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DYNAMIC CONSUMPTION AND PORTFOLIO CHOICE WITH AMBIGUITY ABOUT STOCHASTIC VOLATILITY

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Dynamic Consumption and Portfolio Choice with Ambiguity about Stochastic Volatility*

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Abstract

We introduce ambiguity about the variance of the risky asset's return in the model of Chacko and Viceira (2005) for dynamic consumption and portfolio choice with stochastic variance. We find that, with investors being able to update their portfolio continuously (as a function of the instantaneous variance), ambiguity has no impact. To shed some light on the case in which continuous portfolio updating is not possible, we also evaluate the effect of ambiguity when investors must use their expectation of future variance for their portfolio decision. In the latter scenario, demand for the risky asset can be decomposed into three components: myopic and intertemporal hedging demands (as in Chacko and Viceira (2005)) and ambiguity demand. Using long-run US data, Chacko and Viceira (2005) found that intertemporal hedging demand is empirically small, suggesting a low impact of stochastic variance on portfolio choice. Using the same calibration, we find that ambiguity demand may be very high, much more than intertemporal hedging demand. Therefore, stochastic variance can be very relevant for portfolio choice, not because of the variance risk, but because of investors' ambiguity about variance.

Keywords: Asset Allocation, Stochastic Volatility, Ambiguity.

JEL Classification: C61, D81, E21, G11.

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1 Introduction

We study optimal dynamic consumption and portfolio choice, in the presence of stochastic variance, of an investor that is averse both to risk and ambiguity. In our setting, the variance of the risky asset's return is simultaneously the source of the risk and the ambiguity that are perceived by the investor.

There is a large literature on portfolio choice (see Campbell and Viceira (2002) for a survey), but only a few works study the optimal dynamic portfolio choice with stochastic variance of the risky asset's return. Two exceptions are Chacko and Viceira (2005) and Liu (2007). In those two papers, potential adverse changes in the investment opportunity set are associated with the stochastic variance of the risky asset's return, which therefore represents a source of risk to investors. From Merton (1973), this implies that the investor's optimal intertemporal hedging demand is driven by the stochastic variance.[1] However, both in Chacko and Viceira (2005) and in Liu (2007), there is only risk, and no ambiguity.

Ambiguity is uncertainty that cannot be represented by a single probability distribution. Risk, on the contrary, is uncertainty that is susceptible of being described by a probability distribution. This conceptual distinction, first explored by Knight (1921), has relevant implications for the behavior of economic agents, and therefore for economic theory in general. Ellsberg (1961) disclosed experimental evidence supporting the Knightian distinction between risk and ambiguity. This evidence became known as the Ellsberg paradox, and motivated a large body of empirical studies, surveyed in Camerer and Weber (1992).

Notwithstanding this, the mainstream theory of choice under uncertainty in economics in the last 60 years ignored ambiguity, being based on the expected utility theories of von Neumann and Morgenstern (1944) and Savage (1954).[2] But, gradually, ambiguity is being incorporated in decision theory. Two major approaches are being used: (i) the multiple priors (MP) approach, associated with the ambiguity aversion concept, whereby the single probability measure of the standard expected utility model is replaced by a set of probabilities or priors; (ii) the robust control (RC) approach, more associated to the model mispecification concept.[3]

In studies of portfolio choice with ambiguity, Gollier (2006) and Garlappi et al. (2007) concluded that, by introducing ambiguity aversion in a static MP approach, the optimal demand for the risky asset decreases versus the standard mean-variance and Bayesian models.[4] The same conclusion was reached by Chen et al. (2009) in a dynamic MP setting, and by Maenhout (2004) in a dynamic RC model. Uppal and Wang (2003) studied the implications of ambiguity aversion in portfolio diversification. In all these works, the source of ambiguity is the expected risky asset's return or the model of the risky asset's return.[5]

[1] In Merton (1973), following Samuelson (1969) and Merton (1969, 1971), it is showed that when investors time-horizon exceeds one period, their optimal demand for the risky asset differs from that of "myopic" investors in one extra component: intertemporal hedging demand. This extra component is used to hedge investors against adverse changes in future investment opportunities.

[2] In the Expected Utility Theory of von Neumann and Morgenstern (1944), the probabilities of the possible states of nature are known, while in the Subjective Expected Utility Theory of Savage (1954), although probabilities are not necessarily known, the choice behavior of an agent coincides with maximization of expected utility according to some subjective probability beliefs.

[3] See Hansen and Sargent (2001a), Hansen et al. (2002) and Epstein and Schneider (2003) for a discussion about the relationship between these two types of models.

[4] However, Gollier (2006) also demonstrates that, in his setting, this result requires some restrictions on the set of priors and on the investor's attitude towards risk.

[5] Another active branch of literature working with ambiguity is the asset pricing literature. Under the MP approach, examples of papers focused on equilibrium asset pricing are Epstein and Wang (1994, 1995), Chen and Epstein (2002),

We extend the model of Chacko and Viceira (2005) for optimal dynamic portfolio choice with stochastic variance, by introducing ambiguity about the expected value of the variance of the risky asset's return. Motivation for this is provided by Chacko and Viceira (2005) themselves:

"*An important caveat of our empirical analysis is that we have counterfactually assumed that investors observe volatility (or precision), and that they take as true parameters our empirical estimates of the joint process for returns and volatility. In practice, however, investors do not observe volatility, and they do not know the parameters of the process for volatility, or even the process itself.*"

It has been advocated in the literature (e.g. Cao et al. (2005), Garlappi et al. (2007) and Ui (2009)) that it is reasonable to assume that investors estimate the variance of the risky asset's return without ambiguity, and that it is preferable to assume ambiguity about expected returns. Reasons invoked for this are analytical tractability, empirical evidence on the predictability of the variance of stock returns (e.g. Bollerslev et al. (1992)), higher difficulty in estimating the expected returns versus expected variance (Merton (1980)) and higher costs associated with errors in estimating expected returns versus expected variance (Chopra and Ziemba (1993)).

Nevertheless, we introduce ambiguity about the variance of the risky asset's return because: (1) there is no "a priori" reason to assume that investors are not ambiguous about variance, particularly considering that they do not observe it; (2) we are able to find analytical solutions; and (3) the expectation of variance under statistical-econometric methods isn't the sole relevant indicator of variance in the financial world.[6]

We consider ambiguity only about the variance of the risky asset's return to keep a parsimonious model and to isolate its effects on the optimal dynamic consumption and portfolio choices. The reason to choose the expected value of variance as the source of ambiguity (besides analytical tractability) is that it seems more intuitive to assume that investors think more about that parameter of the variance process than about other parameters (like the variance of variance or the reversion parameter in the case of a mean-reverting variance process).

We introduce ambiguity through the MP approach, adopting the Maxmin Expected Utility framework of Gilboa and Schmeidler (1989), which enables us to introduce ambiguity aversion without changing significantly the setting of Chacko and Viceira (2005). The choice behavior is as if the investor has a set of possible values of the expected value of variance, and considers the worst possible of these values when evaluating each portfolio and consumption policy.[7]

Epstein and Miao (2003) and Ju and Miao (2008), which deal with the well-known "equity premium" and "home-bias" puzzles. Dow and Werlang (1992), Cao et al. (2005) and Ui (2009) deal with the issue of limited market participation. Still under the MP approach, Miao and Wang (2007) analyze the option exercise problem when there is a distinction between risk and ambiguity. Under the RC approach, equilibrium asset pricing was studied, for e.g., by Hansen and Sargent (2001b), Anderson et al. (2003) and Liu et al. (2005).

[6] Option implied variance frequently differs both in level and dynamics from the statistical measure (e.g. Drechsler and Yaron (2008) and Todorov (2009)), and embeds relevant information about investors perception about the future distribution of the risky asset's return. Which part of this spread between statistical and option-implied expectations of variance is due to investor's ambiguity about variance is still an interesting open question in the literature.

[7] The MP approach includes three major settings for agent's attitude towards ambiguity: (i) Non-additive Expected Utility (Schmeidler (1989)) that makes use of Choquet Integration (CEU) to work with non-additive priors; (ii) Maxmin Expected Utility (MEU) (Gilboa and Schmeidler (1989)) that works with a set of additive priors in such a way that an ambiguity averse agent considers the worst prior associated with each decision; and more recently, (iii) the Smooth Ambiguity Aversion Theory (KMM) (Klibanoff et al. (2005)), that distinguishes ambiguity from ambiguity aversion and allows for smooth indifference curves, avoiding the infinite ambiguity aversion implied in the MEU approach. Further extensions of those works have been made recently in order to adapt them to dynamic problems, either through non-recursive settings (e.g. Gilboa and Schmeidler (1993) and Klibanoff and Hanany (2007)) and recursive settings within CEU framework (e.g. Nishimura and Ozaki (2003) and Eichberger et al. (2005)), MEU framework (e.g. Chen and

We find that, in our setting, the variance of the risky asset's return is always bad to the investor, in the sense that it decreases his utility. However, we conclude that ambiguity about variance has no impact whatsoever.

A crucial assumption of the model is that the investor is able to continuously update his portfolio as a function of the observed instantaneous variance, which means that the variance of the portfolio's return is always known by the investor. Without such continuous portfolio adjustment (which is not observed in reality, due to transaction costs), the variance of the portfolio's return becomes uncertain, even if investors can observe the instantaneous variance. To shed some light on this scenario, we also consider an alternative case in which investors use their expectation about the future variance in their portfolio decision. We find that, in this case, the investor's ambiguity aversion about variance reduces the demand for the risky asset.

In this latter scenario, the demand for the risky asset can be decomposed into three components: myopic and intertemporal hedging demands (as in Chacko and Viceira (2005)) and ambiguity demand, which is the novelty. Using the same calibration of Chacko and Viceira (2005) with long-run US data, we find that ambiguity demand has a relevant empirical dimension, much more than intertemporal hedging demand. We therefore conclude that, when investors use their expectation about future variance (and not the instantaneous variance) in their portfolio decision, the "ambiguity dimension" of the uncertainty about the stochastic variance is relevant, more than its "risk dimension".

In our view, this paper brings three major contributions: (1) it is the first to introduce ambiguity aversion within a dynamic optimal portfolio choice setting with an explicit process for the stochastic variance; (2) it is the first to introduce ambiguity aversion specifically about the variance of the risky asset return; (3) it suggests that, in some scenarios, ambiguity about the variance of the risky asset's return is important for portfolio choice.

The paper is organized as follows. In section 2, we state the problem to be solved. In section 3, we present the analytical solution to the problem and the key results. In section 4, we show simulation outputs. In section 5, we conclude the paper with some remarks.

2 The Dynamic Consumption and Portfolio Choice Problem

In section 2.1, we describe the investment opportunity set, which is the same as in the work of Chacko and Viceira (2005). In section 2.2, we disclose the preferences of the investor, extending the framework of Chacko and Viceira (2005) by introducing ambiguity about the expected value of the precision (which is the reciprocal of variance) of the risky asset's return. In section 2.3, we present the dynamic optimization problem to be solved by the investor.

Epstein (2002), Epstein and Schneider (2003), Hayashi (2005) and Maccheroni et al. (2006a,b)) and KMM framework (Klibanoff et al. (2009)). Additionally, Miao (2001), Wang (2003) and Epstein and Schneider (2007, 2008) extend the recursive multiple-priors approach in order to incorporate learning.

2.1 Investment Opportunity Set

It is assumed that all wealth is allocated between a riskless asset with price B_t and a risky asset with price S_t. The instantaneous return of the riskless asset is described by:

$$\frac{dB_t}{B_t} = r dt\,, \tag{1}$$

where r stands for the risk free interest rate.

The instantaneous return of the risky asset is given by:

$$\frac{dS_t}{S_t} = \mu dt + \sqrt{\frac{1}{y_t}} dW_S\,, \tag{2}$$

where μ is the expected return of the risky asset, W_S is a standard Brownian motion and y_t is the instantaneous precision of the risky asset's return process (the instantaneous variance is $v_t = \frac{1}{y_t}$). From (2), the expected excess return of the risky asset versus the riskless asset, $\mu - r$, is constant over time.

The precision, y_t, follows a mean-reverting, square-root process given by:

$$dy_t = \kappa\left(\theta - y_t\right) dt + \sigma\sqrt{y_t} dW_y\,, \tag{3}$$

where the expected value of the precision is $E\left[y_t\right] = \theta$, the reversion parameter is $\kappa > 0$, and, thus, $Var\left[y_t\right] = \frac{\sigma^2\theta}{2\kappa}$ (Cox et al. (1985), pp. 392). W_y is a standard Brownian motion. To guarantee standard integrability conditions, it is assumed that $2\kappa\theta > \sigma^2$(Cox et al. (1985), pp. 393).

Applying Itô's Lemma to (3), a mean-reverting, square-root process for proportional changes in variance is obtained (Appendix 6.1) :

$$\frac{dv_t}{v_t} = \kappa_v\left(\theta_v - v_t\right) dt - \sigma\sqrt{v_t} dW_y\,, \tag{4}$$

where $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa}\right)^{-1}$ and $\kappa_v = \kappa\left(\theta - \frac{\sigma^2}{\kappa}\right) = \frac{\kappa}{\theta_v}$.

Taking expectations of the second-order Taylor expansion of v_t around θ, the approximate unconditional mean of instantaneous variance is (Appendix 6.2):

$$E\left[v_t\right] \approx \frac{1}{\theta} + \frac{1}{2}\frac{\sigma^2}{\theta^2\kappa} = \frac{1}{\theta} + \frac{Var(y_t)}{\theta^3}\,. \tag{5}$$

As the expected return of the risky asset, μ, is assumed to be constant, (5) is also the expected unconditional variance of the risky asset's return.[8]

It is assumed that shocks to precision (W_y) are correlated with shocks to the return on the risky asset (W_S), with $dW_y dW_S = \rho dt$ and $\rho > 0$. From (4), this implies that the instantaneous correlation

[8] Chacko and Viceira (2005) perform a Monte Carlo simulation that validates this statement and the accuracy of the approximation (5). They conclude that this approximation understates the true variance by 0.27%.

between proportional changes in variance and the risky asset's return is given by:

$$Corr_t\left(\frac{dv_t}{v_t}, \frac{dS_t}{S_t}\right) = -Corr_t\left(dy_t, \frac{dS_t}{S_t}\right) = -\rho dt\,. \tag{6}$$

This investment opportunity setting incorporates three of the main stylized facts about the variance of the return of risky assets: the mean reversion property, the "leverage effect" property (given by the negative correlation between returns and its variance), and the clustering property (as proportional changes in variance are higher when variance is high).

2.2 Investor's Preferences

The investor faces ambiguity about the expected value of precision, θ (equivalently, from (5), about the expected value of the unconditional variance of the risky asset return, $E[v_t]$). Regarding the values of the remaining parameters of the investment opportunity set, it is assumed that there is no uncertainty.

Following Gilboa and Schmeidler (1989), we assume that the θ-ambiguity averse investor has a set of priors, the interval $\left[\underline{\theta}, \overline{\theta}\right]$, with $0 < \underline{\theta} \leq \theta \leq \overline{\theta}$, and makes his choice considering the minimal expected utility over all priors in the set. The higher the difference between $\underline{\theta}$ and $\overline{\theta}$, the higher the level of ambiguity. No learning process about θ is considered. It would enrich the analysis at expense of increased complexity.[9]

The preferences of investors are described by the stochastic differential utility (SDU) function introduced by Duffie and Epstein (1992b) and applied to asset pricing theory by Duffie and Epstein (1992a). This is a continuous-time form of recursive utility, analogous to the discrete-time parametrization of Epstein and Zin (1989, 1991), that exhibits intertemporal consistency, admits Bellman's characterization of optimality, and separates risk aversion from elasticity of intertemporal substitution by not constraining to be reciprocals of one another (as in standard additive intertemporal utility function).

The utility process that defines the SDU function is represented by:

$$J = E_t\left[\int_t^{\infty} f(C_s, J_s)\, ds\right], \tag{7}$$

where C_s represents current consumption and J_s is the continuation utility for C at time $t = s$, with infinite time horizon. The function $f(C_s, J_s)$ is the normalized aggregator that generates J, which, as in Chacko and Viceira (2005), defines a SDU function that represents the preferences introduced by Kreps and Porteus (1978). An explicit closed-form expression for that SDU utility function is not available. Following Duffie and Epstein (1992a), the normalized aggregator $f(C, J)$ is given by (Appendix 6.3):

$$f(C, J) = \frac{\beta}{1 - \frac{1}{\psi}}(1 - \gamma) J\left[\left(\frac{C}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - 1\right], \tag{8}$$

where $\gamma > 0$ is the coefficient of relative risk aversion, $\psi > 0$ is the elasticity of intertemporal substitution and $\beta > 0$ is the rate of time preference. With $\gamma = \frac{1}{\psi}$, (8) becomes the standard power utility

[9] See comments in Garlappi et al. (2007), pp.73, for some reasons to ignore the effect of learning under an ambiguity context.

representation.

When $\psi \to 1$, the normalized aggregator $f(C, J)$ takes the form:

$$f(C,J) = \beta(1-\gamma) J \left\{ log(C) - \frac{1}{1-\gamma} log\left[(1-\gamma) J\right] \right\}. \tag{9}$$

Two remarks on the preferences of the representative investor. The first regards the different dynamic nature of the two types of uncertainty - ambiguity and risk - in our setting. We introduce ambiguity through the approach of Gilboa and Schmeidler (1989), assuming that the ambiguity problem has a "once-for-all" solution. Conversely, the attitude towards risk is settled through a recursive dynamic setting given by (7), where the dynamic consistency is guaranteed (Duffie and Epstein (1992a)). This approach is justified by two main reasons: (i) it is a parsimonious way of extending the framework of Chacko and Viceira (2005) to incorporate θ-ambiguity;[10] (ii) although simple, it permits the study of the relevance of the dichotomy ambiguity-risk when assessing the impact of stochastic precision on the portfolio choice - we obtain an analytical solution that disentangles the ambiguity and risk components of investor's portfolio choice (section 3), enabling an elucidating empirical testing (section 4).

The second remark regards the preference for the timing of the resolution of risk. With the Kreps and Porteus (1978) preference structure, agents can have preference for early or late resolution of risk (as well as indifference), while the standard additive intertemporal utility function implies that agents are indifferent to the temporal resolution of risk. In the framework of Epstein and Zin (1989), the preference for temporal resolution of risk depends on the relationship between ψ and γ: if $\gamma > \frac{1}{\psi}$ $(<, =)$ investors have preference for early (late, indifferent) resolution of risk. Our specification (7) from Duffie and Epstein (1992a), being the continuous-time limit of that of Epstein and Zin (1989), inherits this property. However, on the contrary of other streams of literature with Epstein and Zin (1989) preferences, for e.g. the "long-run risk" literature (from seminal work of Bansal and Yaron (2004)), we do not restrict investor to have preference for early resolution of risk. Two main reasons support this decision: (i) as our model evolves in a long-run setting, it should not be excluded the possibility of the "cost" (of opportunity) becoming higher than the "benefit" of planning advantages brought by the early resolution of risk (e.g. Arai (1997)) and (ii) there is evidence that agents may have preference for late resolution of risk (e.g. Epstein and Zin (1991)).

[10] A more elaborate way of doing it would be to work with a recursive multiple priors utility specification. The setting of Chen and Epstein (2002) would be a good candidate, as it extends the SDU function of Duffie and Epstein (1992b) by replacing the single prior by a set of priors. In this case, θ would be a new state variable, alongside X_t and y_t, following its own stochastic process. The state of nature would be described by (θ_t, X_t, y_t), and there would be ambiguity about θ_t. This means that the set of priors under which the intertemporal optimization problem would be solved would be exclusively defined by a density generator associated to the θ_t process. This setting is richer than ours as it considers intertemporality in both sources of uncertainty (ambiguity and risk), but: (i) analytically, it implies a more complex stochastic optimal control problem to be solved, as there is a third stochastic differential equation (associated with the dynamics of θ_t) to be included in the deduction of the Bellman Equation and respective value function; and (ii) for the particular case of the SDU function specification for Kreps and Porteus (1978) preferences, it is not ensured that an utility function exists, as the correspondent aggregator ((8) or (9)) violates the Lipschitz condition (see Chen and Epstein (2002) section 3 for further details). Notwithstanding this added complexity, we see it as an interesting topic for future work.

2.3 Dynamic Optimization Problem

The dynamic optimization problem of the investor is to maximize the θ-minimized expected utility (7), subject to the precision process (3) and to the intertemporal budget constraint (11).

$$\max_{\pi,C} \left\{ \min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]} E_{t_0}\left[\int_{t_0}^{\infty} f\left(C_s, J_s\right) ds\right]\right\} \tag{10}$$

$$s.t.$$

$$dX_t = \left[\pi_t\left(\mu - r\right)X_t + rX_t - C_t\right]dt + \pi_t\sqrt{\frac{1}{y_t}}X_t dW_S \ , \tag{11}$$

$$dy_t = \kappa\left(\hat{\theta} - y_t\right)dt + \sigma\sqrt{y_t}dW_y \ ,$$

where C_t, X_t and π_t represent instantaneous consumption, wealth and fraction of wealth invested in the risky asset, respectively. Is is assumed that total wealth at the initial moment ($t_0 = 0$) is positive, i.e., $X_{t_0} > 0$.

Following Kamien and Schwartz (1991) (ch. 22), the Bellman Equation of this problem is:

$$\begin{aligned} 0 \ = \ & \max_{\pi,C} \min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]} \left\{ f\left(C_s, J_s\right) + J_X\left(\pi_t\left(\mu - r\right)X_t + rX_t - C_t\right) + J_y\kappa\left(\hat{\theta} - y_t\right) + \right. \\ & \left. + \frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2 + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Xy}\pi_t\rho\sigma X_t \right\} , \end{aligned} \tag{12}$$

where $f\left(C_s, J_s\right)$ is the normalized aggregator given in (8) and (9) for general values of ψ and for $\psi \to 1$, respectively, and J_X, J_y, J_{XX}, J_{yy} and J_{Xy} are partial derivatives.

The order of minimization and maximization in (12) can be changed by applying the Saddle Point Theorem (Fan (1953), Sion (1958)), which is possible as: (i) the domain of $\hat{\theta}$ is compact; (ii) the argument is concave on π and on C (the expressions for J will be disclosed in the next two subsections);[11] and (iii) the argument is convex on $\hat{\theta}$.[12] We obtain:

$$\begin{aligned} 0 \ = \ & \min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]} \max_{\pi,C} \left\{ f\left(C_s, J_s\right) + J_X\left(\pi_t\left(\mu - r\right)X_t + rX_t - C_t\right) + J_y\kappa\left(\hat{\theta} - y_t\right) + \right. \\ & \left. + \frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2 + \frac{1}{2}J_{yy}\sigma^2 y_t + J_{Xy}\pi_t\rho\sigma X_t \right\} , \end{aligned} \tag{13}$$

which is the Bellman Equation of the problem in which the order of the maximization and the minimization is exchanged. Therefore, the dynamic consumption-portfolio problem with stochastic precision

[11] There are two parcels that are linear on π and one that is quadratic but is associated with a minus sign (from $J_{XX} < 0$), therefore the argument is concave on π. Concavity on C also holds, as there is one parcel that is linear and, for the expressions for J given in the next two subsections, $f\left(C_s, J_s\right)$ ((8) or (9)) is concave on C.

[12] Note that J is not a function of $\hat{\theta}$, as it already considers the specific value that solves the minimization problem. This implies that the only relevant parcel of the argument of (12) to study convexity on $\hat{\theta}$ is $J_y\kappa\left(\hat{\theta} - y_t\right)$. The latter is linear, and therefore convex, on $\hat{\theta}$.

faced by the investor that is both θ-ambiguity and risk averse can be written as:

$$\min_{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}\left\{\max_{\pi,C} E_{t_0}\left[\int_{t_0}^{\infty} f\left(C_s,J_s\right)ds\right]\right\}$$

$$s.t. \tag{14}$$

$$dX_t=\left[\pi_t\left(\mu-r\right)X_t+rX_t-C_t\right]dt+\pi_t\sqrt{\frac{1}{y_t}}X_t dW_S\ ,$$

$$dy_t=\kappa\left(\hat{\theta}-y_t\right)dt+\sigma\sqrt{y_t}dW_y\ .$$

3 Optimal Dynamic Consumption and Portfolio Choices

We solve problem (14) in two steps. First, for each value of $\hat{\theta}$, we find the value function, $J_{t_0}(\hat{\theta})$, of the maximization problem, as in Chacko and Viceira (2005). Then, we minimize this value function with respect to $\hat{\theta}$ to obtain the value of $\hat{\theta}$ that is used by the ambiguity averse investor:

$$\theta^*=\underset{\hat{\theta}\in[\underline{\theta},\overline{\theta}]}{argmin}\, J_{t_0}(\hat{\theta}).$$

The maximization problem, for each possible value of $\hat{\theta}$, is a stochastic continuous-time optimal control problem with two state variables, wealth (X_t) and precision of the risky asset's return (y_t), and two control variables, consumption (C_t) and fraction of wealth invested in the risky asset (π_t). This implies that the value function, $J_{t_0}(\hat{\theta})$, that solves the problem is a function of X_t and y_t. The corresponding Bellman equation is:

$$\begin{aligned}0 \;=\; \max_{\pi,C}\Big\{ & f\left(C_s,J_s\right)+J_X\left(\pi_t\left(\mu-r\right)X_t+rX_t-C_t\right)+J_y\kappa\left(\hat{\theta}-y_t\right)+\\ & +\frac{1}{2}J_{XX}\pi_t^2\frac{1}{y_t}X_t^2+\frac{1}{2}J_{yy}\sigma^2y_t+J_{Xy}\pi_t\rho\sigma X_t\Big\}\ .\end{aligned} \tag{15}$$

Chacko and Viceira (2005) found an exact solution of this maximization problem for $\psi=1$, and an approximate solution for the general case of $\psi\neq 1$. We study θ-ambiguity aversion in both scenarios.

3.1 Exact solution ($\psi=1$)

When the investor has unit elasticity of intertemporal substitution of consumption ($\psi=1$), the value function that solves (15), for any value of $\hat{\theta}$, is given by (Chacko and Viceira (2005), Proposition 1):

$$J\left(\hat{\theta},X_t,y_t\right)=exp\left\{Ay_t+B(\hat{\theta})\right\}\frac{X_t^{1-\gamma}}{1-\gamma}\,, \tag{16}$$

where A and $B(\hat{\theta})$ are given by

$$A = \frac{\gamma\left(1-\gamma\right)\left[\left(\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma}\right) \pm \sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2[\gamma(1-\rho^2)+\rho^2]}{\gamma^2(1-\gamma)}}\right]}{\sigma^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}, \tag{17}$$

$$B(\hat{\theta}) = \frac{\left(1-\gamma\right)\left(\beta log\beta + r - \beta\right)}{\beta} + \frac{\kappa\hat{\theta}}{\beta}A\,. \tag{18}$$

The sign of the square-root in A is "+" for $\gamma > 1$ and "-" for $\gamma \leqslant 1$ (see Appendix 6.4).

This implies the following optimal consumption (C_t) and portfolio rules (π_t):

$$C_t = \beta X_t\,, \tag{19}$$

$$\pi_t = \frac{1}{\gamma}\left(\mu - r\right)y_t + \frac{\sigma\rho}{\gamma}Ay_t\,. \tag{20}$$

When $\psi = 1$, optimal consumption is a constant fraction of wealth to be consumed throughout time (19). This means that the income and substitution effects on consumption that result from a change in the investment opportunity set are always exactly canceled out.

The optimal portfolio rule (20) has two components: (i) mean-variance portfolio demand, $\frac{1}{\gamma}\left(\mu - r\right)y_t$ (myopic demand[13]); and (ii) intertemporal hedging demand, $\frac{\sigma\rho}{\gamma}Ay_t$ (Merton (1973)). As highlighted by Chacko and Viceira (2005), the latter is zero (and therefore the portfolio myopic demand is optimal) when: the investor has unit coefficient relative risk aversion ($\gamma = 1$); investment opportunities are constant ($\sigma = 0$) or, being time-varying, it is not possible to use the risky asset to hedge against those changes ($\rho = 0$). As both components of portfolio demand are linear functions of y_t, their ratio does not depend on y_t. From $E\left[y_t\right] = \theta$ and (20), the mean optimal allocation in the risky asset is:

$$\pi_\theta = \frac{1}{\gamma}\left(\mu - r\right)\theta + \frac{\sigma\rho}{\gamma}A\theta. \tag{21}$$

An ambiguity averse investor considers, from his set of priors, $\theta^* = \underset{\hat{\theta}\in[\,\underline{\theta}\,,\,\overline{\theta}\,]}{argmin}\, J_{t_0}(\hat{\theta})$, where:

$$J_{t_0}(\hat{\theta}) = exp\left\{Ay_{t_0} + B(\hat{\theta})\right\}\frac{X_{t_0}^{1-\gamma}}{1-\gamma}, \tag{22}$$

with y_{t_0} and X_{t_0} representing the instantaneous precision and the total wealth at the present moment ($t = t_0$). From (22), the solution of the ambiguity problem will depend on the investor's risk preferences and on the characteristics of the investment opportunity set.

Proposition 1. *When $\psi = 1$ and $\gamma \geqslant \omega$, where $\omega = \frac{\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2+\sigma^2(\mu-r)^2+2\rho\sigma(\mu-r)(\beta+\kappa)} < 1$, the solution of the ambiguity problem is:*[14]

[13] The intuition for this designation is that when intertemporal hedging demand in (20) becomes zero, investors with a multiperiod problem decide as if they were facing a sequence of identical one-period problem (Merton (1973)).

[14] For $\gamma < \omega$, A in (17) is a complex number. From (16) this implies a complex value function J. As J is the optimized intertemporal utility, and utility functions are defined in the real space, the domain of the problem is restricted to values of parameters such that $\gamma \geqslant \omega$.

$$\theta^* = \underline{\theta}\,.$$

Proof. *Appendix 6.5.*

From Proposition 1, we conclude that the domain of the solution to the θ-ambiguity problem depends on the combination of the level of investor's risk aversion[15] and on the characterization of the investment opportunity set dynamics (represented by ω). Under that domain ($\gamma \geqslant \omega$), it results that precision is always good, implying that $\theta^* = \underline{\theta}$.

Two further comments related to Proposition 1. Firstly, although in our setting precision ends up to be always good, the hypothesis of precision being bad should also be considered, as we did. This follows from the conclusion of Rothschild and Stiglitz (1970, 1971) that some risk averse investors may be better off with an increase in the variance (decrease of precision) of returns.[16] If under the domain of analysis ($\gamma \geqslant \omega$), there were scenarios where precision is bad, then the solution of the ambiguity problem would be $\theta^* = \overline{\theta}$.

Secondly, regarding investor's preferences for the temporal resolution of risk, the domain of analysis $\gamma \geqslant \omega$ includes scenarios where the investor has preference for late resolution of risk ($\omega \leqslant \gamma < 1$), for early resolution of risk ($\gamma > 1$) and is indifferent to that timing ($\gamma = 1$). Only scenarios where the investor has a strong preference for late resolution of risk ($\gamma < \omega$) are excluded.

Using Proposition 1, problem (14) becomes:

$$\max_{\pi, C} E_{t_0}\left[\int_{t_0}^{\infty} f\left(C_s, J_s\right) ds\right]_{|\hat{\theta}=\underline{\theta}}, \; with\, \gamma \geqslant \omega,\, \psi = 1,$$

subject to the same restrictions as before, but with $\hat{\theta} = \underline{\theta}$. The solution is given by equations (16)-(20), for a specific value of $\hat{\theta} = \underline{\theta}$ and a consistent set of remaining parameters values (such that $\gamma \geqslant \omega$).

It is immediate from (19) and (20) that both the optimal consumption and portfolio rules are not affected by ambiguity about θ, as neither depend on $\underline{\theta}$. If the θ-ambiguity averse investor follows the instantaneous optimal portfolio policy (20), his mean optimal allocation to the risky asset is given by (21), the same as that of an investor who faces no ambiguity.

However, if the θ-ambiguity averse investor could not instantaneously adjust his portfolio following the observation of the instantaneous precision (as in (20)), then his expectation of the future precision

[15] Most empirical studies on γ conclude its value is higher than one. However there are also some few studies that obtain a γ estimate lower than one. See, for example, Table 7 in Bliss and Panigirtzoglou (2004) for a quick review of estimated values of γ in the literature.

[16] In the finance literature, particularly on portfolio choice, the mean-variance (MV) approach, under which risk averse agents make their choices considering only the first two moments of the return distribution, has been extensively used. This despite the fact that the expected utility theory (EU) approach is more consistent. AS a consequence, the risk-return trade-off is frequently and erroneously treated as the mean-variance trade-off. Although variance is probably the best scalar measure of risk, relevant probability distributions aren't exclusively characterized by their first two moments. At least their third and fourth moments - skewness and kurtosis - may be also very informative and relevant. Rothschild and Stiglitz (1970) demonstrate this by introducing a richer definition of risk, based on the concept of a "mean-preserving spread". For the portfolio problem, Rothschild and Stiglitz (1971) show that an increase in the riskiness of the risky asset does not necessarily reduce its demand by risk-averse investors. For the MV and EU approaches to give the same results, it is necessary to assume the quadratic utility specification, which is both theoretically and empirically implausible, or to restrict the choice space in a way that the mean and variance contain all the relevant information (e.g. all distributions are Normal).

would become relevant for the choice of portfolio. Note that due to the existence of transaction costs and other market frictions, investors in the real world do not adjust their portfolios continuously. In this case, θ-ambiguity aversion becomes relevant as, from Proposition 1, the expectation of the future precision differs between the ambiguity averse and the ambiguity neutral investor ($\underline{\theta}$ and θ, respectively). For an ambiguity neutral investor, the mean allocation to the risky asset continues to be given by (21). Proposition 2 elaborates on the portfolio choice of a θ-ambiguity averse investor.

Proposition 2. *When $\psi = 1$, $\gamma \geqslant \omega$, and the θ-ambiguity averse investor considers the expected precision of the risky asset return instead of the instantaneous precision, the demand for the risky asset is:*

$$\pi_{\underline{\theta}} = \frac{1}{\gamma}(\mu - r)\,\underline{\theta} + \frac{\sigma\rho}{\gamma} A\underline{\theta}\,, \tag{23}$$

which can be decomposed into three components:

$$\begin{aligned} myopic\,demand &= \frac{1}{\gamma}(\mu - r)\,\theta & (24)\\ intertemporal\,hedging\,demand &= \frac{\sigma\rho}{\gamma} A\theta & (25)\\ ambiguity\,demand &= \left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma} A\right](\underline{\theta} - \theta)\,. & (26) \end{aligned}$$

In this "expectation-driven" scenario, the mean demand for the risky asset is therefore decomposed into three components: myopic and intertemporal hedging demands (as in Chacko and Viceira (2005)) and ambiguity demand. The sum of (24) and (25) gives the mean optimal risky asset demand under no ambiguity (21) and under ambiguity when investor instantaneously changes his asset allocation. By adding the ambiguity demand (26), one obtains the mean risky asset demand under ambiguity (23) when the investor cannot instantaneously adjust his portfolio.

The main conclusions about the impact on π_θ, in this "expectation-driven" scenario, of the existence of θ-ambiguity are stated in Proposition 3.

Proposition 3. *When $\gamma \geqslant \omega$ and $\psi = 1$, (i) the mean allocation to the risky asset is lower when investors are θ-ambiguity averse (ambiguity demand is always negative); (ii) a higher (lower) level of ambiguity (measured by $\theta - \underline{\theta}$) implies a smaller (greater) allocation to the risky asset; (iii) the intertemporal hedging demand for the risky asset is negative if $\gamma > 1$ and positive if $\omega \leqslant \gamma < 1$.*

Proof: *Appendix 6.6.*

The result that ambiguity aversion reduces the demand for the risky asset is the standard result within the still recent literature on portfolio choice under ambiguity. We extend this result to a setting where precision, and not the expected return, is the source of ambiguity.

However, note that from (26), if there were scenarios with precision being bad, implying $\theta^* = \overline{\theta}$,

then ambiguity aversion would increase investor's demand for the ambiguous risky asset.[17]

The conclusion that an investor with $\gamma > 1$ has a negative intertemporal hedging demand, and the opposite when $\omega \leqslant \gamma < 1$, is consistent with the findings of Chacko and Viceira (2005) and applies to all scenarios and not only to the "expectation-driven" one. When risk aversion is low ($\omega \leqslant \gamma < 1$), the investor is ready to support a worse performance when precision is low for extra performance when precision is high (recall that $\rho > 0$). An investor with high risk-aversion ($\gamma > 1$) is not willing to accept this trade-off.

3.2 Approximate solution ($\psi \neq 1$)

The general case in which the investor's elasticity of intertemporal substitution of consumption (ψ) may differ from unity is relevant because most empirical studies suggest that $\psi \neq 1$, either higher (Hansen and Singleton (1982), Attanasio and Weber (1989) and Guvenen (2001)) or lower (Hall (1988), Epstein and Zin (1991), Campbell (1999) and Vissing-Jorgensen (2002)). However, the Bellman equation (15) has no exact solution when $\psi \neq 1$. Chacko and Viceira (2005) found an approximate analytical solution for $\psi \neq 1$ that converges to the exact solution determined above when $\psi \to 1$. This approximate solution may be interpreted as describing the behavior of an investor with bounded rationality.

The accuracy of the approximate solution is insufficient for $\gamma < 1$ and also for higher levels of γ when coupled with low values of ψ (Chacko and Viceira (2005), table 6). Therefore, we extend the analysis of the previous section to the general case of $\psi \neq 1$, but only for $\gamma > 1$.

The Bellman equation continues to be given by (15), for any $\hat{\theta}$ value, but now the normalized aggregator is given by (8). The value function that solves (15) in this scenario is given by (Chacko and Viceira (2005)):

$$J\left(\hat{\theta}, X_t, y_t\right) = exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)\left(A_1 y_t + B_1(\hat{\theta})\right)\right\}\frac{X_t^{1-\gamma}}{1-\gamma}\,, \tag{27}$$

where A_1 and $B_1(\hat{\theta})$ are given by:

$$A_1 = \frac{\frac{\rho\sigma(\mu-r)(1-\gamma)}{\gamma} - (h_1+\kappa) + \sqrt{\left((h_1+\kappa) - \frac{\rho\sigma(\mu-r)(1-\gamma)}{\gamma}\right)^2 - \frac{\sigma^2(\mu-r)^2(1-\gamma)[\gamma(1-\rho^2)+\rho^2]}{\gamma^2}}}{\frac{\sigma^2}{\gamma}\left(\frac{1-\gamma}{1-\psi}\right)[\gamma(1-\rho^2)+\rho^2]}\,, \tag{28}$$

$$B_1(\hat{\theta}) = \psi log\beta + \frac{h_0 - \psi\beta - r(1-\psi)}{h_1} + \frac{\kappa\hat{\theta}}{h_1}A_1\,, \tag{29}$$

which implies the following optimal consumption (C_t) and portfolio rules (π_t):

$$C_t = \beta^{\psi} X_t exp\left\{-A_1 y_t - B_1(\hat{\theta})\right\}\,, \tag{30}$$

$$\pi_t = \frac{1}{\gamma}(\mu - r)\, y_t + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1 y_t\,. \tag{31}$$

[17] This is a result reached by Gollier (2006) within a setting with ambiguity aversion over expected excess return, with constant precision and under the specification of Klibanoff et al. (2005), for certain choices of multiple priors for the risky asset's return.

Two notes on expressions (28) and (29). Firstly, in (28), only the positive sign on the square root guarantees that this approximate solution converges to the exact solution when $\psi \to 1$. Secondly, h_1 represents the unconditional mean of the consumption-wealth ratio around which the linear approximation of the Bellman Equation (15) is made in order to obtain this approximate solution.[18]

Regarding the investor's optimal policies, the main difference between the set of equations (19)-(20) and (30)-(31) concerns the consumption policy. From (30), the optimal consumption-wealth ratio is not fixed (β when $\psi = 1$). It is a decreasing function of precision when $\psi > 1$ and an increasing function of precision when $\psi < 1$ (Appendix 6.8). This reflects the relative importance of the intertemporal income and substitution effects of precision on consumption: when $\psi > 1$, the intertemporal substitution effect dominates the income effect, and the contrary happens when $\psi < 1$. Regarding (31), its structure is the same as in (20): it has a myopic component and an intertemporal hedging component that, for $\gamma > 1$, is always negative (Appendix 6.8). Chacko and Viceira (2005) show that for empirically plausible characterizations of the process for precision, expressions (20) and (31) are very close.

From $E[y_t] = \theta$ and (31), the mean optimal allocation in the risky asset is

$$\pi_\theta = \frac{1}{\gamma}(\mu - r)\,\theta + \frac{\sigma\rho}{\gamma}\frac{(\gamma - 1)}{(1 - \psi)}A_1\theta. \tag{32}$$

An ambiguity averse investor considers, from his set of priors, $\theta^* = \underset{\hat{\theta}\in[\underline{\theta},\,\overline{\theta}]}{argmin}\, J_{t_0}(\hat{\theta})$. The expression of $J_{t_0}(\hat{\theta})$ is now given by (27) at $t = t_0$:

$$J_{t_o}(\hat{\theta}) = exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)\left(A_1 y_{t_0} + B_1(\hat{\theta})\right)\right\}\frac{X_{t_0}^{1-\gamma}}{1-\gamma}, \tag{33}$$

with y_{t_0} and X_{t_0} representing the instantaneous precision of the risky asset's return and the wealth at the present moment ($t = t_0$). Proposition 4 states the conclusions on the ambiguity problem solution within this approximate formulation:

Proposition 4. *When $\psi \neq 1$ and $\gamma > 1$, the solution of the ambiguity problem is:*

$$\theta^* = \underline{\theta}.$$

Proof. *Appendix 6.7.*

Considering Proposition 4, the problem (14) to be solved becomes:

$$\max_{\pi, C} E_{t_0}\left[\int_{t_0}^{\infty} f(C_s, J_s)\,ds\right]_{|\hat{\theta}=\underline{\theta}}, \; with\, \gamma > 1,\, \psi \neq 1,$$

subject to the same restrictions as above, but with $\hat{\theta} = \underline{\theta}$. The solution of this problem is given by

[18] $h_1 = exp\{\overline{c - x}\}$, with $c_t - x_t = log\left(\frac{C_t}{X_t}\right)$, is therefore an endogenous variable. h_0 is established as $h_0 = h_1(1 - log(h_1))$.

expressions (27)-(31), with $\hat{\theta} = \underline{\theta}$.

In this case, the optimal consumption rule is affected by θ-ambiguity as expression (30) depends explicitly on $\hat{\theta}$: for each level of instantaneous precision y_t, the consumption-wealth ratio is higher (lower) when investors are θ-ambiguity averse if $\psi > 1$ ($\psi < 1$) (Appendix 6.8).

Regarding the optimal portfolio rule, it happens the same thing as in the exact solution. If the θ-ambiguity averse investor follows the optimal portfolio policy (31), his mean optimal allocation to the risky asset is given by (32), the same of an ambiguity neutral investor.

However, by analogy to the exposed for the exact solution, θ-ambiguity becomes relevant if the θ-ambiguity averse investor does not adjust continuously his portfolio for a given level of instantaneous precision (as in (31)). In this "expectation-driven" scenario, the investor's expectation of the risky asset return's precision becomes a key driver of the portfolio decision. In this scenario, while the demand of an investor who faces no ambiguity continues to be given by (32), the choice of a θ-ambiguity averse investor is described by Proposition 5.

Proposition 5. *When $\psi \neq 1$, $\gamma > 1$, the mean consumption-wealth ratio and the mean allocation to the risky asset are given, respectively, by*

$$\left(\frac{C}{X}\right)_{\underline{\theta}} = c_{\underline{\theta}} = \beta^{\psi} exp\left\{-A_1\underline{\theta} - B_1(\underline{\theta})\right\}, \tag{34}$$

$$\pi_{\underline{\theta}} = \frac{1}{\gamma}(\mu - r)\,\underline{\theta} + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\underline{\theta}\,. \tag{35}$$

The demand for the risky asset (35) can be decomposed into three components:

$$myopic\,demand = \frac{1}{\gamma}(\mu - r)\,\theta \tag{36}$$

$$intertemporal\,hedging\,demand = \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\theta \tag{37}$$

$$ambiguity\,demand = \left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right](\underline{\theta} - \theta)\,. \tag{38}$$

As in the exact solution, the risky asset demand in the "expectation-driven" scenario, is decomposed into three components: myopic and intertemporal hedging demands (as in Chacko and Viceira (2005)) and ambiguity demand. The sum of (36) and (37) gives the mean optimal risky asset demand under no ambiguity (32) and under ambiguity with instantaneously portfolio updates. By adding the ambiguity demand (38), one obtains the mean risky asset demand with ambiguity, when the investor cannot continuously update his portfolio allocation, under the approximate solution (35).

The main conclusions about the impact, in this "expectation-driven" scenario, on the consumption-wealth ratio and on π_θ of the existence of θ-ambiguity are stated in Proposition 6.

Proposition 6. *When $\gamma > 1$ and $\psi \neq 1$: (i) the mean allocation to the risky asset is lower when investors are θ-ambiguity averse (ambiguity demand is always negative); (ii) a higher (lower) level of*

ambiguity, measured by $(\theta - \underline{\theta})$*, implies a smaller (greater) mean allocation to the risky asset; (iii) the intertemporal hedging demand is negative; (iv) the mean consumption-wealth ratio is higher (lower) when investors are* θ*-ambiguity averse if* $\psi > 1$ $(\psi < 1)$*.*

Proof. *Appendix 6.8.*

4 Simulation

Chacko and Viceira (2005) found that, calibrating their model to long-run US data, the optimal intertemporal hedging demand has a small empirical dimension. This suggests that the "risk dimension" of stochastic variance is empirically not very relevant to the dynamic optimal portfolio decision. However, in their concluding remarks, they acknowledge that an important caveat of their analysis is that they have counterfactually assumed that investors observe variance and take as true the empirical estimates of the parameters of the variance process.

Following this lead, we have extended their model to account for ambiguity about one of the parameters of the variance process: the expected value of precision (inverse of variance). As a result, under a scenario where the ambiguity averse investor decides about his portfolio allocation considering his expectation of precision (and not the instantaneous precision), a third component of the demand for the risky asset appears. In addition to the "myopic demand" and the "intertemporal hedging demand", we obtain an "ambiguity demand" component. We denominated the referred scenario as "expectation-driven", and it is the one under which we run these simulations.

Using the calibration of Chacko and Viceira (2005), we find that the ambiguity demand component of the allocation in the risky asset has a relevant empirical dimension, much higher than that of the intertemporal hedging demand. Stochastic variance may therefore have a significantly higher impact on the portfolio choice than what is suggested by the results of Chacko and Viceira (2005).

The reference parameter values used in the simulation are those estimated in Chacko and Viceira (2005) based on monthly excess stock returns on the CRSP value-weighted portfolio over the T-Bill rate from January 1926 through December 2000:

$$\begin{aligned} \mu - r &= 0.0811, \\ \kappa &= 0.3374, \\ \theta &= 27.9345, \\ \sigma &= 0.6503, \\ \rho &= 0.5241, \\ r &= 0.015, \\ \beta &= 0.06. \end{aligned} \tag{39}$$

It is important to note that (5) is equivalent to (Appendix 6.9):

$$\theta \approx \frac{2\kappa + \sqrt{4\kappa^2 + 8\kappa\sigma^2 E\left[v_t\right]}}{4\kappa E\left[v_t\right]}. \tag{40}$$

Expression (40) sets the relation between θ and the expected variance of the risky asset's return, for a given pair of values for κ and σ. This relation is important for simulation purposes, because $\underline{\theta}$ is set for specified variance values[19], subject to the restriction that $2\kappa\theta > \sigma^2$ (section 2.1). Note that, for a given pair of values for κ and σ, and from (5):

$$\theta \in \left[\underline{\theta}, \overline{\theta}\right] \Longrightarrow E[v_t] \in \left[E[v_t]_{\theta=\overline{\theta}}, E[v_t]_{\theta=\underline{\theta}}\right]. \tag{41}$$

4.1 Simulation Results

Implications for the mean allocation in the risky asset from the introduction of θ-ambiguity are exemplified in Table 1, being consistent with conclusions in Proposition 3 and 6. The first column presents the results for the scenario without θ-ambiguity, in which the expected annual standard deviation of the risky asset's return is approximately equal to 19.1%. The other three columns represent scenarios with θ-ambiguity, in which the upper bound of the expected annual standard deviation value is increased to 20%, 25% and 30%.[20] From (40) we obtain the related $\underline{\theta}$ values. The implied ambiguity level in each scenario, $1 - \underline{\theta}/\theta$ (in percentage), is also reported.[21]

Simulations are run for different levels of risk aversion ($\gamma = 0.75$, 1, 2, 4, 20, 40) both using the exact solution ($\psi = 1$) and the approximate solution ($\psi = 1/0.75$ and $\psi = 1/1.5$).[22] In panel A we show the mean allocation to the risky asset (percentage). In panels B and C, the intertemporal hedging demand and the ambiguity demand are shown as a percentage of the myopic demand.

[19] From (40), it is equivalent to consider a value for θ or $E[v_t]$. However, we believe it is more natural for investors to think in terms of $E[v_t]$ than θ. For the set of parameters in (39), the implied expected standard deviation of returns is 19.1314%.

[20] From (41), those three values, 20%, 25% and 30%, are $\sqrt{\left(E[v_t]_{\theta=\underline{\theta}}\right)}$ values, i.e, they correspond (in each scenario) to the upper bound of intervals for $E[v_t]$ "built" by the θ-ambiguity averse investor.

[21] Implementation using Dynare version 3.065 and MatLab version 7.0.0.19920 R14.

[22] Recall discussion in section 3.2 about the scenarios under which the approximate solution is accurate.

Table 1: Mean allocation to risky asset, intertemporal hedging demand and ambiguity demand.

	Expected Annual Standard Deviation of Risky Asset Returns											
	19,1314%			20%			25%			30%		
implied θ	θ = 27.935			θ = 25.612			θ = 16.604			θ = 11.706		
implied ambiguity level	0%			8%			41%			58%		
ψ	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5
	A - Mean allocation to risky asset (%)											
R.R.A.												
0.75	305,93	305,66	305,42	280,45	280,24	280,05	181,73	181,68	181,62	128,09	128,09	128,08
1.00	226,55	226,55	226,55	207,71	207,71	207,71	134,66	134,66	134,66	94,94	94,94	94,94
2.00	111,38	111,37	111,37	102,12	102,11	102,11	66,22	66,20	66,18	46,69	46,67	46,65
4.00	55,26	55,24	55,21	50,67	50,64	50,62	32,85	32,83	32,81	23,16	23,15	23,13
20.00	10,99	10,98	10,97	10,08	10,07	10,05	6,53	6,52	6,52	4,60	4,60	4,59
40.00	5,49	5,48	5,48	5,03	5,03	5,02	3,26	3,26	3,26	2,30	2,30	2,30
	B - Ratio of hedging demand over myopic demand (%)											
R.R.A.												
0.75	1,28	1,19	1,11	1,27	1,19	1,12	1,22	1,19	1,16	1,20	1,19	1,18
1.00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00	0,00
2.00	-1,68	-1,68	-1,68	-1,67	-1,68	-1,68	-1,65	-1,68	-1,70	-1,64	-1,68	-1,71
4.00	-2,43	-2,47	-2,52	-2,43	-2,47	-2,52	-2,41	-2,47	-2,54	-2,40	-2,47	-2,55
20.00	-3,00	-3,09	-3,19	-2,99	-3,09	-3,19	-2,99	-3,09	-3,20	-2,99	-3,09	-3,20
40.00	-3,06	-3,17	-3,28	-3,06	-3,17	-3,28	-3,06	-3,17	-3,28	-3,06	-3,17	-3,28
	C - Ratio of Ambiguity demand over myopic demand (%)											
R.R.A.												
0.75	0,00	0,00	0,00	-8,42	-8,41	-8,41	-41,06	-41,04	-41,03	-58,79	-58,79	-58,78
1.00	0,00	0,00	0,00	-8,32	-8,32	-8,32	-40,56	-40,56	-40,56	-58,10	-58,10	-58,10
2.00	0,00	0,00	0,00	-8,18	-8,18	-8,18	-39,89	-39,88	-39,87	-57,14	-57,12	-57,10
4.00	0,00	0,00	0,00	-8,11	-8,11	-8,11	-39,58	-39,56	-39,53	-56,70	-56,66	-56,62
20.00	0,00	0,00	0,00	-8,07	-8,06	-8,05	-39,35	-39,31	-39,27	-56,36	-56,30	-56,24
40.00	0,00	0,00	0,00	-8,06	-8,05	-8,04	-39,32	-39,28	-39,23	-56,32	-56,26	-56,19

Note 1 - Panel A: $E[\pi_t(y_t)] = \pi_\theta \times 100$. Panel B: for $\psi = 1$, (ratio of (25) over (24))×100; for $\psi \neq 1$, (ratio of (37) over (36))×100. Panel C: for $\psi = 1$, (ratio of (26) over (24))×100; for $\psi \neq 1$, (ratio of (38) over (36))×100.

Note 2 - Implied ambiguity level (%): $(1 - \underline{\theta}/\theta) \times 100$.

Note 3 - **R.R.A.**=γ.

Let us give an example based on panel A of Table 1. Consider a risk-averse investor, with $\gamma = 2$ and $\psi = 1$, that is θ-ambiguity neutral. His mean optimal allocation to the risky asset corresponds to 111.4% of his wealth. If the investor becomes ambiguous about θ, under the "expectation-driven" scenario disclosed in previous sections, with the implied upper bound of the interval for annual expected standard deviation (from (41)) being equal to 25%, his mean allocation to the risky asset declines to 66,20% of his wealth.

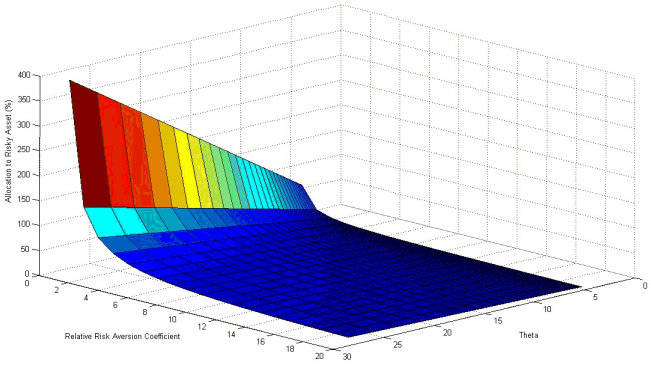
Panel B reports the estimates of the intertemporal hedging demand, measured as a ratio of myopic demand. The overall results with θ-ambiguity are similar to the ones in the scenario without ambiguity (the same for $\psi = 1$ and very close in the other cases). The main results are: (i) this ratio does not vary with the precision of the return; (ii) intertemporal hedging demand is positive when $\gamma < 1$ and negative when $\gamma > 1$; (iii) intertemporal hedging demand is small - even for a very high risk averse investor ($\gamma = 40$).

Panel C presents the estimated ratios of ambiguity demand versus myopic demand. It shows that ambiguity demand is always negative, with an importance that increases with the level of ambiguity and decreases with the level of risk aversion, showing much less sensitivity to γ.

Results for the exact and the approximate solution are very close, denoting low sensitivity to the elasticity of intertemporal substitution (ψ).

In panel A, it is shown that, the demand for the risky asset is decreasing with the risk aversion, γ, and with the level of ambiguity, $\theta - \underline{\theta}$. This is graphically highlighted in Figure 1.

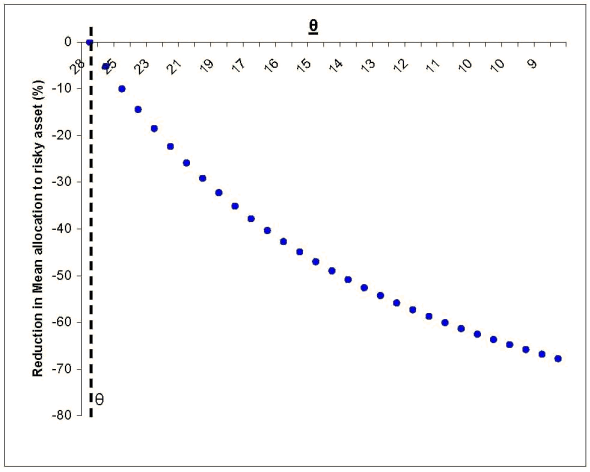
Figure 1: Mean allocation to risky asset as a function of risk aversion and ambiguity level



Note - Simulation run with $\psi = 1$.

It is clear that, under our setting and using calibration (39), the allocation to risky asset reacts strongly to changes in $\underline{\theta}$. Figure 2 further highlights this for a larger set of $\underline{\theta}$ values within a reasonable range of expected annual standard deviation of returns (from 19,1% to 35%).

Figure 2: Mean allocation to risky asset as a function of $\underline{\theta}$.



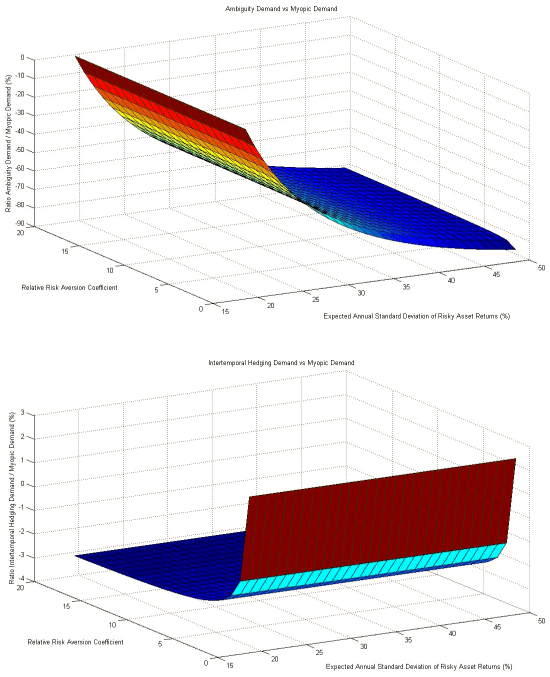
Note - Simulation run with $\psi = 1$.

The ambiguity effect on the demand of the risky asset is empirically expressive: even for a low level of ambiguity (second column in Table 1), where expected annual standard deviation is slightly adjusted from 19,1% (assumed to be the true value from calibration (39)) to 20%, the ambiguity effect corresponds to 8% of the investors myopic demand.[23] This is further illustrated in Figure 3 for a wider range of values of expected annual standard deviation of returns and of γ. Figure 3 also highlights the fact of ambiguity demand being empirically much more relevant than intertemporal hedging demand

[23] In each θ-ambiguity scenario in Table 1, the percentage values in Panel C are very close to the respective implied ambiguity level. This is due to the small empirical dimension of the intertemporal hedging demand (Panel B).

(note the difference of scale in the vertical axis).

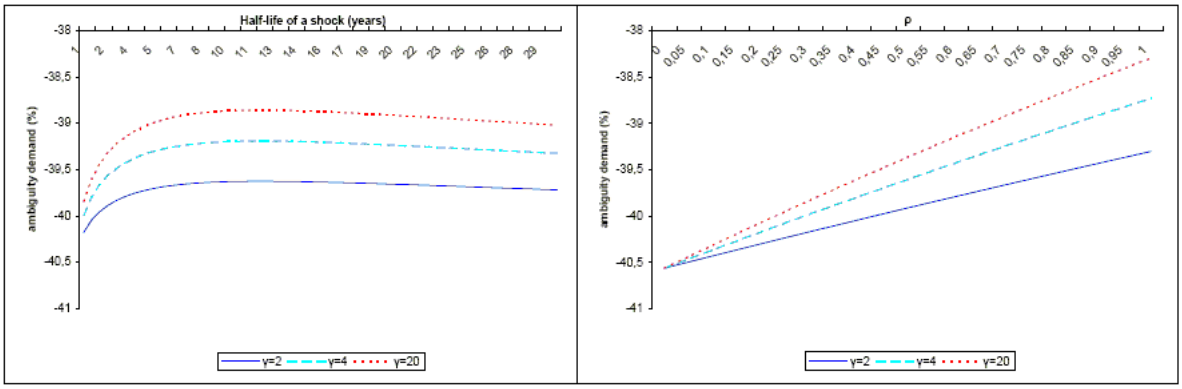
Figure 3: Ambiguity demand and intertemporal hedging demand versus myopic demand



Note - Simulations run with $\psi = 1$.

In Figure 4 we study the sensitivity of ambiguity demand to variations of two other parameters of the precision process (3): the reversion parameter (κ), which determines the persistence of shocks to precision, and the instantaneous correlation between shocks to precision and to risky asset return (ρ).

Figure 4: Effect on ambiguity demand of changes in κ and ρ, with $\gamma = 2$, 4 and 20.



Note 1 - Simulations run with $\psi = 1$.

Note 2 - Half-life of a shock (years) to precision: $\log(2)/\kappa$.

Note 3 - Ambiguity demand (%) measured as a ratio of myopic demand.

Until a certain level of half-life of the persistence of a shock to precision (around 7 years), the more persistent is the shock in precision, the lower is the absolute value of the ambiguity demand. Above that level, this effect disappears. The intuition for this can be that the higher is the persistence of shocks in precision, the larger is the period of time that it stays away from its expected value (θ), and, therefore, the less relevant is this parameter and the ambiguity about it.

The higher is the instantaneous correlation between shocks to precision and to risky asset returns, the lower is the absolute value of the ambiguity demand. This means that the easier it is to hedge variations in precision, the less relevant is the ambiguity over its expected value. However, Figure 4 also shows that both effects have a small empirical dimension.

Table 2 shows the impacts on the mean consumption-wealth ratio (Panel A) and on the long term expected return on wealth (Panel B) from the introduction of θ-ambiguity. Panel A shows that: (i) in the case of exact solution ($\psi = 1$), the consumption-wealth ratio is constant and equal to β as in (19); (ii) in the case of approximate solution ($\psi \neq 1$), when $\psi > 1$ ($\psi < 1$) the consumption-wealth ratio is higher (lower) when there is θ-ambiguity, in consistency with Proposition 6, and the same happens with the level of risk aversion. Figure 5 illustrate well this conclusion. Moreover, both Table 2 and Figure 5 show that changes in the consumption-wealth ratio from changes in the level of ambiguity and risk aversion are of small empirical dimension[24].

Table 2: Consumption-wealth ratio and long-term expected return on wealth.

	Expected Annual Standard Deviation of Risky Asset Returns											
	19,1314%			20%			25%			30%		
implied θ	θ = 27.935			θ = 25.612			θ = 16.604			θ = 11.706		
implied ambiguity level	0%			8%			40,6%			58%		
ψ	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5	1/0.75	1	1/1.5
	A - Consumption-wealth ratio (%)											
R.R.A.												
0.75	3,30	6,00	8,68	3,65	6,00	8,33	5,01	6,00	6,99	5,75	6,00	6,25
1.00	4,44	6,00	7,56	4,69	6,00	7,31	5,68	6,00	6,32	6,22	6,00	5,78
2.00	6,02	6,00	5,98	6,15	6,00	5,85	6,62	6,00	5,38	6,88	6,00	5,12
4.00	6,77	6,00	5,23	6,83	6,00	5,16	7,07	6,00	4,93	7,20	6,00	4,80
20.00	7,36	6,00	4,64	7,37	6,00	4,63	7,41	6,00	4,58	7,44	6,00	4,56
40.00	7,43	6,00	4,57	7,43	6,00	4,57	7,46	6,00	4,54	7,47	6,00	4,53
	B - Long-term expected return on wealth (%)											
R.R.A.												
0.75	26,31	26,29	26,27	24,25	24,23	24,21	16,24	16,23	16,23	11,89	11,89	11,89
1.00	19,87	19,87	19,87	18,35	18,35	18,35	12,42	12,42	12,42	9,20	9,20	9,20
2.00	10,53	10,53	10,53	9,78	9,78	9,78	6,87	6,87	6,87	5,29	5,29	5,28
4.00	5,98	5,98	5,98	5,61	5,61	5,61	4,16	4,16	4,16	3,38	3,38	3,38
20.00	2,39	2,39	2,39	2,32	2,32	2,32	2,03	2,03	2,03	1,87	1,87	1,87
40.00	1,95	1,94	1,94	1,91	1,91	1,91	1,76	1,76	1,76	1,69	1,69	1,69

Note 1 - Panel A: $c_t/x_t = exp\{E[c_t - x_t] \times 100\}$; Panel B: $(\pi_\theta(\mu - r) + r) \times 100$

Note 2 - Implied ambiguity level (%): $[1 - \underline{\theta}/\theta] \times 100$.

Note 3 - **R.R.A.**=γ.

In Panel B, it is possible to see that the long-term expected return on wealth, measured by $(\pi_\theta(\mu - r) + r)$, of an investor that is both risk averse and θ-ambiguity averse is a decreasing function

[24] An exception is for the scenario $\gamma = 0.75$. However, we recall that for $\gamma < 1$ the accuracy of approximate solution is lower.

of both risk aversion and the level of ambiguity, which was expectable considering results from Panel A of Table 1. This is further illustrated in Figure 6.

Figure 5: Consumption-wealth ratio as a function of $\underline{\theta}$ (for different values of γ and ψ).

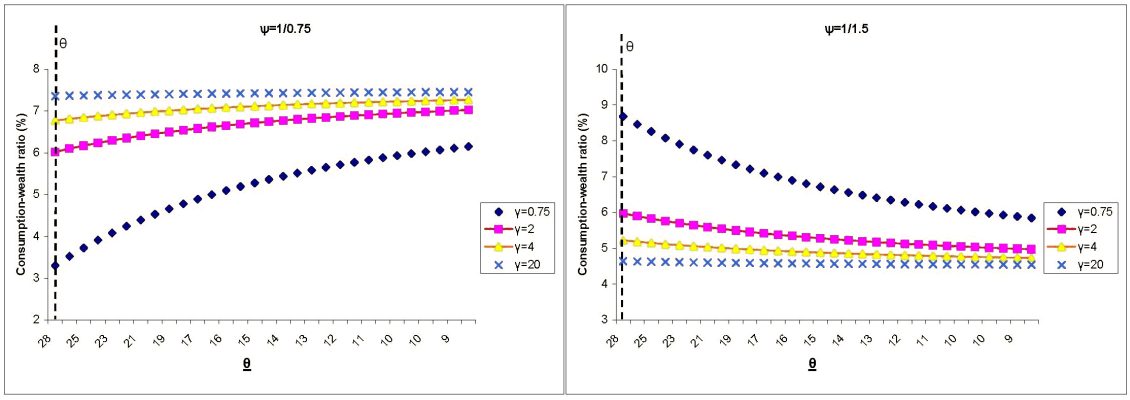
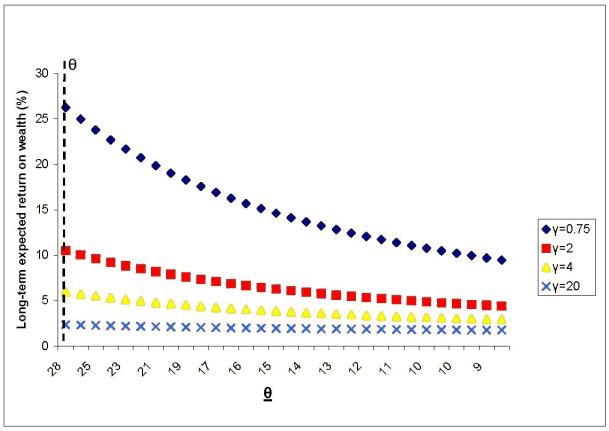


Figure 6: Long-term expected return on wealth as a function of $\underline{\theta}$ and γ.



5 Concluding Remarks

We presented an extension of the model of Chacko and Viceira (2005) for optimal dynamic consumption and portfolio choice with stochastic variance, by introducing ambiguity about the expected precision (inverse of variance) of the risky asset's return (parameter θ). In our setting, precision of the risky asset return is therefore simultaneously the source of risk and ambiguity perceived by the risk averse and θ-ambiguity averse investor.

Long-horizon investors with recursive preferences (Duffie and Epstein (1992a) with the specification of Kreps and Porteus (1978)) have two assets to invest in, a risk-free asset and a risky asset. The precision of the risky asset return is stochastic and the investor is ambiguous about its expected value, with ambiguity aversion in the spirit of the Maxmin Expected Utility model of Gilboa and Schmeidler

(1989). Not knowing the true value of precision, investors consider the worst possible value in a given interval.

A preliminary conclusion is that precision is always good, in the sense that it increases the utility attainable by the investor. This implies that the solution of the ambiguity problem is given by the lower bound ($\underline{\theta}$) of the assumed interval for possible θ values.

The main conclusions concern the impact on optimal dynamic policies from ambiguity about the expected value of precision. Regarding the optimal consumption policy, when the intertemporal elasticity of consumption (ψ) is different from one, the optimal consumption-wealth decision is always affected by the θ-ambiguity: consumption is higher when investors are ambiguity averse compared to when they are ambiguity neutral if $\psi > 1$ and the contrary when $\psi < 1$. When $\psi = 1$, the mean optimal consumption-wealth ratio does not depend on the level of precision, as found by Chacko and Viceira (2005).

Regarding the optimal demand for the risky asset, we conclude that ambiguity about θ has no impact. This result should be viewed having in mind that the model assumes that investors can adjust their portfolios continuously, as a function of the instantaneously observed precision. In the real world, investors cannot update their portfolios continuously, for reasons that are not easy to model (such as transaction costs or human limitations). Incorporating such market frictions in the model is an interesting task for future research.

To provide an additional perspective in the context of the present model, we also studied an "expectation-driven" scenario in which case investors use their expectations of future precision instead of the instantaneous precision. In this scenario, the mean allocation to the risky asset decreases with ambiguity, i.e., ambiguity demand is negative.

The last conclusion of this paper respects to the empirical relevance of ambiguity demand, as determined in this latter scenario. Chacko and Viceira (2005) concluded that the variance of the risky asset's return generates a small intertemporal hedging demand, suggesting low relevance of the stochastic variance (precision) in the dynamic portfolio decision. Using the same calibration, we conclude that the ambiguity demand component of the risky asset demand is relevant and has a much higher empirical dimension than that of intertemporal hedging demand. This indicates that stochastic variance (precision) may have a much higher impact on investors portfolio decision than found in Chacko and Viceira (2005). Recovering Knight (1921) conceptual dichotomy of ambiguity versus risk, we conclude that under those circumstances the "ambiguity dimension" of variance seems to be much more relevant than its "risk dimension" for dynamical optimal portfolio decisions.

6 Appendix

6.1 Deduction of equation (4)

We start by stating Itô's lemma (see, for e.g., Hull (2006) chapter 12, pp.273). Let X_t be a variable that follows the Itô process:

$$dX_t = \xi\left(X_t, t\right) dt + \nu\left(X_t, t\right) dBM_t\,, \tag{42}$$

where BM_t is a standard Brownian motion and ξ and ν are functions of X_t and t. Let $g\left(t,\, X_t\right)$ be a continuous differentiable function as regards its two arguments. Then,

$$Y = \left\{Y_t = g\left(t,\, X_t\right)\right\},$$

is a stochastic process that verifies:

$$dY_t = \frac{\partial g}{\partial t}\left(t,\, X_t\right) dt + \frac{\partial g}{\partial X}\left(t,\, X_t\right) dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\left(t,\, X_t\right)\left(dX_t\right)^2.$$

Applying the rule $dt^2 = dt\, dBM_t = 0$ and $\left(dBM_t\right)^2 = dt$, then:

$$dY_t = \left(\frac{\partial g}{\partial t}\left(t,\, X_t\right) + \frac{\partial g}{\partial X}\left(t,\, X_t\right)\xi + \frac{1}{2}\frac{\partial^2 g}{\partial X^2}\left(t,\, X_t\right)\nu^2\right) dt + \frac{\partial g}{\partial X}\left(t,\, X_t\right)\nu dBM_t\,, \tag{43}$$

i.e., the variable Y_t also follows an Itô Process.

In our setting, considering $v_t = g\left(y_t\right) = \frac{1}{y_t}$, then $g_y = -\frac{1}{y_t^2}$ and $g_{yy} = \frac{2}{y_t^3}$, where g_y and g_{yy} are the first and second derivatives of $g\left(\cdot\right)$ in order to y_t. Moreover, making the parallel with (42), in our setting we have:

$$\xi = \kappa\left(\theta - y_t\right),$$

$$\nu = \sigma\sqrt{y_t}.$$

Using these inputs and applying (43) to (3), we find that the stochastic process v_t is given by:

$$\frac{dv_t}{v_t} \quad = \quad \left(\kappa + v_t\sigma^2 - v_t\kappa\theta\right) dt - \sigma\sqrt{v_t} dW_y\,. \tag{44}$$

Defining $\theta_v = \left(\theta - \frac{\sigma^2}{\kappa}\right)^{-1}$ and $\kappa_v = \frac{\kappa}{\theta_v}$ (which imply that $\kappa + v_t\sigma^2 - v_t\kappa\theta = \kappa_v\left(\theta_v - v_t\right)$), we find (4).

6.2 Deduction of equation (5)

Applying the second-order Taylor expansion of $v_t = \frac{1}{y_t}$ around θ:

$$v_t \approx \frac{1}{\theta} - \frac{1}{\theta^2}\left(y_t - \theta\right) + \frac{1}{2}\cdot\frac{2}{\theta^3}\left(y_t - \theta\right)^2. \tag{45}$$

Taking expectations:

$$E\left[v_t\right] \approx \frac{1}{\theta} - \frac{1}{\theta^2}\left(E\left[y_t\right] - \theta\right) + \frac{1}{\theta^3}E\left[\left(y_t - \theta\right)^2\right]. \tag{46}$$

From Cox et al. (1985) it is known that $E[y_t] = \theta$ and $E\left[(y_t - \theta)^2\right] = Var[y_t] = \frac{\sigma^2\theta}{2\kappa}$. Substituting those results in (46), we obtain (5).

6.3 Stochastic Diffusion Utility (SDU) Function

Duffie and Epstein (1992a) define the stochastic differential utility (SDU) function $U : D \to \mathbb{R}$ by two primitive functions, $\bar{f} : C \times \mathbb{R} \to \mathbb{R}$ and $\bar{A} : \mathbb{R} \to \mathbb{R}$. The function $\bar{A}(\cdot)$ is called variance multiplier, as it applies a penalty (or reward) as a multiple of the utility "volatility". For deterministic consumption process, $\bar{A}(\cdot)$ is therefore irrelevant (without uncertainty, only $\bar{f}(\cdot)$ matters).

Duffie and Epstein (1992a) state that if, for each consumption process C, there exists a well-defined utility process $\bar{J}$, then the SDU function U is defined by $U(C) = \bar{J}_0$, the initial value of that utility process. The pair $\left(\bar{f}, \bar{A}\right)$ generating $\bar{J}$ is called an aggregator.

Two aggregators $\left(\bar{f}, \bar{A}\right)$ and (f, A) are said to be ordinally equivalent if they generate ordinally equivalent utility functions, i.e. represent the same preference ordering of consumption processes. Duffie and Epstein (1992a) present a method through which for any aggregator $\left(\bar{f}, \bar{A}\right)$ an ordinally equivalent aggregator (f, A), with variance multiplier $A(\cdot)$ equal to zero, is obtained. (f, A) is called the normalized aggregator of $\left(\bar{f}, \bar{A}\right)$ and generates the utility process J given by:

$$J_t^T = E_t\left[\int_t^T (f_s, J_s)\, ds\right]. \tag{47}$$

Note how (47) is close to (7). The difference between the expressions is the infinite time horizon in the later which Duffie and Epstein (1992a) also considers by defining $J_t = \lim_{T\to\infty} J_t^T$.

The method to obtain (f, A) from $\left(\bar{f}, \bar{A}\right)$, consists in a change of variables φ that satisfies the differential equation $\varphi''(x) = \bar{A}(x)\,\varphi'(x)$, which implies:

$$\varphi(J) = \delta_2 + \delta_1 \int_{J_0}^{J} exp\left[\int_{J_0}^{u} \bar{A}(x)\, dx\right] du, \tag{48}$$

where J_0 is arbitrary, δ_2 and δ_1 are constants, with $\delta_1 > 0$, defined so that $\varphi(0) = 0$. Using φ, the relationship between the two ordinally equivalent aggregators $\left(\bar{f}, \bar{A}\right)$ and (f, A) is given by:

$$\begin{aligned} \bar{f}(c, z) &= \frac{f(c, \varphi(z))}{\varphi'(z)}, \quad (c, z) \in C \times \mathbb{R}, \\ \bar{A}(x) &= \varphi'(x)\, A[\varphi(x)] + \frac{\varphi''(x)}{\varphi'(x)}. \end{aligned} \tag{49}$$

In this paper, following Chacko and Viceira (2005), the utility process J generated by the normalized aggregator $f(C_s, J_s)$ is set to define a SDU function ordinally equivalent to Kreps and Porteus (1978) utility. The aggregator $\left(\bar{f}, \bar{A}\right)$ for this particular utility function is defined (Duffie and Epstein (1992a)) as:

$$\bar{f}(c, J) = \frac{\beta}{\xi}\frac{c^\xi - J^\rho}{J^{\xi-1}}, \qquad \bar{A}(J) = \frac{\alpha - 1}{J}, \tag{50}$$

with $C = \mathbb{R}_+$, $0 \neq \xi \leq 1$, $0 \leq \beta$, $0 \neq \alpha \leq 1$, and $\nu > 0$.

Deduction of expression (8)

Replacing (50), $\bar{A}(x) = \frac{\alpha - 1}{x}$, in (48):

$$\begin{aligned} \varphi(J) &= \delta_2 + \delta_1 \int_{J_0}^{J} exp\left[\int_{J_0}^{u} \frac{\alpha - 1}{x} dx\right] du \\ &= \delta_2 + \frac{\delta_1}{J_0} \int_{J_0}^{J} u^{\alpha - 1} du \\ &= \delta_2 + \delta_1 \left(\frac{J^\alpha}{\alpha}\right). \end{aligned} \tag{51}$$

To have$\varphi(0) = 0$, we set $\delta_2 = 0$. Assuming $\delta_1 = 1$, possible as Duffie and Epstein (1992a) only require $\delta_1 > 0$, expression (51) yields $\varphi(J) = \frac{J^\alpha}{\alpha}$ and:

$$\varphi'(J) = J^{\alpha - 1}. \tag{52}$$

Using (50) and (52) in (49), gives:

$$\frac{\beta}{\xi} \frac{C^\xi - J^\xi}{J^{(\xi - \alpha)}} = f(C, \varphi(J)). \tag{53}$$

From $\varphi(J) = \frac{J^\alpha}{\alpha}$ we get $(\alpha\varphi(J))^{\frac{1}{\alpha}} = J$.

Introducing this result in (53):

$$\begin{aligned} \frac{\beta}{\xi} \frac{C^\xi - (\alpha J)^{\frac{\xi}{\alpha}}}{(\alpha J)^{\frac{\xi - \alpha}{\alpha}}} &= f(C, J) \\ \frac{\beta}{\xi} \alpha J \left(\left(\frac{C}{(\alpha J)^{\frac{1}{\alpha}}}\right)^\xi - 1\right) &= f(C, J). \end{aligned} \tag{54}$$

Expression (8) follows simply by changing notation: $\xi = 1 - \frac{1}{\psi}$ and $\alpha = 1 - \gamma$.

6.4 Sign of the square-root in (17)

Regarding (17), an issue to be addressed is the sign of the square root. With $\psi = 1$, as $\gamma \to 1$, the utility representation (9) converges to the log-utility representation. The exact solution of (20) in the special case of log-utility ($\gamma = \psi = 1$) is well-known (Merton (1969, 1971, 1973)):

$$\pi_t = (\mu - r) y_t,$$

i.e., the intertemporal hedging demand component disappears (if $\psi = \gamma = 1$, then $A = B = 0$). It is therefore necessary to guarantee that with $\psi = 1$, $\lim_{\gamma \to 1} A = 0$[25] as the limit of (20) as $\gamma \to 1$ is given by:

$$\lim_{\gamma \to 1} \pi_t = (\mu - r)\, y_t + \left(\lim_{\gamma \to 1} A \right) \rho \sigma y_t \,.$$

From (17), $\lim_{\gamma \to 1} A$ is:

$$\lim_{\gamma \to 1} A = \frac{(\beta+\kappa) \pm \lim\limits_{\gamma \to 1} (1-\gamma)\gamma \sqrt{\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma} \right)^2 - \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}}}{\sigma^2}. \tag{55}$$

If $\gamma \to 1^+$, i.e., $\gamma > 1$, then $(1-\gamma) < 0$ and the discriminant of the square root in (55) is always > 0. By assumption, $\beta + \kappa > 0$, which implies that, in order to have $\lim\limits_{\gamma \to 1^+} A = 0$, the " $+$ " sign must be considered.

The same rational implies that when $\gamma < 1$, the " $-$ " sign of the square root guarantees $\lim\limits_{\gamma \to 1^-} A = 0$ (it can be easily shown that the discriminant of the square root in (55) is positive as γ approaches 1 from below).

When $\gamma = 1$, only the "-" sign of the square root gives $A = 0$, as:

$$A \,|_{\gamma=1} = \frac{(\beta+\kappa) \pm (\beta+\kappa)}{\sigma^2}.$$

6.5 Proof of Proposition 1

The domain of analysis is set so that A in (17) is a real number, i.e., its discriminant is non-negative. Consequently the condition to be satisfied is

$$\left(\frac{\rho\sigma\,(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma} \right)^2 \;\geqslant\; \frac{\sigma^2\,(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\,(1-\gamma)}. \tag{56}$$

For $\gamma > 1$ it is straightforward to conclude that $\left(\frac{\rho\sigma(\mu-r)}{\gamma} - \frac{\beta+\kappa}{1-\gamma} \right)^2 > \frac{\sigma^2(\mu-r)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2(1-\gamma)}$, and therefore (56) is always true.
For $\gamma < 1$, (56) is true as long as:

$$\begin{aligned}
\frac{\gamma}{1-\gamma} &\geqslant \frac{\sigma^2\,(\mu-r)^2}{(\beta+\kappa)} + \frac{2\rho\sigma\,(\mu-r)}{(\beta+\kappa)} \\
\Leftrightarrow \gamma &\geqslant \frac{\sigma^2\,(\mu-r)^2 + 2\rho\sigma\,(\mu-r)\,(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2\,(\mu-r)^2 + 2\rho\sigma\,(\mu-r)\,(\beta+\kappa)} \\
\Leftrightarrow \gamma &\geqslant \omega,
\end{aligned}$$

by making $\omega = \frac{\sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}{(\beta+\kappa)^2 + \sigma^2(\mu-r)^2 + 2\rho\sigma(\mu-r)(\beta+\kappa)}$. Note that $\omega < 1$ as $(\beta+\kappa)^2 > 0$.

[25] From (18) $\lim\limits_{\gamma \to 1} A = 0 \implies \lim\limits_{\gamma \to 1} B = 0$.

The domain of analysis is therefore $\gamma \geqslant \omega$.

From (22) and (17)-(18): $\frac{dJ_{t_o}}{d\theta} = \frac{\kappa exp\left\{Ay_{t_0}+B\right\}X_{t_o}^{1-\gamma}}{\beta}\frac{A}{1-\gamma}$. This implies that:

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = Sign\,\frac{A}{1-\gamma}\,, \tag{57}$$

as $\frac{\kappa exp\left\{Ay_{t_0}+B\right\}X_{t_o}^{1-\gamma}}{\beta} > 0$. Further development of (57) gives:

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = Sign\left\{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma\left(\mu-r\right)}{\gamma} \pm \sqrt{\left(\frac{\rho\sigma\left(\mu-r\right)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2\left(\mu-r\right)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\left(1-\gamma\right)}}\right\}, \tag{58}$$

since $\frac{\gamma}{\sigma^2[\gamma(1-\rho^2)+\rho^2]} > 0$ as, by assumption, $\gamma > 0$ and $0 < \rho < 1$.

Recovering conclusions regarding the sign of the square root in Appendix 6.4, (58) becomes:
For $\gamma > 1$:

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = Sign\left\{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma\left(\mu-r\right)}{\gamma} + \sqrt{\left(\frac{\rho\sigma\left(\mu-r\right)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2\left(\mu-r\right)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\left(1-\gamma\right)}}\right\}.$$

Although $\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} < 0$, when $\gamma > 1$ it results:

$$\sqrt{\left(\frac{\rho\sigma\left(\mu-r\right)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2\left(\mu-r\right)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\left(1-\gamma\right)}} > \left|\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma\left(\mu-r\right)}{\gamma}\right|,$$

implying that $Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = " + " \,|_{\gamma>1}$.

For $\omega \leqslant \gamma < 1$:

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = Sign\left\{\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma\left(\mu-r\right)}{\gamma} - \sqrt{\left(\frac{\rho\sigma\left(\mu-r\right)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2\left(\mu-r\right)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\left(1-\gamma\right)}}\right\}.$$

For $\gamma < 1$, one gets $\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} > 0$ if $\gamma > \frac{\rho\sigma(\mu-r)}{(\beta+\kappa)+\rho\sigma(\mu-r)}$.

As $\frac{\rho\sigma(\mu-r)}{(\beta+\kappa)+\rho\sigma(\mu-r)} < \omega$ it results that $\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma(\mu-r)}{\gamma} > 0\,|_{\omega\leqslant\gamma<1}$. This implies:

$$\frac{\beta+\kappa}{1-\gamma} - \frac{\rho\sigma\left(\mu-r\right)}{\gamma} > \sqrt{\left(\frac{\rho\sigma\left(\mu-r\right)}{\gamma} - \frac{\beta+\kappa}{1-\gamma}\right)^2 - \frac{\sigma^2\left(\mu-r\right)^2\left[\gamma\left(1-\rho^2\right)+\rho^2\right]}{\gamma^2\left(1-\gamma\right)}}$$

when $\omega \leqslant \gamma < 1$, and consequently $Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = " + " \,|_{\omega\leqslant\gamma<1}$.

Thus overall:[26]

$$Sign\left(\frac{dJ_{t_o}}{d\theta}\right) = \text{"}+\text{"}\ , \ with\, \gamma \geqslant \omega.$$

Following the spirit of the Maxmin Expected Utility model of Gilboa and Schmeidler (1989), as an increase in the expected precision of the risky asset return increases utility, then the lowest possible value for the expected precision is considered, i.e., $\theta^* = \underline{\theta}$.

6.6 Proof of Proposition 3

Proof of (i): $\underline{\theta} < \theta \Rightarrow \pi_{\underline{\theta}} < \pi_{\theta}$.

The difference between the mean allocation to the risky asset when ambiguity is considered $\left(\pi_{\underline{\theta}}\right)$ versus when there is no ambiguity (π_{θ}), is given by (26):

$$\pi_{\underline{\theta}} - \pi_{\theta} \quad = \quad \left(\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A\right)(\underline{\theta} - \theta)\ .$$

We need only to show that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$ in the domain under analysis $(\gamma \geqslant \omega)$:
For $\gamma > 1$, it is algebraically easy to show that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$, by substituting the expression for A (17) and considering the positive sign of its square root in line with conclusions in Appendix 6.4.

For $\gamma = 1$, it results that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A = (\mu - r) > 0$, by substituting the expression for A (17) and considering the negative sign of its square root in line with conclusions in Appendix 6.4.

For $\omega \leqslant \gamma < 1$, from (57) and conclusions in Appendix 6.5, it is immediate to conclude that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$, as $A > 0$.

Since $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$ and $\underline{\theta} < \theta$ (when $\underline{\theta} = \theta$ there is no ambiguity), it is immediate to conclude that $\left(\pi_{\underline{\theta}} - \pi_{\theta}\right) < 0$ and that ambiguity demand (26) is always negative.

Proof of (ii): $\pi_{\underline{\theta}}$ *is decreasing with* $(\theta - \underline{\theta})$.

The relation between the mean allocation to the risky asset when there exists ambiguity $(\pi_{\underline{\theta}})$ and changes in the level of ambiguity, measured by $(\theta - \underline{\theta})$, is:

$$\frac{d\pi_{\underline{\theta}}}{d(\theta - \underline{\theta})} = -\left(\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A\right).$$

As $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}A > 0$ (see previous proof of (i)), it results that an increased level of ambiguity implies a reduction of the mean allocation to the risky asset.

Proof of (iii): *Sign of the intertemporal hedging demand.*

[26] By applying l'Hôpital rule to the calculation of $\lim_{\gamma \to 1}\left(\frac{dJ_{t_o}}{d\theta}\right)$ one obtains $\lim_{\gamma \to 1}\left(\frac{dJ_{t_o}}{d\theta}\right) = \frac{\sigma^2(\mu - r)^2}{2(\beta + \kappa)}$, which is > 0.

From (25), and since $\sigma, \rho, \gamma, \theta > 0$:

$$Sign(HD) = Sign(A).$$

From (57) and conclusions in Appendix 6.5:

$$\gamma \gtrless \omega: \quad \begin{cases} \gamma > 1: & Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = " + " \Rightarrow Sign\left(A\right) = " - " \\ \\ \omega \leqslant \gamma < 1: & Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = " + " \Rightarrow Sign\left(A\right) = " + " \end{cases}$$

When $\gamma = 1$, intertemporal hedging demand is null because $A = 0\,|_{\gamma=1}$ (see Appendix 6.4).

6.7 Proof of Proposition 4

From (28)-(29) and (33):

$$\frac{dJ_{t_0}}{d\theta} = -\frac{1}{1-\psi}\frac{A_1\kappa}{h_1} exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)\left(A_1 y_{t_0} + B_1\right)\right\} X_{t_0}^{1-\gamma}.$$

As $exp\left\{-\left(\frac{1-\gamma}{1-\psi}\right)\left(A_1 y_{t_0} + B_1\right)\right\} > 0$ and $X_{t_0}^{1-\gamma}$, κ, $h_1 > 0$:

$$Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = Sign\left(-\frac{1}{1-\psi}A_1\right). \tag{59}$$

Expression (28) for A_1 is further simplified to:

$$A_1 = \frac{(1-\psi)\left[b + \sqrt{b^2+u}\right]}{(1-\gamma)\,e}, \; with \tag{60}$$

$$\begin{aligned} b &= \rho\sigma\left(\mu - r\right)(1-\gamma) - \gamma\left(h_1 + \kappa\right) \\ u &= -\sigma^2\left(\mu - r\right)^2(1-\gamma)\left[\gamma\left(1-\rho^2\right) + \rho^2\right] \\ e &= \sigma^2\left[\gamma\left(1-\rho^2\right) + \rho^2\right], \quad always > 0 \; as \, 0 < \rho < 1, \, \gamma > 0. \end{aligned}$$

Substituting (60) into (59), for $\gamma > 1$ one obtains:

$$Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = Sign\left(b + \sqrt{b^2+u}\right).$$

As $\sqrt{b^2+u} > b$, because $b < 0$ and $u > 0$ when $\gamma > 1$, then

$$Sign\left(\frac{dJ_{t_0}}{d\theta}\right) = " + ",$$

which, following the same rational as in Appendix 6.5, implies $\theta^* = \underline{\theta}$.

6.8 Proof of Proposition 6

Proof of (i): $\underline{\theta} < \theta \Rightarrow \pi_{\underline{\theta}} < \pi_{\theta}$.

The difference between the mean allocation to the risky asset under ambiguity aversion $\left(\pi_{\underline{\theta}}\right)$ versus under ambiguity neutrality (π_{θ}), is given by (38):

$$\pi_{\underline{\theta}} - \pi_{\theta} \quad = \quad \left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right](\underline{\theta} - \theta) \ . \tag{61}$$

Substituting expression for A_1 (28) into $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1$, after some algebra the latter expression is simplified to:

$$\frac{(\mu - r)\sigma^2\gamma + \sigma\rho\left[\gamma(h_1 + \kappa) - \sqrt{\left[(h_1 + \kappa)\gamma - \rho\sigma(\mu - r)(1-\gamma)\right]^2 - \sigma^2(\mu - r)^2(1-\gamma)\left[\gamma(1-\rho^2) + \rho^2\right]}\right]}{\gamma\sigma^2\left[\gamma(1-\rho^2) + \rho^2\right]},$$

which is always positive as both the numerator and denominator are positive.

Going back to (61), and considering that $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1 > 0$ and $\underline{\theta} < \theta$, it is immediate to conclude that $\pi_{\underline{\theta}} - \pi_{\theta} < 0$, and that ambiguity demand is always negative.

Proof of (ii): *$\pi_{\underline{\theta}}$ is decreasing with$(\theta - \underline{\theta})$.*

The relation between the mean allocation to the risky asset under ambiguity aversion and changes in the level of ambiguity, measured by $(\theta - \underline{\theta})$, is given by:

$$\frac{d\pi_{\underline{\theta}}}{d(\theta - \underline{\theta})} = -\left[\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1\right]$$

As $\frac{1}{\gamma}(\mu - r) + \frac{\sigma\rho}{\gamma}\frac{(\gamma-1)}{(1-\psi)}A_1 > 0$ (see previous proof of (i)), it results that increased level of ambiguity implies a reduction of the mean allocation to the risky asset.

Proof of (iii): *Sign of intertemporal hedging demand* (HD).

From (37) and considering that $\sigma, \rho, \theta > 0$ and $\gamma > 1$, it results that the sign of intertemporal hedging demand (HD) is:

$$Sign(HD) \quad = \quad Sign\left(\frac{A_1}{1-\psi}\right).$$

From (59) and conclusions on Appendix 6.7 it is known that $\frac{A_1}{1-\psi} < 0$.

Proof of (iv): *$c_{\underline{\theta}}$ is higher (lower) under θ-ambiguity aversion if $\psi > 1$ (< 1).*

Under the "expectation-driven" scenario, from (34) and (28)-(29):

$$\begin{aligned} \frac{dc_{\underline{\theta}}}{d\underline{\theta}} &= -A_1\left(1+\frac{\kappa}{h_1}\right)\beta^{\psi}exp\left\{-\left(A_1\underline{\theta}+B_1\left(\underline{\theta}\right)\right)\right\} \\ \Rightarrow \quad Sign\left(\frac{dc_{\underline{\theta}}}{d\underline{\theta}}\right) &= Sign\left(-A_1\right), \end{aligned} \tag{62}$$

as $\left(1+\frac{\kappa}{h_1}\right)\beta^{\psi}exp\left\{-\left(A_1\underline{\theta}+B_1\left(\underline{\theta}\right)\right)\right\}>0$. From (60), for $\gamma>1$:

$$\begin{aligned} Sign\left(-A_1\right) &= \left(1-\psi\right)Sign\left(\frac{b+\sqrt{b^2+u}}{\left(\gamma-1\right)e}\right) \\ Sign\left(-A_1\right) &= Sign\left(1-\psi\right), \end{aligned}$$

as $\frac{b+\sqrt{b^2+u}}{(\gamma-1)e}>0$ (see Appendix 6.7), and therefore:

$$\begin{aligned} \psi<1 \quad &: \quad \frac{dc_{\theta}}{d\theta}>0, \\ \psi>1 \quad &: \quad \frac{dc_{\theta}}{d\theta}<0. \end{aligned}$$

Consequently, when $\psi<1$, θ-ambiguity aversion decreases the optimal consumption wealth ratio ($\underline{\theta}<\theta$). The contrary happens when $\psi>1$. The same conclusion is obtained when studying $\frac{dc_t}{dy_t}$, with c_t given by (30), as $Sign\left(\frac{dc_t}{dy_t}\right)=Sign\left(-A_1\right)$.

6.9 Expression (40)

From (5):

$$E\left[v_t\right]2\theta^2\kappa-2\theta\kappa-\sigma^2\approx 0$$

which is a quadratic equation for θ with roots:

$$\theta\approx\frac{2\kappa\pm\sqrt{4\kappa^2+8\kappa\sigma^2E\left[v_t\right]}}{4\kappa E\left[v_t\right]}.$$

As $\theta>0$, $2\kappa<\sqrt{4\kappa^2+8\kappa\sigma^2E\left[v_t\right]}$ and $4\kappa E\left[v_t\right]>0$, only the positive sign of the square root is relevant.

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