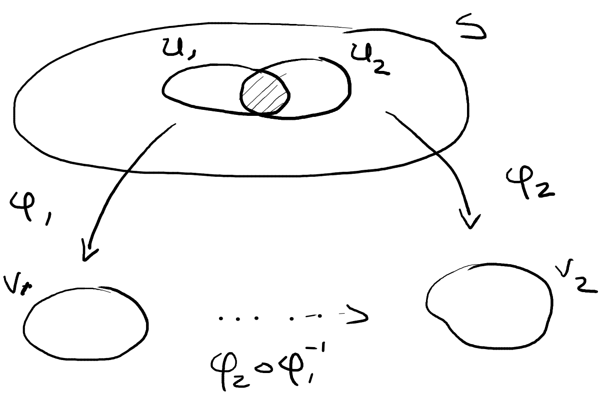
$\boxed{\varphi \text{ called isothermal coordinates}}$

Prop :

The Riemaniann metric $\langle\, , \,\rangle$ naturally determines a Riemann surface structure on S.

Proof / Construction :

Define an atlas of a Riemann surface structure for S as follows: for every point $p \in S$, consider isothermal coordinates φ around p.



φ_1, φ_2 isothermal coordinates

$$\varphi_{1*} [\langle\, , \,\rangle] = f \cdot (\, , \,)_{Eucl} \quad \text{where } f : V_1 = \varphi(U_1) \to \mathbb{R}^2$$

$$\varphi_{2*} [\langle\, , \,\rangle] = g \cdot (\, , \,)_{Eucl.} \quad \text{where } g : V_2 = \varphi(U_2) \to \mathbb{R}^2$$

Thus $\quad (\varphi_2 \circ \varphi_1^{-1})_* \, [f \cdot (\, , \,)_{Eucl}] = g \cdot (\, , \,)_{Eucl.}$

it sends the Euclidean metric to a function times itself. In other words, it preserves angles

$\Rightarrow \quad \varphi_2 \circ \varphi_1^{-1}$ is conformal with respect to the Euclidean metric

$$\measuredangle (u, v) = \measuredangle \left(D(\varphi_2 \circ \varphi_1^{-1}) . u, D(\varphi_2 \circ \varphi_1^{-1}) v \right)$$

$\Rightarrow$ $\varphi_2 \circ \varphi_1^{-1}$ is holomorphic or anti-holomorphic

Since S is orientable, we can choose coordinates so as to be holomorphic.

Question: when two Riemannian metrics on S yield the same Riemann surface structure on S?

Summary:

S Riemann surface

$\rightsquigarrow$

S can be covered by $\mathbb{CP}(1)$, $\mathbb{C}$ or $\mathbb{D}$

- $\mathbb{CP}(1)$ only covers $\mathbb{CP}(1)$
- $\mathbb{C}$ covers $\mathbb{C}$, $\underbrace{\text{cylinders, torus}}_{\mathbb{C}/\Gamma}$

$\Gamma = \langle z \mapsto z + c \rangle \quad c \in \mathbb{C}^*$
(cylinder)

or

$\Gamma = \langle z \mapsto z + a,\ z \mapsto z + b \rangle$
$a, b \in \mathbb{C}^*, \ a/b \notin \mathbb{R}$
(tori)

- any other Riemann surface is covered by $\mathbb{D}$

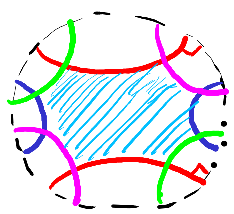
Furthermore: Every Riemann surface S is biholomorphically equivalent to D/Γ where $D = \mathbb{C}$, $\mathbb{CP}(1)$ or $\mathbb{D}$ and Γ is a group acting freely and properly discontinuously on D.

$\rightsquigarrow$

we gave an example of a genus-2 Riemann surface

How we did it?

- we classified elements of $\mathrm{Aut}(\mathbb{D})$
  - * elliptic (fixed point in $\mathbb{D}$)
  - * hyperbolic (2 fixed points in $\partial\mathbb{D}$)
  - * parabolic (just one fixed point in $\partial\mathbb{D}$)
- we looked to its dynamics



$S = \mathbb{D} / \langle f_1, f_2, f_3, f_4 \rangle$

f_i hyperbolic

acting freely and properly discontinuously on $\mathbb{D}$

M = diff. manifold

TM = total space of tangent bundle

$p \in M$, $T_pM \rightsquigarrow$ structure of vector space

$\rightsquigarrow$

Riemannian metric $\quad p \mapsto \langle \, , \, \rangle_p \quad$ scalar product on T_pM
(positive definite and symmetric bylinear form)
depending smoothly on p

* Every diff. manifold may be equipped with a Riemannian metric

$$\gamma \text{ curve on } M \rightsquigarrow l(\gamma) = \int_a^b \left\| \frac{d\gamma}{dt}(t) \right\| dt$$

M manifold
$p, q \in M$
$d(p,q) = \inf \{ L(\gamma) : \gamma \text{ smooth paths joining } p \text{ to } q \}$

$\rightsquigarrow$

metric on M

- $d(p,q) \geq 0$ and $d(p,q) = 0 \Leftrightarrow p = q$
- $d(p,q) = d(q,p)$
- $d(p,q) \leq d(p,w) + d(w,q) \quad \forall w \in M$

- Notion of geodesic

  $\hookrightarrow$ curve having minimum distance among the curves joining two points

(1) $\| \gamma'(t) \| = 1$

(2) $d(\gamma(t_1), \gamma(t_2)) = L(\gamma)_{[t_1, t_2]} = t_2 - t_1$

when $|t_1 - t_2| < \epsilon$

<u>Isometry</u> : $f: M \to M$ such that

$$d(f(p), f(q)) = d(p,q) \quad \forall p,q \in M$$

$$\left(\Leftrightarrow \quad \langle u, v \rangle_p = \langle D_p f . u , D_p f . v \rangle_{f(p)} \right)$$
$$\forall p \in M \quad \forall u,v \in T_pM$$

<u>Prop</u> : $\exists$ Riemannian metric on $\mathbb{D}$: $\boxed{Iso(\mathbb{D}) = Aut(\mathbb{D})}$

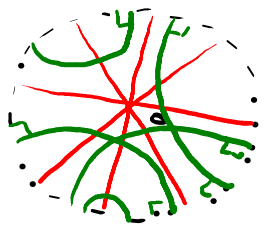
hyperbolic metric

$$\langle u, v \rangle_{z_0} = \frac{4}{1-|z_0|^2} (u,v)$$

$$Aut(\mathbb{D}) = \left\{ e^{i\theta} \cdot \frac{z - z_0}{-\bar{z}_0 z + 1} : \theta \in [0, 2\pi[, z_0 \in \mathbb{D} \right\}$$

$$= S^1 \times \mathbb{D}$$

<u>Geodesics</u> :



Everything deduced in the upper-half plane model

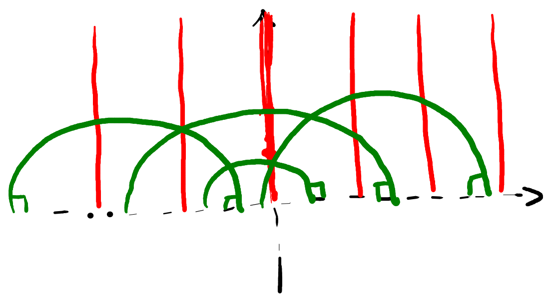
$\exists$ Riemannian metric on $\mathbb{H}$: $Iso(\mathbb{H}) = Aut(\mathbb{H})$

$$\cong PSL(2, \mathbb{R}) = SL(2,\mathbb{R}) / \{\pm Id\}$$

$$\langle u, v \rangle_z = \frac{1}{y^2} (u, v)$$

$(z = x + iy)$

<u>Geodesic</u> :



α : $\alpha(0) = i$

$\alpha'(0) = i$

Distance

$p, q \in \mathbb{H}$

$$\mathrm{dist}_{\mathbb{H}}(p,q) = 2\ln\left(\frac{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}+\sqrt{(x_1-x_2)^2+(y_1+y_2)^2}}{2\sqrt{y_1 y_2}}\right)$$

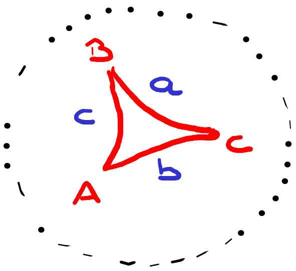
$p = x_1 + iy_1 \quad , \quad q = x_2 + iy_2$

$\boxed{x_1 = x_2}$

$\Downarrow$

$$\mathrm{dist}_{\mathbb{H}}(p,q) = \left| \ln(y_2) - \ln(y_1) \right|$$

Rmk



geodesic triangle
(all edges are geodesics)

hyperbolic metric is conformal to the Euclidean metric : they measure the same angles.

Law of sinus :
$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}$$

Law of cosinus I :
$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(A)$$

Law of cosinus II :
$$\cos(B) = -\cos(A)\cos(C) + \sin(A)\sin(C)\cdot\cosh(b)$$

Consequence: Tow geodesic triangles with the same angles are congruent !!

Many examples of Riemann surfaces

$\langle\, , \rangle_R$ Riemannian metric $\rightsquigarrow$ it determines on

S oriented surface $\qquad$ S a Riemann surface structure

Idea : Gauss : $\exists$ isothermal coordinates

: $\langle\, , \rangle_R = f \cdot \varphi^*[(\, ,)_{Eucl}]$

Question : when two Riemannian metrics on S (oriented surface) yields the same Riemann surface structure on S ?

Def :

Two metrics $\langle\, , \rangle_1$ and $\langle\, , \rangle_2$ are said to be conformal if they yield the same notion of angle between vectors. In other words, if there is a function $h: S \to \mathbb{R}$ such that

oriented surface

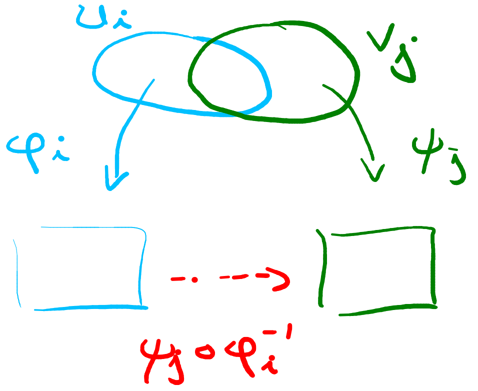
$$\langle\, , \rangle_{1,p} = h(p) \langle\, , \rangle_{2,p}$$

Lemma :

If $\langle\, , \rangle_1$ and $\langle\, , \rangle_2$ are conformal metrics on the (oriented) surface S, then they induce the same Riemann surface structure on S.

Proof:

Let $\mathcal{A}_1 = \{(U_i, \varphi_i)\}$ be a Riemann surface atlas induced by $\langle , \rangle_1$, while $\mathcal{A}_2 = \{(V_j, \psi_j)\}$ is the Riemann surface atlas induced by $\langle , \rangle_2$. To check that both Riemann surfaces are the same, it is enough to check that the union of the two atlas' is still a Riemann surface atlas. More precisely,



if U_i and V_j are such that $U_i \cap V_j \neq \emptyset$, then the map

$$\psi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap V_j) \longrightarrow \psi_j(U_i \cap V_j)$$

is holomorphic.

Rmk:
$$\varphi_i^{*}(\, , \,)_{Eucl} = \mu_i \langle \, , \, \rangle_1 \qquad \mu_i \text{ function on } U_i \cap V_j$$

$$\psi_j^{*}(\, , \,)_{Eucl} = \nu_j \langle \, , \, \rangle_2 \qquad \nu_j \text{ function on } U_i \cap V_j$$

Since $\langle , \rangle_1$ and $\langle , \rangle_2$ are conformal, there exists h such that $\langle \, , \, \rangle_2 = h \langle \, , \, \rangle_1$. Hence

$$\psi_j^{*}(\, , \,)_{Eucl.} = \nu_j h \langle \, , \, \rangle_1$$

$$\Rightarrow \quad (\, , \,)_{Eucl} = \nu_j h . \psi_{j*} \langle \, , \, \rangle_1$$

$$\Rightarrow \quad (\psi_j \circ \varphi_i^{-1})_* (\, , \,)_{Eucl} = \frac{\nu_j h}{\mu_i} (\, , \,)_{Eucl}$$

In other words, $\psi_j \circ \varphi_i^{-1}$ is conformal. Moreover, they are both compatible with the orientation of S so that the composition $\psi_j \circ \varphi_i^{-1}$ preserves the orientation. Hence it is holomorphic. ▨

Def:

Two Riemann surfaces $(S, \langle , \rangle_1)$ and $(S, \langle , \rangle_2)$ are said to be conformally equivalent if there is a diffeomorphism $f : S \to S$ such that

$$f^* \langle , \rangle_2 \quad \text{and} \quad \langle , \rangle_1 \quad \text{are conformal}$$

In other words,

$$f^* \langle , \rangle_2 = h \langle , \rangle_1 \quad \text{for some function } h : S \to \mathbb{R}$$

The diffeo f is said to be a conformal diffeo between the (Riemann) surfaces in question.

Rmk:

Two metrics $\langle , \rangle_1$ and $\langle , \rangle_2$ are conformal in the sense of the previous definition if and only if the identity is a conformal diffeo between them

Given S, let us define the class of conformal Riemannian structures on S as the collection of Riemannian metrics $(S, \langle , \rangle)$ up to conformal equivalence. In other words

$$(S, \langle\, , \rangle_1) \cong (S, \langle\, , \rangle_2)$$

$$\Updownarrow$$

there is a conformal diffeo

$$f : (S, \langle\, , \rangle_1) \longrightarrow (S, \langle\, , \rangle_2)$$

<u>Proposition:</u>

The collection of Riemann surface structures on S are in 1-1 correspondence with the collection of conformal Riemannian structures on S.

(everything is oriented with ¢ and f respect the orientation)

<u>Proof:</u>

Let f be a conformal diffeomorphism

$$\begin{tikzcd}
(S, \langle\, , \rangle_1) \arrow[r, "f"] \arrow[d, "\varphi_i"'] & (S, \langle\, , \rangle_2) \arrow[d, "\psi_j"] \\
(\ ,\ )_{Eucl}. & (\ ,\ )_{Eucl}.
\end{tikzcd}$$

φ_i, ψ_j are isothermal coordinates

Since f is a conformal diffeo, we have

$$f_* \langle\, , \rangle_1 = h \langle\, , \rangle_2$$

h function

So, we have

- $(\varphi_i)^* (\ ,\ )_{Eucl} = (\varphi_i^{-1})_* (\ ,\ )_{Eucl} = \mu \langle\, , \rangle_1$
- $f_* \langle\, , \rangle_1 = h \langle\, , \rangle_2$

u, v, h functions

- $(\psi_j)_* \langle\, , \rangle_2 = \nu (\, ,)_{Eucl}$

<u>Therefore</u> :

$$(\psi_j \circ f \circ \varphi_i^{-1})_* (\, ,)_{Eucl} = (\psi_j \circ f)_* \left[(\varphi_i^{-1})_* (\, ,)_{Eucl} \right]$$
$$= (\psi_j \circ f)_* (\mu \langle\, , \rangle_1)$$
$$= (\psi_j)_* (f_* (\mu \langle\, , \rangle_1))$$
$$= (\psi_j)_* (\mu h \langle\, , \rangle_2)$$
$$= \mu h \nu (\, ,)_{Eucl}.$$

We have then that $\psi_j \circ f \circ \varphi_i^{-1}$ is holomorphic or anti-holomorphic. Since orientation is preserved, it is holomorphic.

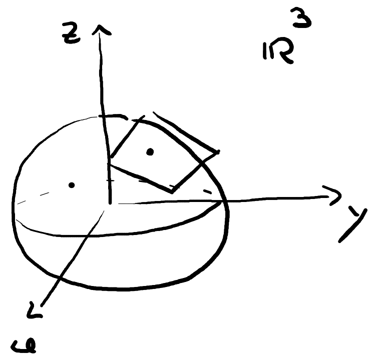
$\boxtimes$

$$\left\{ \begin{array}{c} \text{conformal structures} \\ \text{on } S \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} \text{Riemann surface} \\ \text{structures on } S \end{array} \right\}$$

Prop:
There exists a prefered (canonical) metric on S where conformal classes induces the initial Riemann surface structure on S

Proof:
Let $\tilde{S}$ be the universal covering of S

(1) $\tilde{S} = \mathbb{C}\mathbb{P}(1)$ $(\simeq S^2)$ $\Longrightarrow$ $S = \mathbb{C}\mathbb{P}(1)$ $(\simeq S^2)$



The scalar product on $T_p S^2$ is the restriction of the (global) Euclidean scalar product on $\mathbb{R}^3$.

$u, v \in T_p S^2$
(u, v)
$\Downarrow$
Euclidean s.p.

Consider the stereographic proj. $\rightsquigarrow$ used for defining the complex atlas on S^2

$\downarrow$

this is a conformal map
(preserves angles)

$\Downarrow$

hence, they are isothermal coordinates

$\Downarrow$

they induce the original R.s. structure on $S^2 \simeq \mathbb{C}\mathbb{P}(1)$

(2) $\tilde{S} = \mathbb{C}$

$\Rightarrow$ $S = \mathbb{C}/_{\Gamma}$ where Γ is deck transformation group.

In particular

$\Gamma \subseteq \{ z \mapsto z + c \}$ (translations of $\mathbb{C}$)

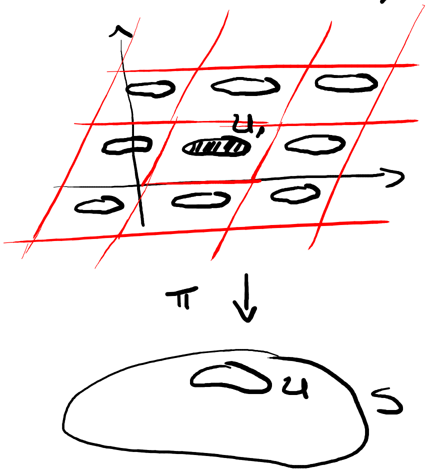
$\Rightarrow$ Γ preserves the Euclidean structure of $\mathbb{C}$

$\Rightarrow$ the Euclidean metric on $\mathbb{C}$ descends to $S = \mathbb{C}/\Gamma$

$\Rightarrow$ S has a Euclidean metric

flat Riemaniann metric
(curvature zero)

Conversely, this Euclidean metric yields the same R.S. structure on S. More precisely:



$\pi : \mathcal{U}_1 \to \mathcal{U}$ local diffeo

Let $\pi_1^{-1} : \mathcal{U} \to \mathcal{U}_1$

$\Downarrow$

the atlas they provide is the atlas for $\mathbb{C}/\Gamma \simeq S$. It therefore coincides with the initial structure on S

Furthermore :

$\hookrightarrow$ they are isothermal coordinates

(as said before, the Eucl. metric descends to S)

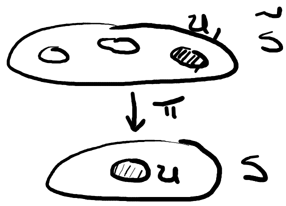
(3) $\tilde{S} = \mathbb{D}$ $\Rightarrow$ $S = \mathbb{D}/\Gamma$ Γ is a certain discrete subgroup of $\mathrm{Aut}(\mathbb{D})$

$\mathrm{Aut}(\mathbb{D})$ preserves the hyperbolic metric on $\mathbb{D}$.

$\Rightarrow$ $S = \mathbb{D}/\Gamma$ inherits a Riemannian metric of curvature -1
(hyperbolic metric)
$\downarrow$
Poincaré

Claim : The hyperbolic metric induces the same R.S. structure on $S = \mathbb{D}/\Gamma$

$\Downarrow$



$\pi : \mathcal{U}_1 \to \mathcal{U}$ local diffeo

$\Downarrow$

$\pi_1^{-1} : \mathcal{U} \to \mathcal{U}_1$ coordinate

(the atlas they define $\mathbb{D}/\Gamma$)

Furthermore the coordinate is isothermal

$$\langle \ , \ \rangle_{z_0} = \frac{4}{1-|z_0|^2} (\ , \ )_{Eucl.}$$

(conformal w.r.t. Euclidean metric)

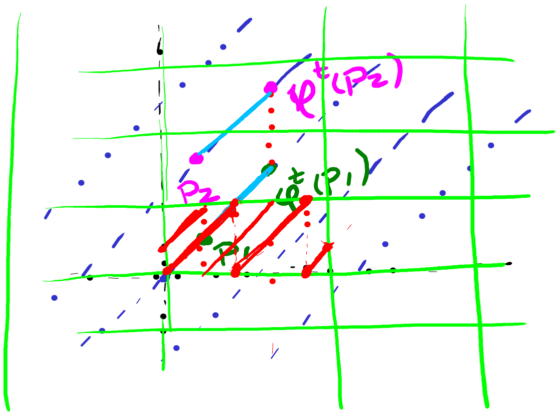
Why Riemann surfaces?

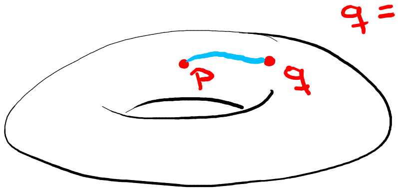
$\mathbb{R}^2/\mathbb{Z}^2$

$v = (v_1, v_2)$

$v \in \mathbb{R}^2 \quad (v \neq 0)$

$$\varphi^t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(p_1, p_2) \longmapsto (p_1, p_2) + tv$$

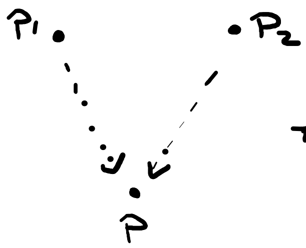




$q = \varphi^t(p_1)$
$= \varphi^t(p_2)$

Kronecker system

Rmk The flow commutes with the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$



$$\pi[\varphi^t(p_1)] = \pi[\varphi^t(p_2)]$$

$\Rightarrow$ the flow descends to a flow on $\mathbb{R}^2/\mathbb{Z}^2$

Furthermore: Dynamics on $\mathbb{R}^2/\mathbb{Z}^2$ is rather simple

(1) All orbits are dense (if the slope of v is irrational)

(2) Otherwise all orbits are periodic with the same period (if the slope of v is rational)

From the perspective of ergodic theory. } Lebesgue measure is invariant and, if the slope is irrational, then the flow is uniquely ergodic.

Generalization of this result

Setting : Homogeneous spaces

↪ G/Γ

G Lie group

Γ lattice

(lattice : discrete subgroups such that G/Γ has finite Haar measure)

G = Lie group ($\mathbb{R}^2$)

Γ = lattice

G/Γ ⇝ manifold endowed with a finite volume measure

Given $H < G$

H acts on G/Γ on the left

$\boxed{SL(2,\mathbb{R})}$

Q1 : Are ther lattices on $SL(2,\mathbb{R})$? Many, few ?

Q2 : what is the nature of the subgroups H that will act on $SL(2,\mathbb{R})/\Gamma$?

⇝ H ⇝ 1-parameter subgroup (flow)

$$H = \exp(tM)$$

$t \in \mathbb{R}$; $M \in sl(2,\mathbb{R})$

↪ matrix with $tr(M) = 0$

$$\left(\exp(M) = I + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots \right)$$

To generalize the Kronecker system, we shall ask the orbits to split as slow as possible.

↳ The eigenvalues of M will be zero

Ex: $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \exp(tM) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$

(polynomial behavior)

↳ the split of the trajectories is polynomial in t as opposed to exponential.

__Def__:
In a matrix group (Lie group) a unipotent flow is the 1-parameter subgroup of the form $\exp(tM)$ where M is a matrix all of whose eigenvalues are equal to zero.

__Thm__ (Roughly speaking)
Dynamics of unipotent flows in homogeneous spaces G/Γ are as in the Kronecker system.

__Our aim__: To understand the situation for $G = SL(2,\mathbb{R})$

$\left\{ \begin{array}{l} * \text{ Describe } SL(2,\mathbb{R}) \text{ homogeneous spaces} \\ \quad \text{essentially equivalent to Riemann surfaces} \\ * \text{ Then try to understand the dynamics} \end{array} \right.$

$$PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \{Id, -Id\}$$

$$SL(2,\mathbb{R}) \xrightarrow[\text{degree } 2]{\text{covering}} PSL(2,\mathbb{R})$$

w.r.t. hyperbolic metric

$$PSL(2,\mathbb{R}) \simeq Aut(\mathbb{H}) \;\; (= Iso(\mathbb{H}))$$

Lattices in $SL(2,\mathbb{R})$ $\rightsquigarrow$ Lattices in $\mathbb{P}SL(2,\mathbb{R})$

$\overline{\Gamma}$ $\qquad\qquad\qquad\qquad$ $\underbrace{\qquad\qquad}_{\Gamma}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Gamma \ni M$

$M, -M \quad \longleftarrow\!\sim\!\sim \quad M$

$$SL(2,\mathbb{R})/\overline{\Gamma} = PSL(2,\mathbb{R})/\Gamma$$

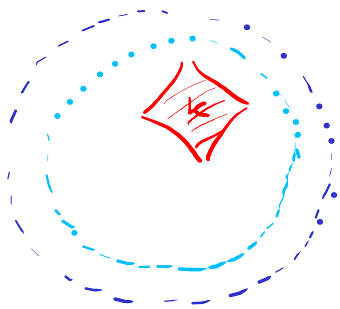
Prop:

$\Gamma \subseteq SL(2,\mathbb{R})$ is discrete $\iff$ Γ acts on $\mathbb{H}$ in a properly discontinuously manner.

$\Updownarrow$

Given K compact of $\mathbb{H}$, the set $\{\gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset\}$ is finite

$$\mathbb{P}SL(2,\mathbb{R}) = Aut(\mathbb{D}) = S^1 \times \mathbb{D}$$



References :

1. Farkas & Kra , Riemann surfaces, Springer
2. Elon Lages Lima , Fundamental groups and covering maps
3. Einsiedler & Ward , Ergodic theory with a view towards number theory , Springer

—//—

$Iso(\mathbb{H}) = Aut(\mathbb{H})$

$\hat{\Updownarrow}$

we have constructed a Riemannian metric on $\mathbb{H}$ and the construction allowed us to prove that

$$Aut(\mathbb{H}) \subseteq Iso(M)$$

$\Downarrow$

The converse was not proved

Use $\mathbb{D}$ instead of $\mathbb{H}$

$\Downarrow$

Take $f \in Iso(\mathbb{D}) \rightsquigarrow f(0) = q \in \mathbb{D}$

Take now $f_q \in Aut(\mathbb{D})$: $f_q(q) = 0 \in \mathbb{D}$ $\qquad f_q(q) = \frac{z-q}{-\bar{q}z+1}$

$\Downarrow$

$$\underbrace{f_q \circ f}_{\text{Isometry}}(0) = 0$$

$(f_q \circ f)'(0) \rightsquigarrow R_\theta$

$$(\underbrace{R_\theta^{-1} \circ f_q \circ f}_{\text{Isometry}})(0) = 0 \quad , \quad (R^{-1} \circ f_q \circ f)'(0) = 1$$

$\Downarrow$

$$R_\theta^{-1} \circ f_q \circ f = id$$

$\Downarrow$

$$f = f_q^{-1} \circ R_\theta \in Aut(\mathbb{D})$$

G = group

Γ = subgroup of G

$G/\Gamma \rightsquigarrow$ set of the equivalence classes

$$\{ g_1 = g_2 \iff \exists\, \gamma \in \Gamma : g_2 = g_1 \gamma \}$$

$\Gamma \backslash G$

$$g_1 = g_2 \iff \exists\, \gamma \in \Gamma : g_2 = \gamma g_1$$

<u>Rmk</u>: If G is a Lie group and Γ is discrete, then G/Γ is a <u>manifold</u>

Let H be another subgroup of G

we have that H acts on the left on G/Γ and the action is given as follows

$$\rho : H \times G/\Gamma \longrightarrow G/\Gamma$$
$$(h, \underset{[g]}{g}) \longmapsto \underset{[hg]}{h.g}$$

<u>Claim</u>: ρ is well defined in the sense that it does not depend on the choice of the representative g. In fact, if $g.\gamma$ is another representative ($\gamma \in \Gamma$) then

$$\rho(h, g\gamma) = h(g\gamma) \underset{\text{associativity}}{=} (hg).\gamma$$

and $(hg)\gamma$ belongs to the same classe of hg on G/Γ

<u>Let now</u>

$$\left.\begin{array}{l} G = \text{Lie group} \\ \Gamma = \text{lattice} \end{array}\right\} \Rightarrow G/\Gamma \text{ manifold}$$

to precise letter

<u>Volume on G</u> : Haar measure

Take $e \in G$ and $T_e G$ $\quad$ (e = identity of G)

⇝

- every n-dimentional vector space possesses a unique (up to multiplicative constant) n-linear alternate and non-identically zero form

$$\omega : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R}$$

(1) Linear in each coordinate

(2) alternate in the sense that

$$\omega(v_1, \dots, v_i, v_{i+1}, \dots, v_n) = -\omega(v_1, \dots, v_{i+1}, v_i, \dots, v_n)$$

- choose a non-identically zero alternate n-form ω and normalize it by taking $\omega(e_1, \dots, e_n) = 1$

$\Rightarrow$ ω is the determinant, i.e

$$\omega_1(v_1, \dots, v_n) = \det \begin{array}{c} \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \\ \text{colums} \end{array}$$

Choose a base $(u_1, \dots, u_n)$ of $T_e G$ and take a n-form. Then we normalize by taking $\omega(u_1, \dots, u_n) = 1$

Note that ω is just defined on $T_e G$ and we need to extend it to a volume on G.

⇝

we will extend ω to a volume form on G by making it invariant by left/right translations

⇝

ω becomes a differential n-form

$$p \in G : \quad \omega(p) : T_p G \times \dots \times T_p G \longrightarrow \mathbb{R} \qquad (n = \dim G)$$
$$(u_1, \dots, u_n) \longmapsto$$

$$\omega(p)[u_1, \dots, u_n] = \omega_e(w_1, \dots, w_n)$$

where $w_1, \dots, w_n$ are the elements in $T_e G$ defined by

$$D_e T_p \cdot w_i = u_i$$

being T_p the translation map that sents e to p.

T_p can be $\quad T_p : G \longrightarrow G$

(*) $\quad z \longmapsto p.z$ or $z.p$

Haar measure:

$\mathcal{U} \subseteq G$, open set. The Haar measure of $\mathcal{U}$ is defined by
$$\int_{\mathcal{U}} \omega$$

with ω as above.

(*) we will more generally denote the mentioned translations by
$$r_p : g \longmapsto gp \quad \text{or} \quad l_p : g \longmapsto pg$$

we will then have
$$r_p^* \omega = \omega \quad \text{or} \quad l_p^* \omega = \omega$$

Summarizing:

Let G Lie group and consider a non-degenerated alternate multilinear n-form det on $T_e G$
$$\det : \underbrace{T_e G \times \dots \times T_e G}_{n \text{ times}} \longrightarrow \mathbb{R}, \quad n = \dim(G)$$

we know that, det can uniquely be extended into a (right) invariant volume form ω on G. In fact, this extension is determined by letting
$$\omega(p)(u_1, \dots, u_n) = \det\left((D_e r_p)^{-1} u_1, \dots, (D_e r_p)^{-1} u_n \right)$$

where $r_p : G \to G$ is defined by $r_p(g) = g.p$ (right translation by p)

The (right-invariant) Haar measure is defined by setting
$$\mu(\mathcal{U}) = \int_{\mathcal{U}} \omega$$

for every open set $\mathcal{U} \subseteq G$ (and then μ has a unique extension as measure on G)

In particular, we can define lattices on G as follows:

Def :

A subgroup $\Gamma < G$ is called a lattice if

- Γ is discrete
- the right quotient G/Γ has finite Haar measure

The manifold G/Γ is defined as the equivalence classes on G by the right action of Γ. Since the Haar measure on G is invariant by right translations, ω - the volume form - is, in particular, invariant by the action of Γ on G and descends to a volume form $\hat{\omega}$ on the right quotient. The quotient G/Γ is said to have finite Haar measure if

$$\int_{G/\Gamma} \hat{\omega} < \infty$$

Rmk : This is always the case if G/Γ is compact.

Example : $G = \mathbb{R}^2$ ($\mathbb{R}^2$ is an additive Lie group that acts on itself by translations)

$\Gamma = \mathbb{Z}^2$ = lattice

the quotient $\mathbb{R}^2/\mathbb{Z}^2$ is compact so that the condition on finiteness of the Haar measure is automatically satisfied.

The problem of bi-invariance

Recalling that ω was constructed as a right-invariant volume form, we can ask weather or not ω happens to be invariant by left translations as well.

$$\left.\begin{aligned} r_p^* \omega &= \omega \\ l_p^* \omega &= \omega \end{aligned}\right\} \text{ simultaneously}$$

this leads us to the notion of modular function on G.

Let me start with a simple (yet important) lemma :

Lemma:

For every fixed $p \in G$, the left translations $l_p : G \to G$ defined as $l_p(g) = p.g$ is such that $l_p^* \omega = \text{const.}\, \omega$, i.e. $l_p^* \omega$ is a constant multiple of ω

this constant may depend on the point p.

Proof:

Note that the associativity of a Lie group implies that the automorphisms l_p and r_q commute for arbitrary $p, q \in G$.
In fact, we have

$$(l_p \circ r_q)(g) = l_p(gq) = p.(g.q) = (p.g).q = r_q(pg)$$
$$= (r_q \circ l_p)(g) \qquad \forall g \in G$$

$$\Rightarrow \quad l_p \circ r_q \equiv r_q \circ l_p$$

$$(l_p \circ r_q)^* \omega = r_q^*(l_p^* \omega)$$
$$\shortparallel$$
$$(r_q \circ l_p)^* \omega = l_p^*(r_q^* \omega) = l_p^* \omega$$

Thus

$$r_q^*(l_p^* \omega) = l_p^* \omega$$

$\Updownarrow$

in other words, the volume form $l_p^* \omega$ is right invariant on G. Thus, it is fully determined by the alternate n-linear form

$$(l_p^* \omega)(e) : \underbrace{T_e G \times \ldots \times T_e G}_{n\text{-times}} \longrightarrow \mathbb{R}$$

Since on a n-dimensional vector space there is only one alternate n-linear form, up to multiplication by a constant, there follows that $l_p^* \omega$ is a constant multiple of ω

$\blacksquare$

The constant in question will be denoted by $\dfrac{l_p^* \omega}{\omega}$

Def

The modular function J on a Lie group G is defined as the function $J : G \to \mathbb{R}$ given by

$$J(p) = \frac{l_p^* \omega}{\omega}$$

Clearly, the Haar measure of a group G is bi-invariant if and only if the resulting modular form is constant equal to 1. For this reason, Lie groups with a bi-invariant Haar measure are called unimodular groups.

The existence of lattices ensure that a group must be unimodular. More precisely, we have

Lemma:

If G contains a lattice Γ, then G is unimodular.

Proof:

Let Γ be a lattice and set $M = G/\Gamma$ which is equipped with the (projection of) the Haar measure μ satisfying $\mu(M) < \infty$. Now, consider an element $p \in G$ and the associated left translation $l_p : G \to G$. We know already that l_p commutes with the right action of Γ so that l_p descends to de diffeo

$$\hat{l}_p : M \to M$$

(In particular, $\mu(\hat{l}_p(M)) = \mu(M)$). However, as previously seen, l_p, and hence $\hat{l}_p$, multiply the Haar measure by the constant $J(p)$. Thus, we obtain

$$\mu(\hat{l}_p(M)) = \int_M l_p^* \omega = J(p) \cdot \int_M \omega = J(p) \cdot \mu(M)$$

However, since $\mu(\hat{l}_p(M)) = \mu(M)$, we conclude that

$$J(p) = 1 .$$

The lemma is proved. ▨

The actions of Γ

To begin to answer the above questions we consider two actions of Γ:

(1) the action of Γ on $\mathbb{H}$ (by projective transformations)

(2) the "tangent action" of Γ on $T^1\mathbb{H} \simeq S^1 \times \mathbb{H}$ given by

$$\gamma(p, v) = (\gamma(p), \gamma'(p).v)$$

for $(p.v) \in T^1\mathbb{H}$

Recall the identification

$$PSL(2,\mathbb{R}) \longrightarrow T^1\mathbb{H} \simeq S^1 \times \mathbb{H}$$

given by

$$\sigma(p,v) = \gamma \quad , \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Updownarrow$$

$$\frac{ai+b}{ci+d} = p$$

$$\gamma'(i).i = v$$

$$\gamma'(i) = \frac{1}{(ci+d)^2}$$

Furthermore, σ is equivariant in the following sense

if $\sigma(p,v) = \gamma_0$, then for every $\gamma \in PSL(2,\mathbb{R})$

$$\gamma.\gamma_0 = \sigma(\gamma(p,v))$$

Aim : to "see" the quotient $SL(2,\mathbb{R})/\Gamma$

$\Updownarrow$

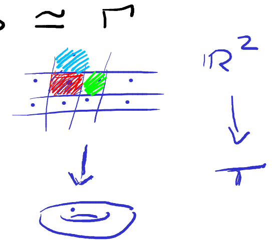
we begin by considering the simpler quotient $\mathbb{H}/\Gamma$. As mentioned

$$p: \mathbb{H} \longrightarrow \mathbb{H}/\Gamma$$

is a covering map such that deck transformation group $\simeq \Gamma$

$\Rightarrow \quad \mathbb{H}/\Gamma \simeq$ Fundamental domain

(tiling $\mathbb{H}$ by the action of Γ which coincides with the Deck transformations)

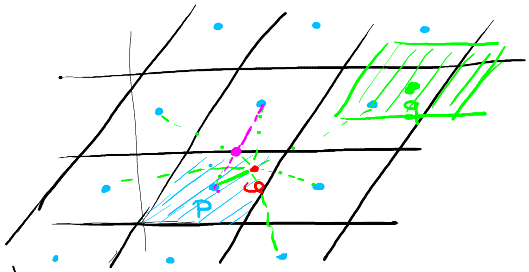


Determining a fundamental domain for $\mathbb{H}/\Gamma$

The Dirichlet region

Def: The Dirichlet region centered at a point $p \in \mathbb{D}/\mathbb{H}$ is defined by

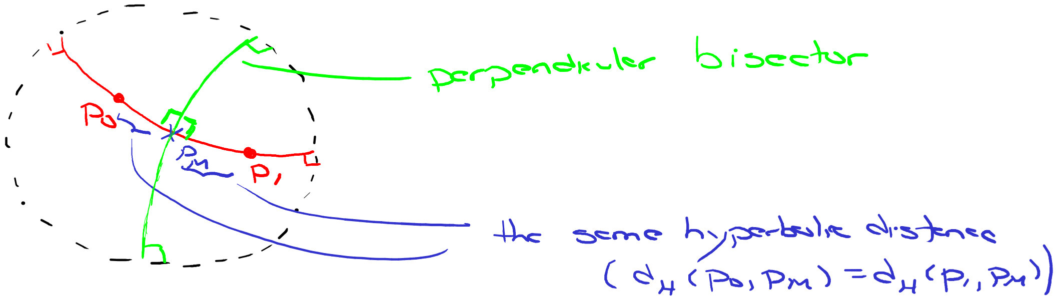
$$D_\Gamma(p) = \{ \omega \in \mathbb{D}/\mathbb{H} \ : \ d_{\mathbb{H}}(\omega, p) \leq d_{\mathbb{H}}(\omega, \gamma.p) \text{ for every } \gamma \in \Gamma \}$$



⟵ exemplification in the case of $\mathbb{R}^2$ with the Euclidean distance

Return to $\mathbb{D}$

Consider a segment of geodesic limited by points $[p_0, p_1]$. The perpendicular bisetor of the segment $[p_0, p_1]$ is by definition the geodesic orthogonal to $[p_0, p_1]$ and passing through its mid point



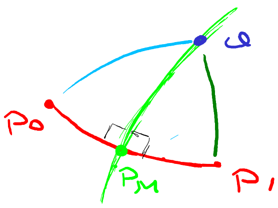
Lemma:

the perpendicular bisector of the geodesic segment $[p_0, p_1]$ is nothing but the set of point $\omega \in \mathbb{D}$ satisfying

$$d_H(\omega, p_0) = d_H(\omega, p_1)$$

Proof:

It is clear that a point ω lying in the bisector of $[p_0, p_1]$ satisfies $d_H(\omega, p_0) = d_H(\omega, p_1)$. Indeed, we have the geodesic triangles



Hyperbolic cosino law:

$$\cosh(d_H(\omega, p_0)) = \cosh(\underbrace{d_H(p_M, p_0)}_{/2}) \cdot \cosh(d_H(p_M, \omega)) - \sinh(\underbrace{d_H(p_M, p_0)}_{/2}) \cdot \sinh(d_H(p_M, \omega)) \cdot \cos\frac{\pi}{2}$$

and the same holds for $d_H(\omega, p_1)$ and hence

$$d_H(\omega, p_0) = d_H(\omega, p_1)$$

For the converse, just note that the geodesic triangles

$$\omega p_0 p_M \quad \text{and} \quad \omega p_1 p_M$$

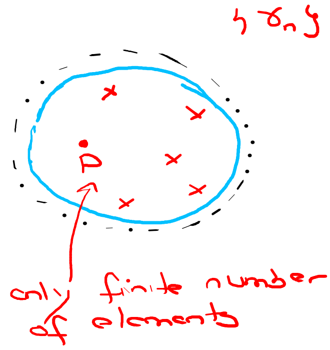
have equal sides. This determines the cosines of the angles $p_0 \widehat{p_M} \omega$ and $p_1 \widehat{p_M} \omega$ so that the angles must be equal (and equal to $\pi/2$) ∎

Thm :

If Γ is a discret subgroup of $PSL(2,\mathbb{R})$ as above, then $D_\Gamma(p)$ is a fundamental domain for the action of Γ on $\mathbb{H}$ ($\mathbb{D}$). Furthermore, $D_\Gamma(p)$ is connected.

Lemma:

If Γ is a discrete group, then for every point $p \in \mathbb{H}$ ($\mathbb{D}$) and every non-trivial sequence $\{\gamma_n\} \subseteq \Gamma$, the sequence $\{\gamma_n(p)\}$ escapes of every compact set of $\mathbb{H}$ ($\mathbb{D}$).



Proof:

Recall that $PSL(2,\mathbb{R}) \simeq S^1 \times \mathbb{D}$. thus, if there is a compact set $K \subseteq \mathbb{D}$ such that $\{\gamma_n(p)\} \subseteq K$ ($\gamma_n(p) \in K$, $\forall n \in \mathbb{N}$), it follows that $\{\gamma_n\}$ is contained in a compact subset of $PSL(2,\mathbb{R})$. this contradits the assumption that Γ is discrete.

Proof of the theorem:

Consider a point ω and denote by $\Gamma.\omega$ its orbit. Since $\{\Gamma.\omega\} \subseteq \mathbb{D}$ is a discrete set (previous lemma), there exists $\omega_0 \in \Gamma.\omega$ such that

$$d(p,\omega_0) \leq d(p,\gamma.\omega) \qquad \forall \gamma \in \Gamma$$

(i.e. ω_0 has minimum distance to p among points in $\Gamma.\omega$). By definition, there follows that $\omega_0 \in D_\Gamma(p)$ and hence $D_\Gamma(p)$ contains at least one point for every orbit under Γ.

To prove that $D_\Gamma(p)$ is a fundamental region, it only remains to check that the interior of $D_\Gamma(p)$, $\mathrm{Int}(D_\Gamma(p))$ contains at most, point of an orbit (in general the minimal point $\omega_0 \in \Gamma.\omega$ is unique, but when it is not unique, they all lie in $D_\Gamma(p) \setminus \mathrm{Int}(D_\Gamma(p))$)

Suppose that q_0, q_1, $q_1 = \gamma_0(q_0)$ ($\gamma_0 \in \Gamma$), are both in $\mathrm{Int}(D_\Gamma(p))$

$$\Longrightarrow d(q_0,p) = d(q_1,p)$$

$$\Longrightarrow p \text{ lies in the bissector of } [q_0,q_1]$$

However, if $\{\omega_n\} \to q_0 \Longrightarrow \gamma_0.\{\omega_n\} \to q_1$

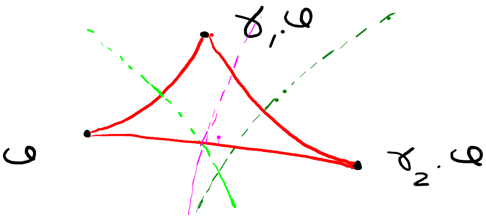
$\Longrightarrow$ in particular, they are still in $\mathrm{Int}(D_\Gamma(p))$

Thus $\forall n$ suff. large, p lies in the bisector of ω_n and $\gamma_0\omega_n$

This intersection being empty, the theorem follows.

Exercise: check that $D_\Gamma(p)$ is connected!

Rmk
The boundary $\partial D_\Gamma(p) = \overline{D_\Gamma(p)} \setminus \mathrm{Int}(D_\Gamma(p))$ contains in general two, and exactly 2, points of each orbit.

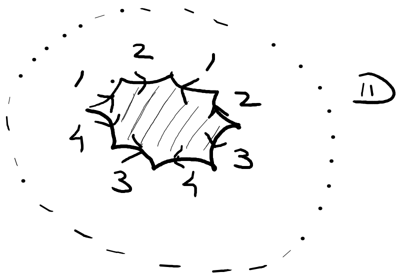


They do not intersect in general (intersect, for example, when the geodesic

triangle has sides
with the same lenght)

Identifying these points (i.e. gluing these pairs of points allows us to recover the Riemann surface)

Example :



Fundamental region for $PSL(2,\mathbb{R})$

Since $PSL(2,\mathbb{R}) \simeq T^1\mathbb{D} = S^1 \times \mathbb{D}$ and the action is fibered (tangent) we immediately see that the homogeneous space

$$PSL(2,\mathbb{R})/\Gamma \simeq S^1 \times \underbrace{D_\Gamma(p)}_{\text{Dirichelet region for the action of } \Gamma \text{ on } \mathbb{D}}$$

This gives a "geometric vision" of the quotient $PSL(2,\mathbb{R})/\Gamma$.

<u>Question</u> : when is Γ a lattice ?

To answer this question, we need to know when the Haar measure on $PSL(2,\mathbb{R})/\Gamma$ is finite. For this, the key is to provide the formula for the Haar measure (or, more presisely, of the invariant volume form) on the coordinate $T^1\mathbb{D}$.

<u>Rmk</u> : when the fundamental domain $D_\Gamma(p)$ of the action of Γ on $\mathbb{D}$ is compact, then the Haar measure in question is the integral of some volume form on the compact region $S^1 \times D_\Gamma(p)$. It is therefore necessarily finite

The above question aims therefore at deciding on the existence of discret groups Γ for which $D_\Gamma(p)$ is not compact but still $S^1 \times D_\Gamma(p)$ has finite Haar measure.

The formula below clarifies this issue (it is nicer to work on $\mathbb{H}$)

Consider the map

$$\sigma : T^1\mathbb{H} \longrightarrow PSL(2,\mathbb{R})$$
$$(p,v) \longmapsto \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\gamma(z) = \frac{az+b}{cz+d}$$

$$\gamma(i) = \frac{ai+b}{ci+d} = p$$

$$\gamma_i'(i) = v$$

Denote by ω the (bi)-invariant volume form on $PSL(2,\mathbb{R})$ defined up to a multiplicative constant.

Prop:

The volume form $\sigma^*\omega$ is given in coordinates (θ, ω, y) for $S^1 \times \mathbb{H}$ by

$$\frac{1}{y^2}\, d\omega\, dy\, d\theta$$

(up to a multiplicative constant)

Proof: calculations depending on diff. geometry

Corollary:

By Fubini's theorem, we have

$$\int_{S^1 \times D_\Gamma(p)} \frac{1}{y^2}\, d\omega\, dy\, d\theta = 2\pi \int_{D_\Gamma(p)} \frac{1}{y^2}\, d\omega\, dy$$

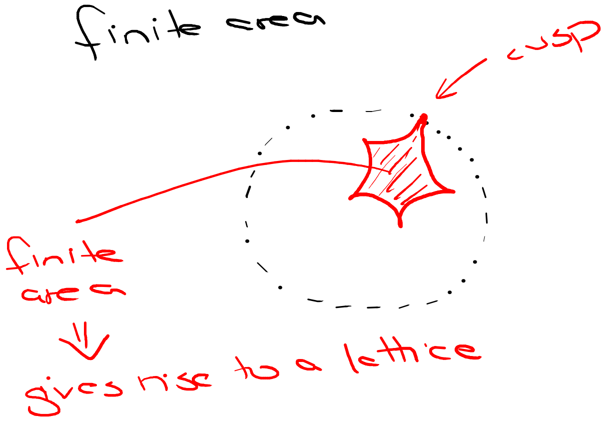
hyperbolic area form

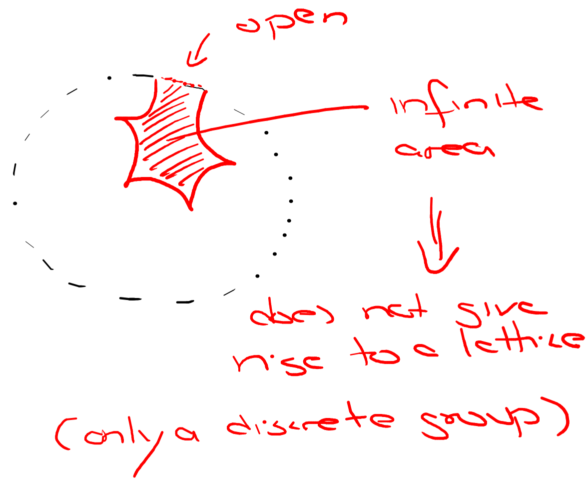
Prop: (summarizing)

A discrete subgroup $\Gamma \subseteq PSL(2,\mathbb{R})$ is a lattice if and only if the hyperbolic area of a fundamental domain $D_\Gamma(p)$ is finite. $\boxtimes$

Complement (follows from Gauss-Bonnet)

Non-compact regions with "cusps" at infinity are of finite area



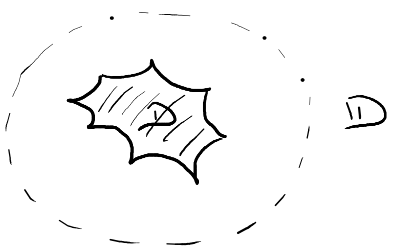


Lattices on $SL(2,\mathbb{R})$ - Introduction to Teichmüller spaces

Consider a lattice $\Gamma \subseteq SL(2,\mathbb{R})$ (or $PSL(2,\mathbb{R})$). We have seen that Γ acts properly discontinuously on $\mathbb{D}$ ($\mathbb{H}$) and that $\mathbb{D}/\Gamma$ is a Riemann surface. In particular, the fundamental domain (Dirichelet region) for $\mathbb{D}/\Gamma$ contains some information about the lattice / Riemann surface.

⇓

For example, whether or not it is compact and, in the compact cases, the genus of the Riemann surface in question
⇓
(and hence topological / C^∞ - classification)



Question: fixed these topological constants, how many "different" lattices in $SL(2,\mathbb{R})$ realizing them there exists?

To make the question more accurate, recall that a lattice is, in particular, a subgroup. $\Gamma \subseteq SL(2,\mathbb{R})$ (or $PSL(2,\mathbb{R})$) If g is an arbitrary element of $SL(2,\mathbb{R})$ (or $PSL(2,\mathbb{R})$) then the conjugate

$$\Gamma' = g \Gamma g^{-1}$$

is again a discret subgroup of $SL(2,\mathbb{R})$

Since, moreover, the Haar measure is bi-invariant, it immedictly follows that the volume of $SL(2,\mathbb{R})/\Gamma'$ equals the volume of $SL(2,\mathbb{R})/\Gamma'$. Thus, Γ' is a lattice as well, which basically shares all geometric characteristics of Γ.

Def:

Two lattices Γ, Γ' in $SL(2,\mathbb{R})$ are conjugate if there exists $g \in SL(2,\mathbb{R})$ such that $\Gamma' = g \Gamma g^{-1}$.

we will look for describing lattices on $SL(2,\mathbb{R})$ up to conjugation in the above sense.

Rmk: If $\Gamma' = g \Gamma g^{-1}$, then the actions of Γ and Γ' on $\mathbb{D}$ (or $\mathbb{H}$) are also conjugate by automorphismus of $\mathbb{D}$ (or $\mathbb{H}$) induced by g. In particular, the Riemann surfaces

$$\mathbb{D}/\Gamma \quad \text{and} \quad \mathbb{D}/\Gamma'$$

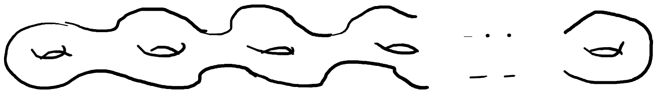
are the same (i.e. there exists a local holomorphic diffeomorphism from $\mathbb{D}/\Gamma$ to $\mathbb{D}/\Gamma'$).

$\rightsquigarrow$ Somehow we will look for a converse for this statement.

To simplify, let us only consider the case of co-compact lattices, i.e. the Riemann surface $\mathbb{D}/\Gamma$ is compact.

$\rightsquigarrow$

In terms of topological / smooth classification, we have

$\mathbb{D}/\Gamma$ =  genus g surface

we fix then g (the genus).

moduli space : $\left\{ \begin{array}{l} \text{Parametrize the space of all} \\ \text{distinct Riemann surfaces structures} \\ \text{on a genus } g \text{ surface} \end{array} \right\}$

- $g=0$ $\rightsquigarrow$ this space is reduced to a single point (the Riemann sphere)

- $g=1$ $\rightsquigarrow$ this space is parametrized as in the proposed exercise

$\rightsquigarrow$

Exercise : $\Gamma = \langle z \mapsto z+1, \ z \mapsto z+\tau \rangle$

$\tau \in \mathbb{C} \setminus \mathbb{R}$

$\mathrm{Im}(\tau) > 0$

τ_1 and τ_2 define the same tori

$\Updownarrow$

$$\exists\, M \in \underset{SL(2,\mathbb{Z})}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} : \frac{a\tau_1 + b}{c\tau_1 + d} = \tau_2$$

- $g \geq 2$: more complicated

$\rightsquigarrow$ related to the problem of classifying lattices in $SL(2,\mathbb{R})$ up to conjugation

The mapping class group

From now on we fix the genus g and we talk about compact surfaces Σ_g of genus g.

$\Downarrow$

Let $\mathrm{Diff}_+(\Sigma_g)$ the group of C^∞-diffeomorphisms of Σ_g preserving orientation

Def:
An isotopy between two elements $f_1, f_2 \in \mathrm{Diff}(\Sigma_g)$ is a continuous map

$$I : [0,1] \times \Sigma_g \longrightarrow \Sigma_g$$

Satisfying the following conditions

(1) $\quad I(0,\cdot) = f_1 : \Sigma_g \to \Sigma_g$

$\quad\;\; I(1,\cdot) = f_2 : \Sigma_g \to \Sigma_g$

(2) $\quad \forall t \in [0,1]$, the map $I(t,\cdot) : \Sigma_g \to \Sigma_g$ defines an element of $\mathrm{Diff}_+(\Sigma_g)$

It is straightforward to see that isotopy defines an equivalence relation on $\mathrm{Diff}_+(\Sigma_g)$ and we consider the classes of equivalence for this relation.

$\Downarrow$

More precisely, denote by $\mathrm{Diff}_0(\Sigma_g)$ the set of elements on $\mathrm{Diff}_+(\Sigma_g)$ that are isotopic to the identity. Then we have:

Lemma:

(1) $\mathrm{Diff}_0(\Sigma_g)$ is a subgroup of $\mathrm{Diff}_+(\Sigma_g)$

(2) $\mathrm{Diff}_0(\Sigma_g)$ is actually a normal subgroup of $\mathrm{Diff}_+(\Sigma_g)$

Proof:
To check (1) we have to show that $f_1 \circ f_2 \in \mathrm{Diff}_0(\Sigma_g)$ provided that $f_1, f_2 \in \mathrm{Diff}_0(\Sigma_g)$. Consider then isotopies

$$I_1 : f_1 \sim id$$

$$I_2 : f_2 \sim id$$

we define an isotopy I_3 between $f_1 \circ f_2$ and the identity by letting

$$I_3(t,\cdot) = \begin{cases} f_1(I_2(2t,\cdot)) & , \text{if } t \leq 1/2 \\ I_1(2t-1,\cdot) & , \text{if } t \geq 1/2 \end{cases}$$

$$I_3(0,\cdot) = f_1(I_2(0,\cdot)) = (f_1 \circ f_2)(\cdot)$$

$$I_3(1/2,\cdot) < \begin{matrix} f_1(I_2(1,\cdot)) = f_1 \circ id = f_1(\cdot) \\ I_1(0,\cdot) = f_1(\cdot) \end{matrix}$$

$$I_3(1,\cdot) = I_1(1,\cdot) = id(\cdot)$$

$\Rightarrow$ I_3 is an isotopy map between $f_1 \circ f_2$ and id and (1) is proved

To check (2), we need to show that $g \circ f \circ g^{-1} \in \mathrm{Diff}_0(\Sigma_g)$ for any $f \in \mathrm{Diff}_0(\Sigma_g)$ and any $g \in \mathrm{Diff}_+(\Sigma_g)$. Denote by

$$I : [0,1] \times \Sigma_g \to \Sigma_g$$

the isotopy map starting at f and going to the identity. The map

$$\begin{aligned} I_g : [0,1] \times \Sigma_g &\to \Sigma_g \\ (t,\omega) &\longmapsto g(I(t, g^{-1}(\omega))) \end{aligned}$$

is an isotopy starting at $g \circ f \circ g^{-1}$ and ending at $g \circ id \circ g^{-1} = id$. Hence $g \circ f \circ g^{-1}$ lies in $\mathrm{Diff}_0(\Sigma_g)$ and this proves the lemma. $\boxtimes$

<u>Corollary / Definition</u>:

The set of equivalence classes $\mathrm{Diff}_+(\Sigma_g) / \mathrm{Diff}_0(\Sigma_g)$ is itself a group. It is called the <u>Mapping class group</u> of the genus g surface.

Teichmüller spaces

is a simplification of the moduli space ; where moduli problem admit a much nice answer

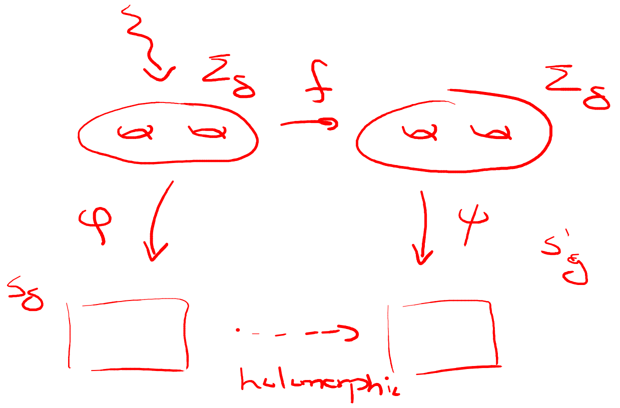
it is well addapt to our initial problem about conjugate lattices.

Def:

Fixed g (genus), the Teichmüller-space T_g associated with Σ_g consists of all Riemman surface structures on Σ_g modulo the following equivelence relation: two structures S_g and S_g' on Σ_g are said to be equivalent if there exists a holomorphic diffeomorphism

$$f: S_g \longrightarrow S_g'$$

that is isotopic to the identity



Rmk: There are distinct points in T_g that actually define the same strure of Riemann surface on Σ_g and, hence, a unique point in the moduli space.

$$f: \Sigma_g \longrightarrow \Sigma_g$$ such that f is not isotopic to the identity

So, what is the advantage brought by imposing that diffeomorphisms should be isotopic to the identity ?

Here is the answer:

$$\begin{tikzcd} \mathbb{D} \arrow[r, dashed, "F"] \arrow[d, dashed, "\pi"'] & \mathbb{D} \arrow[d, dashed, "\pi"] \\ \Sigma_g \arrow[r, "f"'] & \Sigma_g \end{tikzcd}$$

Can we lift $f: \Sigma_g \to \Sigma_g$ to a diffeomorphism $F: \mathbb{D} \to \mathbb{D}$?

yes, but...

$$\begin{tikzcd} & \mathbb{D} \arrow[d, "\pi"] \\ \mathbb{D} \arrow[ur, dashed] \arrow[r, "f\circ\pi"] & \Sigma_g \end{tikzcd}$$

simply connected

there is $\widetilde{f\circ\pi} = F$
$\Leftrightarrow$
lifting a map through a covering map π

However, there are plenty of lifts
$\Leftrightarrow$
they depend on choosing an initial point in $\mathbb{D}$
$\Leftrightarrow$
This poses a problem to lift groups: if F_1 and F_2 are lifts of f_1 and f_2 (respectively) then $F_1 \circ F_2$ may not be "prescribed" lift of $f_1 \circ f_2$
$\Leftrightarrow$
one needs to find a way to "select a prefered lift"

$\Downarrow$

This problem is solved by the condition of being isotopic to the identity: given $f : \Sigma_g \to \Sigma_g$ isotopic to the identity, we proceed as follows:

$$\begin{tikzcd} \mathbb{D} \arrow[r, "Id"] \arrow[d, "\pi"] & \mathbb{D} \arrow[d, "\pi"] \\ \Sigma_g \arrow[r, "f \simeq id"'] & \Sigma_g \end{tikzcd}$$

canonical lift of Id
$\Leftrightarrow$
then choose a lift $\tilde{f}$ of f by lifting through π the homotopy/isotopy between f and id.

$\Leftrightarrow$
this yields a lift of f that does not depend on any choice: with this lift we obtain in particular

$$\widetilde{f_1 \circ f_2} = \tilde{f}_1 \circ \tilde{f}_2$$

and this enables us to work with group actions!!

This notion is well adapted to the "lattice problem".

Lemma: If $\Gamma, \Gamma' \subseteq SL(2,\mathbb{R})$ are two conjugated lattices, then $\mathbb{D}/\Gamma$ and $\mathbb{D}/\Gamma'$ define the same point in the Teichmüller space $\mathcal{T}_g$. The converse also holds.

Proof:

Given $g \in SL(2,\mathbb{R})$, let $g_{hom} : \mathbb{D} \to \mathbb{D}$ denote the corresponding automorphism of $\mathbb{D}$. Next, note that $SL(2,\mathbb{R})$ is connected (or $PSL(2,\mathbb{R})$). Thus, every element $g \in SL(2,\mathbb{R})$ can be linked to the identity by a path

$$\sigma : [0,1] \to SL(2,\mathbb{R})$$

$$\sigma(0) = id$$
$$\sigma(1) = g$$

Setting

$$\sigma^t = \sigma(t) \quad , \quad t \in [0,1]$$

we get that

$$g^t_{hom} : \mathbb{D} \to \mathbb{D} \qquad , \; t \in [0,1]$$

(the corresponding element in automorphisms of $\mathbb{D}$ associated with g^t)

defines an isotopy from g_{hom} and id ($id : \mathbb{D} \to \mathbb{D}$). Furthermore, by construction $g_{hom} = g'_{hom}$ is a holomorphic diffeo that is isotopic to the identity on Σ_g

$$\Downarrow$$

they represent the same point on T_g.

The converse: let $f : \mathbb{D}/\Gamma \to \mathbb{D}/\Gamma'$ be holomorphic map that is isotopic to the identity. Then, by lifting an isotopy between f and id to $\mathbb{D}$

$$\begin{tikzcd} \mathbb{D} \arrow[r, dashed, "\tilde{f} \simeq Id"] \arrow[d, "\pi"'] & \mathbb{D} \arrow[d, "\pi"] \\ \mathbb{D}/\Gamma \arrow[r, "f \simeq id"'] & \mathbb{D}/\Gamma' \end{tikzcd}$$

we obtain of maps

$$\tilde{f}_t : \mathbb{D} \to \mathbb{D}$$
$$\tilde{f}_0 = Id$$
$$\tilde{f}_1 = \text{lift of } f$$

that extend to a diffeomorphisms of $S^1 \times \mathbb{D}$ (tangent action)

$$S^1 \times \mathbb{D} \cong PSL(2,\mathbb{R})$$

Identifying $S^1 \times \mathbb{D}$ with $PSL(2,\mathbb{R})$, we produce a path of elements $\bar{\sigma}$ in $PSL(2,\mathbb{R})$ such that $\bar{\sigma}_1 = g$, $\bar{\sigma}_0 = id$

$\Rightarrow \quad \boxed{\Gamma' = g\Gamma g^{-1}}$ $\boxtimes$

Parametrizing Teichmüller space
$g=2$ as example

Let us start by introducing a topology on $\mathcal{T}_g$. For this note that the space of Riemann surface structures on Σ_g is the same as the space of hyperbolic metrics on Σ_g (denote by $\underline{\underline{\mathcal{H}}}$)

$\rightsquigarrow$ this space is a closed subset of the space of sections

$$\Sigma_g \longrightarrow T^*\Sigma_g \otimes T^*\Sigma_g$$

and hence can be endowed with the C^∞ - topology.

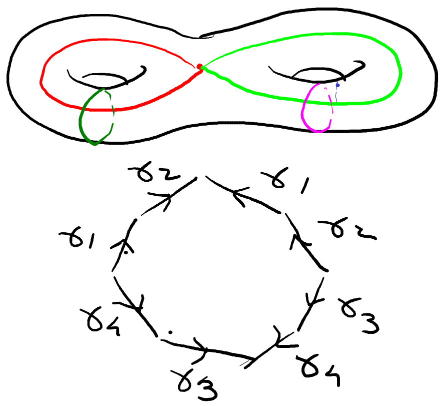
Next, $\mathrm{Diff}_0(\Sigma_g)$ operates in $\mathcal{H}$ as a group of homeomorphisms: each $f \in \mathrm{Diff}_0(\Sigma_0)$ induces a homeomorphism from $\mathcal{H}$ to $\mathcal{H}$ by pulling-back metrics.

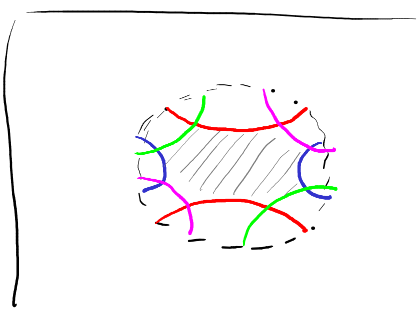
$\rightsquigarrow$ this space $\mathcal{H}/\mathrm{Diff}_0(\Sigma_g)$ can be endowed with the quotient topology.

<u>Our purpose</u> : to construct a homeomorphism from $\mathcal{T}_g$ to $\mathbb{R}_+^{3(g-1)} \times \mathbb{R}^{3(g-1)}$ (without all the details ...)

(Fenchel - Nielsen coordinates)

Set, for example, $g=2$ and look at the standard generators of $\pi_1(\Sigma_g)$





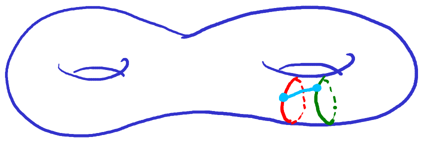
we are going to choose "geometric" representatives for the curves $\gamma_1, \gamma_2, \gamma_3, \gamma_4$, in the following form

Introduce the notion of freely-homotopic loops.

Def :

Two loops $\alpha, \beta : [0,1] \to \Sigma_g$ are said to be freely-homotopic if there is a continuous map $F : [0,1] \times [0,1] \to \Sigma_g$ such that

$$F(t,0) = \alpha(t)$$
$$F(t,1) = \beta(t)$$
$$F(0,s) = F(1,s) \quad \rightsquigarrow \quad \text{i.e. for each } s \in [0,1] \text{, the map } t \mapsto F(t,s) \text{ defines a loop in } \Sigma_g$$



Now : Let γ be a homotopically non-trivial loop in Σ_g (endowed with some (fixed) hyperbolic metric). Consider the set $\mathcal{L} \subseteq \mathbb{R}$ defined by

$$\mathcal{L} = \{ \ell \in \mathbb{R} : \exists \text{ loop } c \subseteq \Sigma_g \text{, } c \text{ freely-homotopic to } \gamma \text{ and } \operatorname{length}(c) = \ell \}$$

- $\mathcal{L}$ is bounded from below by some strictly positive $\lambda > 0$

  * γ non-contractible, γ' freely-homotopic to γ $\Big\} \Rightarrow \gamma'$ not contractible

  * if length is too small $\Rightarrow$ it is contained in a chart $\Rightarrow$ it is contractible

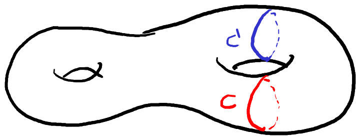
- Set $L = \inf \mathcal{L}$, $L > 0$

  $\rightsquigarrow$ by using uniform convergence it can be shown that L is atained, i.e $\exists\, c \subseteq \Sigma_g$ freely-homotopic to γ and such that $\operatorname{length}(c) = L$.

In particular, $\boxed{c \text{ is a geodesic}}$

$\rightsquigarrow$ if c did not locally minimize distance, then we could replace c by a shorter loop still freely-homotopic to c. This clearly contradicts the assumption that L is atained with c.

$\square$

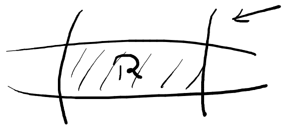


__Claim__ : In fact, $\underline{c}$ is the unique geodesic loop freely homotopic to γ

$\Updownarrow$

This claim follows from applying Gauss-Bonnet theorem

__Rmk__ : General Gauss-Bonnet formula



geodesic polygon

$$\int_R k\, dA = (2-n)\pi + \sum_{i=1}^{n} \theta_i$$

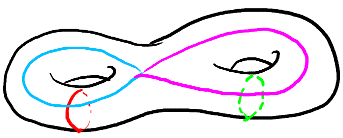
θ_i = internal angles of the polygon

If c and c' are freely-homotopic, they bound a region $R \subseteq \Sigma_g$. Since the integral

$$\int_R k\, dA < 0 \qquad (k=-1)$$

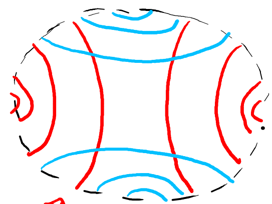
we immediatly obtain a contradiction.

__Now__



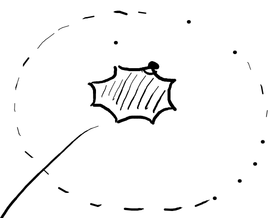
we can choose geodesic representatives for the fundamental group of Σ_g

lift all these 4 $(=2g)$ geodesics to $\mathbb{D}$



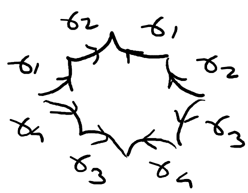
all lifts of the red geodesic

the total geodesic define a tiling of $\mathbb{D}$



choose a fundamental polygon

$\rightsquigarrow$



- the length of sides match in pairs
- sum of the angles is 2π (condition for tiling - they have to wrap up around any vertex)

If two Riemann surfaces S and S' define the same point in the Teichmüller space, i.e. if there exists holomorphic diffeomorphisms $f : S \to S'$ isotopic to the identity, then we have:

(1) f lift to $F : \mathbb{D} \to \mathbb{D}$, which is an isometry

(2) Since f is isotopic to the identity, it sends each geodesic loop to a loop still freely-homotopic to the initial one (i.e. the action of f on $H_1(\Sigma_g)$ is trivial). Thus the polygon representing S' is still an octogon with "the same sides".

Thus, we have obtained:

Prop:
Having fixed the initial loops, two Riemann surfaces structures on Σ_g define the same point in the Teichmüller space if and only if their geodesic polygons are isometric.

In order to obtain the desired paremetrization of T_g, it remains to describe the set of the resulting polygons (arising from Riemann surfaces).

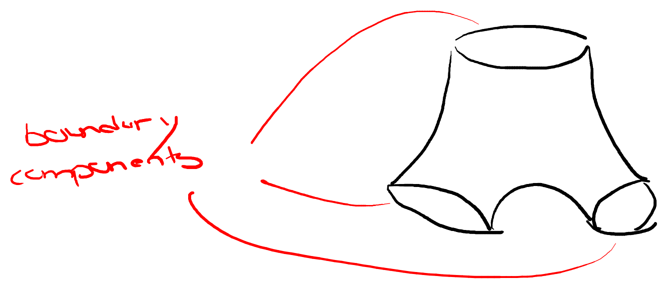
Pants decomposition

A surface Σ_g can be cut in "smaller" pieces called pants (or pair of pants) which, in turn, allows one to split the problem in two parts:

(1) understand hyperbolic structures of pants (i.e. understand the Teichmüller space of a pant)

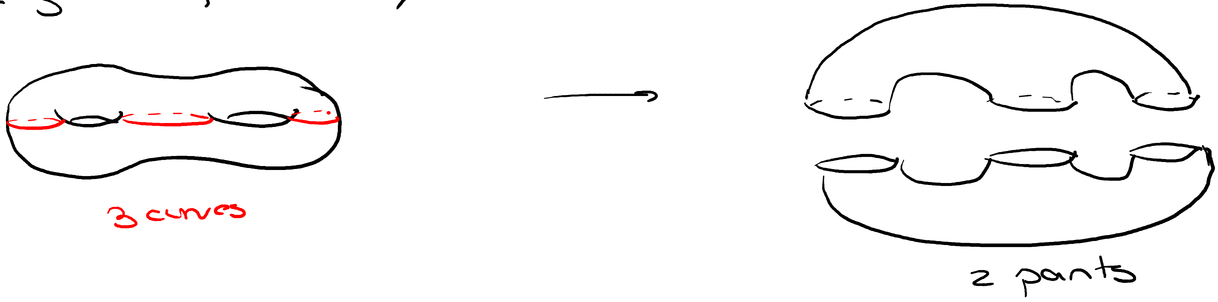
(2) Glue back together the pants with their hyperbolic structure to understand hyperbolic structures on Σ_g.

Def:
A pant (or pair of pants) is a compact surface of genus 0 with 3 boundary components. In other words, it is the surface indicated below:

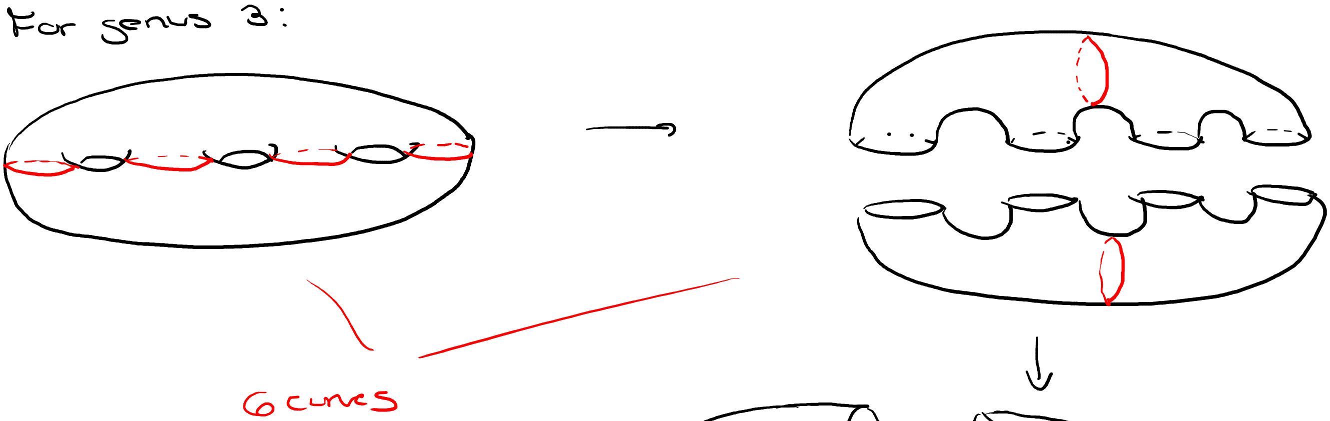


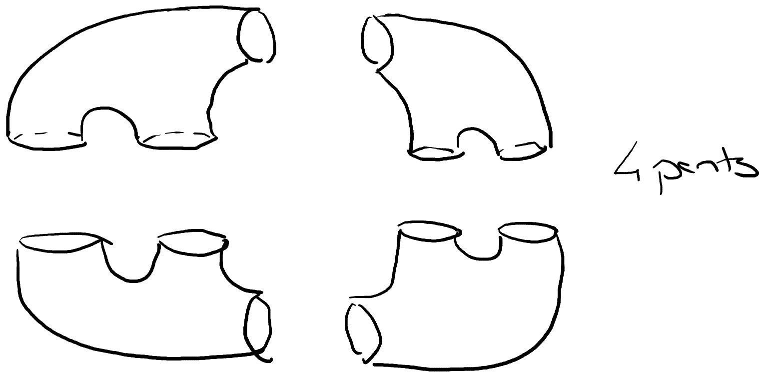
Every surface can be cut up in a number of pants.

- for genus 2, we simply do the following



- For genus 3:



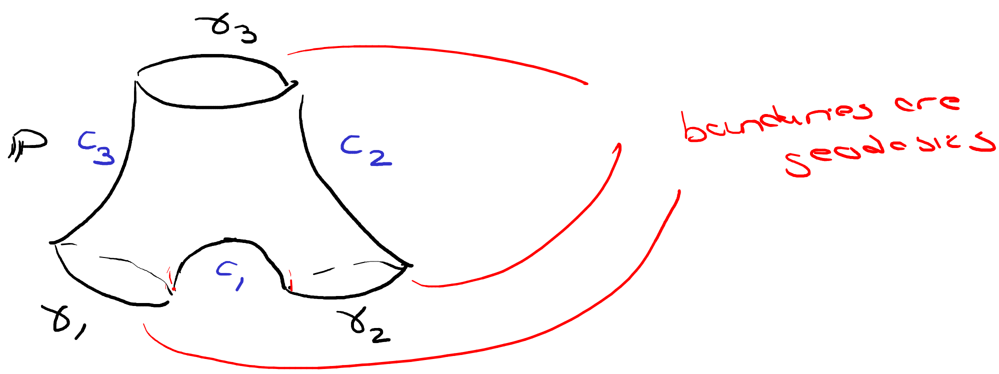


total: 4 pants ($4 = 2g-2$)
6 curves ($6 = 3g-3$)

Now suppose that Σ_g is endowed with a hyperbolic structure. Then choose the representative loop to be the <u>unique</u> geodesic in the corresponding homotopy class.

$\Downarrow$

we get $(2g-2)$ pants each pant has a hyperbolic structure with geodesic boundaries

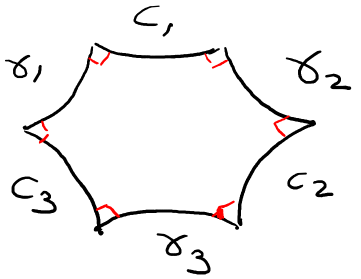


Teichmüller space of pants

Let c_1 be the geodesic segment realizing $\mathrm{dist}(\gamma_1, \gamma_2)$
c_2 be the " " " $\mathrm{dist}(\gamma_2, \gamma_3)$
c_3 " " " " " $\mathrm{dist}(\gamma_1, \gamma_3)$

In particular,
- c_1 is orthogonal to both γ_1, γ_2
- c_2 " " " " γ_2, γ_3
- c_3 " " " " γ_1, γ_3

The polygon defining $\mathbb{P}$ is then a hexagon

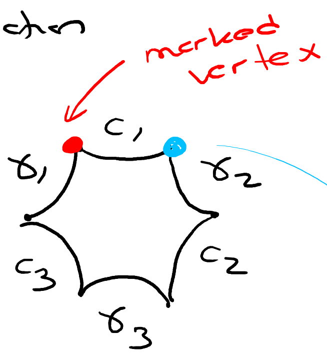


$\rightsquigarrow$ all angles of this hexagon are $\pi/2$

Def:
A marked hexagon means an hexagon as above but having a distinguished vertex. Two marked hexagons are said to be equivalent if there is an isometry of $\mathbb{D}$ sending one hexagon into the other along with the distinguished vertices.

$\mathcal{H} \rightsquigarrow$ the space of marked hexagons.

Prop:
The map $\mathcal{H} \longrightarrow \mathbb{R}^3_+$ defined by taking the length of every other side and starting from the marked vertex is a bijection



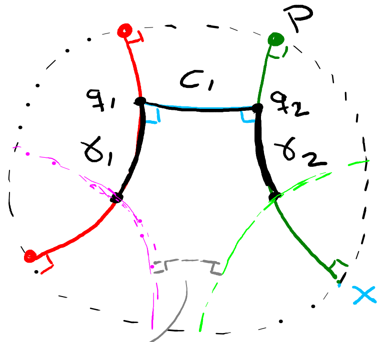
$\longrightarrow (\ell(\gamma_1), \ell(\gamma_3), \ell(\gamma_2))$

$\longrightarrow (\ell(c_1), \ell(c_3), \ell(c_2))$

The hexagon is uniqely determined

up to isometry we can fix a geodesic supporting say γ_1

marked vertex at it



$PSL(2,\mathbb{R})$ transitive in triplex

fix the two points in the red geodesic (where γ_1 will be contained) and we also fix a point in green through wich will pass a geodesic where γ_2 will be contained

this length need not to coincide with $\ell(\gamma_3)$

to be checked: $\exists!$ geodesic throug P such that an hexagon with the corresponding sides may be constructed

the point is that this length varies with the point $\times$ and the variations is monotonous from zero to ∞. So, the length is atained exactly once.

Prop:

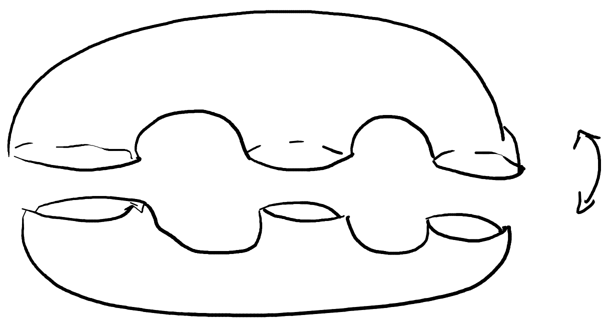
$$\mathrm{Teich}(P) \longrightarrow (\ell(c_1), \ell(c_2), \ell(c_3)) \in \mathbb{R}^3_+ \text{ is homeomorphism}$$

$\boxed{\text{Gluing back pants}}$ (Fenchel-Nielsen coordinates)

Prop:

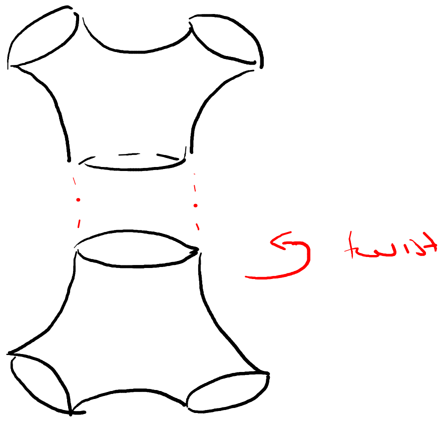
For $g \geq 2$, $\mathcal{T}_g$ isomorphic $\mathbb{R}^{6g-6}$

$\boxed{g=2}$



gluing back pants

Freedom of the gluing procedure:



$\varphi : \mathbb{H} \to \mathbb{D}$ given by $\varphi(z) = \dfrac{z-i}{z+i}$

$\varphi^{-1} : \mathbb{D} \to \mathbb{H}$ $\varphi^{-1}(z) = \dfrac{i(z+1)}{1-z}$

Hyperbolic metric $\mathbb{H}$ is

$$\langle v_1, v_2 \rangle_{(x+iy)} = \frac{1}{y^2} (v_1, v_2)_{Eucl.}$$

v_1, v_2 viewed as elements of $\mathbb{R}^2$

In particular, the vertical semi-line $\{ i\alpha \}_{\alpha \in \mathbb{R}^+}$ is a geodesic

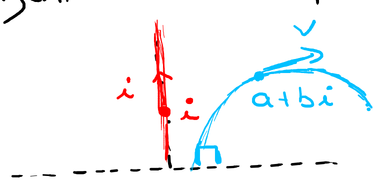
An explicit parametrization (with speed of constant modulus equal to 1) is given by

$$\gamma(t) = i e^t \qquad (\gamma(0) = i \ ; \ \gamma'(0) = i)$$

($\underline{\text{Note}}$ that $\gamma'(t) = ie^t$ so that

$$\|\gamma'(t)\| = \sqrt{\langle \gamma'(t), \gamma'(t) \rangle_{\gamma(t)}} = \sqrt{\frac{1}{(e^t)^2} \big((0,e^t), (0,e^t) \big)} = 1 \ )$$

φ sends the vertical semi-line $\{ i\alpha \}_{\alpha \in \mathbb{R}^+}$ to the horizontal diameter of $\mathbb{D}$



<u>Summarizing the identification $PSL(2,\mathbb{R}) \cong T^1\mathbb{H}$</u>

The map $z \mapsto a + bz$ sends i to $a+bi$ and in terms of matrices in $PSL(2,\mathbb{R})$ becomes

$$\begin{bmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{bmatrix} \qquad \boxed{b > 0}$$

$$bz + a = \frac{\boxed{\sqrt{b}}\, z + \boxed{a/\sqrt{b}}}{\boxed{0}\, z + \boxed{\frac{1}{\sqrt{b}}}}$$

$f(z) = \dfrac{\alpha z + \beta}{\lambda z + \gamma} \in Aut(\mathbb{H}) = Iso(\mathbb{H})$ $\alpha, \beta, \lambda, \gamma \in \mathbb{R}$

$$\Updownarrow$$

$$\begin{bmatrix} \alpha & \beta \\ \lambda & \gamma \end{bmatrix} \sim \begin{bmatrix} \alpha/c & \beta/c \\ \lambda/c & \gamma/c \end{bmatrix}$$

The stabilizer of i, $Stab(i)$, consists of the homographies

$$g(z) = \frac{az+b}{cz+d} \quad \text{such that} \quad g(i) = i \qquad a,b,c,d \in \mathbb{R}$$

$$g(i) = i \iff \frac{ai+b}{ci+d} = i \iff ai+b = di - c$$

$$\iff \begin{cases} a = d \\ b = -c \end{cases}$$

So $g(z) \in \mathrm{Stab}(i)$ iff g takes on the form $\boxed{g(z) = \dfrac{az+b}{-bz+a}}$

Furthermore, in terms of matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and since we can assume the determinant to be equal to 1, we have $\boxed{a^2+b^2 = 1}$

Thus $\begin{array}{l} a = \cos\theta \\ b = \operatorname{sen}\theta \end{array}$ for some $\theta \in [0, 2\pi]$

i.e. $g(z) \in \mathrm{Stab}(i) \rightsquigarrow g(z) = \dfrac{\cos\theta\, z + \operatorname{sen}\theta}{-\operatorname{sen}\theta\, z + \cos\theta}$

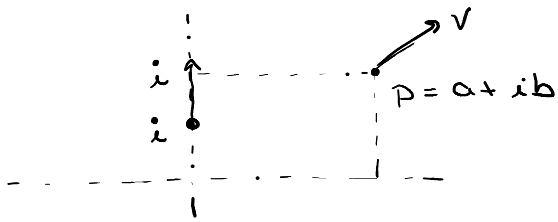
$$(\operatorname{tg}\theta = b/a)$$

In particular,

$$g'(z) = \frac{a^2+b^2}{(-bz+a)^2} = \frac{1}{(-\operatorname{sen}\theta\, z + \cos\theta)^2}$$

$$\Rightarrow g'(i) = e^{2i\theta}$$

We can now write an identification of $T^1\mathbb{H}$ with $PSL(2,\mathbb{R})$ explicitly as follow:



The pair $(p, v) \in T^1\mathbb{H}$ will be associated with the unique matrix $\{m_{ij}\} \in PSL(2,\mathbb{R})$ satisfying the following two conditions:

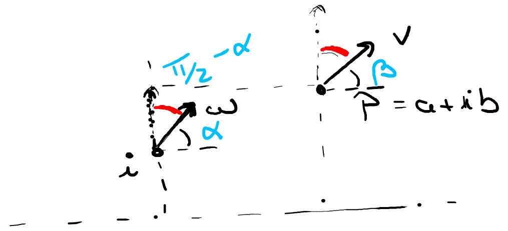
(i) $f(i) = p \; (= a+ib)$

(ii) $f'(i) \cdot i = v$

$$f(z) = \frac{m_{11} z + m_{12}}{m_{21} z + m_{22}}$$

- To construct $f \simeq \{m_{ij}\}$, recall first the map $z \mapsto bz + a$ sends i to $a + ib$

• Furthermore, its derivative preserves the vertical direction



$$f'(i) \cdot i = ib$$

$$\boxed{f'(i) = b}$$

Thus, $f'(i) = b$ sends the vector ω to v if and only if the angles α and β are equal. Thus, we have to take an element $g \in \mathrm{Stab}(i)$ so that ω is the image of i by the derivative of this element $\leadsto$ we have to rotate i by an angle $\pi/2 - \beta$ in the clockwise direction

Now, since

$$g(z) = \frac{\cos\theta \, z + \sin\theta}{-\sin\theta \, z + \cos\theta}$$

and

$$g'(i) = e^{2i\theta}$$

we must have

$$e^{2i\theta} = e^{-i\left(\frac{\pi}{2} - \beta\right)}$$

$$\Longrightarrow \boxed{\theta = \frac{\beta}{2} - \frac{\pi}{4}}$$

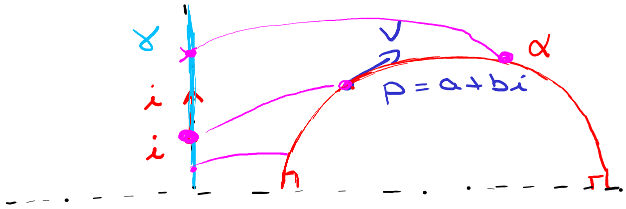
For this value of θ we obtain

$$(p, v) \cong \underbrace{\{ z \mapsto bz + a \} \circ g}_{\text{in terms of matrices of } PSL(2,\mathbb{R}) \text{ is given by}}$$

$$\underbrace{\begin{bmatrix} \sqrt{b} & a/\sqrt{b} \\ 0 & 1/\sqrt{b} \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{M = [m_{ij}]}$$

The geodesic flow on $PSL(2,\mathbb{R})$ coordinates

Let us now try to express the action of the geodesic flow on the Lie group $PSL(2,\mathbb{R})$ by means of the identification previously described.



$\gamma(t) = i e^t$

$\Updownarrow$

geodesic $\begin{cases} \gamma(0) = i \\ \gamma'(0) = i \end{cases}$

In terms of matrices

$$\gamma \rightsquigarrow \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

(it is associated to the map $z \longmapsto z e^t$)

α geodesic s.t. $\begin{cases} \alpha(0) = p \\ \alpha'(0) = v \end{cases}$

If f is the automorphism of $\mathbb{H}$ such that $\begin{cases} f(i) = p \\ f'(i) \cdot i = v \end{cases}$

then

$$\alpha(t) = f(\alpha(t))$$

Hence, the homography $f \circ \{z \mapsto z e^t\}$ sends (i, i) to $(\alpha(t), \alpha'(t))$

In terms of matrices, the point $(\alpha(t), \alpha'(t)) \in T'\mathbb{H}$ is identified with the matrix

$$\underbrace{[m_{ij}]}_{\text{represents initial point } (p,v)} \cdot \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

point in $T'\mathbb{H}$ after time t

Thus, in $PSL(2,\mathbb{R})$, the geodesic flow corresponds simply to the flow

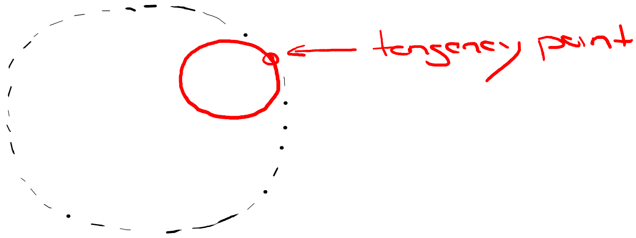
$$(t, [m_{ij}]) \longmapsto [m_{ij}] \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$$

$\boxed{\text{The horocyclic flow}}$

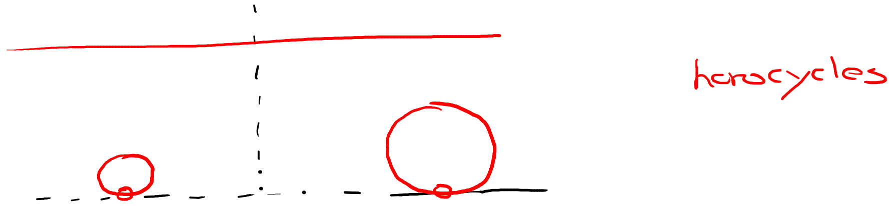
Along similar lines, we are going to introduce the horocyclic flows on $T'\mathbb{H}$ and then to comput it expression in $PSL(2,\mathbb{R})$-coordinates.

Def:

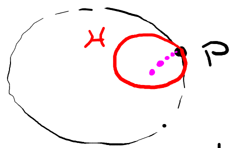
On the disc model, horocycles are Euclidean circles that are tangent to the boundary of $\mathbb{D}$.



In the upper half-plane model, they become either circles tangent to the real axis, or horizontal lines (tangency point at ∞)



$\boxed{\text{Rmk}}$ horocycles are also natural limits of hyperbolic circles (so-called Hausdorff-Gromov topology). More precisely, let $\mathcal{H} \subseteq \mathbb{D}$ be a horocycle tangent to $S^1 = \partial\mathbb{D}$ at a point p



then, there is a sequence $\{z_n\} \to p$ and a sequence of radii $\{R_n\} \to \infty$ such that the hyperbolic circles

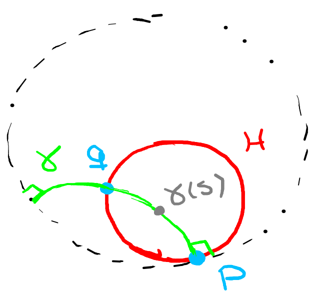
$$S_{R_n}(z_n) = \{ z \in \mathbb{D} : d_{\mathcal{H}}(z, z_n) = R_n \}$$

converges to $\mathcal{H}$

$\rightsquigarrow$ p is often referred to "the center of the horocycle" (and we do not talk about the radii of the horocycle)

The choice of $\{z_n\}$ and $\{R_n\}$ can be made explicitly as follows.

Give: the center $p \in \partial\mathbb{D}$
a point $q \in \mathcal{H} \subset \mathbb{D}$



Note that there exists a unique geodesic γ satisfying

$$\gamma(0) = q$$

$$\lim_{t \to +\infty} \gamma(t) = p$$

For $s \in \mathbb{R}^+$, consider the point $\gamma(s)$ and note that

$$\mathrm{dist}(\gamma(s), q) = \underbrace{\mathrm{Length}\left(\gamma(t)\right)_{t \in [0,s]} = s}_{\text{geodesic in } \mathbb{D} \text{ are globally minimizing}}$$

set : $\left\| \begin{array}{l} z_n = \gamma(n) \\ r_n = n \end{array} \right.$ (taking $s = n \in \mathbb{N}$)

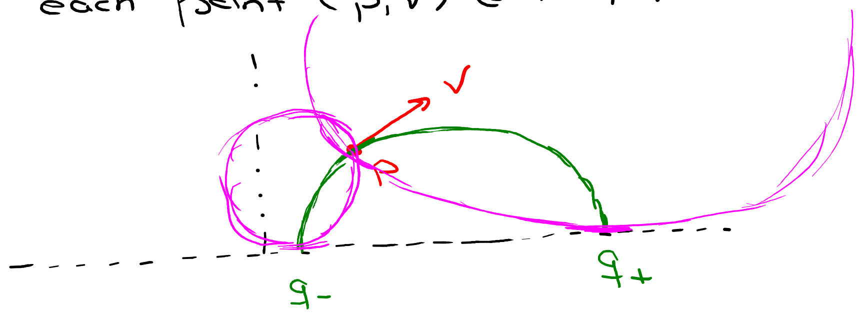
For every $n \in \mathbb{R}^+$, the point q belongs to the hyperbolic circle

$$S_n(z_n) = \{ z \in \mathbb{D} : d(z, z_n) = n \}$$

As $n \to \infty$, the hyperbolic circles $S_n(z_n)$ naturally converges towards H.

Horocycles determined by points $(p, v) \in T^1\mathbb{H}$

There a is a natural pair of horocycles that can be associated with each point $(p,v) \in T^1\mathbb{H}$. This is as follows



γ geodesic satisfying $\begin{array}{l} \gamma(0) = p \\ \gamma'(0) = v \end{array}$, if we set

$$q_+ = \lim_{t \to +\infty} \gamma(t) \quad ; \quad q_- = \lim_{t \to -\infty} \gamma(t)$$

<u>Def</u>:
The <u>positive horocycle</u> H^+ defined by (p,v) is the horocycle of center q_+ and passing through p. Analogously, the <u>negative horocycle</u> H^- is the horocycle of center q_- and passing throug p.

<u>Rmk 1</u>: Clearly the negative horocycle defined by (p,v) becomes the positive horocycle defined by $(p,-v)$ and conversely.

Rmk 2:

Every horocycle $\mathcal{H} \in \mathbb{H}$ has two natural lifts in $T^1\mathbb{H}$, namely the ones defined by unit vectors orthogonal to $\mathcal{H}$ and points outwards of $\mathcal{H}$ (which will be called unstable horocycles) and pointing inwards it (which will be called stable horocycles)

Lemma:

The family of all stable horocycles (resp. unstable horocycles) in $T^1\mathbb{H}$ defines a foliation on $T^1\mathbb{H}$.

Proof: Exercise: it amounts to showing the following

(i) two stable horocycles in $T^1\mathbb{H}$ cannot intersect unless they coincide.

(ii) every point $(p, v) \in T^1\mathbb{H}$ belongs to one and only one stable horocycle.

Lemma:

An isometry of $\mathbb{H}$ takes horocycles to horocycles.

Proof: Its immediate since they preserve Euclidean circles and angle/tangency points.

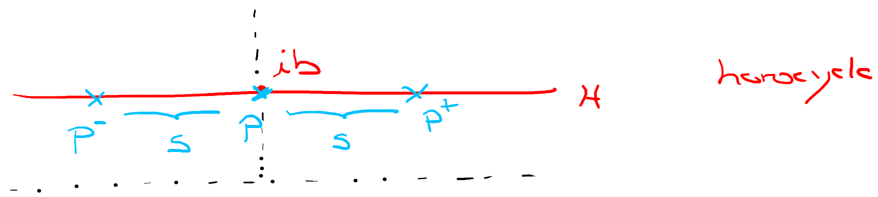
Lemma:

The metric on a horocycle H that is obtained by restricting the hyperbolic metric is a constant multiple of the Euclidean metric of $\mathbb{R}$.

Proof

Since the statement is invariant by isometries, we can consider a particular holocycle and prove the statment for this particular horocycle.

Consider the horocycle H given by an horizontal line



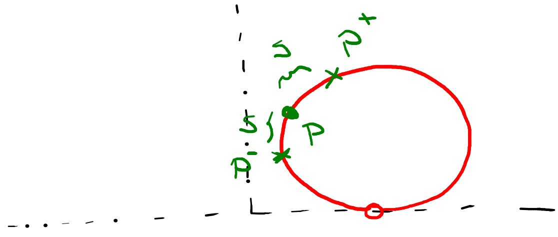
The metric along H is nothing but $\frac{1}{b^2}(\ ,\ )_{Eucl.}$, so it differs from the Euclidean metric by a constant multiple. ☒

Corollary:

Given any point p in a horocycle H and given any $s \in \mathbb{R}^+$, there are points p^+ and p^- on the right and on the left o p on H such that

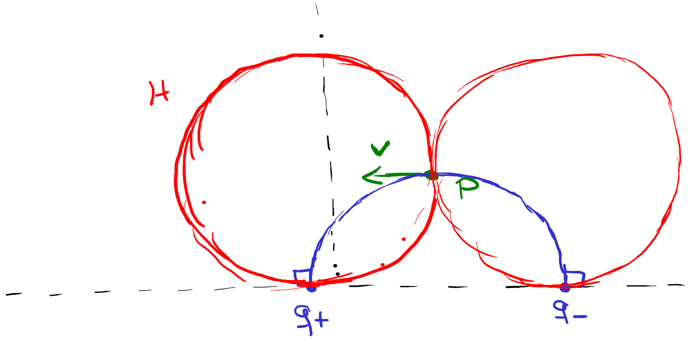
$$d_I(p^+, p) = d_I(p^-, p) = s$$

where d_I stands for the induced distance on H



Geometric definition of the horocycle flows (along with the correct orientation)

Let $(p,v) \in T^1\mathbb{H}$, we want to define $h_+^s(p,v) \in T^1\mathbb{H}$, i.e. the image of (p,v) by the positive horocyclic flow at the time s



Let $\mathcal{H}$ be the positive horocycle defined by (p,v). Then "slide" (p,v) along $\mathcal{H}$ in the indicated direction until the point that has distance s to the initial point

induced distance on $\mathcal{H}$

$$\boxed{h_+^s(p,v)}$$

Similarly $h_-^s(p,v)$ is conjugate to h_+ by sending v to $-v$. In other words, to compute

$h_-^s(p,v)$ we compute $h_+^s(p,-v) = (q,w) \in T^1\mathbb{H}$

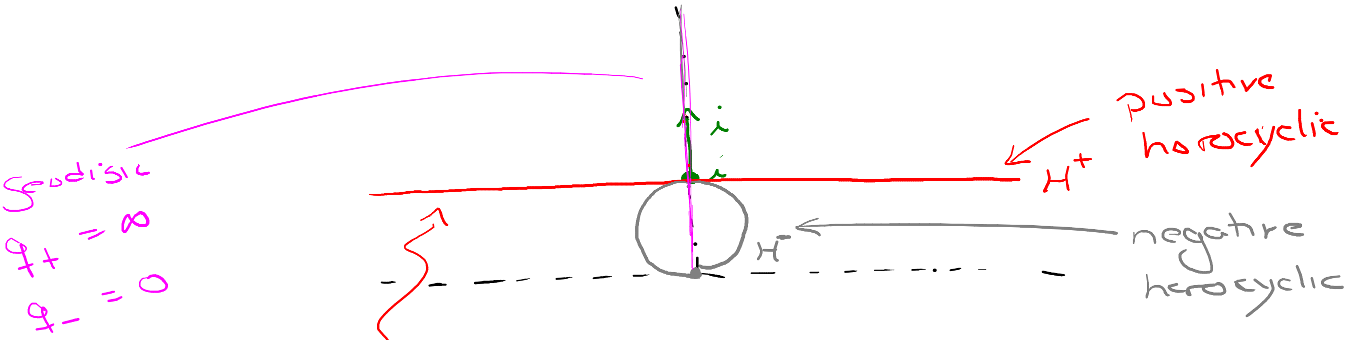
and thus $h_-^s(p,v) = (q,w)$

The horocyclic flow in $PSL(2,\mathbb{R})$ - coordinates

Recall that points $(p,v) \in T^1\mathbb{H}$ are associated with the unique matrix $[m_{ij}] \in PSL(2,\mathbb{R})$ which corresponds to the homographic map f satisfying
$$\left| \begin{array}{l} f(i) = p \\ f'(i).i = v \end{array} \right.$$

and we want to compute $h_+^s(p,v)$.

- f sends the positive horocycle defined by (i,i) to the positive horocycle defined by (p,v)



the induced metric on H^+ coincides with the Euclidean metric

so $\quad h_s^+(i,i) = (\underbrace{i+s}_{z \mapsto z+s}, i) \quad \forall s \in \mathbb{R}$

Thus the homography f taking (i,i) to (p,v) will take $(i+s, i)$ to $h_+^s(p,v)$

Thus, the homography taking (i,i) to $h_+^s(p,v)$ is

$$\underbrace{f \circ \{z \mapsto z+s\}}_{[m_{ij}]\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Positive

Horocyclic flows in $PSL(2,\mathbb{R})$ becomes : $\quad (t, [m_{ij}]) \longmapsto [m_{ij}]\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$ ∎

As to the negative horocyclic flow, we proceed in the same way. By noticing that the inversion $(z \mapsto 1/z)$ takes the positive horocyclic to the negative horocyclic, we set

negative horocyclic flow : $\quad (t, [m_{ij}]) \longmapsto [m_{ij}]\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$

$\boxed{\text{The fundamental relation between the geodesic and the horocyclic flows.}}$

geodesic flows $\longleftrightarrow$ hyperbolic nature

$\downarrow$

spaces of negative curvature $\rightsquigarrow$ origin of the notion of "Anosov systems"

In $PSL(2,\mathbb{R})$-coordinates, all the flows g, h_+, h_- act on the right side of $PSL(2,\mathbb{R})$

Starting with a matrix $M = [m_{ij}]$; fix $t, s \in \mathbb{R}$ and we apply successively g^t, then h_+^s and g^{-t} to M

this amount to multiply M on the right by

$$M \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{bmatrix}$$

$$= M \underbrace{\begin{bmatrix} 1 & se^{t} \\ 0 & 1 \end{bmatrix}}_{\text{this is the same then applying } h_+^{se^t} \text{ to } M}$$

we have then

$$\boxed{g^t h_+^s g^{-t} = h_+^{se^t}}$$

with respect to the negative horocyclic flow we set

$$\boxed{g^t h_-^s g^{-t} = h_-^{se^{-t}}}$$

geodesic flows contracts the positive horocycles and dilates the negative horocycles

VI