Viana, V., Nagy, D., Xavier, J., Neiva, A., Ginoulhiac, M., Mateus, L. & Varela, P. (Eds.). (2022). Symmetry: Art and Science | 12th SIS-Symmetry Congress [Special Issue]. *Symmetry: Art and Science*. International Society for the Interdisciplinary Study of Symmetry. 292-299. https://doi.org/10.24840/1447-607X/2022/12-37-292

SYMMETRY IN ENCODING INFORMATION SEARCH FOR COMMON FORMALISM

MARCIN J. SCHROEDER

Name: Marcin J. Schroeder (b. Wroclaw, 1953)
Profession: University Professor
Fields of interest: Mathematics and Philosophy
Affiliation: Tohoku University at Sendai, Japan
E-mail: mjs@gl.aiu.ac.jp

Major publications and/or exhibitions:

Schroeder, M.J. (2022). Multidisciplinarity, Interdisciplinarity, and Transdisciplinarity: The Tower of Babel in the Age of Two Cultures. *Philosophies, 7*, 26.

Schroeder, M.J. (2020). Contemporary Natural Philosophy and Contemporary *Idola Mentis*. *Philosophies,* **2020**, *5*, 19.

Schroeder, M.J. (2019). Theoretical Study of the Concepts of Information Defined as Difference and as Identification of a Variety. In *Philosophy and Methodology of Information—The Study of Information in a Transdisciplinary Perspective.* G. Dodig-Crnkovic & M. Burgin (eds.) World Scientific Series in Information Studies, Vol. 10; World Scientific: Singapore, 2019; pp.289-314.

Abstract: *The concept of symmetry has become the main tool for structural analysis in mathematics and science. There were many attempts to provide a quantitative methodology for research on information (e.g., Shannon's entropy) but the structural aspects of information were always investigated within the limits of specific applications without any common structural methodology. This paper has as its main objective the development of a uniform formalism incorporating the general concepts of information and symmetry allowing the study of structural characteristics of the encoding of information. It starts from the justification for the description of the encoding of information as a filter in information logic formalized as the lattice of closed subsets in a closure space. Then, symmetry of the encoding of information is associated with the invariance of the filter representing this encoding with respect to a group of automorphisms of the logic of information.*

Keywords: Symmetry; Information; Encoding; Structure; Closure spaces.

INTRODUCTION

Symmetry and its breaking became the main conceptual tool of science and mathematics for the study of structural characteristics. Felix Klein's Erlangen Program initiated or accelerated the process of synthesis integrating inquiries of diverse forms of geometric structures. However, even in Klein's vision of unified geometry, we can find the limits for synthesis. The recognition of the unifying role of symmetry identified with the invariance with respect to the action of groups of geometric transformations and its fundamental role in all contexts involving the geometric description of the subject of inquiries promoted geometric conceptualization across diverse disciplines of science but inhibited attempts to transcend the confines of geometry in the study of symmetry in matters such as symmetry breaking or symmetry-structure relationship. This motivated me to explore a more general formalization of the concept of symmetry in terms of closure spaces where symmetry is understood as the invariance of configurations of closed subsets instead of the invariance of configurations of elements (points in geometry) (Schroeder, 2017). In geometry formulated in terms of closure spaces, all one-element sets are closed and therefore the orthodox symmetry is a special case. However, even in geometry, the generalization can offer some additional insight into the symmetry–structure relationship. In the orthodox approach, very different geometric structures such as cubes and octahedrons have the same symmetry (i.e. the same symmetry group), while their symmetries understood in terms of the invariance configurations of closed subsets are different.

I will not elaborate here on the well-known reasons why at present there is no commonly accepted comprehensive, mathematically sound theory of information, in particular why the so-called information theory initiated by Shannon is not a theory of information but its transmission. It is enough to mention the absence of semantic aspects of information or conceptual tools for its structural analysis. The critique of the existing attempts to develop such a theory and exploration of alternative formulations can be found in many papers, including my own (Schroeder, 2011). My original proposal of the description of the encoding of information as a filter in the logic of information was based on the analogy with the description of the state of a physical system in terms of filters in quantum logic.

The present paper has as its main objectives a presentation of the justification for this description of the encoding of information as a filter in information logic independent from the analogy to quantum theory or other applications and a demonstration that this formalization allows for the application of the concept of symmetry. Symmetry in the encoding of information is associated with the invariance of the filters with respect to automorphisms of the logic of information represented by a complete lattice of closed subsets of the closure space.

ALGEBRAIC PRELIMINARIES

The key concept here, both for the formalism of symmetry and the description of information is a *closure space*, i.e. a set S with a closure operator f: $2^S \rightarrow 2^S$ which satisfies the following conditions: (i) $\forall A \subseteq S$: $A \subseteq f(A)$, (ii) $\forall A, B \subseteq S$: If $A \subseteq B$, then $f(A) \subseteq f(B)$, (iii) $\forall A \subseteq S$: $f(f(A)) = f(A)$.

Every closure space can be defined in an equivalent (cryptomorphic) way by a Moore family of subsets of S, i.e. family closed with respect to arbitrary intersections and including the set S. Every Moore family $\mathcal{M}$ defines a transitive operator: $f(A) = \cap \{M \in \mathcal{M} : A \subseteq M\}$ and in turn the family f-Cl $= \{M \subseteq S: f(M) = M\}$ is a Moore family.

The set-theoretical inclusion defines a partial order on f-Cl with respect to which it is a complete lattice $\mathcal{L}_f$. To this structure, we will refer to as the *logic* $\mathcal{L}f$ *of a closure space* <*S,f*>.

Closure operators play important roles in the context of Galois connections of special importance for my inquiry of symmetries. A Galois connection between two partially ordered sets (in short posets) $<P, \leq_P>$ and $<Q, \leq_Q>$ is defined by two antitone mappings $\varphi:P \to Q$ and $\psi:Q \to P$ (i.e. such that $\forall x,y \in P$: $x \leq_P y \Rightarrow \varphi(y) \leq_Q \varphi(x)$ and $\forall s,t \in Q$: $s \leq_Q t \Rightarrow \psi(t) \leq_P \psi(s)$) which satisfy the additional condition: $\forall x \in P$: $x \leq_P \psi(\varphi(x))$ and $\forall t \in Q$: $t \leq_Q \varphi(\psi(t))$. The three conditions for a Galois connection (two for the mappings to be antitone and third linking them) are equivalent to the single condition: $\forall x \in P \forall t \in Q$: $t \leq_Q \varphi(x)$ *iff* $x \leq_P \psi(t)$.

It is easy to see that whenever P and Q are complete lattices, the mappings which are compositions of φ and ψ defined on P and Q respectively $\psi\varphi:P \to P$ and $\varphi\psi:Q \to Q$ satisfy the generalized conditions for a closure operator on a complete lattice. Thus, if both posets P and Q are power sets 2^S and 2^T respectively for some sets S and T ordered by inclusion $\subseteq$ of their subsets, then $\psi\varphi$ is a closure operator on 2^S and $\varphi\psi$ is a closure on 2^T. Moreover, both mappings φ and ψ define anti-isomorphisms (or inverse isomorphisms) of the complete lattices for these closure operators. More details can be found in the classic monograph on the lattice theory by Birkhoff (1967).

ENCODING OF INFORMATION

Information, in general, is defined in my approach as an identification of variety considered within the context of the categorical opposition of one and many. The identification is understood as that which makes one out of many either by the selection of one out of many or by supplying a structure that binds many into a whole. This duality results in the two co-existing manifestations of information: the selective and the structural. Based on this general definition we can consider a set-theoretic model of information system and encoding of information using the closure space formalism.

We will consider a closure space $<S,f>$ with its corresponding Moore family $\mathcal{M}$ of closed subsets as an information system. The specific choice of closure space depends on the choice of the type of information system. For instance, we can consider geometric, topological, logical information, etc. The family of closed subsets $\mathcal{M}$ = f-Cl is equipped with the structure of a complete lattice $\mathcal{L}_f$ which we can consider to be the *logic of information*. It plays a role in the generalization of logic for (not necessarily linguistic) information systems, although it does not have to be a Boolean algebra. In many cases, it maintains all the fundamental characteristics of a logical system (Schroeder, 2011).

Encoding of information is a distinction of a subfamily $\mathfrak{I}$ of $\mathcal{M}$, such that it is closed with respect to the (pair-wise) intersection and is dually hereditary, i.e., with each subset A of S belonging to $\mathfrak{I}$, all subsets of S including A belong to $\mathfrak{I}$ (i.e., $\mathfrak{I}$ is a filter in the lattice $\mathcal{L}_f$). The Moore family $\mathcal{M}$ can represent a variety of substructures of a structure of a particular type (e.g., geometric, topological, algebraic, logical, etc.) defined on S. This corresponds to the structural manifestation of information. Filter $\mathfrak{I}$ in turn, in many mathematical theories associated with localization, can be used as a tool for identification, i.e., selection of an element within the family $\mathscr{M}$, and under some conditions within the set S. Thus, filters can be interpreted in both the structural and selective way.

There is a legitimate question about the reason for the use of closure spaces as models for information systems and filters in the lattice of closed subsets as models for encoding information.

The first question has a rather simple answer. The most important stages of the development of civilizations can be associated with the methods of encoding, storage, and processing of information. We know about a great variety of distinct methods or ways in which information can be encoded by human intervention or by natural processes. In each case of encoding, we have some method or mechanism of its realization and this is called here an information system. Closure spaces can be found in virtually all mathematical theories either as an original fundamental concept (e.g. the case of topology) or as a fundamental concept of a derived, alternative formulation (e.g. in the finite geometries). Of course, nobody can claim that the choice of closure spaces for the description of information systems is the only possible or even the best.

The second question about the identification of filters of closed subsets as instances of information is more complex. The original reason for this choice was to formulate a theory of information for which we can develop semantics. I will give here only an example of a special case that can support intuitive understanding.

When we have this extreme case of $\mathscr{M}$ being the entire power set of S we can relate the idea of distinction of sets in terms of the linguistic information as it is done in the Axiom (schema) of Reduced Comprehension: Given any set A, there exists a set B (a subset of A) such that, given any set x, x is a member of B *iff* x is a member of A and φ holds for x (where φ is a formula of set theory with a free variable x). The axiom tells us that to identify a set we need to use a set from which we select elements (i.e, all elements of the collection have to be elements of some predefined set A to avoid Russell's Paradox), and then we can identify the elements of our set by the description of its properties in the language of the set theory.

In this axiom, the set-theoretical formula φ plays a central role. Intuitively, it is information that allows us to select elements for the set B formulated as their property. However, we want to consider models of information which is independent of the particular restrictions of linguistic encoding of information, and additionally, we want to have information characterized relative to the diverse types of encoding, i.e. diverse information systems. Moreover, we want to consider multiple levels of information related to complete identification or an only partial one. Here comes the idea that properties are not corresponding to all possible subsets but only to closed subsets and the identification is achieved by the distinction of the filter of closed subsets.

The distinction of all closed subsets of S is equivalent to the unique determination of the closure operator on S, i.e., the determination of the closure space. Information in the form of an ultrafilter (maximal filter) corresponds to the former case of complete information, the trivial filter which consists of only the entire set S is another extreme case of minimal information and there are many possible levels of information in between.

The additional gain is that the identity of the elements is associated with the structure formed by the subsets in the filter. If they correspond to the structure of properties, then we have the structural characteristic parallel to the selective one. The meaning of the information is not determined by just a singular property but by a structure of properties. This gives us a link between denotation which is related to the identification (or localization) and connotation which is related to the structure of descriptive characteristics. We regain what was lost in Shannon's (in)famous exclusively selective focus in which the meaning of information is not a subject of concern for the engineering problem of information transmission.

Shannon's approach has the advantage of providing a quantitative methodology. Is the quantitative methodology lost when we model information as a filter in the lattice of closed subsets? At the higher level of generality, the quantitative description becomes more complex. However, at the level corresponding to Shannon's finite case (i.e. when the set S is finite with n elements) and we do not make any restriction on the subsets corresponding to properties (the closure is trivial $\forall A \subseteq S$: $f(A) = A$) the task is quite simple. First, we have to observe that in the finite case every filter is *principal*, i.e. for every filter $\mathfrak{I}$, there is a subset A of S such that this filter consists of all subsets that include A (i.e. $\mathfrak{I} = \mathfrak{I}_A = \{B \subseteq S\colon A \subseteq B\}$). Let's assume that A has k elements. Then we can use reasoning similar to the reasoning used by Hartley (1928) in establishing his quantitative description of information. Since S has n elements and A has k elements, the least numbers of elementary units of any type within an auxiliary set used for encoding that is necessary to encode all elements of S and A respectively is the rounding to the nearest integers of $\log_2 n$ and $\log_2 k$, thus we can consider $H_A = \log_2 n - \log_2 k = \log_2 n/k$ to be a quantitative description of information represented by $\mathfrak{I}$ when it is understood as the degree of selection. The maximum value of $\log_2 n$ is for A with one element only and the minimum value of 0 for A = S.

Notice that thus far we don't use here probability but simple enumerating reasoning about the construction involved in encoding information. Also, we do not restrict our reasoning to letters, characters, or messages. Moreover, we do not assume here that encoding has to be in the form of ordered sequences of a distinguished set of units (positional system).

The transition to the probabilistic formulation in the finite case can be facilitated by the interpretation of the above reasoning in terms of characteristic functions of subsets. The subset A is associated with its characteristic function $\chi_A(x)$ on S such that $\chi_A(x) = 1$ when $x \in A$, and $\chi_A(x) = 0$ when $x \notin A$. It is easy to see that $\mathfrak{I}_A$ consists of all subsets B of S such that $\forall x \in S$: $\chi_A(x) \leq \chi_B(x)$. The characteristic function $\chi_A(x)$ can be interpreted as a random variable X with Bernoulli distribution with parameter p = 1. In the next step, we can consider a more general case with the parameter p belonging to the interval [0,1]. Then we have that the probability P(A) of A considered as an event is p and for all B $\in \mathfrak{I}_A$ the probability $P(B) \geq p$. However, $P(B) \geq p$ does not entail $B \in \mathfrak{I}_A$. The necessary and sufficient condition is that the conditional probability $P(B|A) = 1$.

This leads us to the next step. If we have a family $\mathfrak{I}$ of events for the finite probability space S with a probability measure P defined by $\mathfrak{I} = \{ B \subseteq S: P(B) = 1\}$, then $\mathfrak{I}$ is a filter as $p(A \cap B) = 1$ when $p(A) = p(B) = 1$, and $\forall A,B \subseteq S$: If $A \subseteq B$ and $p(A) = 1$, then $p(B) = 1$. This of course applies to every conditional probability generated by P too. Now, we have that there is a correspondence linking filters in 2^S and probability measures in the probability space S with probability measure P which determines conditional probabilities $P(\bullet\,|A)$ for every A such that $P(A) \neq 0$. Information in the form of a filter can be defined by $\mathfrak{I}_A = \{ B \subseteq S: P(B|A) = 1\}$.

Finally, when we have the very special case of the uniform (classical) probability distribution for P our magnitude of information considered above $H_A = \log_2 n/k = - \log_2 P(A)$. Shannon's entropy $H(X) = -\sum p_i \log_2 p_i$ for the random variable X partitioning S into subsets A with the values $x_1 = - \log_2 p_1$, $x_2 = - \log_2 p_2, \ldots, x_m = - \log_2 p_m$ is simply the expected value for the random variable X with values $x = H_A$, whenever As are elements of some partition of S.

This was just an example that shows that in the very special and simple case of the trivial closure ($\forall A \subseteq S$: $f(A) = A$) the corresponding model of information in the form of a filter in the logic of information is not very far from the approach of Shannon's information theory. However, the general case is much more complex. For instance, there are non-prime filters in the infinite case and the logic of information (the complete lattice of closed subsets) may be non-Boolean. The original rather ad hoc definition of encoding of information (Schroeder, 2011) was guided by such a non-Boolean case in the quantum logic formulation of quantum mechanics where the filters are states of the physical systems.

SYMMETRY IN A CLOSURE SPACE

The concept of symmetry can be defined in an arbitrary closure space <*S,f*>. Let $G = \mathrm{Aut}(\mathscr{L}_f)$ be the group of automorphisms of the logic $\mathscr{L}_f$ of a closure space <S,f> and $H \blacktriangleleft G$ be a subgroup of the group G = Aut($\mathcal{L}_f$); $\mathcal{B} \subseteq \mathcal{L}_f$ be a configuration of closed subsets (e.g. points and lines in geometry on a plane); φ* be an automorphism induced on the lattice $\mathcal{L}_f$ by a bijection φ on S.

We get a correspondence between subgroups H of the group of automorphisms of <*S,f*> *a*nd invariant families of configurations $\mathcal{B}$ as follows.

Let H ◄ G = Aut($\mathcal{L}_f$). Define the family $\mathcal{J}_H$ of subsets of $\mathcal{L}_f$ by $\forall \mathcal{B} \subseteq \mathcal{L}_f$: $\mathcal{B} \in \mathcal{J}_H$ iff $\forall A \in \mathcal{B} \forall \varphi \in H$: $\varphi^*(A) \in \mathcal{B}$. Then $\mathcal{J}_H$ is a complete lattice with respect to the order of set inclusion.

The following two functions Φ and Ψ form a Galois connection between $\Phi(H) = \mathcal{J}_H$ defined by $\forall \mathcal{B} \subseteq \mathcal{L}_f$: $\mathcal{B} \in \mathcal{J}_H$ *iff* $\forall A \in \mathcal{B} \forall \varphi \in H$: $\varphi^*(A) \in \mathcal{B}$ and $\Psi(\mathcal{J}) = H$ defined by $H = \vee\{K ◄ G: \mathcal{J} \subseteq \mathcal{J}_K\} = \{\varphi \in G: \varphi(\mathcal{J}) \subseteq \mathcal{J}\}$, where the last equality is a consequence of the fact that $\{\varphi \in G: \varphi(\mathcal{J}) \subseteq \mathcal{J}\}$ is a subgroup of G.

We will consider configuration $\mathcal{B}$ of closed subsets and conditions for its invariance. Let G = Aut($\mathcal{L}_f$) be the group of automorphisms of the logic $\mathcal{L}_f$ of a closure space <S,f> and H◄G be a subgroup of the group G = Aut($\mathcal{L}_f$). Let $\mathcal{B} \subseteq \mathcal{L}_f$ be a configuration of closed subsets. Then there is a mutual correspondence between subgroups H of the group of automorphisms of <*S,f*> and invariant families of configurations $\mathcal{B}$ defining a Galois connection between the lattice of subgroups of *G = Aut(*$\mathcal{L}_f$*), i.e.* $\mathcal{L}_G$ or $\mathcal{L}_{Aut(\mathcal{L}_f)}$ and the lattice of families of closed subsets of the closure space <*S,f*>. The Galois connection is defined by two mappings:

$\Phi: \mathcal{L}_G \rightarrow \mathcal{J}$ & $\Psi: \mathcal{J} \rightarrow \mathcal{L}_G$: $\Phi(H) = \mathcal{J}_H$ and $\Psi(\mathcal{J}) = H = \{\varphi \in G: \varphi(\mathcal{J}) \subseteq \mathcal{J}\}$

This Galois connection defines anti-isomorphism of the lattice of subgroups of G and the lattice of invariant families of closed subsets of <*S,f*>.

Thus, the invariant families $\mathcal{B}$ are symmetric with respect to the corresponding subgroup of the group of automorphisms of the logic of closure space. The symmetric configurations are distinguished as those closed with respect to Galois closure (different from *f* of course).

We can observe that in this approach it is necessary to consider a hierarchy of three levels of sets: sets, families of such sets, and families of families of such sets. Let $\wp(S)$ indicate the power set of the set S, $\wp(\wp(S))$ the power set of $\wp(S)$, etc. Then we have here S, $\wp(S)$, and $\wp(\wp(S))$. The reason is that the concept of symmetry involves two concepts: that of a fixed subset of a group of transformations (set of elements which are not changed by the transformations) and that of an invariant subset (set which remains the same, although its elements are permuted by the transformations). The two concepts are very different, although they are related across the distinction between sets and their power sets. Invariant subsets at the lower level are fixed points at the higher level.

At the level of $\wp(\wp(S))$, we have the Galois connection defined by $\Phi: \mathcal{L}_G \rightarrow \mathcal{J}$ & $\Psi: \mathcal{J} \rightarrow \mathcal{L}_G$ which links symmetric configurations of closed subsets of <S,f> with subgroups of Aut($\mathcal{L}_f$).

The level of $\wp(S)$: $\Phi(H) = \mathcal{J}_H$ & $\Psi(\mathcal{J}) = H$, where $\Phi(H) = \mathcal{J}_H$ defined by $\forall \mathcal{B} \subseteq \mathcal{L}_f: \mathcal{B} \in \mathcal{J}_H$ *iff*

$\forall A \in \mathcal{B} \forall \varphi \in H: \varphi^*(A) \in \mathcal{B}$ and $\Psi(\mathcal{J}) = H$ where $H = \bigvee\{K \blacktriangleleft G: \mathcal{J} \subseteq \mathcal{J}_K\} = \{\varphi \in G: \varphi(\mathcal{J}) \subseteq \mathcal{J}\} \blacktriangleleft G$

The level of S: There is a more general bijective correspondence between group G acting on the set S which preserves closure *f* and the group *Aut($\mathcal{L}_f$).*

So, at the lowest level, we have simply a group G acting on S. The group G does not have to be the entire symmetric group Sym(S). It is a subgroup of Sym(S) selected by the choice of closure *f*. The distinction of these three levels serves the distinction between subsets of fixed points and invariant subsets. The approach is based on the group *Aut($\mathcal{L}_f$).* There could be a legitimate concern that for the closure operation *f* of subalgebras of a given algebra, $\mathcal{L}_f$ does not determine uniquely the algebra, e.g. non-isomorphic groups can have isomorphic lattices of subgroups. However, the symmetry is not described as a distinction of this lattice, but it is described by the Galois connection that adds additional specification. Similarly, symmetry is not simply giving a privileged position to the lattice $\mathcal{L}_G$ of subgroups of a given group, but it is involving it in the Galois connection.

CONCLUSION

We have now both concepts of information encoding, and symmetry formulated in terms of a closure space that defines our information system. The symmetry is determined by a subgroup of the group of automorphisms of the information logic (the lattice of closed subsets). Thus, we have a uniform formalism for both information and symmetry and our main objective has been achieved.

REFERENCES

Birkhoff, G. (1967). *Lattice Theory, 3rd. ed.* American Mathematical Society Colloquium Publications, Vol XXV, Providence, R. I.: American Mathematical Society.

Hartley, R.V.L. (1928). Transmission of information. *Bell System Technical Journal*, 7, 535-563.

Schroeder, M.J. (2017). Exploring Meta-Symmetry for Configurations in Closure Spaces. In Horiuchi, K. (Ed.) Developments of Language, Logic, Algebraic system and Computer Science, RIMS Kokyuroku, Kyoto: Research Institute for Mathematical Sciences, Kyoto University, No. 2051, pp. 35-42.

Schroeder, M.J. (2011). From Philosophy to Theory of Information, *Intl. J. Information Theories and Applications,* 18(1), 56-68.

Marcin SCHROEDER

Marcin J. Schroeder, Ph.D. is a Specially Appointed Professor at Tohoku University at Sendai, Japan. He is also a Professor Emeritus at the Akita International University in Akita, Japan where he was a professor for fifteen years, most of this time serving as the Dean of Academic Affairs. He was educated as a physicist and mathematician, but his research interests include information studies and philosophy. He is the founding Editor-in-Chief of the journal Philosophies. In the years 2019-2021, he was the President of the International Society for the Study of Information.