

## UNEXPECTED SENSE OF SYMMETRY INSIDE THE PROJECTION

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Major publications:

Cocchiarella, L. (2022). Orthographic Anamnesis on Piero's Perspectival Treatise. *Nexus Network Journal*, 1-17. Springer Nature. DOI 10.1007/s00004-022-00593-0. Available at <https://rdcu.be/cIYkO>

Caglioti, G., Thouvilleva, T., Cocchiarella, L. (2020). Odi et Amo. Ambiguità percettive e pensiero quantistico. *Mimesis*. Milano.

Hemmerling M., Cocchiarella L. (eds.). (2018). *Informed Architecture. Computational Strategies in Architectural Design*. Springer. Cham

Major exhibitions:

40 Years of Geometry and Graphics. Exhibition organized as the Executive Chair of the 18<sup>th</sup> International Conference on Geometry and Graphics (ICGG2018), at the Politecnico di Milano (*conference link* <http://www.icgg2018.polimi.it/> *conference Photogallery* [http://www.icgg2018.polimi.it/?page\\_id=1745](http://www.icgg2018.polimi.it/?page_id=1745) *exhibition's introductory materials* [http://www.icgg2018.polimi.it/wp-content/uploads/2019/07/ICGG\\_2018\\_40-Years-on-Geometry-and-Graphics\\_ExhibitionReport.pdf](http://www.icgg2018.polimi.it/wp-content/uploads/2019/07/ICGG_2018_40-Years-on-Geometry-and-Graphics_ExhibitionReport.pdf)). Proceeding published by Springer in 2 Volumes and indexed as Scopus records.

The Visual Languages of Technique Between Science and Art: Heritage and Expectations in Research and Teaching.

Three Exhibitions related to Three International Seminars, Proposed and Chaired on the occasion of the 150<sup>th</sup> Anniversary of the Politecnico di Milano (May, June, July 2013). Proceedings are published in 3 Volumes by Springer.

**Abstract:** *Aim of this work is to focus on some aspects of symmetry in the light of a projective approach not strictly metrics oriented. What seems to be permitted by etymology, since as we know, more than to metrics itself, the word symmetry refers to relations (syn = together with) by, or, in metrics (metron = measure). Regarding architectural design and construction, despite the dominant idea (and practice) for long mainly based on axial and polar metric mirroring, emphasis on symmetry as a relationship among components of architectural spaces considered as systems was already at the base of the classic compositional approach. From this side, metric-based symmetry may be seen as a subset of the overall symmetry. Inspiration came from Projective Geometry and its applications to architectural survey and design, where the relationship between space and image gets the stage. And the research question arose: if thousands of images (projections) can be generated from one space (physical or theoretical), is it allowed, and in which way, to look for a 'sense of symmetry' inside those interrelated graphic representations? Answering this question would have implied thinking about symmetry as a transformation in a wider sense and going beyond the classic Euclidean borders according to the modern concept of space. Then we entered the projective world looking at its relational more than metric side, where homology seemed to offer a valid vessel to sail onboard.*

Keywords: Symmetry; Homology; Euclidean vs Projective Space.

## DISCUSSION

Strictly speaking, moving from Euclidean to Projective space, we find symmetry in special homological cases, namely, under the particular conditions when the centre or axis is at infinity (both at infinity define a translation): in the first case we recognize *axial*, in the second *polar* symmetry. In the two cases, metrics are preserved.

However, we have to admit that those Euclidean-like results are generated from projective transformations. In other words, we are working with projections inside our – locally supposed – Euclidean world. That is, we do not have direct visual access to the Euclidean world, which is a well-known condition, and has an impact on the way of thinking of an architect as well.

On the other hand, the Euclidean has for a long time been assumed as the only *real* reference world in architecture, where construction can rely on secure metrical bases. Drawings were consequently limited to specific sets of projections, basically, plans and sections, where the highest number of elements could comfortably appear in true proportions *as if* translated and flattened from real space on the paper. A smart and efficient representational strategy indeed, referring to space by planar diagrams like in a kind of *ante litteram* tomography system, so appropriate to facilitate the design process. But its intensive manneristic use has probably contributed to weakening architects' sensitivity to the overall real three-dimensional space, over time.

Treatises and manuals, sometimes presenting fascinating symmetric drawings and patterns including sophisticated proportional settings (more or less *golden*), have for centuries fed the idea that those were the only graphic algorithms at the base of an architectural design process. The same considerations could be further confirmed by looking at the drawings of facades, where spatial vibrations resulted often omitted in favor of calibrated interplays among planar symmetry patterns. Which often disappeared in the real three-dimensional body of the built architecture. Cross-sections might deserve a comment apart, since they necessarily require a more complex spatial understanding, as students (and teachers) involved in traditional drawing know. However, the final perceptual impact of the built architectural space hardly corresponds to the visual impact of these graphic patterns, usually organized in separated views, which today we know are generated from a centre of projection theoretically located at infinity. What would ancient Greek architects, so sensitive to optical inflections, have thought about this?

Renaissance perspective, in fact, brought a not inconsiderable disturbance in the field, since perspective distortions seemed not to work following the apparently unassailable measurable symmetry shown by plan, front, and sections. Consequently, *trouble with numbers*, so clearly discussed by Robin Evans arose (Evans, 1995, pp. 241-271), of which the *abacus* masters, including Piero Della Francesca, had been among the earliest and smartest interpreters. The principle of Euclidean *Optics* seemed to contrast with the Euclidean *Elements*. The geometry of vision and geometry of measure seemed unable to get along, in the real space and on the canvas. Painters first dealt with this contrast, sometimes amending some graphic constructions to make them more acceptable to the eye. Then perspective has for a long time been considered for showing projects more than for making projects.

A turning point, initiating a change of paradigm in the concept of space was, in our opinion, that of the projective investigation of the planets' orbits, carried out by Johannes Kepler in *Ad Vitellionem Paralipomena*. Admirably synthesized in the well-known image shown below (Figure 1), it opened

up a new season in the universe of conics, thanks to the clear recognition of the points at infinity as proper items to work with and manage by a projective process.

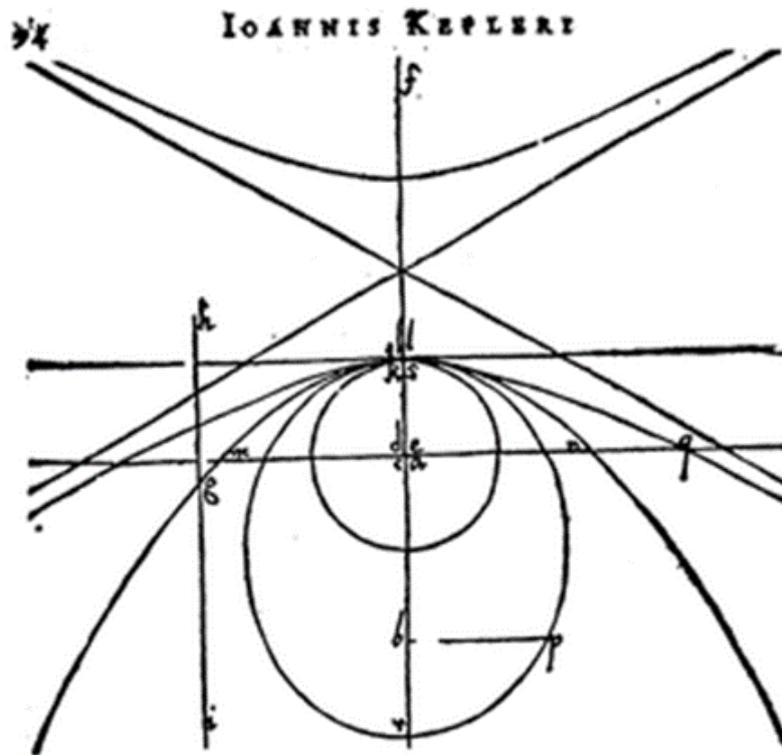


Figure 1 Johannes Kepler, *de conic sectionibus* (Kepler, 1604, Cap. IV, p. 94).

More than the axial symmetry appearing in the diagram, that is, its visible part, the extension to the infinity and its correspondence with the finitude in real space (Figure 2) is the new astonishing and revolutionary issue (De Rosa, Sgrosso, Giordano, 2001, Vol. 2, pp. 218-220). Beyond the tangible limits of our everyday space, infinity itself tends to arise as the other half of it, let's say, as its *symmetric counterpart*, despite its non-visible evidence on paper, because of the projective distortions occurring. This, in our opinion, in a broad sense, suggests a more extensive and flexible approach to symmetry. This happens if we consider the mentioned graphic diagram inside its projective context of origin, without giving priority to metrics, but focusing on the intrinsic and ineradicable point-to-point relationships among the four curves representing the three conics as *projective transformations* of a circle (which is a conic itself) (Glaeser, Stachel, Odehnal, 2016, pp. 177-206). Back to the remote meaning mentioned in the abstract, *symmetry* concerns relationships and structures indeed.

Another paradigm shift having significant relevance to the mentioned image, is established by its appearance as a *homology*, which emphasizes the relationships among the curves, so that, one can imagine them as frames of an ideal animation. Let's quite popularly say, like several pictures of a round door, as it opens. Homology provided what we can call the unification of the projective forces, as far as its basic pattern was fully understood and generalized (Cocchiarella, 2015, pp. 237-256). It took a considerable amount of time. Even though finally theorized by Jean-Victor Poncelet in the 19<sup>th</sup> century (Poncelet, 1865), including relief-perspective, examples of direct and inverse homological constructions had already been developed by Brook Taylor in the first half of the 18<sup>th</sup> century (Taylor, 1749), while its initial intuition in terms of *homography* between top and perspective view dates back even to Piero Della Francesca in the 15<sup>th</sup> century! (Nicco-Fasola, 1984, pp. 37-

38, pp. 78-79, Tav. IV-V), and its early establishment in form of germinal theorem is due to Girard Desargues in the 17<sup>th</sup> century (Kemp, 1990, pp. 119-123). As several treatises published during this period demonstrate, architecture offered itself as an extraordinary text field, either for proving geometrical properties by taking advantage of the tangible three-dimensional built spaces, or for developing appropriate graphic strategies, supporting more and more complex theoretical elaborations.

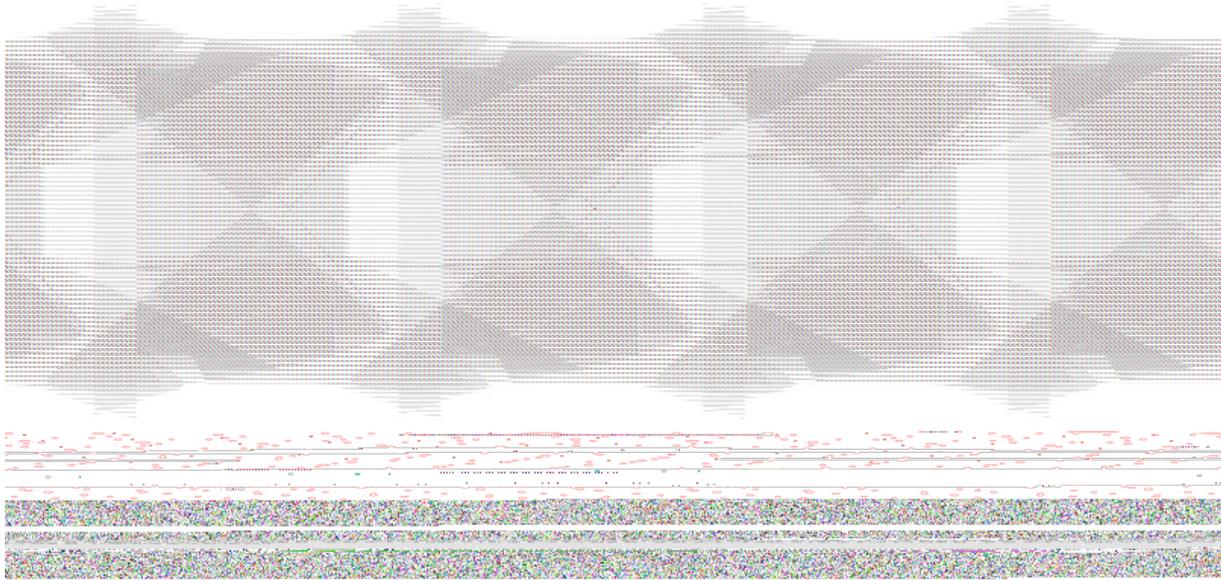


Figure 2 Diagrams showing conic sections in space (image by author, digital editing by Matteo Cavagliá).

What is new in the homological approach is the chance to simultaneously generate, or think of, space visualizations from more than one point of view, synchronically combining and relating multiple projections inside a unitary representation. As declared in the title of this paper, this can somehow *remind* the symmetry. In the diagram below (Figure 3), a series of rings, generated as homological rotations of a given element about the axis of homology, is represented in perspective. Apart from those appearing perfectly opposed left-right, any generic couple of rings does not contain symmetric figures. But all the figures show the geometric feature of a ring, rotated at a certain angle about the axis, as well as in the case of the Kepler's conics. In this way the overall system is strongly evocative of symmetry, that is, of a *sense* of symmetry.

Even if the same figure is considered as a (quasi-) stereographic view of a system of meridians, each couple of rings can be considered in homological connection. Then, again the *projective* sense of symmetry emerges, and in the elements directly opposed left-right, a properly said Euclidean symmetry, too. As if, in the projective space, projective invariance could assure a wider kind of symmetry, by geometrically linking each view to the corresponding given configuration.

In this light, we can say that computer graphics itself inherited this benefit, since each *viewport* frames projective-alike views of a certain properly 3D digitized spaces, in real-time. In our opinion, this profound structural connection, and the consequent *sense of symmetry* perceived, experienced in a projective way and metrically based on *cross-ratio*, contributes to making even the most severe graphic distortion familiar to the architects, allowing them to think and control the analysed, as well as the designed space, by projective-based visual representations, be it analogue or digital.

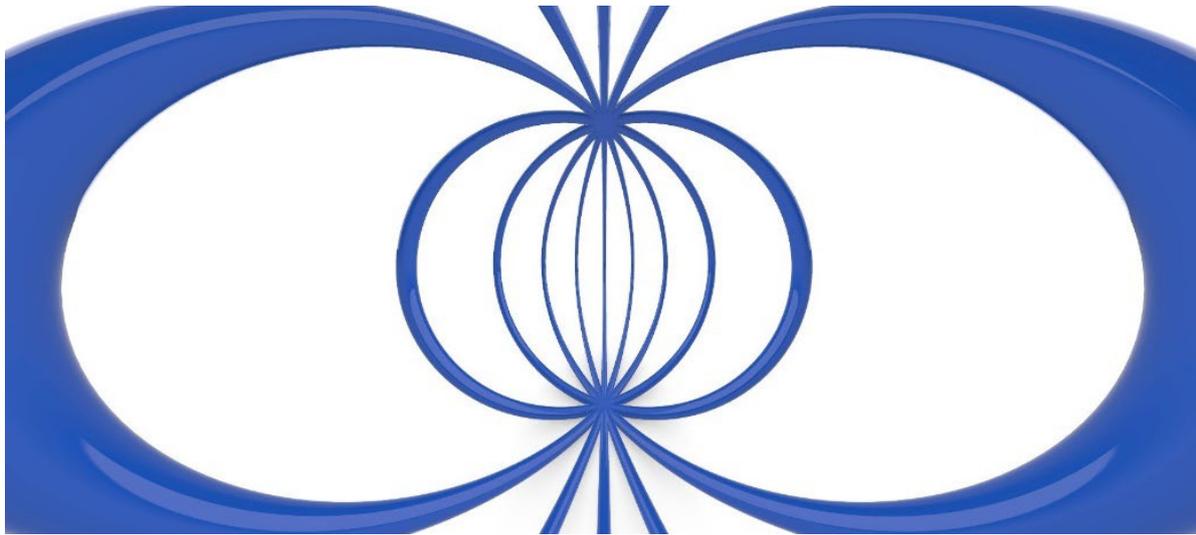


Figure 3 Homology series in a quasi-stereographic bundle of rings (diagram by author).

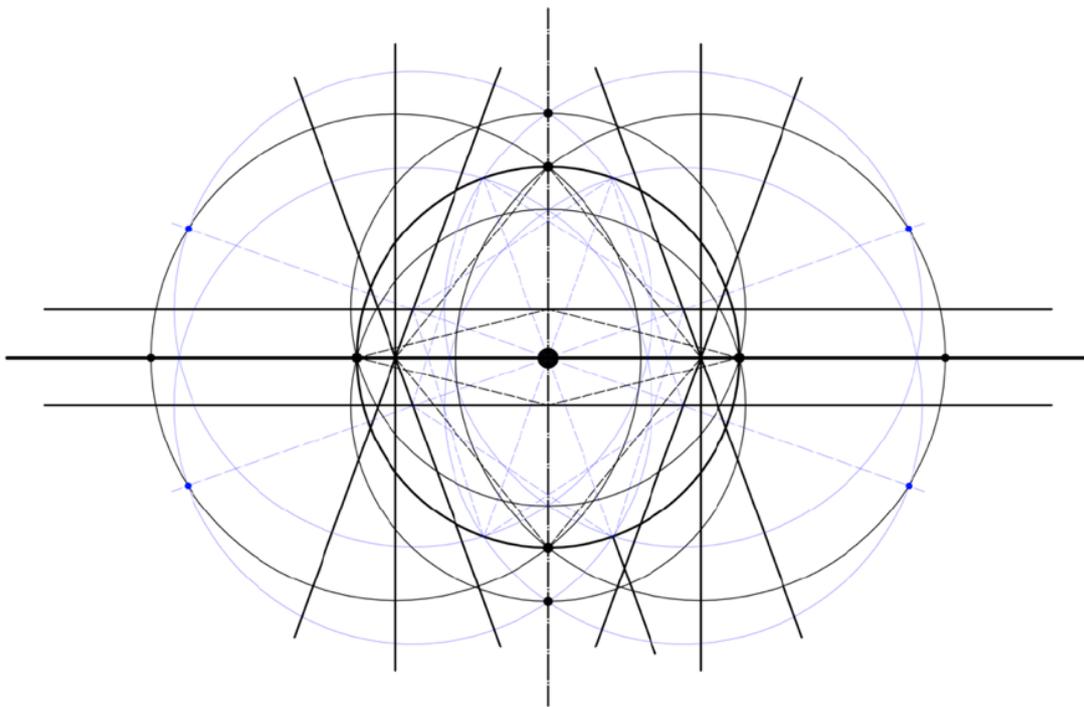


Figure 3 Varying the centre of homology in perspective according to viewing distance (Cocchiarella, 2014, p.11).

Up to now, we focused on the final graphic result, that is, on the graphic representation of the *object*. It is interesting to note that similar comments also arise when we focus on the other side of the canvas, or, on the *subject*, that is, on the viewing point as the centre of projection, which is the main controller of the projective process. In the image above (Figure 4), some spatial excursions of the viewing point generating graphic effects similar to those shown in Figure 3, are summarized. Again, a symmetric map is easily recognizable when looking at the various positions acquired by the sight point, to provide all the possible visualizations according to viewing distance, as the distance circle in the middle of the figure reminds us. As well as when looking at the plenty of circular paths available on canvas by the effect of the rotation of the viewing point about the vanishing axes of the

planes, to be rotated on the same canvas to visualize the corresponding figures generated by the homological process. Just to stay in our obsession (!), a further symmetry seems established between the *object* and the *subject*.

Although non-symmetric in the traditional way, these interlinked projections look consistent with that relational meaning of *symmetry* implicit in the etymology of the term, which we mentioned in the abstract. In addition, these visualizations are also genuinely consistent with our visual perception, which makes our surrounding living environment projective at the end, including metric symmetries as special cases. However, as we saw, in graphic representations the correspondence is mathematically based on *cross-ratio*, the fundamental metric *invariant* of Projective Geometry (Boyer, 1968, pp. 413-418). A valuable operator, whose solid foundations not for nothing rest on scientific bases also arising from observations made at the infinite scale of the Universe.

## CONCLUSION

Based on the etymological meaning of *symmetry* as a relations system, a focus on *homology* has been proposed, regarding architecture. Looking beyond the special cases of axial and polar symmetries, which are familiar to Euclidean metrics, we referred to the Kepler's conics, as a genuine projective graphic diagram where *symmetry* itself can be seen in a new light, showing the traditional metrics, based on *ratio*, as a subset of the projective metrics, more generally based on *cross-ratio*. Something unexpected at the origin, and still at work in a sublimated way, even in the present world of visual graphics and digital graphics.

## REFERENCES

- Boyer, C.B. (1968). *History of Mathematics*. John Wiley & Sons.
- Cocchiarella, L. (2014). Surfing projective space homology in graphic education. *Abstract – 3<sup>rd</sup> Croatian Conference on Geometry and Graphics*, 10-11.
- Cocchiarella, L. (2015). Perspective Between Fiction and Function: Pattern Mutations Through Science and Art. *Journal for Geometry and Graphics*. Vol. 19(2), 237-256.
- De Rosa, A., Sgrosso, A. & Giordano, A. (2000). *La Geometria nell'Immagine. Storia dei metodi di rappresentazione* (Vol. 2). Utet.
- Evans, R. (1995). *The Projective Cast. Architecture and Its Three Geometries*. The MIT Press.
- Glaeser, G. & Stachel, H. & Odehnal, B. (2016). *The Universe of Conics. From the ancient Greeks to 21<sup>st</sup>-century developments*. Springer Spektrum.
- Nicco-Fasola G. (ed.) (1984). Piero della Francesca: *De Prospectiva Pingendi. Le Lettere*.
- Kemp, M. (1990). *The Science of Art. Optical themes in western art from Brunelleschi to Seurat*. Yale University Press.
- Kepler, J. (1604). *Ad Vitellionem paralipomena*. Apud Claudium Marnium & Haeredes Ioannis Aubrii in Francofurti.
- Poncelet, J-V. (1865; 1<sup>st</sup> ed 1822). *Traité des Propriétés projective des figures*. Gauthier-Villars.
- Taylor, B. (1749). *The New Principles of Linear Perspective: or the Art of Designing on a Plane, the Representations of all sorts of objects, in a more general and simple Method than has been hitherto done*. The Third Edition. John Ward.

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