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SYMMETRY IN MATHEMATICS AND THE MATHEMATICS OF SYMMETRY

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Abstract: *On the one hand, symmetry is frequently used in mathematical arguments as an effective tool to produce simple and elegant proofs. On the other hand, the mathematical concept of group is an important tool to classify objects with respect to symmetry.*

Keywords: Symmetry; Groups.

INTRODUCTION

What is a symmetry of an object? When is a given object more or less symmetric than another? What does Mathematics have to do with these matters? We outline here some answers to these questions, starting by noticing how symmetry is used in mathematical arguments, and then describing how the concept of symmetry is given a precise sense in Mathematics that was carved into a sharp tool to deal with problems within Mathematics, as well as in other fields.

SYMMETRY IN MATHEMATICS

Symmetric objects or patterns have a natural aesthetic appeal, being usually a visual characteristic of things. But there is another sense of symmetry that is extremely useful in crafting mathematical arguments that are elegant and relatively concise.

As an example, we mention here a particular case of the result known as Fermat's Last Theorem, namely the result found by Pierre de Fermat (1607–1665) that there are no cubes of natural numbers that add up to another cube. This means that the equation $x^3 + y^3 = z^3$ has no solution in natural numbers. The usual proof is divided into several cases depending on whether one of the numbers is divisible or not by 3. By considering negative integers, this statement is implied by the more symmetrical statement that the equation $x^3 + y^3 + z^3 = 0$ has no solutions in integers except in the case where at least one of them is zero. This more general statement avoids dealing with so some cases, since using the fact that the numbers x, y, and z play symmetric roles in the second equation, one may assume, without loss of generality, that it is a specific number, say z, that is divisible by 3, avoiding considering the other two cases. For other examples see Merrots (2008), and for a bit more sophisticated examples see Hilton & Pedersen (1986).

MATHEMATICS OF SYMMETRY

The following observation played an important role in the attempt to find formulas for the solutions of polynomial equations, in the 18th and 19th century: without knowing the roots (solutions) of a given polynomial, one can compute the values of symmetric polynomials in those roots. To make the meaning of this clear, let us just give a simple example: if r_1, r_2, r_3, are the (unknown) roots of the cubic polynomial $x^3 + ax^2 + bx + c$, then one can compute, knowing only the numbers a, b and *c,* the value of any expression of those roots that is symmetric, like $r_1^2 + r_2^2 + r_3^2 + r_1 r_2 r_3$, in the sense that the expression is unchanged by any permutation of the indices 1, 2, 3.

Eventually (see Tignol (2001)) this led to the development of the concept of *group*. Roughly speaking, while numbers measure the size of a collection of objects, groups are mathematical objects that measure the "amount" of symmetry of an object. A bit more precisely, given an object that is structured in some way, a *symmetry* is a mapping of the object onto itself which preserves that structure the object has. The set of those symmetries make up a group, and it is this group that gives a measure of how much symmetry the object has.

Detailed accounts of this mathematical approach to symmetry are given in Weyl (1980), Martin (1982), and Armstrong (1988). Examples of the classification of objects according to symmetry can be seen in Conway *et al*. (2008), as well as in the link https://www.atractor.pt/simetria/matematica of the site of *Atractor*, a portal dedicated to the dissemination of Mathematics.

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