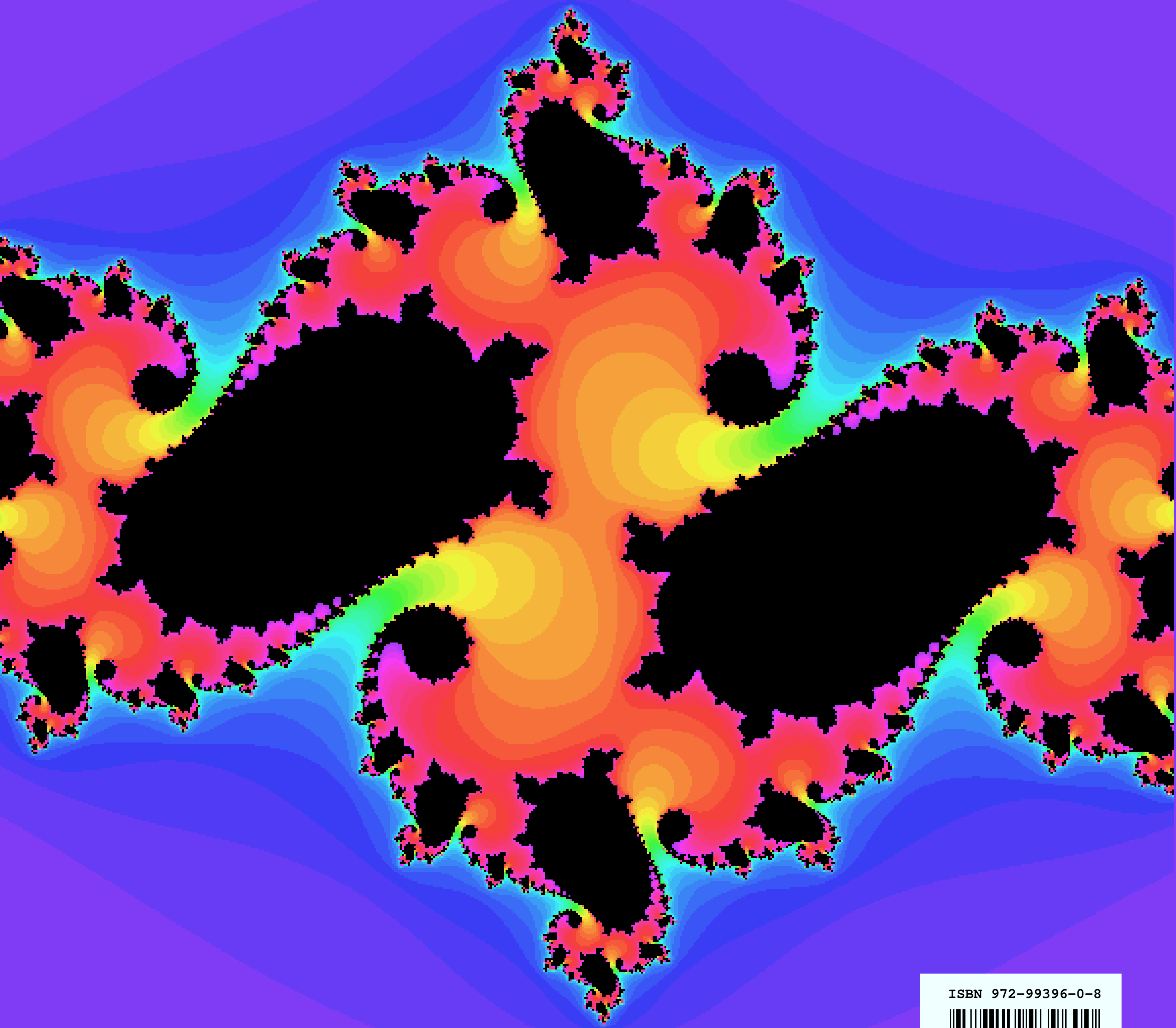
Introduction to Dynamical Systems

A HANDS-ON APPROACH WITH MAXIMA



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A Hands-on Approach with Maxima

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This is a partial translation of the Portuguese version 1.2 of February 27, 2007.

The cover figure is the Julia set for the complex number $-0.75 + i\,0.1$, with 48 iterations, as explained in chapter 12.

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Preface

This book is updated very often. The number of the current version can be found on the second page and the most recent version can always be found on the Web at http://fisica.fe.up.pt/maxima/dynamicalsystems. This version has been written to be used with Maxima's version 5.11. (http://maxima.sourceforge.net).

This book started as the lecture notes for a one-semester course on the physics of dynamical systems, taught at the College of Engineering of the University of Porto, since 2003. The subject of this course on dynamical systems is at the borderline of physics, mathematics and computing, and it substituted a course on classical mechanics that we used to teach to students majoring in computing engineering.

Since dynamical systems is usually not taught with the traditional axiomatic method used in other physics and mathematics courses, but rather with an empiric approach, it is more appropriate to use a practical teaching method based on projects done with a computer.

The study of dynamical systems advanced very quickly in the decades of 1960 and 1970, giving rise to a whole new area of research with an innovative methodology that gave rise to heated debates within the scientific community. The innovative boost was fueled by the rapid development of computers.

A new generation of researchers rose, who used their computers as laboratories for experimenting with equations discovering new phenomena. The traditional mathematicians criticized that approach for the lack of a rigorous mathematical foundations for those new results. Many of those results were found within the framework of physical problems: non-linear dynamics, condensed-matter physics and electromagnetism. However, many physicists regarded that new research field simply as a computer simulation of old physical concepts long established, without any new physics in it. A usual comment would be: "this is all very nice, but where is the physics in it?".

Thus, the pioneers in this new field of dynamical systems would be often confronted with rejected publications in scientific journals and negative assessments of their work. On the other hand, their activities awakened a strong interest that increased very quickly and were viewed by some as a refreshing new trend; the methods used in the study of dynamical systems match well with the working environment of modern scientists.

The new paradigm spread to teaching and the traditional courses on physics and mathematics have been gradually "infected" with this new experimental/ computational methodology, contrasting with the traditional axiomatic method. As it happened in the scientific community, the new methodology has also led to some debate among teachers; at the same time, it has awakened big interest as a better way to motivate today's students. Subjects such as chaos and fractals are very appealing to them.

In this book we intend to explore some topics on dynamical systems, using an active teaching approach, supported by computing tools and trying to avoid too may abstract details. The use of a Computer Algebra System (CAS) does not eliminate the need for mathematical analysis from the student; using a CAS to teach an engineering course does not turn it into a purely technical subject either. One of the difficulties inherent to any Computer Algebra System is the fact that there are no unique solutions to the problems it solves. Different methods to solve a problem may lead to solutions that look very different but might be equivalent. Or the solutions can be really different and only equivalent within some domain. In some cases the system does not give any answer or it might even give a wrong answer.

It is necessary to gain some experience to be able to use CAS tools successfully and to be able to test the validity of its results. In the process of gaining that experience, the user will also gain a better insight into the mathematical methods implemented in the system.

Nowadays the great majority of engineering and exact sciences professionals depend on a calculator to calculate the square root of a real number, for instance, 3456. Some of us were taught in School how to do that with pencil an paper, in a time when there were no calculators. I do not believe that this new dependency is a serious handicap, and I'm not in favor of teaching kids how to calculate square roots with paper and pencil before they are allowed to use the calculator. What I find very important is that the algorithm we used to calculate square roots with paper and pencil remains available and well documented in the literature; it is an valuable piece in our legacy of algorithms.

On the other hand, now that students have calculators to compute square roots, they can move faster into other topics such as the study of quadratic equations; and in doing so, they might even gain a deeper insight of the function $\sqrt{x}$, which they did not attain when they had to spend a lot of time learning the algorithm to calculate square roots. In the case of differential equations and difference equations, with the help of Computer Algebra Systems students can advance faster into subjects such as chaos and fractals, instead of dedicating a whole semester to learn several algorithms to obtain analytical solutions for a few types of equations.

I would like to acknowledge the help of my colleagues Helena Braga and Francisco Salzedas, with whom I have taught the course on Physics of Dynamical Systems; I would also like to thank our students in that course throughout the last 3 years; their positive comments have encouraged me to undertake the task of writing this book. The students have been asked to make projects for that course, and some of those projects were very interesting and helped me learn some of the subjects covered in this book. Special thanks go to the student Pedro Martins and to my colleague Francisco Salzedas, who made a careful review of the manuscript.

Jaime E. Villate
Porto, February 2007

1 Introduction

1.1 Differential equations

Differential equations play a very important role in Engineering and Science. Many problems lead to one or several differential equations that must be solved. Most attention has been given to linear equations in the literature; several analytical methods have been developed to solve that type of equations.

Non-linear differential equations are much harder to analyze and there are no general solution techniques for those equations. Problems that lead to linear equations are easier to study. From the last half of the 20th century, the rapid development of the computer made it possible to solve non-linear problems using numerical methods. Non-linear systems lead to a wealth of new and interesting phenomena not present in linear systems.

A new approach, that relies more on geometric interpretation rather than analytical analysis, has gained popularity for the study of non-linear systems. Many of the concepts used in that geometrical approach, such as the phase space, have long be used in dynamics to study the motion of a mechanical system.

In order to give a short introduction to that methodology to study differential equations, in the next chapters we will consider several problems specific dynamics and electrical circuit theory. Before we begin, we will introduce a Computer Algebra System (CAS), Maxima, which will be used extensively throughout the book.

1.2 Solving physics problems with Maxima

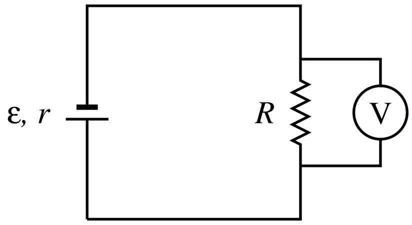
Maxima is a software package in the category of Computer Algebra Systems (CAS), namely, a system that can be used not just for numerical calculation but also to deal with algebraic equations with abstract variables. There are various CAS packages available; we have decided to use Maxima because it is Free Software. That means that it can be installed and used by our students without having to obtain a license for it, and they can even study its source code to get a better understanding of how that system works. Another important advantage is the possibility of modifying the original package to better suit our needs; we took advantage of that facility to add new features needed for this book.

Maxima includes several functions to manipulate mathematical functions, including differentiation, integration, power series approximation, Laplace transforms, solving ordinary differential equations and graph plotting in 2 and 3 dimensions. It can also work with matrices and vectors. Maxima can be used to solve problems numerically and write down programs as done with traditional programming languages.

The following examples should serve to give a first glimpse at the way Maxima can be used. In the next chapters we will go deeper into the subject, but readers who are not familiar with Maxima and would like to have a general overview from the beginning can start by going through appendix A. The examples that we will solve in this section are in the area of dynamics of a particle and direct-current circuits, which are the main subjects in this book. A minimum knowledge in those two subjects will be necessary in order to follow those examples.

Example 1.1

A battery is connected to an external resistor with resistance R and the voltage across the resistor is measured with a voltmeter V. To find the electro-motive force ε and the internal resistance r of the battery, two external resistors of 1.13 kΩ and 17.4 kΩ were used. The voltage drop in both cases were 6.26 V e 6.28 V. Find the intensity of the current in both cases. Obtain the values of ε and r. Plot a graph of the power dissipated in the external resistance, as a function of R, for values of R within 0 and $5r$.



Solution: The current through R is found from **Ohm's law**:

$$I = \frac{\Delta V}{R} \tag{1.1}$$

With the values given for the potential difference, ΔV, and the resistance, R, we can use Maxima to find the currents:

```
(%i1) 6.26/1.13e3;
(%o1)                         .005539823008849558
```

When Maxima's console is started, the `(%i1)` label appears indicating that the system is ready to accept a command; the letter "i" stands for *input*. The expression 1.13e3 is the form used to represent the number 1.13×10^3 in Maxima. Each command must end with a semi-colon. When the "Enter" key is pressed, the system responds with a label `(%o1)` followed by the result of the first command `(%i1)`; "o" stands for *output*.

The current in the second case is computed in a similar way:

```
(%i2) 6.28/17.4e3;
(%o2)                        3.609195402298851E-4
```

Thus, the current in the 1.13 kΩ resistor is 5.54 mA, and in the 17.4 kΩ resistor is 0.361 mA.

To obtain the battery's electro-motive force and internal resistance we should use the **voltage-current characteristic** for a battery:

$$\Delta V = \varepsilon - rI \tag{1.2}$$

replacing the two set of values given for ΔV and R we will get a system of two equations with two variables. We will save those two equations in two Maxima variables that we will dub as `eq1` and `eq2`

```
(%i3) eq1: 6.26 = emf - r*%o1;
(%o3)                  6.26 = emf - .005539823008849558 r
(%i4) eq2: 6.28 = emf - r*%o2;
(%o4)                  6.28 = emf - 3.609195402298851E-4 r
```

notice that the symbol used to save a value in a variable is a colon and not an equal sign. A maxima variable can store a numerical value or something more abstract as a mathematical equation in this case. The equal sign makes part of the equation that is being stored. To avoid having to write the numerical values of the currents obtained previously, we used the symbols `%o1` and `%o2` that stand for the value of those previous results.

The last two equations constitute a linear system of equations with two variables. That kind of system can be solved in Maxima, using the command `solve`:

```
(%i5) solve([eq1,eq2]);
                                  983100         79952407
(%o5)                     [[r = ------, emf = --------]]
                                  254569         12728450
(%i6) %,numer;
(%o6)           [[r = 3.861821352953423, emf = 6.281393806787158]]
```

The syntax `[eq1,eq2]` was used to create a **list** with two elements, which is what the command `solve` expects when there are more than one equation to be solved. Some warning messages given by Maxima were omitted above. The command `solve` gives an exact result, in the form of two rational numbers. The command in `%i6` was used to approximate those rational numbers with fixed-point numbers. The symbol `%` stands for the output of the last command executed; in this case it is equivalent to `%o5`. We thus conclude that the electro-motive force is approximately 6.2814 V and the internal resistance is 3.8618 Ω.

The electric power dissipated in the resistance R is

$$P = RI^2$$

the current I across the external resistor can be calculated in terms of the electro-motive force and the resistances r and R

$$I = \frac{\varepsilon}{R + r}$$

therefore, the power dissipated in the external resistor is

$$P = R\left(\frac{\varepsilon}{R+r}\right)^2$$

to draw the plot of P, as a function of R, we can use the following command

```
(%i7) plot2d(R*(6.2814/(R+3.8618))^2, [R, 0, 5*3.8618]);
```

the result is shown in figure 1.1. Moving the cursor in the graphic window, it is possible to read the coordinates of the point where the cursor is. We can check that the power dissipated in an external resistor reaches its maximum value when the external resistance equals the internal one.

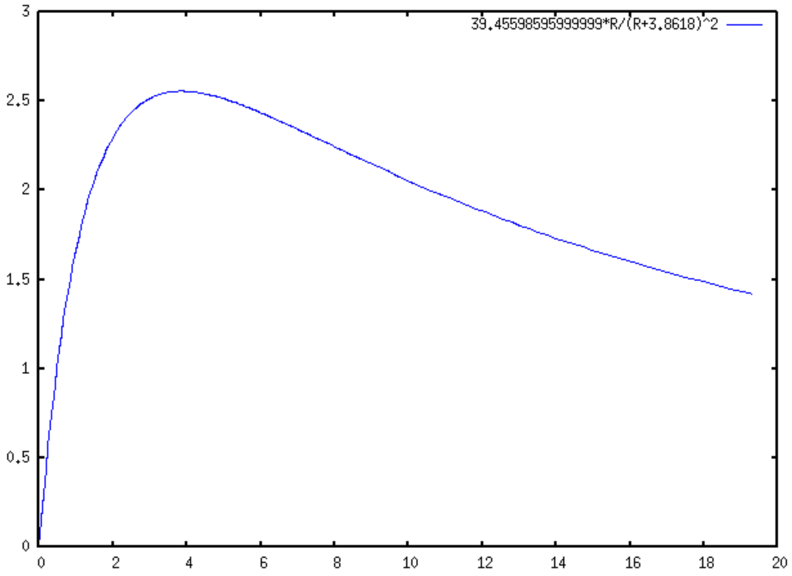


Figure 1.1: Power dissipated in the external resistor as a function of the external resistance.

Two frequent errors should be avoided:

- An expression such as

  ```
  a = 3;
  ```

  will not assign any value to the variable a. After that expression is written, variable `a` remains as an undefined variable. To assign a value of 3 to the variable `a`, the following command must be used:

  ```
  a : 3;
  ```

- New users of Maxima should become aware of the distinction between equations and expressions. An example of an equation is

  ```
  x^2 - 3*x = 2*x + 5
  ```

while an expression is something like

```
2*x + 5
```

Some commands in Maxima accept only equations or expressions as their input values. For instance, the command `plot2d` used in the previous example accepts only expressions and not equations. The command `solve` requires one equation, or a list with several equations, but it will also accept expressions instead of equations: each expression given will be automatically converted into an equation by equating it to zero; for instance, the command

```
solve(x^2 - 5*x + 5);
```

will find the two values of x that will solve the equation

$$x^2 - 5*x + 5 = 0$$

Example 1.2
The position vector of a particle, as a function of time t, is given by the equation:

$$\vec{r} = \left(5 - t^2\, e^{-t/5}\right)\vec{e}_x + \left(3 - e^{-t/12}\right)\vec{e}_y$$

in MKS units. Find the vectors for the position, velocity and acceleration at $t = 0$, $t = 15$ s, and when time approaches infinity. Plot the trajectory of the particle during the first 60 seconds of its motion.

Solution: We start by representing the position vector as a list with two elements; the first element will be the x coordinate and the second one will be the y coordinate. We will save that list in a variable named r, so we can use it later on.

```
(%i8) r: [5-t^2*exp(-t/5),3-exp(-t/12)];
                               2   - t/5        - t/12
(%o8)                    [5 - t  %e     , 3 - %e      ]
```

the vector velocity equals the derivative of the position vector and the vector acceleration is the derivative of the vector velocity. The command used in Maxima to find the derivative of an expression is `diff`. The input to that command can also be a list with several expressions; thus, the velocity and acceleration are:

```
(%i9) v: diff(r,t);
                        2   - t/5                   - t/12
                       t  %e              - t/5   %e
(%o9)                 [---------- - 2 t %e      , --------]
                           5                         12
(%i10) a: diff(v,t);
                  2   - t/5         - t/5                   - t/12
                 t  %e         4 t %e            - t/5    %e
(%o10)        [- ---------- + ----------- - 2 %e      , - --------]
                     25            5                        144
```

the constant `%e` in Maxima represents the Euler number, e. To find the position, velocity and acceleration at $t = 0$, we use the following commands

```
(%i11) r, t=0, numer;
(%o11)                              [5, 2]
(%i12) v, t=0, numer;
(%o12)                      [0, .08333333333333333]
(%i13) a, t=0, numer;
(%o13)                   [- 2, - .006944444444444444]
```

The argument `numer`, was used to get the result in floating-point form. In vector notation, the results we obtained are:

$$
\begin{aligned}
\vec{r}(0) &= 5\vec{e}x+2\vec{e}_y \\
\vec{v}(0) &= 0.08333\vec{e}_y \\
\vec{a}(0) &= -2\vec{e}x-0.006944\vec{e}_y
\end{aligned}
$$

For $t = 15$ s the results are obtained in a similar way

```
(%i14) r, t=15, numer;
(%o14)              [- 6.202090382769388, 2.71349520313981]
(%i15) v, t=15, numer;
(%o15)              [.7468060255179592, .02387539973834917]
(%i16) a, t=15, numer;
(%o16)                     [0.0497870683678639, - .001989616644862431]
```

The limiting values when times goes to infinity can be calculated with Maxima's command `limit`; the symbol used in Maxima to represent infinity is `inf`

```
(%i17) limit(r,t,inf);
(%o17)                              [5, 3]
(%i18) limit(v,t,inf);
(%o18)                              [0, 0]
(%i19) limit(a,t,inf);
(%o19)                              [0, 0]
```

Thus, a particle will approach the point $5\vec{e}_x+3\vec{e}_y$, where it will remain at rest.

To plot the graph of the trajectory we will use the option `parametric` of the command `plot2d`. the x and y components of the position vector will be given separately; the command `plot2d` will not accept them inside a list as we have been using them. To get the first element of the list `r` (x component) is labelled as `r[1]` and the second element `r[2]`.

```
(%i20) plot2d([parametric,r[1],r[2],[t,0,60],[nticks,100]]);
```

The time domain, from 0 to 60, is defined with the notation `[t,0,60]`. The option `nticks` was used to increase the number of intervals of t, because the default value of 10 intervals would render a broken curve instead of a continuous trajectory. The graph obtained is shown in figure 1.2.

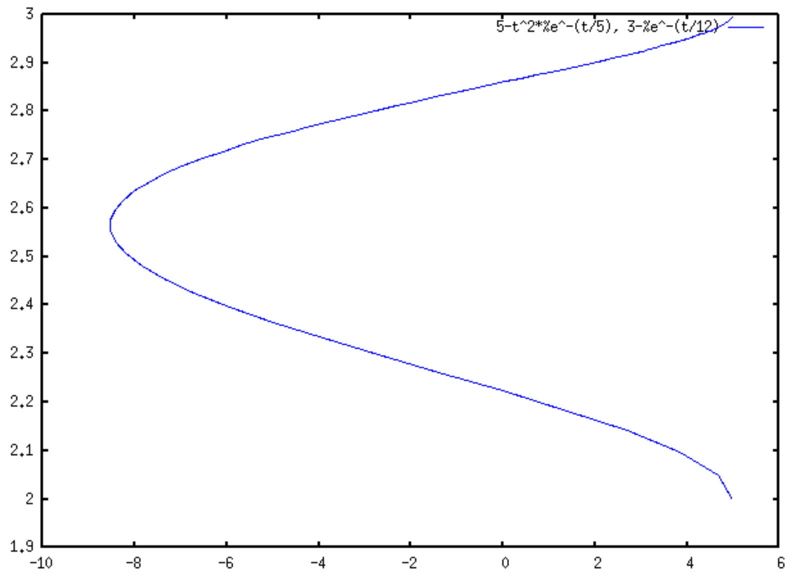


Figure 1.2: Trajectory of the particle during the first 60 seconds, from the initial instant when *t* was equal to 5.2 .

1.3 References

For more information about Maxima, see appendix A and the Maxima Book (de Souza et al., 2003).

1.4 Multiple-choice questions

1. Only one of the following Maxima commands is correct. Which one?

A. `solve(t-6=0,u-2=0,[t,u]);`

B. `solve(t+4=0,u-4=0,t,u);`

C. `solve([x^3+4=2,y-1],[x,y]);`

D. `solve(x-6=0,y-2=0,[x,y]);`

E. `solve([t+3,u-4],[t,u]);`

2. Newton's second law was defined in Maxima with:

```
(%i6) F = ma;
```

which Maxima command should be used to compute the value of the force corresponding to a mass of 7 with an acceleration of 5 (SI units).

A. `solve(F); m=7, a=5);`

B. `solve(F, [m=7, a=5]);`

C. `solve(%o6); m=7, a=5);`

D. `%o6, m=7; a=5;`

E. `solve(F; m=7, a=5)`

3. If we input the following commands in Maxima:

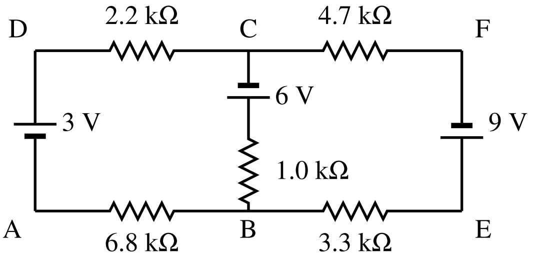
```
(%i1) x:3$
(%i2) x=5$
(%i3) x;
```

which will be the output (%o3)?

A. 5
B. x
C. 3
D. true
E. 0

1.5 Problems

1. An ammeter was used to measure the current at points D and F in the circuit which diagram is shown in the figure. At point D the value obtained for the current was 0.944 mA, in the direction ADC, and at point F it was 0.438 mA, in the direction CFE. (*a*) Store the equation for Ohm's law, $V = IR$, in a Maxima's variable "ohm". (*b*) Give a value to variable I equal to the current at point D, and substitute the resistance in Ohm's law with each of the values 2.2 kΩ and 6.8 kΩ, to compute the potential difference in each of the resistors; repeat the same procedure to calculate the potential difference in each resistor.



2. The position of a particle moving along the x axis is given by the equation $x = 2.5t^3 - 62t^2 + 10.3t$, where x is measured in meters and t in seconds. (*a*) Find the expressions for the velocity and the acceleration as a function of time. (*b*) Find the values of the time, position and acceleration in all the instants when the particle is at rest ($v = 0$). (*c*) Draw the plots for the position, velocity and acceleration as a function of time, for t in the interval between 0 and 20 seconds.

3. The position vector of a particle, as a function of time, is given by the equation:

$$\vec{r} = \left(5.76 - \mathrm{e}^{-t/2.51}\right)\vec{e}_x + \mathrm{e}^{-t/2.51}\cos(3.4t)\,\vec{e}_y$$

in SI units. (*a*) Compute the position, velocity and acceleration in the instants t=0, t=8 s, and when time goes to infinity. (*b*) Plot the graphs of the x and y components of the position, as a function of time, for t in the interval between 0 and 15 seconds. (*c*) Plot the graph of the trajectory, on the plane xy, in the interval of t between 0 and 15 s.

2 Discrete dynamical systems

A discrete dynamical system is a system with a state that only changes at a discrete sequence of instants $\{t_0, t_1, t_2, \ldots\}$. In the interval between two of those instants the state remains constant.

In this chapter we will only consider first-order discrete systems. In the following chapters the methods used in this chapter will be extended to the case of continuous dynamical systems. In the last two chapters of the book we will resume the study of discrete systems and we will introduce second-order systems and systems on the complex plane.

A first-order system is a system in which only one variable y is needed to describe its state. The value of that variable at the instants $\{t_0, t_1, t_2, \ldots\}$ will be a sequence $\{y_0, y_1, y_2, \ldots\}$. The interval of time between a consecutive pair of instants t_n and t_{n+1} does not have to be constant, for different values of n.

The **evolution equation** allows us to compute the state y_{n+1}, at an instant t_{n+1}, from the value of the state y_n at the previous instant t_n:

$$y_{n+1} = F(y_n) \tag{2.1}$$

where $F(y)$ is a known function. The equation above is a first-order **difference equation**. Given an initial state y_0, successive applications of the function F will generate the sequence of states y_n which determine the evolution of the system. In some cases it might be possible to find a general expression for y_n as a function of n.

Example 2.1

Find the first four terms in the evolution of the system $x_{n+1} = \cos x_n$*, with initial state* $x_0 = 2$

Solution: Applying the difference equation three times, we obtain the first four terms in the sequence:

$$\{2,\ \cos(2),\ \cos(\cos(2)),\ \cos(\cos(\cos(2)))\} \tag{2.2}$$

Example 2.2
A loan of \$ 500 is obtained from a bank, which charges a 5% yearly interest rate. The loan is to be paid in 20 months, with monthly payments of \$ 26.11. What will be the amount in debt after 10 months?

Solution: During the month n the amount in debt, y_n, will be equal to the amount in debt in the previous month, y_{n-1}, plus the interests due for that month, minus the monthly payment p:

$$y_n = y_{n-1} + j\,y_{n-1} - p \tag{2.3}$$

where j is the monthly interest rate (in this case, 0.05/12). Using Maxima, the sequence of amounts in debt y_n can be obtained by applying the above recurrence relation several times:

```
(%i1) j: 0.05/12$

(%i2) y: 500$

(%i3) y: y + j*y - 26.11;

(%o3)                          475.9733333333333
(%i4) y: y + j*y - 26.11;

(%o4)                          451.8465555555555
(%i5) y: y + j*y - 26.11;

(%o5)                          427.619249537037
```

it would be necessary to repeat the command `(%i3)` ten times. The answer can be obtained in an easier way if we define a Maxima function depending on an integer variable, using the recurrence relation, and we use that function to calculate y_{10} directly:

```
(%i6) y[0]: 500$

(%i7) y[n] := y[n-1] + j*y[n-1] - 26.11;

(%o7)                   y  := y      + j y      - 26.11
                         n     n - 1      n - 1
(%i8) y[10];

(%o8)                          255.1779109580579
```

Some care should be taken in Maxima when using functions of an integer argument. In the previous example, when we calculated `y[10]`, the values of `y[9]`, `y[8]`,..., `y[1]`, were also implicitly calculated and stored in memory. If we changed the recurrence relation, those values that were already calculated and stored would not be updated. Thus, before we change the recurrence relation, or the initial value `y[0]`, it is necessary to erase the previously calculated sequence, by using the command `kill`.

For example, if the value of the loan was duplicated to $ 1000, and the monthly payment was also duplicated, will the amount in debt after the tenth payment would also duplicate? let us see:

```
(%i9) kill(y)$

(%i10) y[0]: 1000$

(%i11) y[n] := y[n-1] + j*y[n-1] - 52.22;

(%o11)                  y  := y      + j y      - 52.22
                         n     n - 1      n - 1
(%i12) y[10];

(%o12)                        510.3558219161157
```

thus, the amount in debt also doubles.

Another question that we might ask in the original example is: what will the monthly payment should be if instead of 20 months we would want to pay the loan in 40 months?

To answer that question, we use a variable p to represent the monthly payment, we calculate the amount in debt after 40 months, as a function of p, and we equal that expression to zero to calculate the value of p.

```
(%i13) kill(y)$

(%i14) y[0]: 500$

(%i15) y[n] := expand(y[n-1] + j*y[n-1] - p)$

(%i16) solve(y[40] = 0, p);
                                     72970398
(%o16)                          [p = --------]
                                     5366831
(%i17) %, numer;

(%o17)                      [p = 13.59655222979818]
```

The monthly payment should be $ 13.60. The function `expand` was used to force Maxima to calculate the products in every step, avoiding large expressions with several parenthesis in the calculation of y_n. Some additional messages written by Maxima were omitted, which explain that some floating point numbers were represented as fractions, to prevent numerical errors.

2.1 Discrete systems evolution

The evolution of a first-order discrete system:

$$y_{n+1} = F(y_n) \tag{2.4}$$

is obtained by applying successively a function F to the initial state $y_0 = c$:

$$\{c, F(c), F(F(c)), F(F(F(c))), \ldots\} \tag{2.5}$$

or in a more compact form:

$$\{c, F(c), F^2(c), F^3(c), \ldots\, y_n = F^n(c)\} \tag{2.6}$$

2.2 Graphical analysis

A graphical method to represent the evolution of a system consists on plotting a point for each step in the sequence, with x-coordinate equal to the index n and y-coordinate equal to y_n. In Maxima, the function `evolution` in the additional package `dynamics` will plot that kind of graph[1].

Three arguments should be given to the program. The first argument must be an expression that depends only on the variable y; that expression will specify $F(y)$ from the right hand side of the difference equation 2.1. The second argument should be the initial value y_0, and the third argument is the number of sequence elements that should be plotted.

For instance, using y to identify the state variable in example 2.1, we have $F(y) = \cos y$ with initial value $y_0 = 2$. To plot a graph with the first 20 terms, we use the commands:

```
(%i18) load("dynamics")$

(%i19) evolution(cos(y), 2, 20)$
```

The graph obtained in `(%i19)` is shown in figure 2.1.

Another type of diagram which will be very useful to analyze first-order discrete dynamical systems is the so-called **staircase diagram**,[2] which consists in plotting the functions $y = F(x)$ and $y = x$, as well as an alternating sequence of horizontal and vertical segments joining the points (y_0,y_0), (y_0,y_1), (y_1, y_1), (y_1,y_2), and so on. For example, figure 2.2 shows the staircase diagram for the sequence represented in figure 2.1

The function `staircase`, included in the additional package `dynamics`, plots staircase diagrams. That function needs the same three arguments as the function `evolution`;

[1] Maxima's package `dynamics` was added in version 5.10; if you have an older version, you must upgrade it in order to use that package.

[2] Also know as cobweb diagram.

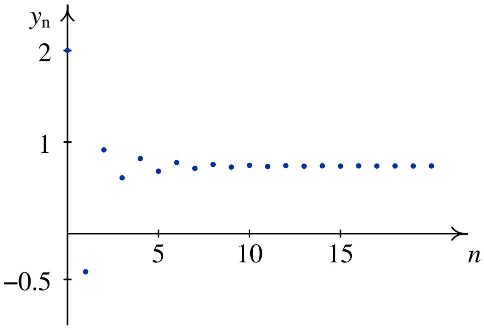


Figure 2.1: Evolution of $y_{n+1} = \cos(y_n)$ with $y_0 = 2$.

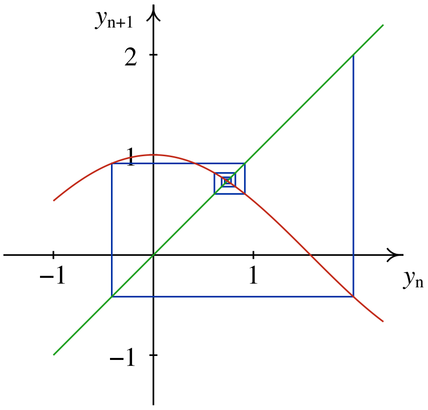


Figure 2.2: Staircase diagram for $x_{n+1} = \cos(x_n)$ with $x_0 = 2$.

namely, function $F(y)$ from the right-hand side of the difference equation 2.1, the initial value y_0 and the number of steps in the sequence. The independent variable in the expression for F should always be y. You might need to make the appropriate change if the state variable is something different from y in your problem.

The graph 2.2 was obtained with the command

```
(%i20) staircase(cos(y),2,8)$
```

Notice that we did not have to load the package `dynamics` again, because it was already loaded in `(%i18)`. the staircase diagram allows us to understand when a sequence will converge or diverge. For instance, consider the system $y_{n+1} = y_n^2 - 0.2$. If the initial value is $y_0 = 1.1$, we obtain the graph on the left side of figure 2.3; we see that the sequence will converge to a negative value y, which is at the intersection of the functions $F(y) = y^2 - 0.2$ and $G(y) = y$, namely, $y = (5 - 3\sqrt{5})/10$.

The two functions intersect in another point, positive, equal to $y = (5 - 3\sqrt{5})/10$. In the graph we can see that even though the initial value was close to the second intersection point, the sequence moved away from it and towards the second intersection point, due to the fact that between the two intersection points the function $y^2 - 0.2$ is under $G(y) = y$. if we used an initial value to the right of the second intersection point, for instance, $y_0 = 1.5$, the sequence grows quickly towards infinity (right-hand side in figure 2.3). To make the sequences converge to the second intersection point, it would have been necessary that between the two intersection points $F(x) > G(y)$; that is to say, the slope of $F(y)$ should be less than 1, rather that greater than 1, at the second intersection point.

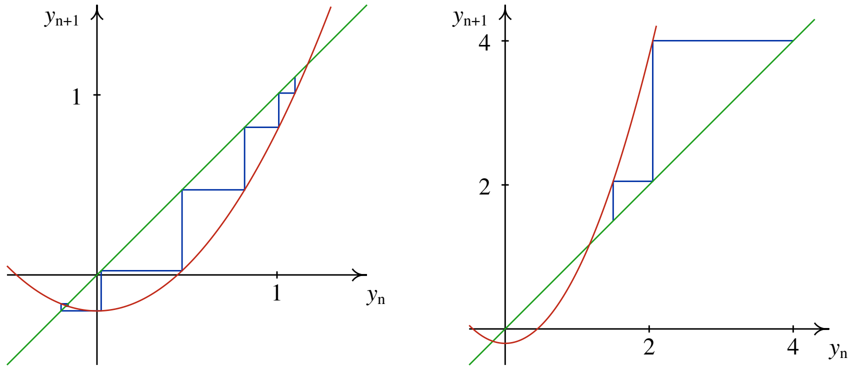


Figure 2.3: Solution to the system $y_{n+1} = y_n^2 - 0.2$ with initial values 1.1 (left) and 1.5 (right).

Example 2.3
Analyze the solutions to the logistic model, which consists in considering a population P with constant natality rate, a, and a mortality rate bP, directly proportional to the population, where a and b are constants.

Solution: The population under study could be a group of specimens from some animal species, where the sequence $\{P_0, P_1, P_2, \ldots\}$ represents the number of specimens in several consecutive years.

Let P_n represent the number of specimens at the beginning of period n. during that period of time, an average aP_n new specimens are born, and bP_n^2 specimens die. Thus, in the beginning of the next period, $n+1$, the population would be

$$P_{n+1} = (a+1)P_n\left(1 - \frac{b}{a+1}P_n\right) \tag{2.7}$$

It is convenient to define a new variable $y_n = bP_n/(a+1)$. Thus, we obtain an equation with a single parameter $c = a+1$

$$y_{n+1} = c\,y_n(1-y_n) \tag{2.8}$$

Figure 2.4 shows the solutions obtained with an initial value $y_0 = 0.1$, in the cases $c = 2$ and $c = 4$. With $c = 2$, the solution converges quickly to the fixed point $y = 0.5$.

With $c = 4$, the state of the system goes through many different values between 0 and 1, and it does not seem to follow any regular pattern. That type of behavior is called **chaotic**. The state in any given period is perfectly determined from the state in the previous period, but a very small modification of the state in an initial period will lead to a completely different state in the next period. The state of the system takes different values within a small interval, without ever repeating a previous value.

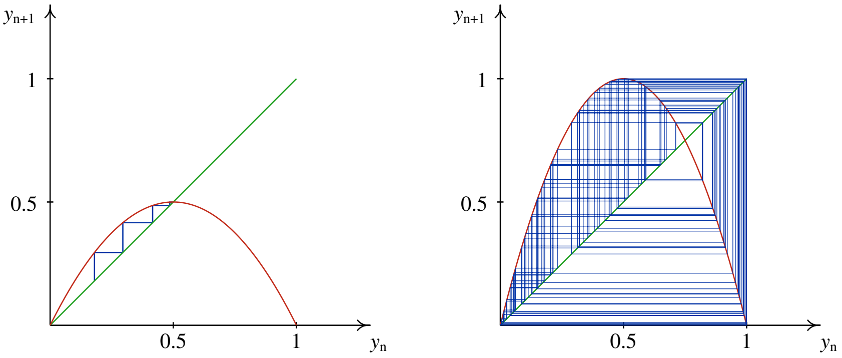


Figure 2.4: Solutions of the logistic model with an initial value 0.1. With $c = 2$ (left) the sequence converges, while with $c = 4$ (right) it becomes chaotic.

2.3 Fixed points

A **fixed point** of the system 2.1 is a value y_0 which, if used as initial value of the state it would lead to a constant state in later periods. That will only be possible if

$$F(y_0) = y_0 \tag{2.9}$$

this implies that successive application of function F will not change the initial value. The solution of the system with initial value y_0 is a constant sequence, $\{y_0, y_0, y_0, \ldots\}$.

From the point of view of the graphical analysis, the fixed points are all those points where the curve $F(x)$ intersects the right line $y = x$ in the stairway diagram. For example, in the case of he logistic model, figure 2.4 shows that in the two cases $c = 2$ and $c = 4$ there are two fixed points, one of them at $y = 0$. We can use Maxima's command `solve` to find the fixed points; in the case $c = 4$, the fixed points can be found in this way:

```
(%i21) flogistic: 4*y*(1-y);

(%o21)                                4 (1 - y) y
(%i22) fixed: solve(flogistic - y);

                                         3
(%o22)                              [y = -, y = 0]
                                         4
```

The two fixed points are 0 and 0.75.

Let us consider a fixed point, where the curve $F(x)$ intersects the straight line $y = x$, such that the derivative $F'(y)$ is bigger than 1 at that point. Namely, at the intersection point between the curve $F(x)$ and the line $y = x$ the curve F crosses from under the line, on the left, to over the line, on the right. Thus, if we plot the staircase diagram starting from a point near the fixed point, the sequence will move away from the fixed point, describing a staircase in the staircase diagram. We call that kind of fixed point a **repulsive node**.

If the derivative is negative and less than -1, the sequences will also move away from the fixed point, but alternating from side to side, describing a "cob web" in the staircase diagram. We call that kind of fixed point a **repulsive focus**.

If the derivative of the function F near the fixed point has a value between 0 and 1, the sequences that start near the fixed point will approach it describing a staircase. That kind of fixed point is called an **attractive node** (an example of this case was already found in the left-hand side of 2.4).

If the derivative of the function F near the fixed point has a value between 0 and -1, the sequences starting near the point will approach it, describing a cob web that alternates from side to side in the staircase diagram. That kind of point is called an **attractive focus** (an example of that kind of point was already encountered in figure 2.2).

In summary, we have the following kinds of fixed points y_0:

1. Attractive node, if $0 \leq F'(y_0) < 1$
2. Repulsive node, if $F'(y_0) > 1$
3. Attractive focus, if $-1 < F'(y_0) < 0$
4. Repulsive focus, if $F'(y_0) < -1$

If $F'(y_0)$ equals 1 or -1, the situation is more complex: the fixed point could be either attractive or repulsive, or even attractive from one side and repulsive from the other side.

Returning to our previous example of the logistic model (see `(%i21)` through `(%o22)` above), the value of the derivative of F at the fixed points is:

```
(%i23) dflogistic: diff(flogistic, y);

(%o23)                          4 (1 - y) - 4 y
(%i24) dflogistic, fixed[1];

(%o24)                                  - 2
(%i25) dflogistic, fixed[2];

(%o25)                                   4
```

Thus, in the case $c = 4$, both fixed points are repulsive. At $y_0 = 0$ there is a repulsive node, and there is a repulsive focus at $y_0 = 0.75$.

2.4 Periodic points

If the sequence $\{y_0, y_1, y_2, \ldots\}$ is a solution of the dynamical system

$$y_{n+1} = F(y_n) \tag{2.10}$$

any element in the sequence can be obtained from y_0, applying the composed function F^n

$$y_n = F^n(y_0) = \underbrace{F(F(\ldots(F}_{n \text{ times}}(y)))) \tag{2.11}$$

A solution is dubbed a **cycle** of period 2, if it is a sequence of only two alternating values: $\{y_0, y_1, y_0, y_1, \ldots\}$. The two points y_0 and y_1 are **periodic points** with period equal to 2. Since $y_2 = F^2(y_0) = y_0$, it is necessary that $F^2(y_0) = y_0$. Furthermore, since $y_3 = F^2(y_1) = y_1$ we also have $F^2(y_1) = y_1$. Finally, since $F(y_0) = y_1 \neq y_0$, it is also necessary that $F(y_0) \neq y_0$, and since $F(y_1) = y_0 \neq y_1$, we also have $F(y_1) \neq y_1$.

Those conditions can be summarized by saying that two points y_0 and y_1 form a cycle of period two if both of them are fixed points of the function $F^2(y)$, but none of them is a fixed point of the function $F(y)$. Explained in a different way, if we calculate the fixed points of $F^2(y)$, all of the fixed points will appear, plus the periodic points of period 2 of functio F.

The cycle of period two will be attractive or repulsive according to the value of the derivative of F^2 at each point in the cycle.

To calculate the derivative of F^2 at y_0, we use the chain rule

$$(F^2(y_0))' = (F(F(y_0)))' = F'(F(y_0))F'(y_0) = F'(y_0)F'(y_1) \tag{2.12}$$

thus, the derivative of F^2 takes the same value in the two points y_0 and y_1 of the cycle, and it is equal to the product of the derivatives of F in those two points.

Generalizing the definition a point y_0 is part of a cycle of period m, if $F^m(y_0) = y_0$, but $F^j(y_0) \neq y_0$ for any $j < m$. The complete set of m points that make part of the cycle are

$$\begin{aligned}
& y_0 & & \\
& y_1 &=& \; F(y_0) \\
& y_2 &=& \; F^2(y_0) \\
& \vdots & & \\
& y_{m-1} &=& \; F^{m-1}(y_0)
\end{aligned}$$

All those points are fixed points of F^m but they cannot be fixed points od F^j, with $j < m$.

If the absolute value of the product of the derivative at the m points of the cycle:

$$\prod_{j=0}^{m-1} F'(y_j) \tag{2.13}$$

is greater than 1, then the cycle is repulsive; if the product is less than 1, the cycle is attractive, and if the product is identical to 1, the cycle could be either attractive or repulsive in different regions.

Example 2.4
Find the cycles with period 2 for the logistic system

$$y_{n+1} = 3.1\, y_n(1 - y_n)$$

and say whether they are attractive or repulsive.

Solution: We start by defining function $F(y)$ and the composed function $F^2(y)$

```
(%i26) flog: 3.1*y*(1-y)$

(%i27) flog2: flog, y=flog;

(%o27)          9.610000000000001 (1 - y) y (1 - 3.1 (1 - y) y)
```

The periodic points of period 2 will be among the solutions of the equation $F^2(y) - y = 0$

```
(%i28) periodic: solve(flog2 - y);

                       sqrt(41) - 41       sqrt(41) + 41       21
(%o28)        [y = - -------------, y = -------------, y = --, y = 0]
                            62                  62             31
```

The last two points, namely 0 and $21/31$ are fixed points (the proof of that is left as an exercise for the reader and can be done by solving the equation $F(y) = y$, or simply by showing the validity of that equation in each case).

The other two points must then form a cycle of period two; if we use any of them as initial value, the sequence will oscillate between those two points.

To find out whether the cycle is attractive or repulsive, we compute the product of the derivative at the two points in the cycle

```
(%i29) dflog: diff(flog, y);

(%o29)                      3.1 (1 - y) - 3.1 y

(%i30) dflog, periodic[1], ratsimp, numer;

(%o30)                       - .3596875787352

(%i31) dflog, periodic[2], ratsimp, numer;

(%o31)                       - 1.640312421264802

(%i32) %o30*%o31;

(%o32)                         .5900000031740065
```

The absolute value of the product is less than 1, which implies that the cycle is attractive.

2.5 Solving equations numerically

Discrete, first-oder dynamical systems can be used for solving one-variable equations numerically. The problem to be solved consists on finding the **roots** of a real function f, namely, the values of x that satisfy the equation

$$f(x) = 0 \tag{2.14}$$

For example, suppose we want to find the values of x that solve the equation:

$$3x^2 - x\cos(5x) = 6$$

That kind of equation cannot be solved analytically; it must be solved by numerical methods. The numerical methods to solve that equation consist on defining a dynamical system with convergent sequences which approach the solutions of the equation. In the following sections we will study two of those methods.

2.5.1 Iteration method

If the equation 2.14 can be written in the form

$$x = g(x) \tag{2.15}$$

Its solutions are the fixed points of the dynamical system:

$$x_{n+1} = g(x_n) \tag{2.16}$$

To find a fixed point, we choose an arbitrary initial point and calculate the evolution of the system.

Example 2.5
Find the solution of the equation $x = \cos x$

Solution: This equation is already given in a form that allows us to use the iteration method. We use the dynamical system with recurrence relation

$$x_{n+1} = \cos(x_n)$$

To find a fixed point, we choose an arbitrary initial point and calculate the evolution of the system

```
 (%i33) x: 1$

 (%i34) for i thru 15 do (x: float(cos(x)), print(x))$

0.54030230586814
0.85755321584639
0.65428979049778
0.79348035874257
0.70136877362276
0.76395968290065
0.72210242502671
0.75041776176376
0.73140404242251
0.74423735490056
0.73560474043635
0.74142508661011
0.73750689051324
0.74014733556788
0.73836920412232
```

The solution of the equation is approximately 0.74. This method was successful in this example, because the fixed point of the dynamical system chosen happened to be attractive. If the point were repulsive, the iteration method would have failed.

Example 2.6
Find the square root of 5, using only additions, multiplications and divisions.

Solution: The square root of 5 is the positive solution of the equation

$$x^2 = 5$$

which can be written as:

$$x = \frac{5}{x}$$

we solve the dynamical system associated to the function

$$f(x) = \frac{5}{x}$$

It can be easily seen that for any initial value x_0, different from $\sqrt{5}$, the solution of that system will always be a cycle with period 2:

$$\left\{ x_0, \frac{5}{x_0}, x_0, \frac{5}{x_0}, \ldots \right\}$$

To escape from that cycle, and approach the fixed point at $\sqrt{5}$, we can try to use the middle point:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$$

That new system will converge quickly to the fixed point at $\sqrt{5}$:

```
(%i35) x : 1$

(%i36) for i thru 7 do (x: float((x + 5/x)/2), print(x))$
3.0
2.333333333333334
2.238095238095238
2.236068895643363
2.236067977499978
2.23606797749979
2.23606797749979
```

2.5.2 Newton's method

Newton's method can be used to find the roots of the equation 2.14. We start by assuming that there is a root of the equation at approximately value x_0 and we improve that approximation by finding the point x_1 where the tangent to the function at $f(x_0)$ intersects the x axis (see figure 2.5)

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{2.17}$$

We can use the same equation to further improve our guess x_1 to a new guess x_2. In general

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{2.18}$$

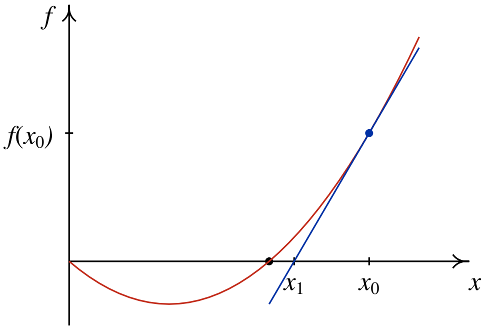


Figure 2.5: Newton's method for finding roots of an equation.

It must be noticed that the roots of a continuous function f, points where f is zero, are fixed points of the dynamical system defined by equation 2.18 [3].

The advantage of this method, over the iteration method, can be seen by using our analysis of the fixed points of a dynamical system. The function that generates the system 2.18 is

$$g(x) = 1 - \frac{f(x)}{f'(x)} \tag{2.19}$$

the derivative of that function is

$$g' = 1 - \frac{(f')^2 - f''f}{(f')^2} = \frac{f''f}{(f')^2} \tag{2.20}$$

at the fixed points, f vanishes. Thus, g' will also vanish at the fixed points. Therefore, the fixed points of 2.18 will always be attractive. It means that if the initial point x_0 is chosen

[3] If there are any regions where f and f' are both equal to zero, the roots will not be isolated points, but there will be a whole interval with an infinite number of roots. In this section we will not study those kind of roots.

close enough to one of the roots of f, the sequence x_n will approach it. The problem consists on finding what *close enough* means in each case.

To illustrate the method, we will solve example 2.6 once again, using Newton's method. The square root of 5 is one of the solutions of the equation $x^2 = 5$. Hence, to find the square root of 5 we can search for the positive root of the function

$$f(x) = x^2 - 5$$

The derivative of that function is

$$f'(x) = 2x$$

substituting it into the recurrence relation 2.18, we obtain

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)$$

which is exactly the same sequence that we have already obtained and solved in the previous section. However, in this case we did not need to introduce any clever tricks; we just applied the standard method.

2.6 References

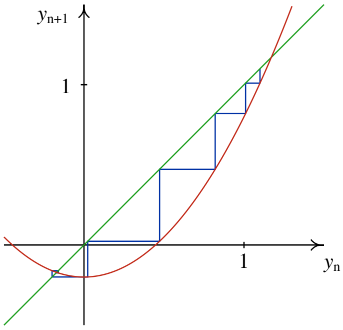
Some useful references, with a level similar to the one used here, are *Chaos* (Alligood et al., 1996), *A First Course in Chaotic Dynamical Systems* (Devaney, 1992) and *Chaos and Fractals* (Peitgen et al., 1992).

2.7 Multiple-choice questions

1. The state variable of a first-order, discrete dynamical system takes on the values from the following sequence:
$\{3.4, 6.8, 7.2, 5.1, 6.8, \ldots\}$
what can be concluded about that system:

A. it does not have any cycles with period less than 5.
B. it has a fixed point.
C. it is a chaotic system.
D. it has a cycle of period 3.
E. it has a cycle of period 2.

2. The figure shows the staircase diagram of the discrete dynamical system $y_{n+1} = y_n^2 - 0.2$, which has two fixed points $y = -0.17$ and $y = 1.17$.



what type of fixed point is $y = 1.17$?

A. repulsive focus.
B. attractive focus.
C. attractive node.
D. part of a cycle with period 2.
E. repulsive node.

3. A first-order discrete dynamical system has a single fixed point at 0.739, and no cycles. Starting with an initial value 2, the evolution of the system is the sequence:
$\{2, 0.915, 0.610, 0.820, 0.683, \ldots\}$
what can be said about that system?

A. it is chaotic.
B. the fixed point is attractive.
C. it has a cycle with period 2.
D. it has a cycle with period 3.
E. the fixed point is repulsive.

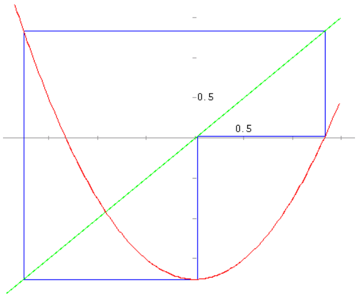
4. A function $F(x)$ has the following properties:

$$F^2(2) = 5 \qquad F(5) = 2$$

thus, we can conclude that the discrete dynamical system $x_{n+1} = F(x_n)$ has a cycle with period equal to:

A. 2
B. 3
C. 4
D. 5
E. 1

5. The figure shows the staircase diagram for the first 40 iterations of a discrete dynamical system.



thus, we can conclude that the system has:

A. an attractive focus.
B. a repulsive focus.
C. a cycle with period 2.
D. a cycle with period 3.
E. a cycle with period 40.

2.8 Problems

1. The sequence obtained in this chapter to calculate square roots,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

was already known by the Sumerians, 4000 years ago. Using that method, calculate $\sqrt{3}$, $\sqrt{15}$ and $\sqrt{234}$. Use any positive initial value and represent the number a as a floating-point number (for instance, 3.0), to force Maxima to give its results also as a floating-point number. In each case, compare the result with the value obtain using function **sqrt()** in Maxima.

2. Assume that the current whale population in the world is 1000 and that every year the normal increase of the population (births minus deaths by natural causes) is 25%. Assuming that the number of whales killed by fishermen every year remained constant at 300 during the next years, how will the whale population evolve during the next 10 years?

3. Compute the first 10 terms of the sequence defined by the equation:

$$x_{n+1} = x_n^2 - 2$$

using the following initial values:

a) $x_0 = 1$
b) $x_0 = 0.5$
c) $x_0 = 2$
d) $x_0 = 1.999$

Discuss the behavior of the sequence in each case.

4. For each function in the following list, the point $y = 0$ makes part of a cycle for the system $y_{n+1} = F(y_n)$. Determine the period of the cycle for each case and calculate the derivative of the function in order to determine whether the cycle is attractive or repulsive. Draw the staircase diagram of the sequence with initial value 0.

a) $F(y) = 1 - y^2$
b) $F(y) = \frac{\pi}{2}\cos y$
c) $F(y) = -\frac{1}{2}y^3 - \frac{3}{2}y^2 + 1$
d) $F(y) = |y-2| - 1$
e) $F(y) = -\frac{4}{\pi}\mathrm{arctg}(y+1)$

5. Find the fixed points and the cycles with period 2 of the dynamical system:

$$y_{n+1} = F(y_n)$$

and classify each point and cycle as attractive or repulsive, for each of the following cases:

a) $F(y) = y^2 - \frac{y}{2}$
b) $F(y) = \frac{2-y}{10}$
c) $F(y) = y^4 - 4y^2 + 2$
d) $F(y) = \frac{\pi}{2}\sin y$
e) $F(y) = y^3 - 3y$
f) $F(y) = \mathrm{arctg}\,(y)$

In each case, start by drawing a staircase diagram using the function **staircase** and use it to find out the position of the fixed points and cycles; use the option `domain` to get a better overview of the position of the fixed points. Then try to find the points analytically. In some cases that will not be possible and the result will have to be obtained approximately from the plots.

6. Considering the sequence $x_{n+1} = |x_n - 2|$

a) Find all the fixed points. Show those points in a plot of the functions $F(x) = |x-2|$ and $G(x) = x$.
b) Explain the kind of sequence that will be obtained if x_0 were a integer, either even or odd.

c) Find the solution for the initial value 8.2.

d) Find all the cycles with period two. Show all the points in those cycles in a plot of the functions $F^2(x)$ and $G(x) = x$.

7. Consider the function

$$F(x) = \begin{cases} 2x & ,\text{if} \quad x \le 1 \\ 4-2x & ,\text{if} \quad x \ge 1 \end{cases}$$

a) Show that $F(x)$ is equivalent to $2-2|x-1|$.

b) Plot, in the same graph, the functions $F(x)$, $F^2(x)$, $F^3(x)$ and $G(x) = x$. What can you conclude about the fixed points and cycles of the system $x_{n+1} = F(x_n)$?

c) Make a table or plot a graph of n against x_n, between $n = 0$ and $n = 100$, for each of the initial values 0.5, 0.6, 0.89, 0.893 and 0.1111111111. Discuss the results obtained.

d) In the previous item, the sequence remains constant, starting at $n = 55$, for each of the initial values. Compute again the sequences obtained in the last item, using the following program, which uses higher numerical precision than the function **evolution** from the **dynamics** module.

```
evolution60(f, x0, n) :=
block([x: bfloat(x0), xlist:[0, x0], fpprec: 60],
for i thru n
do (x: ev(f), xlist: append(xlist, [i, float(x)])),
openplot_curves([["plotpoints 1 nolines"], xlist])
)
```

what can you conclude?

3 Continuous dynamical systems

In this chapter we will use the abbreviated notation $\dot{x}$, $\ddot{x}$, ... to represent the time derivatives of a time-dependent function $x(t)$. The derivatives of a function $y(x)$, with respect to x, will be represented as y', y'', ... Consider for example:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad \ddot{x} = \frac{\mathrm{d}^2x}{\mathrm{d}t^2} \qquad y' = \frac{\mathrm{d}y}{\mathrm{d}x} \qquad y'' = \frac{\mathrm{d}^2y}{\mathrm{d}x^2} \tag{3.1}$$

In Maxima those four derivatives are represented in this form:

```
diff(x,t)     diff(x,t,2)     diff(y,x)     diff(y,x,2)
```

3.1 First-order differential equations

An **ordinary differential equation** (ODE) is an equation involving a function of only one variable, for instance $y(x)$, and its derivatives; the variable x may also appear in the equation. If the only derivative appearing in the equation is the first-order derivative, the equation is called a first-order ordinary differential equation. Hence, first-order ODE's have the general form $F(x,y,y') = 0$, but we will consider only the equations that can be rewritten as one or several equations with the form

$$y' = f(x,y) \tag{3.2}$$

Two examples of first-order ODE's are the following:

$$y'y\cos(x) = 3xy^2\sin(y)$$
$$\dot{y} = 3-5\mathrm{y} - y$$

the function in question, in both cases y, is dubbed **dependent variable**. The variable on which the function depends is dubbed **independent variable**. In the first case above, the independent variable is x. In the second case, the independent variable does not appear in the equation, but from the derivative $\dot{y}$ it becomes clear that the independent variable is t.

A solution of a ODE, in a given interval, is any function f of one variable which satisfies that equation, when substituted for the dependent variable. A solution of the differential equation might also be given in **implicit form**, namely as an implicit function, as we will see in the example that follows.

Example 3.1
Show that the implicit function assim, vemos que a equação obtida é a mesma equação diferencial dada.

$$x+y+e^{xy}=0 \tag{3.3}$$

is a solution of the first-order differential equation

$$\frac{dy}{dx}=-\frac{1+y\,e^{xy}}{1+x\,e^{xy}} \tag{3.4}$$

Solution: from the differential equation, we see that the independent variable is x. In order to prove that the given implicit function is a solution we will differentiate it in terms of x, keeping in mind that y is a function that depends on x, and we will show that the equation obtained is equivalent to the differential equation.

```
(%i2) eq1: x + y + exp(x*y) = 0$
(%i3) depends(y,x);
(%o3)                              [y(x)]
(%i4) diff(eq1,x);
                           x y    dy        dy
(%o4)                   %e    (x -- + y) + -- + 1 = 0
                                  dx        dx
(%i5) solve(%,diff(y,x));
                                         x y
                              dy     y %e    + 1
(%o5)                        [-- = - -----------]
                              dx         x y
                                     x %e    + 1
```

thus, we obtain the same differential equation given in the problem, which proves that the implicit function is a solution of it.

In the previous example it was necessary to use the command `depends` to declare that y is a function that depends on x. If we did not use that command, x and y would be regarded as independent variables and the derivative of y in terms of x would be replaced by zero, rather than dy/dx.

In some cases it is possible to find functions which are solutions of a differential equation. For instance, if the equation had the simple form

$$y'=f(x) \tag{3.5}$$

its solutions would follow from the fundamental theorem of integral calculus

$$y(x)=\int f(x)\,\mathrm{d}x+c \tag{3.6}$$

where c is any arbitrary constant.

There is no general method to find the solutions of any first-order ODE. In some cases the solutions are known but they are special functions which will have to be evaluated with approximate numerical methods. Later in this chapter we will study a simple numerical method to solve differential equations, the Euler method, and at the end of the chapter we will discuss some particular kinds of differential equations which can be solved analytically.

3.2 Direction fields

It is possible to obtain very important information about the solutions of an equation such as 3.2 from a simple geometrical analysis of function $f(x,y)$. Function $f(x,y)$ defines, at each point in the (x,y) plane, the slope that a function $y(x)$ should have in order to be a solution of the ODE.

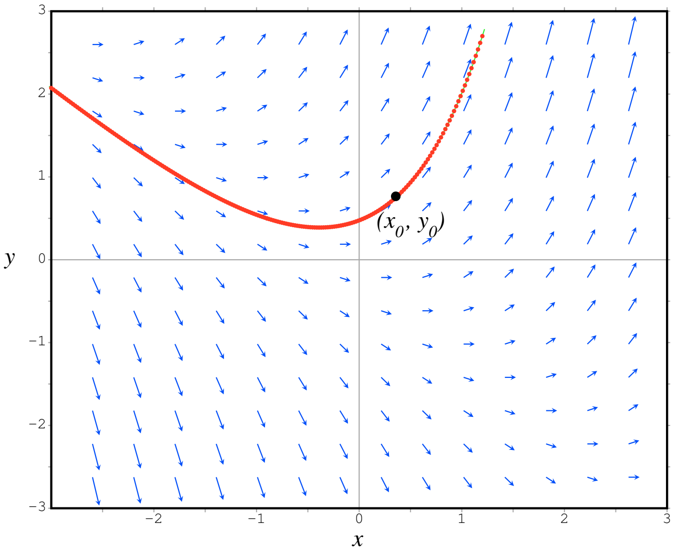


Figure 3.1: Direction field of the equation $y' = y + x$, and the solution that goes though the point (x_0, y_0).

A **direction field** is a plot of the plane (x,y), where at some points it is shown a vector with slope equal to the value of $f(x,y)$ at that point. Hence, the solutions of the differential equation must be the curves which are tangent to those vectors. For exmaple, figure 3.1 shows the direction field of the equation $y' = y + x$, and one of the solutions of the equation.

The direction field (figure 3.1) makes it clear that there is an infinite number of solutions to the differential equation. Every curve that can be traced following the direction of the vectors will be a solution. However, if we fix an initial point $(x_0),y_0)$ on the plane, and if the function $f(x,y)$ has a unique value at every point on the (x,y) plane, there will be only one solution that goes throught that point, because at every point the direction in which the curve must follow is uniquely defined by the function $f(x,y)$.

The solution that goes through a given point (x_0, y_0) is called a **particular solution**. The specification of an given point is amounts to imposing a condition for the value y_0 that the solution must have at an initial value of the independent variable, x_0. Thus, an **initial value problem** consists in finding the particular solution to a differential equation with an initial condition given:

$$y' = f(x,y) \qquad y(x_0) = y_0 \tag{3.7}$$

To plot the direction field corresponding to equation 3.2 we can use Maxima's additional package `plotdf`. For instance, to plot the direction field for the differential equation $y' = y + x$ we use the following command:

```
(%i6) plotdf(y + x)$
```

A new window will be open with a grid of vectors showing the direction field. If you move the cursor over the field, the (x, y) coordinates of the point where the cursor is will be shown in the lower right corner of the window. If you click on a point in the field, the solution that goes through that point will be plotted.

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