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The Footloose Entrepreneur model with 3 regions

José Gaspar*, **Sofia B.S.D. Castro**† **and João Correia-da-Silva**‡

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Abstract: We study a 3-region version of the Footloose Entrepreneur model by Forslid and Ottaviano (J Econ Geogr, 2003). We focus on the analysis of stability of three types of long-run equilibria: agglomeration, dispersion and partial dispersion. We find that the 3-region model exhibits more tendency for agglomeration and less tendency for dispersion than the 2-region model. We show numerical evidence suggesting that equilibria with partial dispersion are always unstable. We also discuss the existence and robustness of bifurcations in the 3-region model.

Keywords: Core-Periphery, Footloose Entrepreneur, Three regions.

JEL Classification Numbers: R10, R12, R23.

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1 Introduction

The secular tendency for spatial agglomeration of economic activity is well known and has always been a matter of profound debate. Recent developments have allowed a more rigorous treatment of such phenomena, with recourse to microeconomic foundations.[1] The benchmark in this literature is the Core-Periphery (CP) model, introduced by Krugman (1991b).

An analytically solvable version of this model, dubbed the Footloose Entrepreneur (FE) model, was introduced by Forslid and Ottaviano (2003). The only difference with respect to the original CP model is that, in the FE model, the variable input in the mobile sector is immobile labor instead of mobile labor. The role of mobile (footloose) labor becomes limited to the fixed input (entrepreneurship) in the mobile sector. This subtle modification renders the model analytically solvable because the marginal production cost becomes exogenous.

Theoretical insights on a 3-region model would be interesting for different reasons. As pointed out by Fujita *et al.* (1999), considering only two regions stems from the attractiveness in dealing with manageable sized problems, although it seems implausible that the geographical dimension of economic activity can be reduced to a 2-region analysis. It is important, therefore, to understand to what extent the main conclusions that were obtained using 2-region models extend to models with more regions.

This motivated Castro *et al.* (2012) to study a 3-region version of the CP model by Krugman (1991b).[2] Comparing the behaviour of the 3-region model relatively to the 2-region model, their main conclusion was that the 3-region model favours the agglomeration of economic activity while the 2-region model favours the dispersion of economic activity.

The inherent technical difficulties that Castro *et al.* (2012) made evident in their analysis of the 3-region CP model call for a base model that is more tractable than the original CP model (Krugman, 1991b). This motivates us to consider (in this paper) a 3-region version of the analytically solvable Footloose Entrepreneur model by Forslid and Ottaviano (2003). Though we are able to obtain closed form solutions for expressions that are crucial for the study of the model, the added complexity that stems from the inclusion of one more region in the aforementioned framework thwarts our efforts to fully assess analytically the dynamic

[1] See, for example, Fujita *et al.* (1999) and Baldwin et al. (2003).

[2] It is also worthwhile mentioning the core-periphery model with n equidistant regions by Puga (1999).

properties of the model.

Our main finding is that, comparing our 3-region model with the original 2-region model by Forslid and Ottaviano, agglomeration is more likely in a 3-region framework while dispersion is more likely in a 2-region framework.[3] We also establish a sufficient condition whereby partial dispersion cannot be stable and present numerical evidence that (even without this condition) it is never stable, whatever the parameter values.

We conclude that the impact of considering an additional region in the FE model is analogous to that of considering an additional region in the CP model. In this sense, the FE model behaves similarly to the CP model (as desired by its creators). However, we also find that the difficulties faced in the analytical treatment of the 3-region CP model are still present in the 3-region FE model. This may be interpreted as a sign that a base model that is even more tractable than the FE model may be necessary to allow significant advances in the analytical study of core-periphery models with more than two regions.

The remainder of the paper is structured as follows. In Section 2, we underline the main assumptions of the FE model with 3 regions. In Section 3, we obtain the general expressions for nominal and real wages as explicit functions of the spatial distribution of the entrepreneurs. In Section 4, we address the dynamics of the model and find the stability conditions for three possible kinds of equilibria: concentration, total dispersion and partial dispersion. We also discuss how each of these outcomes becomes more or less likely as the parameters of the model change. In Section 5, we compare the behaviour of our 3-region model with that of the original 2-region FE model. In Section 6, we make some concluding remarks.

2 Economic environment

The economy is composed by three regions that are assumed to be structurally identical and equidistant from each other. The framework is exactly as that of the 2-region FE model by Forslid and Ottaviano (2003), except for the fact that 3 regions are considered.

The endowments of skilled labour (entrepreneurs) and unskilled labour are, respectively,

[3] More precisely, we conclude that: (i) the set of parameter values for which agglomeration is stable in the 2-region model is a subset of the set of parameters for which it is stable in the 3-region model; and (ii) the set of parameter values for which dispersion is stable in the 3-region model is a subset of the set of parameters for which it is stable in the 2-region model.

H and L. The entrepreneurs can move freely between regions ($H_1 + H_2 + H_3 = H$), while the unskilled workers are immobile and considered to be evenly spread across the three regions ($L_1 = L_2 = L_3 = \frac{L}{3}$).[4]

The representative consumer of region i has the usual utility function:

$$U_i = X_i^{\mu} A_i^{1-\mu}, \tag{1}$$

where A_i is the consumption of agricultural products in region i and X_i is the consumption of a composite of differentiated varieties of manufactures in region i, defined by:

$$X_i = \left[\int_{s \epsilon N} d_i(s)^{\frac{\sigma-1}{\sigma}} ds \right]^{\frac{\sigma}{\sigma-1}}, \tag{2}$$

where $d_i(s)$ is consumption of variety s of manufactures in region i, N is the set of existing varieties, and $\sigma > 1$ is the constant elasticity of substitution between manufactured varieties. From utility maximization, $\mu \in (0, 1)$ is the share of expenditure in manufactured goods.

Production of a variety of manufactures requires, as inputs, α units of skilled labor and β units of unskilled labor for each unit that is produced. Therefore, the production cost of a firm in region i is:

$$C_i(x_i) = w_i \alpha + w_i^L \beta x_i, \tag{3}$$

where w_i is the nominal wage of skilled workers in region i and w_i^L is the nominal wage of unskilled workers in region i.

Trade of manufactures between two regions is subject to iceberg costs $\tau \in (1, +\infty)$. Let τ_{ij} denote the number of units that must be shipped at region i for each unit that is delivered at region j. Since the three regions are equidistant from each other, we have the following trade cost structure:

$$\begin{cases} \tau_{ij} = 1, & \text{if } j = i \\ \tau_{ij} = \tau, & \text{if } j \neq i. \end{cases}$$

[4] Notice that, although similar, the 2-region model cannot be derived from the present one by eliminating skilled workers from a single region. It would also be necessary to redistribute the unksilled workers. Hence, the 3-region model does not contain the 2-region model of Forslid and Ottaviano (2003).

The agricultural good is produced using one unit of unskilled labor for each unit that is produced (constant returns to scale), and is freely traded across the three regions.

3 Short-run equilibrium

Let $p_{ji}(s)$ and $d_{ji}(s)$ denote the price and demand in region i of a variety, s, that is produced in region j. Utility maximization by consumers in region i yields the following aggregate regional demand:

$$d_{ji}(s) = \frac{p_{ji}(s)^{-\sigma}}{P_i^{1-\sigma}}\mu Y_i, \tag{4}$$

where P_i is the regional price index, associated with (2):

$$P_i = \left[\sum_{j=1}^{3}\int_{s\epsilon N} p_{ji}(s)^{1-\sigma} ds\right]^{\frac{1}{1-\sigma}}, \tag{5}$$

and Y_i is the regional income:

$$Y_i = w_i H_i + w_i^L \frac{L}{3}, \tag{6}$$

Turning to the supply side and starting with the agricultural sector, absence of transport costs implies that its price is the same everywhere ($p_1^A = p_2^A = p_3^A$). Furthermore, under perfect competition, we have marginal cost pricing: $p_i^A = w_i^L$. Consequently, there is unskilled workers' wage equalization among regions: $w_1^L = w_2^L = w_3^L$. By choosing the agricultural good as numeraire, we can set $p_i^A = w_i^L = 1$, $\forall i$. We assume that the non-full-specialization (NFS) condition (Baldwin *et al.*, 2003) holds, guaranteeing that agriculture is active in the three regions.[5]

In the industrial sector, given the fixed cost α in (3), the number of varieties manufactured in region i is $n_i = H_i/\alpha$. A manufacturing firm in region i facing the total cost in (3) maximizes the following profit function:

$$\pi_i(s) = \sum_{j=1}^{3} p_{ij}(s) d_{ij}(s) - \beta\left[\sum_{j=1}^{3}\tau_{ij} d_{ij}(s)\right] - \alpha w_i. \tag{7}$$

[5] This condition requires world expenditure on agricultural goods to be greater than the total production of agricultural goods in two regions, i.e., $(1-\mu)(Y_1+Y_2+Y_3) > \frac{2}{3}L$. This is guaranteed if we assume that $\mu < \sigma/(3\sigma-2)$. In the 2-region model by Forslid and Ottaviano (2003), the corresponding assumption is $\mu < \sigma/(2\sigma-1)$.

Total supply to region $j \neq i$ is equal to $\tau d_{ij}(s)$, because it also includes the fraction of the product that melts. The first order condition for maximization of (7) yields the same pricing equation as that of the 2-region model by Forslid and Ottaviano:

$$p_{ij}(s) = \tau_{ij}\beta\frac{\sigma}{1-\sigma}. \tag{8}$$

Note that $p_{ij}(s)$ is independent of s, which implies that $d_{ij}(s)$ also is. All varieties produced in region i are sold at the same price and are equally demanded in region j.

Using (8), the CES price index (5) becomes:

$$P_i = \alpha^{\frac{1}{\sigma-1}}\beta\frac{\sigma}{1-\sigma}\left(\sum_{j=1}^{3}\phi_{ij}H_i\right)^{\frac{1}{1-\sigma}}, \tag{9}$$

where $\phi_{ij} \equiv \tau_{ij}^{1-\sigma} \in (0,1)$ represents the "freeness of trade" between regions i and j.

Absence of entry barriers in the manufacturing industry translates into zero profits in equilibrium. Operating profits must totally compensate fixed costs, which are equal to the wages paid to the entrepreneurs:

$$\alpha w_i = \sum_{j=1}^{3} p_{ij}d_{ij} - \beta\left[\sum_{j=1}^{3}\tau_{ij}d_{ij}\right],$$

which becomes, considering the prices in (8):

$$w_i = \frac{\beta x_i}{\alpha(\sigma-1)}, \tag{10}$$

where $x_i \equiv \sum_j \tau_{ij}d_{ij}$ is total production by a manufacturing firm in region i.

Using (4), (8) and (9), we can derive an expression for x_i that depends on regional incomes and the number of firms in the three regions:

$$x_i = \frac{\mu(\sigma-1)}{\alpha\beta\sigma}\sum_{j=1}^{3}\frac{\phi_{ij}Y_i}{\sum_{m=1}^{3}\phi_{mj}n_m}. \tag{11}$$

Replacing (11) in (10) and knowing that $n_i = H_i/\alpha$ we have:

$$w_i = \frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{\phi_{ij}Y_j}{R_j}, \tag{12}$$

where $R_j \equiv \sum_{m=1}^{3} \phi_{mj} H_m$. By (6), income equals:

$$Y_i = \frac{L}{3} + w_i H_i. \tag{13}$$

The system of three linear equations in w_i, for $i = 1, 2, 3$, using (12) and (13), can be solved to obtain the equilibrium (nominal) wages of the skilled workers as a function of their spatial distribution.

Proposition 1. *The nominal wages of skilled workers in region i are given by:*

$$w_i = \frac{\frac{\mu}{\sigma}\frac{L}{3}\left\{\sum_{j=1}^{3}\frac{\phi_{ij}}{R_j} + \frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{\sum_{k\neq i} H_k}{\prod_{k\neq i} R_k} + \frac{\phi^2-1}{R_i}\sum_{k\neq i}\frac{H_k}{R_k}\right] + \frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{R_i}\prod_{k\neq i}\frac{H_k}{R_k}\right\}}{1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j} + \frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right) - \frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j}}. \tag{14}$$

Proof. See Appendix A. □

The regional distribution of skilled workers can be described, in relative terms, by a pair (h_i, h_j), where $h_i \equiv H_i/H$ and $h_j \equiv H_j/H$. The fraction of skilled workers in the third region is $h_k \equiv H_k/H = 1 - h_i - h_j$.

We can, then, rewrite the nominal wage w_i as function of h_i and h_j:

$$w_i\left(h_i, h_j\right) = \frac{Dw_i\left(h_i, h_j\right)}{D\left(h_i, h_j\right)},$$

where:

$$\begin{aligned} Dw_i &= \frac{\mu L}{3H\sigma}\left\{\sum_{m=1}^{3}\frac{\phi_{im}}{r_m} + \frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{h_j+h_k}{r_j r_k} + \frac{\phi^2-1}{r_i}\left(\frac{h_j}{r_j}+\frac{h_k}{r_k}\right)\right] + \right.\\ &\left. +\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{h_j h_k}{r_i r_j r_k}\right\}, \end{aligned} \tag{15}$$

$$D = 1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{h_j}{r_j} + \frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{h_1h_2}{r_1r_2}+\frac{h_1h_3}{r_1r_3}+\frac{h_2h_3}{r_2r_3}\right) - \frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{h_j}{r_j}, \tag{16}$$

with $r_i \equiv \frac{R_i}{H}$ and $r_k \equiv \frac{R_k}{H} = 1 + (\phi - 1)(h_i + h_j)$.

The price index P_i becomes, after (8):

$$P_i(h_i, h_j) = \beta\frac{\sigma}{\sigma-1}\left(\frac{H}{\alpha}r_i\right)^{\frac{1}{1-\sigma}}. \tag{17}$$

4 Long-run equilibria and stability

Entrepreneurs migrate to the region that offers them the highest indirect utility. For simplicity, we consider that the flow of entrepreneurs to a region is proportional to the difference between the region's real wage and the weighted average real wage in the three regions, and that it is also proportional to the stock of skilled workers in the region. In this dynamical system, migration to empty regions has to be started exogenously.

Given that $h_1 + h_2 + h_3 = 1$, the dynamics is constrained to the 2-simplex and can be described by the following system of two ordinary differential equations:[6]

$$\begin{cases} \dot{h}_i = (\omega_i - \bar{\omega})\, h_i \\ \dot{h}_j = (\omega_j - \bar{\omega})\, h_j \end{cases}, \quad h_i, h_j \in [0,1]\,, \quad i \neq j, \quad i, j \in \{1,2,3\}, \tag{18}$$

where $\omega_i = w_i/P_i^{\mu}$ stands for the real wage in region i and $\bar{\omega}(h_1, h_2) = h_1\omega_1 + h_2\omega_2 + h_3\omega_3$ is the weighted average of real wages in the three regions. The dynamics for the remaining region can be obtained from $\dot{h}_k = -\dot{h}_i - \dot{h}_j$ or, equivalently, from $\dot{h}_k = (\omega_k - \bar{\omega})\, h_k$. Without loss of generality, we set $i = 1$ and $j = 2$ when using coordinates in the 2-simplex.

On the boundary of the 2-simplex at least one of the regions is empty. The corresponding differential equation for the dynamics is $\dot{h}_i = 0$ which, given the initial condition $h_i = 0$, means that the region remains empty. Hence, the boundary of the 2-simplex is invariant for the dynamics.

Direct substitution in equations (18) shows that the configurations:

$$(h_1, h_2, h_3) = (1, 0, 0)\,, \left(\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}\right), \left(0, \tfrac{1}{2}, \tfrac{1}{2}\right)$$

and their permutations are equilibria. The equilibria represented by $(1, 0, 0)$ and its permutations correspond to full agglomeration of industry in one of the regions while the other two remain empty. We call this outcome *concentration* or *agglomeration*. The second configuration describes an even distribution of industry among the three regions. This we call *total dispersion*. The last configuration represents an even distribution of industry between only two of the three regions, while the third remains empty of industry. This is called *partial dispersion*.

[6] Note that, even though the model can be described using just any two coordinates, we sometimes use all three coordinates to convey the full picture.

The description of the dynamics relies on the study of the stability of the aforementioned equilibria. The equilibrium corresponding to total dispersion is fully symmetric, while the other two are partially symmetric.[7] The stability of each equilibrium is preserved by permutation so that the same stability conditions hold for concentration or partial dispersion in any of the regions. Equilibria are stable if, due to occurrence of some marginal exogenous migration of skilled workers to any of the regions, the spatial distribution of skilled workers is pulled back to the initial one.[8]

4.1 Stability of total dispersion

Since total dispersion is an interior configuration, its stability is given by the sign of the real part of the eigenvalues of the Jacobian matrix of the system in (18) at $(h_1, h_2, h_3) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

This matrix has a double real eigenvalue (see Appendix B), given by:

$$\alpha \equiv \frac{\partial \omega_1}{\partial h_1} \left(\tfrac{1}{3}, \tfrac{1}{3}\right), \tag{19}$$

and full dispersion is stable if it is negative. When this occurs, we may write

$$\frac{\partial w_1 / \partial h_1}{w_1} \quad < \quad \frac{\partial P_1^{\mu} / \partial h_1}{P_1^{\mu}},$$

and describe the stability of dispersion in terms of semi-elasticities.

Skilled workers remain equally dispersed across the three regions if a migration of skilled workers to a region induces a percentage change in the nominal wage smaller than the corresponding percentage change in the real prices. In this case, the loss in purchasing power due to an increase of the share of skilled workers, h_i, leads to an exodus of some skilled workers from that region until the initial share of skilled workers is restored, that is, until $h_i = \frac{1}{3}$.

[7] We can permute either the populated or unpopulated regions but not all of the three regions.

[8] For configurations in the interior of the 2-simplex, stability depends on the sign of the real part of the eigenvalues of the jacobian matrix for the dynamics. For configurations on the boundary, such as concentration and partial dispersion, we need to look at the difference between the real wage of the populated regions and the weighted average real wage in the three regions.

Proposition 2. *Total dispersion is a stable configuration if:*[9]

$$\phi < \phi_b \equiv \frac{(\sigma - \mu)(\sigma - \mu - 1)}{\mu^2 + \sigma(\sigma - 1) + 2\mu(2\sigma - 1)}. \tag{20}$$

Note that $\phi_b < 1$.

The proof is straightforward given that the eigenvalue of the Jacobian at full dispersion is given by:

$$BP(\phi) \equiv \mu^2(\phi - 1) + (\sigma - 1)\sigma(\phi - 1) + \mu(-1 + 2\sigma)(1 + 2\phi), \tag{21}$$

which is a linear function of ϕ with positive coefficient.

As in the 2-region model, low transportation costs (high ϕ), discourage dispersion. Following Fujita *et al.* (1999), we call the critical value ϕ_b such that $BP(\phi_b) = 0$ the *break point*.

Note that if ϕ_b is negative, total dispersion is never a stable outcome. We rule out this possibility throughout the paper by assuming that $\sigma > \mu + 1$ (*no black-hole* condition).[10] With $\phi_b > 0$, there always exists a level of transportation costs above which dispersion is stable (for any given values of μ and σ). On the other hand, the fact that $\phi_b < 1$ means that there always exists a level of transportation costs below which dispersion is unstable.

Observing that the derivative of BP with respect to μ is positive, we conclude that a higher fraction of spending on manufacturing discourages total dispersion. In the extreme case in which μ tends to zero, ϕ_b approaches unity. Therefore, dispersion becomes stable.

The effect of σ can be understood by noting that when σ tends to infinity, ϕ_b approaches unity. This means that if the preference for variety is sufficiently low (high σ), dispersion is stable.

[9] For a necessary condition, replace "$<$" with "$\leq$".

[10] The underlying economic interpretation is that if skilled workers have a very strong preference for variety ($\sigma < 1 + \mu$), agglomeration in a single region is always a stable equilibrium, independently of the magnitude of transportation costs.

4.2 Stability of agglomeration

The equilibrium of agglomeration $(h_1, h_2, h_3) = (1, 0, 0)$ is a corner solution placed on a vertex of the 2-simplex. At this equilibrium, by symmetry, the real wages in regions 2 and 3 are the same. Given the absence of population in the other regions, the weighted average real wage is simply $\bar{\omega} = h_1\omega_1$.

Lemma 3. *Full agglomeration is a stable configuration if:*[11]

$$\omega_j < \omega_i,$$

for all j such that $h_j = 0$ and with $h_i = 1$.

Proof. Without loss of generality, assume $h_2 = h_3 = 0$. Then $\bar{\omega} = \omega_1$, that is, the weighted average real wage is the same as the real wage in region 1. That $\omega_j < \omega_i$ is sufficient follows from the fact that skilled workers migrate to regions with higher real wages, together with continuity of real wages with respect to spatial distribution. Thus, if an empty region, j, has a lower real wage than region i, an exogenous migration of skilled workers from region i to region j will be followed by their return to region i. □

If regions 2 and 3 are to remain empty over time, then there can be no incentives for skilled workers to migrate to other regions. That is, skilled workers' indirect utility must be higher in region 1 compared to the other regions:

$$\frac{w_1}{P_1^{\mu}} > \frac{w_2}{P_2^{\mu}} \Leftrightarrow \frac{w_1}{w_2} > \left(\frac{P_1}{P_2}\right)^{\mu} \Leftrightarrow \frac{w_1}{w_2} > \tau^{-\mu} \Leftrightarrow \frac{w_1}{w_2} > \phi^{\frac{\mu}{\sigma-1}}.$$

Note that $\tau^{-\mu} < 1$ so that an easily checked sufficient condition for the stability of concentration is that the nominal wages be higher in the industrial region. Naive inspection of the above inequality also shows that as transport costs decrease ($\tau \to 1$) the sufficient condition $w_1 > w_2$ becomes necessary, enhancing the stability of concentration.

Proposition 4. *Full agglomeration is a stable equilibrium if:*[12]

$$SP(\phi) \equiv \sigma - \mu + (\sigma - \mu)\phi + (\sigma + 2\mu)\phi^2 - 3\sigma\phi^{1-\frac{\mu}{\sigma-1}} < 0. \tag{22}$$

[11] For a necessary condition, replace "$<$" with "$\leq$".

[12] For a necessary condition, replace "$<$" with "$\leq$".

Observe that, since $SP(\phi)$ is a convex function and we have $SP(0) > 0$, $SP(1) = 0$ and $SP'(1) > 0$, the function $SP(\phi)$ has exactly one zero in $\phi \in (0,1)$. This value, denoted ϕ_s, is the 3-region FE model's *sustain point* (name used by Fujita *et al.* (1999)), i.e., it is the threshold of ϕ above which concentration is a stable equilibrium. Strong non-linearity of $SP(\phi)$ means that we are not able to derive an analytical expression for the sustain point ϕ_s. It is expected that, for a high enough "freeness of trade", or, conversely, for low enough transport costs, full agglomeration of industry in one of the regions will be stable. This is because low transport costs imply that price indices become relatively higher in the regions that are deserted and thus real wages become relatively lower. One can also verify that as σ approaches infinity or as μ approaches zero, SP becomes positive.[13] This means that $\phi_s \to 1$. For a given $\phi \in (0,1)$, concentration becomes unstable for a sufficiently high σ or a sufficiently low μ. One limit case ($\mu \to 0$) refers to a situation of absence of the manufacturing sector, because μ is the fraction of expenditure on manufactures. The other ($\sigma \to +\infty$) corresponds to the manufacturing sector operating under perfect competition, since σ close to infinity means that variety in good X is not valued at all, so it is as if X were a homogenous good.

4.3 Stability of partial dispersion

Now we address the stability of a configuration in which all skilled workers are equally dispersed between two of the three regions, leaving the remaining region empty. Without loss of generality, we consider the point $(h_1, h_2, h_3) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Proposition 5. *Partial dispersion is unstable if:*[14]

$$\begin{cases} \xi \equiv \omega_1\left(0, \frac{1}{2}\right) - \omega_2\left(0, \frac{1}{2}\right) & > 0 \\ \beta \equiv \dfrac{\partial \omega_2}{\partial h_2}\left(0, \frac{1}{2}\right) & > 0. \end{cases}$$

Proof. See Appendix B. □

From Proposition 5, we establish the conditions for stability of partial dispersion.

[13] We have $\lim_{\mu \to 0} SP(\phi) = \sigma(1 - 2\phi + \phi^2) > 0$ and $\lim_{\sigma \to \infty} SP(\phi) = \lim_{\sigma \to \infty} \sigma(1 - 2\phi + \phi^2) > 0 = +\infty$.

[14] For necessary conditions, replace ">" with "≥".

Proposition 6. *Partial dispersion is unstable if:*[15]

$$\begin{cases} \xi > 0 \equiv \sigma + \mu(-1+\phi)(1+2\phi) + \sigma\phi \left[1 + 4\phi - 3\left(2\frac{\phi}{1+\phi}\right)^{\frac{\mu}{1-\sigma}}(1+\phi)\right] & > 0 \\ \beta > 0 \equiv 3\mu^2(-1+\phi) + 2(-1+\sigma)\sigma(-1+\phi) + \mu(-2-4\phi+\sigma(5+7\phi)) & > 0. \end{cases} \quad (23)$$

Consider the condition equivalent to $\beta < 0$. If we let ϕ_p be the single zero of the function $\beta(\phi)$[16], then partial dispersion will be unstable if $\phi_p < \phi < 1$, where:

$$\phi_p = \frac{(2\sigma - 2 - 3\mu)(\sigma - \mu)}{3\mu^2 + 2\sigma(\sigma - 1) + \mu(7\sigma - 4)}.$$

The previous Proposition gives us a sufficient condition for the instability of partial dispersion. Further inspection also shows that if $1+\mu < \sigma \leq 1 + \frac{3}{2}\mu$, we have $\beta(\phi) > 0$, rendering partial dispersion unstable. However, we suspect that this configuration is unstable for all parameter values. Numerical inspection of both conditions in (23) suggests that these are never simultaneously met. We can argue that partial stability will certainly never be a stable outcome if $\xi(\phi_p) > 0$. However, because of the non-linearity in ϕ, this may be impossible to prove analytically. Instead, we summarize several numerical results by means of a graphical representation of the regions in (23) in parameter space (Figure 1).

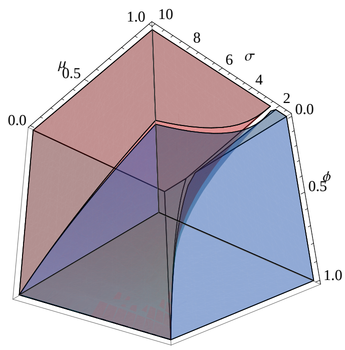

Fig. 1: Depiction of the surfaces $\xi < 0$ and $\beta < 0$ in parameter space. They seem to never overlap each other, indicating that the eigenvalues are never simultaneously negative.

[15] For necessary conditions, replace ">" with "≥".

[16] The function $\beta(\phi)$ is linear in ϕ.

4.4 Bifurcations in the 3-region FE model

One important feature of symmetric 2-region CP models is the existence of a subcritical pitchfork bifurcation.[17] This kind of bifurcation is also present in the 3-region FE model.

There is an open subset in parameter space (ϕ, σ, μ) in which both concentration and total dispersion are stable outcomes. For example, consider the point in parameter space $(\phi, \sigma, \mu) = \left(\frac{3}{5}, 5, \frac{2}{5}\right)$.[18] At this point, we have $SP\left(\frac{3}{5}, 5, \frac{2}{5}\right) < 0$ and $BP\left(\frac{3}{5}, 5, \frac{2}{5}\right) < 0$. Therefore, for $(\phi, \sigma, \mu) = \left(\frac{3}{5}, 5, \frac{2}{5}\right)$, both concentration and total dispersion are stable equilibria. Since SP and BP are continuous functions of (ϕ, σ, μ), we know the signs persist in a open neighbourhood of $\left(\frac{3}{5}, 5, \frac{2}{5}\right)$.

Numerical inspection of the conditions of agglomeration and total dispersion in (21) and (22) suggests that, for every pair (μ, σ), there always exists a $\phi \in (0, 1)$ for which both total dispersion and agglomeration are stable equilibria. If this is true, then it must hold that $\phi_s < \phi_b$, because agglomeration is only stable for $\phi > \phi_s$, while the region for stability of total dispersion implies $\phi < \phi_b$. If it does hold, then we have hysteresis in location, because transport costs have to rise above the corresponding break point in order for total dispersion to be unstable, even if agglomeration is already stable (see Forslid and Ottaviano, 2003). If both equilibria are to be simultaneously stable, we must have $SP(\phi_b) < 0$, $\forall \mu, \sigma$. Again, this seems to be the case, however, nonlinearity of $SP(\phi)$ makes it impossible to fully assess this. Figure 2 suggests that it is always possible to find a value of ϕ, for any pair of μ and σ, such that both agglomeration and total dispersion are stable.

However, the region between the surfaces is very thin, and becomes thinner for a high σ and/or low μ, thus making it harder to visualize simultaneity of concentration and total dispersion. It also appears that the distance between ϕ_s and ϕ_b is bigger for parameter values near the *no black-hole* condition.

Instead of building a three-dimensional bifurcation diagram, we adopt a similar approach to Fujita *et al.* (1999, chap. 6), where the dynamics of an extension of the CP model to three regions are portrayed inside the 2-simplex.

For the simulations that are presented, we have set $\sigma = 5$ and $\mu = 0.4$. Figure 3 depicts the migration dynamics for three different values of ϕ. On the picture to the left we have

[17] This is also known as *tomahawk* bifurcation, though the latter designation might be more suitable to name the corresponding bifurcation diagram, rather than the bifurcation itself.

[18] This point was chosen by numerical inspection.

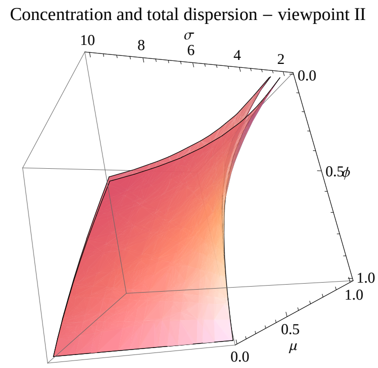

Fig. 2: We have the surfaces $SP = 0$ and $BP = 0$ in the parameter space (resp. top and bottom surfaces). Concentration and total dispersion are both stable in between both surfaces, where we have $SP < 0 \cap BP < 0$.

high transport costs ($\phi = 0.5$) and we can see that total dispersion is the only stable equilibrium, as the vector field exhibits convergence to the middle point of the simplex. The picture in the middle corresponds to a moderate level of transport costs ($\phi = 0.6$), whereby both concentration and total dispersion are stable equilibria. One can see that the decrease in the transport costs gave rise to three new unstable equilibria between total dispersion and the concentration configurations. These were not observed in our analysis as the expressions are too complicated. Finally, when transport costs are low enough ($\phi = 0.9$), total dispersion is no longer stable and the only possible outcome is that of full agglomeration of skilled workers in one of the three regions. This latter case is shown in the picture to the right, where the convergence to either vertices of the simplex is obvious.

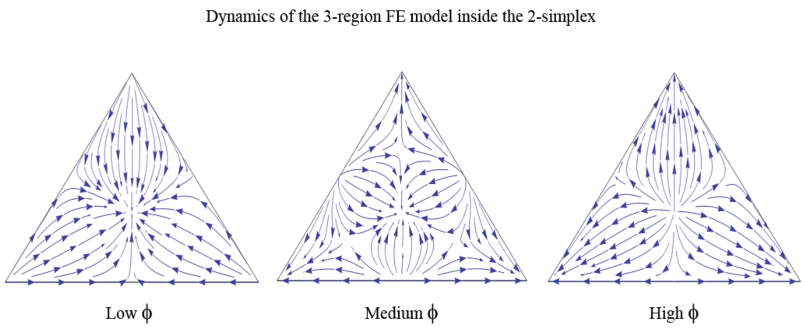

Fig. 3: The dynamics of the FE 3-region model. The pictures from the left to the right depict the change in the stability of equilibria as transport costs fall.

Throughout the three cases described in Figure 3, partial dispersion is unsurprisingly unstable. All these results corroborate those in Fujita *et al.* (1999, chap. 6).

The parametrization used to portray the dynamics is a strategic choice to facilitate the explanation of bifurcations in the 3-region FE model. However, given the numerical evidence presented, it seems plausible that the dynamics implicit in the model will always undergo a "subcritical pitchfork" bifurcation. This excludes, of course, the limit cases of σ tending to infinity and μ, which rule out this possibility.

5 Comparison between the 3-region with the 2-region FE model

A reason to build a 3-region model in the first place is to be able to compare it with the 2-region model in order to draw conclusions on the implications of having more than two regions in Core-Periphery theory. In Castro *et al.* (2012), it is proven that, in an extension of Krugman's CP model to three regions, more regions favour concentration as an outcome. Here, we obtain the following analogous result.

Proposition 7. *The parameter region for which concentration is stable in the* 3*-region FE model contains that of the* 2*-region FE model.*

Proof. See Appendix C. □

In other words, the 3-region model favours concentration over the 2-region model. Figure 4 illustrates the previous proposition.

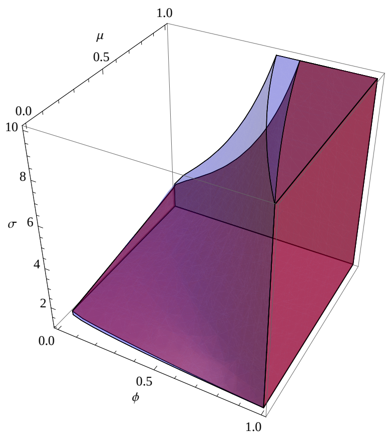

Fig. 4: The red surface illustrates the region in parameter space where $SP_2 < 0$, while the blue surface corresponds to $SP_3 < 0$. Clearly, the blue surface contains the red one, hence concentration in the 3-region model is less restrictive than in the 2-region model.

The following result provides an explanation for why this happens.

Proposition 8. *Comparing agglomeration in the 3-region model with agglomeration in the 2-region model, we find that: (i) the ratio between price indices in the core and in the periphery are the same; (ii) nominal wages in the core are the same; (iii) nominal wages in the periphery are lower in the 3-region model than in the 2-region model.*

Proof. See Appendix C. □

In the 3-region model, an entrepreneur who migrates to the periphery will find the same cost-of-living as in the 2-region model, but an internal market composed by only one third of the unskilled workers (instead of one half). All the other entrepreneurs and the other two thirds of the unskilled workers (instead of the other half) would constitute the external market. Given the existence of transportation costs to the other regions, this enterpreneur will face a lower global demand than in the 2-region model, and, therefore, will have a lower nominal wage than in the 2-region model. This originates the fact that agglomeration is more likely in the 3-region model.

Besides comparing the stability conditions of full agglomeration outcomes in the 2 and 3-region models, we are also able to compare the stability conditions of total dispersion outcomes.

Proposition 9. *The parameter region for which total dispersion is stable in the* 2*-region model contains that of the* 3*-region model.*

Proof. See Appendix C. □

We conclude that the 3-region model indeed favours agglomeration over dispersion when compared to the 2-region model by Forslid and Ottaviano (2003). Of course, total dispersion is always a stable outcome in both models when we are either approaching an economy absent of industry or consumers give almost no value to variety in good X. This fits well with intuition.

6 Conclusion

Building on the 2-region FE model by Forslid and Ottaviano (2003), we have obtained both analytical and numerical results from a FE model with three regions. These results corroborate those already obtained in previous works on 3-region Core-Periphery models.

We have shown that the 3-region FE model favours concentration in comparison with the 2-region one. Furthermore, we have proved analytically that both concentration and full dispersion can be simultaneously stable and provided numerical evidence in that, for every pair (μ, σ), there exists $\phi \in (0, 1)$ where this is possible, though this outcome is very unlikely. This means that, like the 2-region model, the 3-region FE model exhibits a core-periphery pattern based on a "subcritical pitchfork" bifurcation. We have also concluded numerically that the dispersion of skilled workers among two regions is not sustainable in a model with three regions, where it corresponds to an outcome of partial dispersion.

All of these results are tantamount to those in the CP model with three regions in Castro *et al.* (2012), the difference being that, additionally, we were able to obtain explicit solutions for skilled wages and obtain relations between the relevant endogenous variables and the spatial distribution of skilled workers. Additionally, we proved that, when the manufacturing sector becomes irrelevant or it approaches perfect competition, migration decisions of skilled workers are the same in both the 2-region and the 3-region FE models. This, as was previously shown, occurs because there is convergence to zero of the critical values of the transport costs, where the stability of agglomeration and full dispersion changes. However, if this happens, concentration can never be a stable outcome, insofar as transport costs cannot fall below zero, while full dispersion, on the contrary, will always be a stable equilibrium.

Although the FE model is able to give us closed form solutions, the assumptions it makes still do not allow enough simplification to fully assess analytically the dynamic properties when its analysis is applied to three regions. Nonlinearity in transport costs concerning its stability conditions still makes it impossible to analytically exclude the possibility that skilled workers might equally disperse across two regions when there are three regions available to migrate. Whereas the 2-region FE model is useful to address issues beyond the explanation capability of the original CP model by Krugman (1991b), doubts remain about whether it is suitable to tackle the more complex case of n regions, though such an analysis would certainly be of interest.

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Appendix A

Proof of Proposition 1

After (12) and (13), we have the following linear system of equations:

$$\begin{cases} w_1\left(1-\frac{\mu}{\sigma}\frac{H_1}{R_1}\right)-w_2\left(\frac{\mu}{\sigma}\frac{\phi H_2}{R_2}\right)-w_3\left(\frac{\mu}{\sigma}\frac{\phi H_3}{R_3}\right) & =\frac{\mu}{\sigma}\frac{L}{3}\left(\frac{1}{R_1}+\frac{\phi}{R_2}+\frac{\phi}{R_3}\right) \\ w_1\left(-\frac{\mu}{\sigma}\frac{\phi H_1}{R_1}\right)+w_2\left(1-\frac{\mu}{\sigma}\frac{H_2}{R_2}\right)-w_3\left(\frac{\mu}{\sigma}\frac{\phi H_3}{R_3}\right) & =\frac{\mu}{\sigma}\frac{L}{3}\left(\frac{\phi}{R_1}+\frac{1}{R_2}+\frac{\phi}{R_3}\right) \\ w_1\left(-\frac{\mu}{\sigma}\frac{\phi H_1}{R_1}\right)-w_2\left(\frac{\mu}{\sigma}\frac{\phi H_2}{R_2}\right)+w_3\left(1-\frac{\mu}{\sigma}\frac{\phi H_3}{R_3}\right) & =\frac{\mu}{\sigma}\frac{L}{3}\left(\frac{\phi}{R_1}+\frac{\phi}{R_2}+\frac{1}{R_3}\right). \end{cases}$$

This may be written in matrix form as $AW = B$, where A stands for the coefficients matrix, W the vector of nominal wages w_i, while B is the column vector of independent terms in the right-hand side of the system of equations above. Applying Cramer's Rule, the solution to this system is of the following form:

$$w_i = \frac{Dw_i}{D},$$

where the denominator D stands for the determinant of matrix A and Dw_i is the determinant of the matrix obtained by replacing the i-th column of A by the column vector B. Using this method, we only need to solve for a specific nominal wage, e.g., w_1, and easily deduce the remaining solutions applying an argument of symmetry. Finding an expression for D first, we have:

$$D \quad = 1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j}+\frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right)-\frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j},$$

which is invariant under any distribution of skilled workers across the regions, since it is a common denominator for every solution of the nominal wage w_i.

The numerator Dw_1, becomes:

$$\begin{aligned} Dw_1 \quad &= \frac{\mu}{\sigma}\frac{L}{3}\left\{\left(\sum_{j=1}^{3}\frac{\phi_{1j}}{R_j}\right)+\frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{H_2+H_3}{R_2R_3}+\frac{\phi^2-1}{R_1}\left(\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)\right]+\right. \\ &\left.+\frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{H_2H_3}{R_1R_2R_3}\right\} \end{aligned}$$

The expression for the nominal wage in region 1 is:

$$w_1 = \frac{\frac{\mu}{\sigma}\frac{L}{3}\left\{\sum_{j=1}^{3}\frac{\phi_{1j}}{R_j} + \frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{H_2+H_3}{R_2R_3} + \frac{\phi^2-1}{R_1}\left(\frac{H_2}{R_2}+\frac{H_3}{R_3}\right)\right] + \frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{H_2H_3}{R_1R_2R_3}\right\}}{1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j} + \frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right) - \frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j}}.$$

Under a given distribution of H, $w_1(H_1, H_2, H_3) = w_1(H_1, H_3, H_2)$, which is a consequence of the existing symmetry in region 2 and region 3, since there is nothing to distinguish between the two regions. This means that the nominal wage in region i is invariant in the distribution of skilled workers in the other two regions. Symmetry among the regions also asserts that $w_1\left(H_1, H_2, H_3\right) = w_2\left(H_2, H_1, H_3\right) = w_3\left(H_3, H_1, H_2\right)$. Finally, the denominator is invariant in the three regions. Thus, we can easily formulate the following general expression for the nominal wages:

$$w_i = \frac{\frac{\mu}{\sigma}\frac{L}{3}\left\{\sum_{j=1}^{3}\frac{\phi_{ij}}{R_j} + \frac{\mu}{\sigma}\left[\phi\left(\phi-1\right)\frac{\sum_{k\neq i}H_k}{\prod_{k\neq i}R_k} + \frac{\phi^2-1}{R_i}\sum_{k\neq i}\frac{H_k}{R_k}\right] + \frac{\mu^2}{\sigma^2}\left(2\phi^3-3\phi^2+1\right)\frac{1}{R_i}\prod_{k\neq i}\frac{H_k}{R_k}\right\}}{1-\frac{\mu}{\sigma}\sum_{j=1}^{3}\frac{H_j}{R_j} + \frac{\mu^2}{\sigma^2}\left(1-\phi^2\right)\left(\frac{H_1H_2}{R_1R_2}+\frac{H_1H_3}{R_1R_3}+\frac{H_2H_3}{R_2R_3}\right) - \frac{\mu^3}{\sigma^3}\left(2\phi^3-3\phi^2+1\right)\prod_{j=1}^{3}\frac{H_j}{R_j}}.$$

□

Appendix B

As a prerequisite to Proposition 2, establishing the stability of total dispersion, we have the following Lemma.

Lemma 10. *At* $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ *we have* $\frac{\partial \omega_i}{\partial h_j} = 0, \forall i, j \neq i.$

Proof. Assume, by way of contradiction, that $\frac{\partial \omega_1}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right) > 0$. It must follow that, for a small $\varepsilon > 0$, $\omega_1\left(\frac{1}{3}, \frac{1}{3}+\varepsilon, \frac{1}{3}-\varepsilon\right) > \omega_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) > \omega_1\left(\frac{1}{3}, \frac{1}{3}-\varepsilon, \frac{1}{3}+\varepsilon\right)$.

However, the real wage in one region is invariant by the permutation of the share of skilled workers in the other two regions. Therefore, $\omega_1\left(\frac{1}{3}, \frac{1}{3}+\varepsilon, \frac{1}{3}-\varepsilon\right) = \omega_1\left(\frac{1}{3}, \frac{1}{3}-\varepsilon, \frac{1}{3}+\varepsilon\right)$.

But saying that $\omega_1\left(\frac{1}{3}, \frac{1}{3}-\varepsilon, \frac{1}{3}+\varepsilon\right) > \omega_1\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is a contradiction. Hence, $\frac{\partial \omega_1}{\partial h_2}\left(\frac{1}{3}, \frac{1}{3}\right) = 0$.

Symmetry establishes an analogous result for $\frac{\partial \omega_2}{\partial h_1}$. □

This is enough to establish that the Jacobian at total dispersion is a diagonal matrix. The values in the diagonal are equal because of the symmetry of the model.

The following result simplifies the calculation of the diagonal elements of the Jacobian.

Lemma 11. *The weighted real wage average $\bar{\omega}$ attains a critical value when skilled workers are equally dispersed across regions.*

Proof. By lemma 12, we can conclude that both $\frac{\partial \bar{\omega}}{\partial h_1}\left(\frac{1}{3},\frac{1}{3}\right)$ and $\frac{\partial \bar{\omega}}{\partial h_2}\left(\frac{1}{3},\frac{1}{3}\right)$ are zero, since $h_1 = \frac{1-h_2}{2}$ and also $h_2 = \frac{1-h_1}{2}$. Because both partial derivatives are equal to zero, it means that the real wage average is at a critical value at total dispersion. □

The following result concerns partial dispersion and is an auxiliary result for total dispersion.

Lemma 12. *Configurations of the form $h_j = \frac{1-h_i}{2}$, with $0 \leq h_i \leq 1$, satisfy $\frac{\partial \bar{\omega}}{\partial h_j} = 0$.*

Proof. We are looking at configurations of the form (b, a, a) where $a = (1-b)/2$ and $b \in [0, 1]$. Note that if $b = 1/3$ we have full dispersion. Assume, without loss of generality, that $i = 2$. Suppose $\dfrac{\partial \bar{\omega}}{\partial h_2}(b, a, a) \neq 0$. Assume it is positive. Then

$$\bar{\omega}(b, a+\varepsilon, a-\varepsilon) > \bar{\omega}(b, a, a).$$

But, $\bar{\omega}$ is invariant by the permutation that interchanges identically populated regions and therefore

$$\bar{\omega}(b, a-\varepsilon, a+\varepsilon) > \bar{\omega}(b, a, a)$$

indicating that $\dfrac{\partial \bar{\omega}}{\partial h_2} < 0$, which contradicts the assumption and finishes the proof. □

Proof of Proposition 5

First, since $h_1 = 0$, by similar arguments as those in the proof of Lemma 3, we have the following necessary condition:

$$\omega_1 < \omega_2.$$

Second, we need to ensure not only that h_1 will remain zero but also that both h_2 and h_3 will remain at $\frac{1}{2}$. If any of the skilled workers migrates, e.g. to region 2, we need them to want to return to region 3 (symmetry implies the same in the opposite direction). This is achieved when an increase in h_2 leads to a decrease in the difference between the real wage ω_2 and the real wage average $\bar{\omega}$:

$$\frac{\partial \Delta\omega_2}{\partial h_2} = \frac{\partial \omega_2}{\partial h_2}\left(0, \tfrac{1}{2}\right) - \frac{\partial \bar{\omega}}{\partial h_2}\left(0, \tfrac{1}{2}\right) < 0.$$

However, since we have $h_2 = \frac{1-h_1}{2}$ at $\left(0, \frac{1}{2}\right)$, proposition 12 asserts that $\frac{\partial \bar{\omega}}{\partial h_2}\left(0, \frac{1}{2}\right) = 0$. □

Appendix C

Proof of Proposition 7

In the 2-region model by Forslid and Ottaviano (2003), concentration is a stable outcome if:

$$1 - \frac{\mu}{\sigma} + \left(1 + \frac{\mu}{\sigma}\right)\phi^2 - 2\phi^{\frac{\mu}{1-\sigma}+1} \leq 0. \tag{24}$$

The difference between the left-hand sides (LHS) in (24) and (22) is given by:

$$\frac{\mu}{\sigma}\phi(1-\phi) + \phi\left(\phi^{\frac{\mu}{1-\sigma}} - 1\right), \tag{25}$$

which is always positive. Thus, if concentration is stable in the FE model with two regions, then it is also stable in the model with three regions. □

Proof of Proposition 8

In our 3-region model, when $h_1 = 1$:

$$\begin{aligned}
\frac{P_1^\mu}{P_2^\mu} &= \phi^{\frac{\mu}{\sigma-1}}, \\
w_1 &= \frac{\mu L}{(\sigma-\mu)H}, \\
w_2 = w_3 &= \frac{\mu L}{\sigma H}\left[\left(\frac{\mu}{\sigma-\mu} + \frac{1}{3}\right)\phi + \frac{1}{3} + \frac{1}{3\phi}\right].
\end{aligned}$$

While, in the 2-region model of Forslid and Ottaviano (2003), when $h_1 = 1$:

$$\frac{P_1^{\mu}}{P_2^{\mu}} = \phi^{\frac{\mu}{\sigma-1}},$$
$$w_1 = \frac{\mu L}{(\sigma-\mu)H},$$
$$w_2 = w_3 = \frac{\mu L}{\sigma H}\left[\left(\frac{\mu}{\sigma-\mu}+\frac{1}{2}\right)\phi + \frac{1}{2\phi}\right].$$

We conclude the proof by observing that $\frac{1}{2}\phi^2 + \frac{1}{2}\phi > \frac{1}{3}\phi^2 + \frac{1}{3}\phi + \frac{1}{3}$. □

Proof of Proposition 9

In the article by Forslid and Ottaviano (2003), the authors determined the break-point value $\phi_{b,2}$ for the model with two regions which is presented as follows:

$$\phi_{b,2} = \phi_w \frac{\sigma - 1 - \mu}{\sigma - 1 + \mu},$$

where

$$\phi_w \in (0,1) = \frac{\sigma-\mu}{\sigma+\mu}$$

is a threshold value of ϕ above (below) which the region with more skilled workers offers a higher (lower) skilled worker wage w_i. Subtracting the 3-region break-point in (20) to this one yields:

$$D_{BP} = \phi_{b,2} - \phi_b = \frac{\mu(\mu-\sigma)(1+\mu-\sigma)(-1+2\sigma)}{(-1+\mu+\sigma)(\mu+\sigma)\left(\mu^2+(-1+\sigma)\sigma+\mu(-2+4\sigma)\right)}.$$

The denominator is clearly positive and, provided that the "no-black-hole" condition holds, the numerator is also positive. Hence, $D_{BP} > 0$, $\forall \mu, \sigma$. This means that the critical value ϕ_b is lower in the 3-region model compared to the 2-region model. Therefore, dispersion is a more likely outcome in the model with two regions. □

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