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TORRICELLI'S COMPUTATION OF THE AREA UNDER THE CYCLOID

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Abstract: *Like many mathematicians of his era, Evangelista Torricelli (1608-1647) studied the cycloid. This paper examines Torricelli's proof that the area under the cycloid (the quadrature) is three times the area of the generating circle. Torricelli proved this result in three distinct ways. It is worth noting that he carried on an extensive mathematical correspondence with Cavalieri and it is clear that Cavalieri's techniques influenced his proofs. What follows is a discussion of Torricelli's proofs with an emphasis on the use of symmetry and geometric reasoning, as viewed through the lens of graphical representations of the proofs.*

Keywords: Torricelli, Cycloid, Geometry, Calculus, History, Symmetry

INTRODUCTION

Many well-known mathematicians of the seventeenth and eighteenth centuries studied the cycloid. These include Roberval (1693), Descartes (1659), and Pascal (1963). The solutions of Leibniz, Newton, Johann and Jacob Bernoulli were all published in the *Acta Eruditorum* of May 1697 (Mencke, 1697). These mathematicians focused on three types of computation: finding the tangent, the rectification (length of the curve) and the quadrature (area under the curve).

This paper provides a visual representation of Evangelista Torricelli's proofs of the quadrature of the cycloid. His computations were completed sometime before April 1643 when Bonaventura Cavalieri sent a letter to Torricelli congratulating him on his findings (Torricelli *et al*. 1919, p 121). Torricelli's results were published in an appendix in his *Opera Geometrica* (1644).

THE CYCLOID

The cycloid is a simple curve to describe, and was a favorite mathematical example used by many sixteenth, seventeenth, and early 18th century mathematicians. The equation of this curve was not described until the late 17th century; however, knowledge of the equation was not necessary for the curve to be useful in the geometrically-focused proofs of the era. The simple description of the cycloid is that it is the curve that is traced by a point on a circle when the circle is rolled along a horizontal line (see Figure 1).

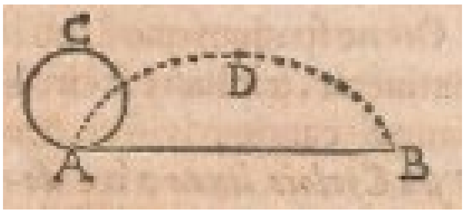

Figure 1. The drawing of the cycloid in Torricelli's *Opera geometrica (1644).* (Torricelli, 1664, p 85 of appendix). Photograph by the author.

Some mathematicians of the period considered the cycloid as an object created by two component motions (Roberval, 1693), the turning of the circle and the horizontal motion of the centre of the circle as it moves parallel to the line. In all proofs discussed in this paper, one critical geometric idea is used: when the circle rolls along the line, the distance moved around the circle is equivalent to the distance travelled along the horizontal line (see Figure 2).

Figure 2. The red marks demonstrate that distance around the circle is the same as the distance traveled in the horizontal line. Drawing by the author.

A GEOMETRIC LOOK AT TORRICELLI'S THREE PROOFS

Torricelli states three times that the area under the cycloid is three times the area of the generating circle. Each of his Theorems I – III represents a unique proof of this fact. Torricelli simplifies his geometric arguments by showing that the quadrature of one half of the cycloid is equivalent to the area contained in three halves of the generating circle. In Theorem I, he links the quadrature to the area of a triangle (Torricelli, 1664, p 86):

> *Theorem I: The space between the cycloid and the straight line of the base is triple the generating circle; or one and a half of the triangle that has the same base and height* (translation from the Latin by the author).

This proof of Torricelli's is fairly well-known since it has been translated into a number of languages. The proof relies on a theorem of Archimedes that states that the area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius and the other to the circumference of the circle. Using parallel lines, the definition of the cycloid, and some simple geometry, Torricelli shows in the proof of Theorem I, that the area of the half cycloid is one and a half times the area of the triangle ACF (Figure 3) (Torricelli, 1664, p 86-87).

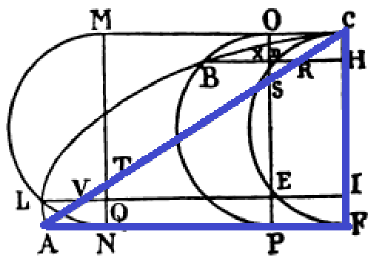

Figure 3. Illustration for Theorem I from *Opera geometrica*.
The blue lines emphasize the triangle used in the proof. Coloring by the author.

Torricelli's second and third Theorems make use of a Lemma (Torricelli, 1664, p 87) (see Figure 4).

> *Lemma I: On opposite sides of a rectangle AEFD, draw two semicircles EIF and AGD. The area between EIF and AGD is equal to the rectangle (translation from the Latin by the author).*

The figure presented with the lemma (Figure 4) shows that if a semicircle is added to the exterior of a rectangle and then a semicircle is subtracted from the resulting shape, the area of the lune (what Torricelli calls an arcuatus) is the same as the area of the rectangle.

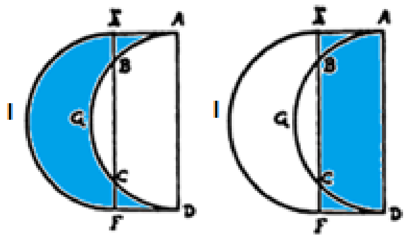

Figure 4. Illustration for Lemma I from *Opera geometrica*.
The blue shading shows the relevant areas. Coloring by the author.

Torricelli's Theorem II simply states that "the area of the cycloid is triple the generating circle" (Torricelli, 1664, p 89 of appendix). Translation by the author. What is needed for this proof is Lemma II (Torricelli, 1664, p 88):

> *Lemma II: Let the cycloid be drawn from the point C on the semicircle CDE which is rolled out along a fixed line AE. Let the rectangle AFCE be completed and make a semicircle AGF on the diameter AF. I say that the cycloid ABC cuts the arcuatus AGFCDE in two halves* (translation from the Latin by the author).

Figure 5 illustrates the conclusion of the Lemma.

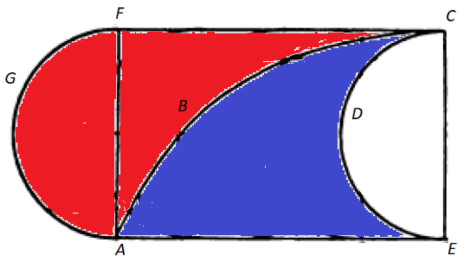

Figure 5. Illustration for Lemma II from *Opera geometrica*.
The blue and red shading illustrate the division into two pieces. Coloring by the author.

The proof of this Lemma is the most complicated argument in Torricelli's discussion of the quadrature of the cycloid, and it is also the argument where the Latin of the proof is not perfectly clear. Torricelli is using a reductio ad absurdum argument and begins by assuming that *ABCDE* is the larger of the two parts in Figure 5. The set up for Torricelli's proof is shown in Figure 6. *AE* is the base of the half cycloid *AC*, it is divided in half at point H, then the segment *EH* is divided in half and the process is continued iteratively until the rectangle *IRCE* with base *IE* has an area smaller than that of the rectangle with predefined area *K*.

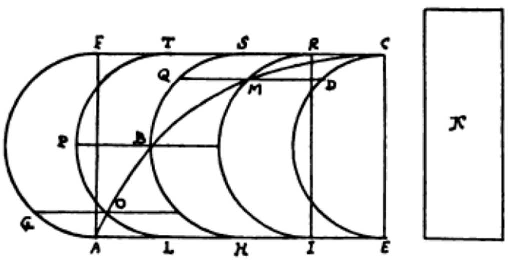

Figure 6. Illustration for Lemma II from *Opera geometrica*.

AE is divided into segments of length *IE*. A tangent semicircle is placed at the endpoint of each *IE*-length segment (e.g., *I, H, L*). These semicircles cross the cycloid at points *O, B* and *M*. The lines *GO*, *PB*, and *QMD* are drawn parallel to the base *AE*. *FC* is divided in the same manner, so segment *RC* is the same length as *IE*. Because *GO* and *AH* are parallel, and the semicircles *FGA* and *TPL* are tangent to *AE* at *A* and *L* respectively, the *arcuatus GAL* and the *arcuatus OLH* must have the same area. Torricelli is using *arcuatus* to give a name to these shapes that are parts of lunes. The process continues and thus pairs of *arcuatus* are given (see Figure 7).

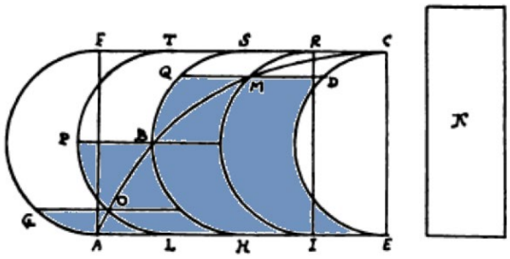

Figure 7. Illustration for Lemma II from *Opera geometrica*.
The slate blue shading illustrates the pairs of areas. Coloring by the author.

The balance of the argument relies on the use of symmetry and the physical definition of a cycloid. In particular, that the length of the segment *AE* is the same as the circumference of the semicircle (*CDE*). This physical definition gives that *arcOL* = *AL* (this can be seen by recognizing that if the semicircle *TPL* was rolled so that point *O* was on the line segment *AE*, the semicircle moves from being tangent at point *L* to being tangent at point *A*). In a similar manner, *arclengthDC*= *RC*. By construction *AL*=*IE*=*RC*=*SR* and because *QMD* is parallel to *FC*, *arclengthDC*=*arclengthMR*. This gives that the areas of *arcuatus OLH* and *arcuatus MRC* are the same (see Figure 8). By use of parallel lines and symmetry (see Figure 7), the areas of *arcuatus OLH* and *arcuatus QSR* are the same (see Figure 8). This now gives two *arcuatus* of equal size, one inscribes in the cycloid (to the right) and one circumscribed on the cycloid (to the left).

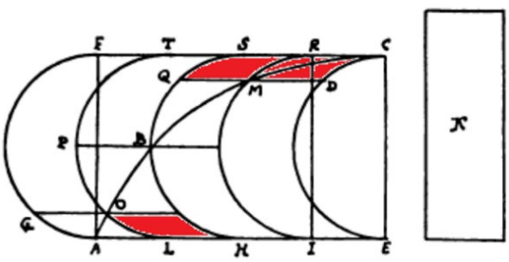

Figure 8. Illustration for Lemma II from *Opera geometrica*.
The red shading indicates areas of the same size. Coloring by the author.

In a similar manner, using the definition of the cycloid, the properties of parallel lines, and symmetry, additional pairs of *arcuatus* can be obtained (see Figure 9). This creates Cavalieri-like divisions of the areas inscribing and circumscribing the cycloid.

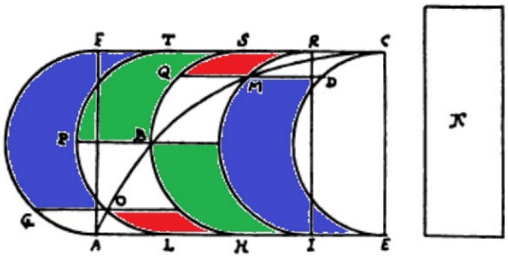

Figure 9. Illustration for Lemma II from *Opera geometrica*.
The blue, green, and red shading illustrate the Cavalieri-like divisions. Coloring by the author.

The Latin in Torricelli's conclusion of this proof is not clear. He says (Torricelli, 1664, p 89):

> *Using the arcuatus, the entire figure inscribed in the triline ABCDE, will be equal to the same figure circumscribed by the triline FGABC, with the arcuatus IMRCDE removed. But if you were to add the circumscribed figures to the same arcuatus IMRCDE, the circumscribed figure will exceed the inscribed by the excess of the given arcuatus, or the rectangle RE, and certainly by an excess less than area K (translation from the Latin by the author).*

An interpretation of this can be found by looking at Figure 10. Consider the area not covered by the pairs of arcuatus, it can be seen that what remains is the area of a full *arcuatus* of the semicircle which has the area of the *arcuatus IMRCDE*. Lemma I shows that the area of the *arcuatus IMRCDE* is the same as the area of the rectangle *RIEC*, and thus smaller than area *K*. The balance of Torricelli's proof of Lemma II states that the argument would be the same if *FGABC* had been chosen to be the larger area, thus *ABC* must divide the shape *AGFCDE* in half.

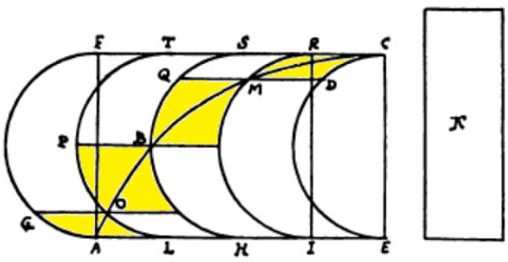

Figure 10. Illustration for Lemma II from *Opera geometrica*.
The yellow shading illustrates the area not used by the pairs of arcuatus. Coloring by the author.

The proof of Theorem II follows from Lemma I and Lemma II. What Torricelli shows is that since the blue area in Figure 11 is the same as the area of the rectangle *AFCE* (Lemma I), the part of the *arcuatus* under the cycloid has the same area as one half of the rectangle (see Figure 5 and Lemma II), thus it is equal to two semicircles. When a third semicircle is added to fill in the white space (semicircle on *CE*), the desired result that the area under the half cycloid is three semicircles is achieved.

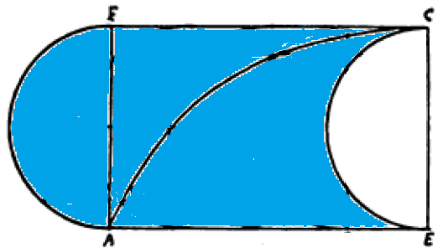

Figure 11. Illustration for Theorem II from *Opera geometrica*.
Coloring by the author.

Torricelli's Theorem III also states that the area under the cycloid is the same as three times the area of the generating circle (Torricelli, 1664, p 90). The proof also relies on showing that the large *arcuatus* is cut in half by the cycloid (as shown in Figure 5), but a different proof for this result is given. Once again, Torricelli uses symmetry and Cavalieri-like techniques. His proof focuses on pairs of parallel lines (*GI* and *LH*) that are symmetric with respect to the centre of the diameter of the generating circle (*CD*) (see Figure 12). Line *GI* intersects the cycloid at point *O* and line *LH* intersects the cycloid at point *B*. Semicircles are drawn through *O* and *B* such that they are tangent to *AD* at points *N* and *Q* respectively.

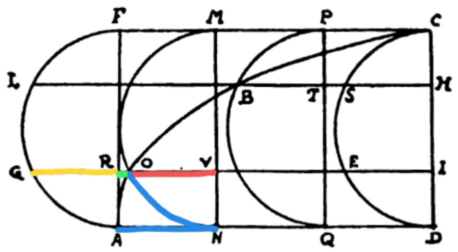

Figure 12. Illustration for Theorem III from *Opera geometrica*.
Coloring by the author.

Once again, Torricelli invokes the physical properties of the cycloid to build his argument. Because *GI* is parallel to *AD*, *GR*=*OV* and thus *GO*=*GR*+*RO*=*OV*+*RO*=*RV*=*AN* (red, yellow, and green lines in Figure 12). By Properties of the cycloid *arcON*=*AN* (blue lines in Figure 12). Similarly in the top of Figure 12, *BT*=*SH* and thus *BS*=*BT*+*TS*=*SH*+*TS*=*TS*=*PC*. Since *GI* and *LH* are symmetric with respect to the centre of the line *CD*, *arcON*=*arcBP*. By properties of parallel lines and circles *arcBP*=*arcCS* and by properties of the cycloid and the rectangle *arcCS*=*QD*=*PC*. Combining these equations yields *GO*=*arcON*=*arcCS*=*PC*=*SH* (see Figure 13). This process can be iterated and thus provides the basis for a Cavalieri-type argument that *ABC* cuts the shape *AGFCSD* in half.

Since the area of *AGFCSD* is the same as the area of the rectangle *AFCD* (Lemma I), then the area of *ABCED* is one half of the rectangle *AFCD* and thus equivalent to two semicircles. When the third semicircle is added to fill in the white space (see Figure 11) the desired result is achieved.

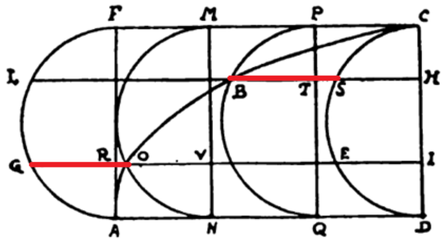

Figure 13. Illustration for Theorem III from *Opera geometrica*.
Coloring by the author.

CONCLUSION

Torricelli had an extensive correspondence with Cavalieri (Torricelli *et al.*, 1919) and Torricelli's three proofs of the quadrature of the cycloid make use of Cavalieri's indivisibles (Cavalieri, 1635). These proofs provide a small window into how 17th century European mathematicians made use of geometry and symmetry in calculations of quadrature well before the "discovery" of calculus.

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