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CROSSINGS
FLYING OVER ANTISYMMETRIC TRUCHET FRIEZES

ANDREIA HALL, JOÃO NUNES and PAOLO VETTORI

Name: Andreia Hall (b. London, 1967), João Nunes (b. Coimbra, 1969), Paolo Vettori (b. Merano, Italy, 1969)
Profession: Associate professor, High School teacher, Assistant professor
Affiliation: CIDMA - Centre for Research and Development in Mathematics and Applications, University of Aveiro
E-mail: andreia.hall@ua.pt, nunes.joaol@gmail.com, pvettori@ua.pt
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Abstract: *Symmetry and antisymmetry are resources used by humans since pre-historic times in their artistic creations. In this work we start by introducing antisymmetry and the 17 possible associated groups. We then present some counts of friezes, built with a particular module, the Truchet tile, which is itself antisymmetric. These counts take into consideration the frieze antisymmetry group and the number of tiles composing the unitary cell which originates the frieze by translation. These Truchet friezes led to the creation of a set of 5 paintings, named Crossings, which illustrate all the possible groups, in an artistic context.*

Keywords: Symmetry; Antisymmetry; Two color symmetry; Truchet tile, GeoGebra.

INTRODUCTION

Symmetry is a universal principal and "it is as much interest to mathematicians as it is to artists" (Wade, 2006, p. 1). Symmetry in itself embeds the notion of repetition, regularity or congruence. However, too much repetition and regularity become tedious and uninteresting and therefore it is not surprising that any notion of symmetry is intrinsically entangled with that of asymmetry or broken symmetry. One way to disturb symmetry without destroying it completely is using antisymmetry. Like symmetry, antisymmetry can be found in human productions since pre-historic times (Radovic & Jablan, 2001, p. 58). Both features can be found in art works. We propose a set of 5 paintings exploring antisymmetry in friezes, using a particular module for the frieze construction, the Truchet tile, which is itself antisymmetric.

In 1704, a Dominican priest named Sébastien Truchet published *Memoir sur les Combinasions*, where he explored patterns made from a simple module composed of a square divided by one of its diagonals into two triangles of different colors, , now known as a Truchet tile. Truchet's work was further developed in 1722 by Father Dominique Doüat, who obtained many other patterns constructed with the same motif. The work of these two priests became known through the publication of Smith & Bouchet (1987) that triggered the interest in patterns made from Truchet tiles.

In geometry, a symmetry of a figure is an isometry that leaves it invariant. The set of symmetries of a figure F together with the operation composition forms a group which is known as the symmetry group of F. In the plane there are only three categories of discrete symmetry groups: rosette groups; frieze groups; and wallpaper groups. There are two types of rosette groups (cyclic groups, C_n, and dihedral groups, D_n), seven types of frieze groups and 17 types of wallpaper groups. More details on symmetry groups may be found in Martin (1982). Several notations have been proposed for the frieze groups, and in this paper, we shall use the crystallographic one: *p111, p1a1, p112, p1m1, pm11, pma2* and *pmm2* (Veloso, 2012). In this notation, the second character is either m or 1, depending on whether there exist reflection symmetries perpendicular to the frieze direction or not. Similarly, the third character is *m, a*, or *1*, depending on whether there exists, along the frieze direction, a reflection symmetry, a glide reflection symmetry, or none. The last character is *2* or *1* depending on whether there exists a half-turn symmetry or not.

Antisymmetry (also known as two-color symmetry) is closely connected to the concept of symmetry and may occur whenever each point of a figure or object has associated a dichotomous characteristic such as one of two colors, one of two electric charges, etc. An antisymmetry may be defined as a symmetry coupled to an exchange of colors (or exchange of the value of the dichotomous variable) that leaves the figure or object invariant. Since there are four possible types of symmetry on the plane, there are also four possible types of antisymmetry on the plane. The set of all antisymmetries and symmetries of a figure also forms a group.

To generate an antisymmetry group we can start from a symmetry group and replace one (or more) of its generators by the corresponding antisymmetric. For example, the group *p111* (friezes with only translation symmetry) can be generated by a vector u. If we replace u by the respective antisymmetric (coupling u with a color exchange), we obtain the antisymmetry group *p111/p111*, where the minimum translation symmetry vector becomes $2u$ and u is an antisymmetry translation vector.

Antisymmetry groups are more diverse than the symmetry ones: there are 17 antisymmetry frieze groups and 46 antisymmetry wallpaper groups (Grunbaum & Shephard,1987, pp. 402-413). Classification and designation of antisymmetry groups can be obtained by analyzing the symmetry group of the uncolored figure (only with contours), G_u, and the symmetry group of the colored figure, G_c. The name of the anti-symmetry group is $G_u|G_c$. Figure 1 illustrates this process through two Truchet friezes (see Radovic & Jablan (2001) for more details on antisymmetry).

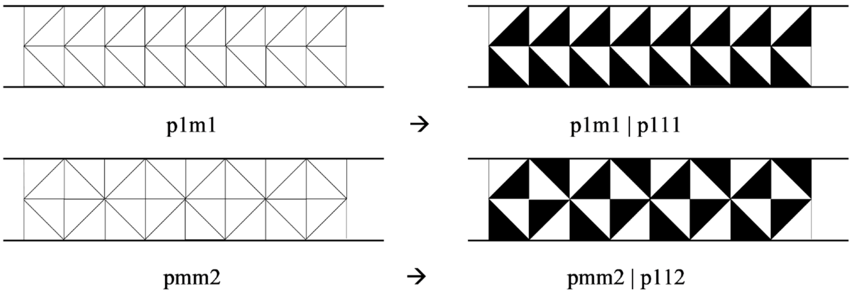

Figure 1 Illustration of the antisymmetry group classification procedure.

Next, we present a summary of all the possible frieze antisymmetry groups. The groups are listed according to the 7 symmetry groups of the uncolored friezes.

- *p111 | p111*
- *p1a1 | p111*
- *p112 | p111* *p112 | p112*
- *pm11 | p111* *pm11 | pm11*
- *p1m1 | p111* *p1m1 | p1a1* *p1m1 | p1m1*
- *pma2 | p1a1* *pma2 | p112* *pma2 | pm11*
- *pmm2 | p112* *pmm2 | pm11* *pmm2 | p1m1* *pmm2 | pma2* *pmm2 | pmm2*

We are presently investigating how many different antisymmetric friezes it is possible to construct withTruchet tiles. This research stemmed from the initial work of Hall *et al*. (2919) who counted all possible Truchet rosettes considering their symmetry properties. Table 1 presents the number of distinct friezes (congruent friezes only count once) which are two tiles high and have period *p = 1, 2, 3 and 4* tiles long. The friezes shown in Figure 1 are examples of such friezes with *p* = 1 and *p* = 2.

Table 1. Counts of Truchet friezes, two tiles high and with periods up to 4 tiles long.

group	*p* = 1	*p* = 2	*p* = 3	*p* = 4
p111 | p111				10
p1a1 | p111				12
p112 | p111	1		15	52
p112 | p112		2		6
pm11 | p111				25
pm11 | pm11				2
p1m1 | p111	1		5	12
p1m1 | p1a1		1		1
p1m1 | p1m1		1		1
pma2 | p1a1		1		2
pma2 | p112		1		2
pma2 | pm11		1		2
pmm2 | p112		1		2
pmm2 | pm11		1		2
pmm2 | p1m1		1		2
pmm2 | pma2				2
pmm2 | pmm2				2

Note that for odd periods only few of the groups exist. This happens because with odd periods (no matter how large they are) no vertical reflexion symmetries (or antisymmetries) may occur. This makes all groups of types *pm11* | *, *pma2* | * and *pmm2* | * impossible, 10 in total. In fact, for odd periods, only groups *p112* | *p111* and *p1m1* | *p111* may exist. All other groups require a translation or glide reflection (antisymmetry) vector half the period long which is not possible since these vectors must be an integer number of tiles long. As can be seen from Table 1, for friezes two tiles high, it is not possible to obtain all 17 types of antisymmetric friezes with periods smaller than 4. Figure 2 shows all friezes of periods p=1 and p=2 referred in Table 1.

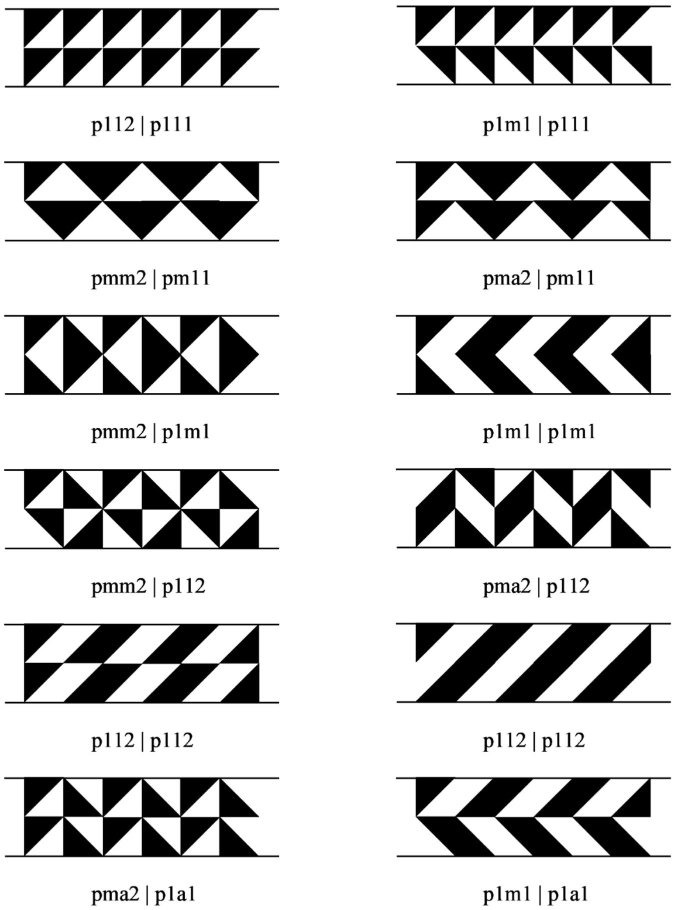

Figure 2 All possible antisymmetric Truchet friezes two tiles high and with periods 1 and 2 tiles long.

CROSSINGS

Crossings is a set of five paintings that illustrate the 17 possible frieze antisymmetry groups. Using GeoGebra, several friezes were built with Truchet tiles. As an artistic/dynamic effect the friezes were "curved" using transformations following particular choices of circular arcs. The background and foreground were adjusted using *Paint.Net* and the resulting digital images were printed and finished using watercolor, soft pastels, and gold appliqué. Each painting has a distinct number of crossings, from 0 to 4, and a distinct set of uncolored symmetry groups. Figures 3 through 7 show the final paintings.

Figure 3 Crossings 1.0. print, watercolor, soft pastels, and gold *appliqué* on paper, 20x29 cm (antisymmetry groups *p112 | p111, p112 | p112* and *p1a1 | p111*; *0* crossings).

Figure 4 Crossings 1.1. print, watercolor, soft pastels and gold *appliqué* on paper, 20x29 cm (antisymmetry groups *pma2 | pm11*, *pma2 | p112* and *pma2 | p1a1*; *1* crossings).

Figure 5 Crossings 1.2. print, watercolor, soft pastels, and gold *appliqué* on paper, 20x29 cm (antisymmetry groups *p111* | *p111*, *pm11* | *p111* and *pm11* | *pm11*; *2* crossings).

Figure 6 Crossings 1.3. print, watercolor, soft pastels, and gold *appliqué* on paper, 20x29 cm (antisymmetry groups *p11m1* | *p111*, *p1m1* | *p1m1* and *p1m1* | *p1a1*; *3* crossings).

Figure 7 Crossings 1.4. print, watercolor, soft pastels, and gold *appliqué* on paper, 39x29 cm (groups *pmm2* | *p112*, *pmm2* | *pm11*, *pmm2* | *p1m1*, *pmm2* | *pma2* and *pmm2* | *pmm2*; *4* crossings).

The frieze designs used in the drawings were selected from all possible friezes, two tiles high and with periods 1, 2 and 4. They are part of the counts shown in Table 1 and to identify which groups were chosen, the respective cells were underlined. Seven of them are explicitly shown in Figure 2. The transformation used to "curve" each frieze is given by

$$\begin{cases} x' = (y - y_0)\, sin \dfrac{x - x_0}{r} + x_0 \\ y' = (y - y_0)\, cos \dfrac{x - x_0}{r} + y_0 \end{cases}$$

where (x_0, y_0) is the centre of the desired circumference and r is its radius.

CONCLUSION

The Truchet tile, , is a simple module that is used to create many different patterns including rosettes, friezes, and wallpapers. These patterns can be randomly generated or intentionally arranged to exhibit symmetry/antisymmetry. In this paper we focus on antisymmetric Truchet friezes and present the counts of all possible friezes that can be built with Truchet tiles, using two tiles for the frieze hight and one to four tiles for the length of the period. As an artistic application we present a set of 5 paintings that illustrate the complete set of 17 antisymmetric groups.

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Andreia HALL

Andreia Hall, PhD in Probability and Statistics, is Associate Professor at the Mathematics Department of the University of Aveiro since 2005. She started her academic career in extreme value theory and later worked in statistical data analysis. Presently she is mostly interested in mathematical education and mathematics and the arts. She teaches mathematics for pre-service primary teachers and for in-service teachers through professional development courses. She is a member of the *Mathematical Circus Project* which promotes the interest in mathematics through shows of mathematical magic. In 2020 she did an exhibition of mathematical quilts - *Between lines: mathematics through patchwork* – at the University of Aveiro.

João NUNES

João Luís da Costa Nunes is a high school teacher since 1992 when he graduated in Mathematics at the University of Coimbra. Twice master's in educational mathematics: *The Mathematics of Identification Systems*, 2009, University of Coimbra; *Polynomial zeros: location, solving formulas and applications*, 2020, University of Aveiro. Since 2021, he is a student of the doctoral program in Educational Mathematics at the University of Aveiro. He is interested in geometry, algebra, arts and their applications in mathematical education. His current research includes *Truchet's tile*, rosettes, friezes and wallpapers.

Paolo VETTORI

Paolo Vettori, PhD in Mathematical Systems Theory, is Assistant Professor at the Mathematics Department of the University of Aveiro since 2008. In his research career he has been studying various aspects of Linear Control Systems and Coding Theory, in particular their symmetries and generalizations to quantum systems. He has been teaching linear algebra, geometry, calculus, informatics and control theory to bachelor and master students. As a member of the Mathematical Circus Project, he performs (supposedly) entertaining and challenging mathematical tricks to a wide variety of audiences (from first graders to elderly people with lung diseases)