The total median in Statistical Quality Control*

Fernanda Figueiredo
CEAUL, Faculdade de Economia, Universidade do Porto
and
M. Ivette Gomes
CEAUL and DEIO (FCUL), Universidade de Lisboa

Abstract. In industry, most of the process observations may be assumed to come from a normal population, but usually we merely want to control the mean value μ. It is thus sensible to find control statistics, which are "robust" to monitor the process mean, giving no kind of false alarm whenever that mean is close to the target value, although not under a normal regime. Simulation studies for a few symmetric and asymmetric distributions allow us to suggest the total median as a robust location estimator, and we shall here analyse the robustness of the total median chart comparatively to the sample mean chart. Some indication is also provided on their comparative out-of-control behaviour.

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1 Introduction and the total median

Control charts are tools widely used in industry to detect abnormal behaviour in manufacturing processes. In general we assume that the process observations are from a normal population with mean μ and standard deviation σ. However, even if it is sensible to assume, on the basis of both theoretical and practical reasons, that the process is normal, there is often a possibility of having disturbances in the data. We then need to find efficient and robust estimators to monitor the process parameters. Simulation studies for some symmetric and asymmetric distributions related to the normal, developed in Figueiredo and Gomes (2000) allow us to suggest the total median as a robust location estimator.

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Let $(X_1, X_2, \cdots, X_n)$ be a random sample of size n from a parent F and, as usual, let us denote $X_{i:n}$, $1 \le i \le n$, the random sample of the associated ascending order statistics (o.s.), and $(X_1^*, X_2^*, \cdots, X_n^*)$ the random bootstrap sample associated to an observed sample $(x_1, x_2, \cdots, x_n)$. We thus mean that X_i^*, $1 \le i \le n$, are obtained from our observed sample through a sampling with replacement, i.e., they are independent, identically distributed (i.i.d.) replicates from X^*, a random variable (r.v.) with distribution function (d.f.)

$$F_n^*(x) = \frac{1}{n} \sum_{i=1}^{n} I_{\{x_i \le x\}},$$

the empirical d.f. of our observed sample. Also, as usual, I_A denotes the indicator function of the set A.

The *total median* is the statistic

$$TMd = \sum_{i=1}^{n} \sum_{j=i}^{n} \alpha_{ij} \, \frac{X_{i:n} + X_{j:n}}{2},$$

$$\alpha_{ij} = P\left(BMd = \frac{x_{i:n} + x_{j:n}}{2}\right), \quad 1 \le i \le j \le n, \quad (1.1)$$

where BMd denotes the median of the bootstrap sample. The probabilities α_{ij} (also denoted $\alpha_{i,j}$) will be explicited in section 2 of this paper, where we shall deal with the total median as a robust location estimator. In section 3 we analyse the robustness of the total median chart comparatively to the sample mean chart, drawing a comparison of this new chart with the classical Shewhart $\overline{X}$-chart in an out-of-control context. Finally, in section 4, we shall draw some overall conclusions.

2 The total median as a robust location estimator

According to the usual definition of sample median we have

$$BMd := \begin{cases} X_{m:n}^* & \text{if} \quad n = 2m-1 \\ \\ \frac{X_{m:n}^* + X_{m+1:n}^*}{2} & \text{if} \quad n = 2m \end{cases}, \quad m = 1, 2, 3, \cdots. \quad (2.1)$$

Following closely the arguments in Efron (1979) and Efron and Tibshirani (1993) we may easily obtain the probabilities α_{ij} in terms of binomial distributions. Indeed, let $N_i^* = \sum_{j=1}^{n} I_{\{X_j^* = x_i\}}$, $1 \le i \le n$. The random vector $(N_1^*, N_2^*, \cdots, N_n^*)$ follows a multinomial scheme, with all parameters equal to $1/n$. Let $N_{(i)}^*$ denote the values of N_i^* induced by the o.s. of our sample. For every integer l, $1 \le l < n$,

$$\begin{aligned} P(X^*_{m:n} > x_{l:n}) &= P(N^*_{(1)} + \cdots + N^*_{(l)} \le m-1) \\ &= P\left(\text{Binomial}\left(n, \tfrac{l}{n}\right) \le m-1\right) \\ &= \sum_{j=0}^{m-1} \binom{n}{j} \left(\frac{l}{n}\right)^j \left(1 - \frac{l}{n}\right)^{n-j}. \end{aligned}$$

For $i \ne j$ the probabilities α_{ij} are obviously equal to 0 if n is odd. For n even, and for $1 \le i < j \le n$, we obtain

$$\begin{aligned} \alpha_{ij} &= \frac{n!}{((n/2)!)^2} \sum_{k=0}^{\frac{n}{2}-1} \binom{n/2}{k} \left(\frac{i-1}{n}\right)^k \sum_{r=1}^{\frac{n}{2}} \binom{n/2}{r} \left(\frac{1}{n}\right)^{\frac{n}{2}-k+r} \left(\frac{n-j}{n}\right)^{\frac{n}{2}-r} \\ &= \frac{n!}{n^n\left((n/2)!\right)^2} \left\{ i^{n/2} - (i-1)^{n/2} \right\} \left\{ (n-j+1)^{n/2} - (n-j)^{n/2} \right\} \\ &= \alpha_{n-j+1,n-i+1}. \end{aligned}$$

For the particular case $i = 1$, $j = n > 1$ we have

$$\alpha_{1n} = \frac{n!}{n^n((n/2)!)^2}.$$

For $i = j$ we have for all i from 1 till n,

$$\alpha_{ii} = \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \sum_{r=[n/2]-k+1}^{n-k} \binom{n-k}{r} \left(\frac{i-1}{n}\right)^k \left(\frac{1}{n}\right)^r \left(\frac{n-i}{n}\right)^{n-k-r},$$

where $[x]$ denotes the integer part of x.

The particular cases $i = 1$ and $i = n$ may thus be written as

$$\alpha_{11} = \sum_{r=[n/2]+1}^{n} \binom{n}{r} \left(\frac{1}{n}\right)^r \left(\frac{n-1}{n}\right)^{n-r}$$

and

$$\alpha_{nn} = \sum_{k=0}^{[(n-1)/2]} \binom{n}{k} \left(\frac{1}{n}\right)^{n-k} \left(\frac{n-1}{n}\right)^k.$$

The *total median* may also be expressed as a linear combination of the sample order statistics, i.e.,

$$TMd = \sum_{i=1}^{n} a_i \; X_{i:n},$$

where the coefficients a_i are obviously related with the previous coefficients α_{ij}, through the relation

$$a_i = \frac{1}{2}\left(\sum_{j=i}^{n} \alpha_{ij} + \sum_{j=1}^{i} \alpha_{ji}\right).$$

These coefficients are "distribution-free", i.e., they are independent of the underlying model F, and depend only on the sample size n. In Table 1 we present the values a_i, with 3 decimal figures, for the most usual rational subgroups n in *Statistical Quality Control* (*SQC*). The missing coefficients in the table are either zero or obtained by symmetrization, i.e., through the condition

$$a_i = a_{n-i+1}, \quad 1 \le i \le n, \quad 0 < a_1 \le a_2 \le \cdots \le a_{[n/2]}, \quad \sum_{i=1}^{n} a_i = 1.$$

Table 1: Values of the coefficients a_i, for sample sizes $n \le 20$.

i	1	2	3	4	5	6	7	8	9	10	15	20
1	1.000	0.500	0.259	0.156	0.058	0.035	0.010	0.007	0.001	0.001	0.000	0.000
2			0.482	0.344	0.259	0.174	0.098	0.064	0.029	0.019	0.000	0.000
3					0.366	0.291	0.239	0.172	0.115	0.078	0.040	0.000
4							0.306	0.257	0.221	0.168	0.021	0.001
5									0.268	0.234	0.063	0.070
6											0.125	0.023
7											0.183	0.055
8											0.208	0.099
9												0.143
10												0.172

A large scale simulation study of size 25×2500 has been undertaken to evaluate the performance of the location estimators $\overline{X}$ and TMd; such evaluation was done in terms of their mean squared error; as a by-product, we have also evaluated the performance of the sample median Md and of the bootstrap median BMd. In this simulation we have considered an adequate set of parent distributions, in order to have different skewness and tail-weight coefficients. The skewness and tail-weight coefficients herewith considered are

$$\gamma_F = \frac{\mu_3}{\mu_2^{3/2}},$$

where μ_r denotes the r-th central moment of F, and

$$\tau_F = \frac{1}{2}\left(\frac{F^{-1}(0.99) - F^{-1}(0.5)}{F^{-1}(0.75) - F^{-1}(0.5)} + \frac{F^{-1}(0.5) - F^{-1}(0.01)}{F^{-1}(0.5) - F^{-1}(0.25)}\right)\left(\frac{\Phi^{-1}(0.99) - \Phi^{-1}(0.5)}{\Phi^{-1}(0.75) - \Phi^{-1}(0.5)}\right)^{-1},$$

respectively. F^{-1} and Φ^{-1} denote the inverse functions of F and of the standard normal distribution function Φ, respectively. The global set of distributions considered includes the following well-known symmetric distributions: the normal (with tailweight 1), the logistic (with tailweight 1.21), Student-*t* models (with tailweights from 1.07 till 1.72) and several contaminated normal models, with high tailweights. The asymmetric distributions considered were the lognormal, the chi-square and the gamma distributions, with the usual parametrization (Johnson et al., 1994, 1995); in this set of distributions we have got values γ_F between 0 and 33.5, and values τ_F between 1 and 5.9. Notice that for all symetric models $\gamma_F = 0$ and that for the normal d.f. $\tau_F = 1$.

In Figure 1 we present the most efficient estimator (the one with smallest mean squared error) among the ones considered, for the estimation of the location of a symmetric distribution. Figures 2 and 3 are equivalent to Figure 1, but for the estimation of the mean and the median, respectively, of an asymmetric distribution.

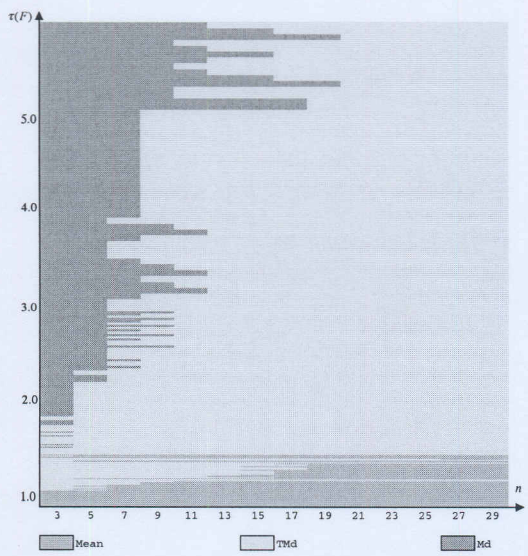

Figure 1: Most efficient estimator for the location of a symmetric distribution.

To obtain the "robust estimator" we have applied a $MaxMin$ criterion:

1. For every model we have obtained the most efficient estimator, among the ones considered.

2. Next, we have computed the efficiency of the other estimators relatively to the best one, selected in 1., retaining the smallest one.

3. The "degree of robustness" is given by that minimum efficiency, and our "robust estimator" is the one with the highest minimum efficiency.

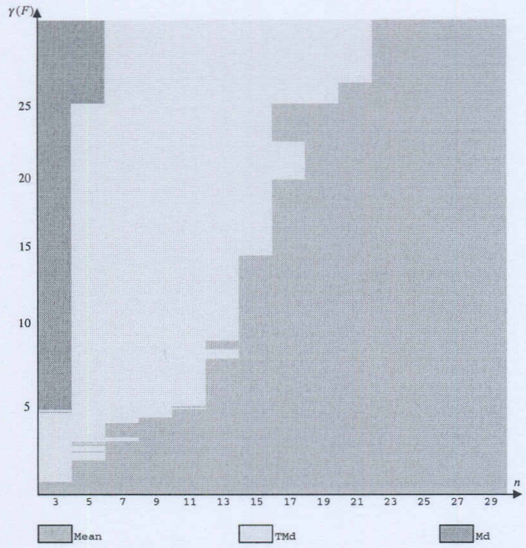

Figure 2: Most efficient estimator for the mean of an asymmetric distribution.

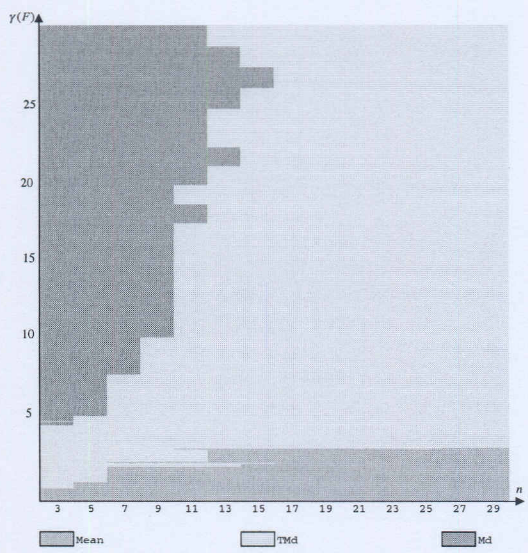

Figure 3: Most efficient estimator for the median of an asymmetric distribution.

The conclusions regarding the robustness of the total median are summarized graphically in Figures 4, 5 and 6. In Figure 4 we represent the minimum relative efficiencies of the location estimators under study, whenever we take into account all the symmetric distributions under consideration. Figures 5 and 6 are analogous to Figure 4, but related to the estimation of the mean and the median of an asymmetric distribution, respectively.

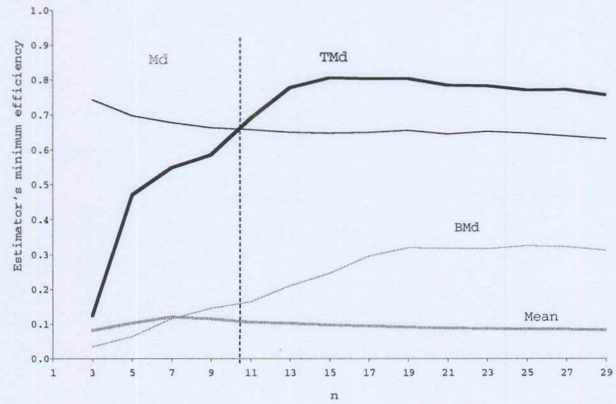

Figure 4: “Degree of robustness” of the location estimators under study, for symmetric parents.

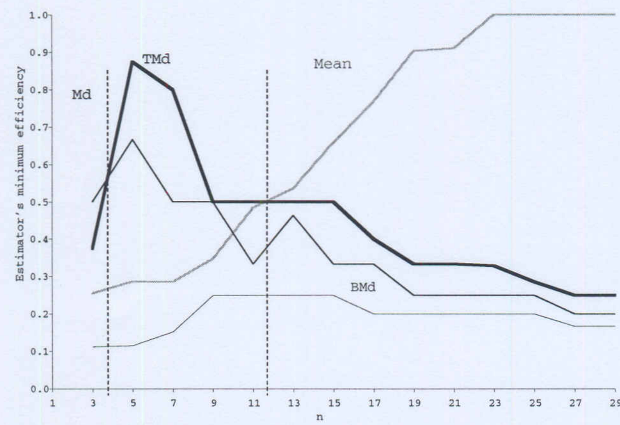

Figure 5: “Degree of robustness” of the mean value estimators under study, for asymmetric parents.

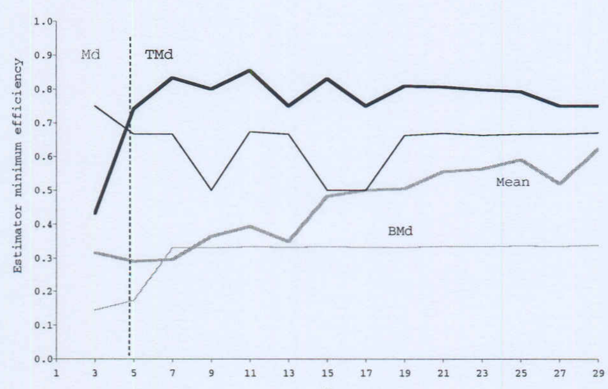

Figure 6: “Degree of robustness” of the median estimators under study, for asymmetric parents.

We have also evaluated the *breakdown point* of the TMd estimator, which depends obviously on the precision of the weights a_i, considered with an adequate fixed number of decimal figures. Informally, the *breakdown point* of an estimator corresponds to the largest fraction of observations we may change in our sample, keeping the estimator under control. For small sample sizes, the breakdown point of the TMd is equal to 0, such as happens to the breakdown point of the mean estimator $\overline{X}$. However, such a value increases with n, and tends to 1/2, the breakdown point of the median Md, as n increases to infinity.

In Figure 8 we represent a *"quasi-change-value scatterplot"* for the TMd statistic, similar to the *change-value curve*. The *change-value curve* measures the rate of change in an estimator whenever an extra observation x is added to our sample (Hampel et al., 1986). Here, the *"Quasi-Change-Value scatterplot"*, $QCV_x(i)$, measures the rate of change in our estimator when one of the o.s. in the sample, say $x_{i:n}$, changes to a position $x \in (x_{i-1:n}, x_{i+1;n})$, $1 \leq i \leq n$ $(x_{0:n} = -\infty,\ x_{n+1:n} = +\infty)$. It is thus possible to say that the introduction of a new observation in a small sample has a small influence on the TMd, being such an influence negligible for large sample sizes and whenever such a new observation is an extreme order statistic.

The total median also enables us to obtain non-parametric confidence intervals for the quantiles with a smaller size than those based on the sample median, for the same coverage probability. Details on robust estimators may be found in Hoaglin et al. (1983), Lax (1985) and Tatum (1997). Cox and Iguzquiza (2001) provide an application of the total median in metrology, in the area of intra-laboratory comparisons, providing a robust estimate of a reference value, which enables the comparison of different laboratories' nominal measures of the same quantity.

3 Robustness of the $\overline{X}$ and TMd control charts

Whenever we are controlling the mean process at μ_0, assuming an underlying normal parent for our data, a control chart based on a statistic W is said to be "robust" if the alarm rate is as small as possible whenever the model changes but the mean is kept at the target μ_0.

To investigate the robustness of the $\overline{X}$ and TMd control charts we have thus computed either analitically or through Monte Carlo simulation techniques the alarm rates of both charts given that there are deviations from normality, always maintaining the mean at $\mu_0 = 0$, without loss of generality. In our study we have considered 3-*sigma* control charts both for $\overline{X}$ and TMd. We are however aware that it would have been preferable for the TMd statistic to use control limits based on its sampling distribution. For the

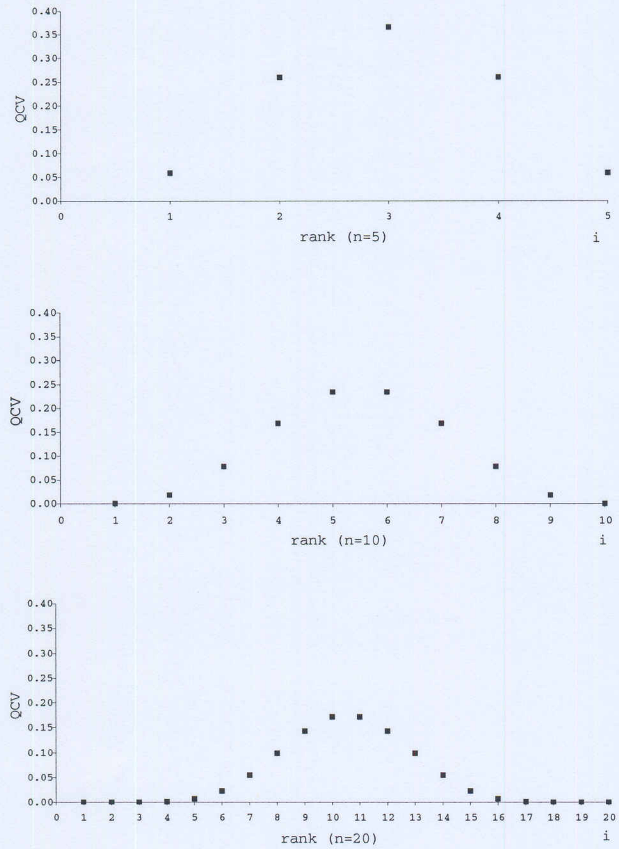

Figure 7: "Quasi-change-value scatterplots" of the total median, TMd.

standard normal parent we have $E[TMd] = 0$ and $Var[TMd]$ is given in Table 2.

Table 2: Variance of the total median in a standard normal model.

1	—-	6	0.18062	11	0.10757	16	0.07647
2	0.50000	7	0.16038	12	0.09881	17	0.07286
3	0.33907	8	0.14066	13	0.09267	18	0.06886
4	0.25681	9	0.12853	14	0.08628	19	0.06592
5	0.21476	10	0.11592	15	0.08157	20	0.06274

In Tables 3 and 4 we present, for sub-rational groups of size $n = 3, 4, 5, 6, 7$, 10, 15 and 20, those control limits and the alarm rates for the $\overline{X}$ and the TMd charts, respectively, whenever we consider standardized data from several well-known distributions with different skewness and tail-weight. Apart from the ones considered before, we have also included several chi-squared models, $\chi^2_k \overset{d}{=}$ Gama$(0, 2, k/2)$, the Inverse Gaussian model, with d.f. $F_{GI}(x; \mu, \zeta) = \Phi\left(\sqrt{\zeta/x}\ (x/\mu - 1)\right) + e^{2\zeta/\mu}\Phi\left(-\sqrt{\zeta/x}\ (x/\mu + 1)\right)$, $x \geq 0$, with Φ the standard Gaussian d.f., and the standard Weibull model.

Table 3: Alarm rates of the 3-sigma $\overline{X}$-chart.

Distribution	γ	τ	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=10$	$n=15$	$n=20$
$Normal$	0.000	1.000	0.0027	0.0027	0.0028	0.0028	0.0027	0.0028	0.0027	0.0028
$D_1 : t_{20}$	0.000	1.067	0.0034	0.0032	0.0032	0.0031	0.0030	0.0030	0.0030	0.0028
$D_2 : t_{10}$	0.000	1.145	0.0046	0.0043	0.0039	0.0037	0.0037	0.0034	0.0032	0.0031
$D_3 : Log(1)$	0.000	1.212	0.0051	0.0046	0.0041	0.0040	0.0039	0.0036	0.0033	0.0031
$D_4 : t_3$	0.000	1.721	0.0117	0.0112	0.0107	0.0104	0.0100	0.0091	0.0084	0.0079
$D_5 : W(2)$	0.631	0.911	0.0038	0.0035	0.0032	0.0032	0.0031	0.0030	0.0029	0.0028
$D_6 : \chi^2_{20}$	0.632	1.001	0.0043	0.0039	0.0037	0.0037	0.0034	0.0032	0.0030	0.0033
$D_7 : \chi^2_{10}$	0.894	1.004	0.0055	0.0049	0.0044	0.0041	0.0040	0.0036	0.0033	0.0033
$D_8 : Ga(2)$	1.414	1.018	0.0085	0.0075	0.0066	0.0061	0.0059	0.0049	0.0043	0.0038
$D_9 : LN(0.5)$	1.750	1.143	0.0106	0.0093	0.0084	0.0079	0.0073	0.0062	0.0052	0.0046
$D_{10} : Exp(1)$	2.000	1.062	0.0118	0.0105	0.0093	0.0087	0.0080	0.0067	0.0057	0.0049
$D_{11} : \chi^2_1$	2.828	1.218	0.0156	0.0141	0.0128	0.0117	0.0109	0.0093	0.0075	0.0067
$D_{12} : Ga(0.5)$	2.828	1.218	0.0159	0.0140	0.0129	0.0117	0.0112	0.0092	0.0076	0.0067
$D_{13} : GI(1)$	3.000	1.371	0.0164	0.0147	0.0135	0.0126	0.0115	0.0101	0.0082	0.0072
$D_{14} : LN(1)$	6.185	1.658	0.0171	0.0168	0.0161	0.0158	0.0154	0.0141	0.0127	0.0118
$D_{15} : W(0.5)$	6.619	2.260	0.0201	0.0198	0.0192	0.0187	0.0180	0.0169	0.0150	0.0136
Control limits			±1.732	±1.500	±1.342	±1.225	±1.134	$\pm.949$	$\pm.775$	$\pm.671$

Table 4: Alarm rates of the 3-sigma TMd-chart.

Distribution	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=10$	$n=15$	$n=20$
$Normal$	0.00270	0.00262	0.00276	0.00277	0.00271	0.00283	0.00271	0.00277
$D_1 : t_{20}$	0.00325	0.00288	0.00268	0.00252	0.00249	0.00236	0.00206	0.00206
$D_2 : t_{10}$	0.00397	0.00340	0.00264	0.00242	0.00207	0.00181	0.00156	0.00141
$D_3 : Logistic(1)$	0.00449	0.00356	0.00273	0.00247	0.00206	0.00161	0.00129	0.00108
$D_4 : t_3$	0.00814	0.00550	0.00197	0.00113	0.00056	0.00020	0.00005	0.00003
$D_5 : W(2)$	0.00378	0.00339	0.00301	0.00291	0.00274	0.00281	0.00330	0.00410
$D_6 : \chi^2_{20}$	0.00405	0.00346	0.00295	0.00287	0.00244	0.00227	0.00234	0.00237
$D_7 : \chi^2_{10}$	0.00511	0.00415	0.00316	0.00274	0.00247	0.00193	0.00205	0.00268
$D_8 : Ga(2)$	0.00746	0.00572	0.00389	0.00307	0.00238	0.00134	0.00113	0.00210
$D_9 : LN(0.5)$	0.00841	0.00585	0.00360	0.00283	0.00187	0.00096	0.00057	0.00096
$D_{10} : Exp(1)$	0.00950	0.00704	0.00433	0.00329	0.00222	0.00101	0.00039	0.00064
$D_{11} : \chi^2_1$	0.01150	0.00797	0.00431	0.00292	0.00166	0.00057	0.00013	0.00003
$D_{12} : Ga(0.5)$	0.01181	0.00806	0.00427	0.00291	0.00181	0.00060	0.00016	0.00005
$D_{13} : GI(1)$	0.01170	0.00777	0.00390	0.00269	0.00144	0.00050	0.00010	0.00003
$D_{14} : LN(1)$	0.01201	0.00805	0.00302	0.00165	0.00072	0.00018	0.00002	0.00000
$D_{15} : W(0.5)$	0.01353	0.00890	0.0283	0.00163	0.00066	0.00010	0.00001	0.00000
Control limits	±1.747	±1.520	±1.390	±1.275	±1.201	±1.021	±0.857	±0.751

For the most usual rational subgroups in SQC, $n = 5$ and $n = 10$, we present in Figure 7 the alarm rates of both the $\overline{X}$ and the TMd charts. We have separated symmetric and asymmetric distributions, and we have ordered the symmetric distributions by the tail-weight coefficient τ and the asymmetric distributions by the skewness coefficient γ. It is clear from Tables 3 and 4, partially pictured in Figure 8, that there is a reasonably high variability of alarm rates for both charts when the model is no longer normal (much more evident for the $\overline{X}$-chart). Neverthless, for small samples, particularly for the usual rational subgroup size, $n = 5$, the differences to the normal-case are much smaller whenever we consider the TMd-chart, even for asymmetric models with a high tail-weight, like the χ_1^2.

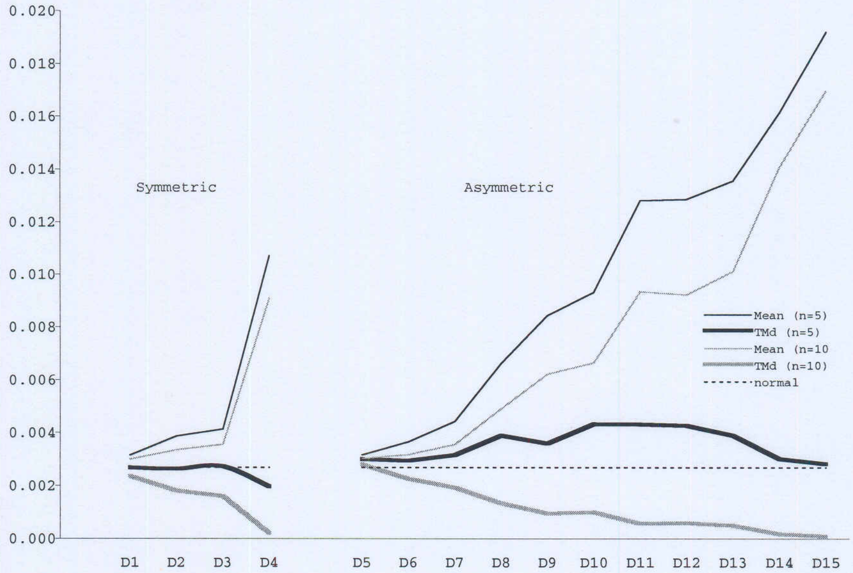

Figure 8: Alarm rates of the 3-sigma $\overline{X}$ and the TMd control charts for $n = 5$ and $n = 10$.

The way the TMd-chart has been devised, with the main objective of providing no kind of false alarm whenever the mean of the process is close to the target value, led us to think that the TMd-chart would have, in an out-of-control situation, a worse performance than the $\overline{X}$-chart. Astonishingly, for a great diversity of models, the TMd-chart is even able to overpass the $\overline{X}$-chart, providing a faster alarm signal when the process is out of control. We shall next compare the efficiencies of an $\overline{X}$ and a TMd control chart, whenever monitoring the mean value of non-normal processes, for rational subgroups of size $n = 5$ and control limits placed at the quantiles $\chi_{0.001}$ and $\chi_{0.999}$ of the respective statistic. In Table 5 we present the power functions of both charts for processes from some of the models previously described, sometimes with specific re-parametrizations. For instance: the $Logistic(\mu, \sigma)$ model, has a d.f. $F_L(x; \mu, \sigma) = \left(1 + \exp\left(-\frac{\pi(x-\mu)}{\sqrt{3}\sigma}\right)\right)^{-1}$, $x \in \mathbb{R}$; the lognormal model, $LN_2(\mu, \sigma, \delta)$, is a model with mean value μ and standard deviation σ such that, with $\xi = \mu - \sigma/\sqrt{\exp(\delta^2) - 1}$, $\ln\{LN_2(\mu, \sigma, \delta) - \xi\}$ is normal with mean value $\left\{\ln\left(\sigma/\sqrt{\exp(\delta^2) - 1}\right) - \delta^2/2\right\}$ and standard deviation δ; the $Exp(0, \delta)$ model has a d.f. given by $F_E(x) = 1 - e^{-x/\delta}$, $x \geq 0$; the $Gamma(\xi, \delta, \theta)$ model has a probability density function (p.d.f.) given by

$g(x;\xi,\delta,\theta) = (x-\xi)^{\theta-1}\ \exp\{-\frac{x-\xi}{\delta}\}/(\delta^\theta\Gamma(\theta))$; and finally, a $Weibull(\xi,\delta,\theta)$ model has a d.f. given by $F_W(x;\xi,\delta,\theta) = 1 - e^{-((x-\xi)/\delta)^\theta}$, $x \geq \xi$.

Table 5: Power functions of the $\overline{X}$ and the TMd charts with control limits at $\chi_{0.001}$ and $\chi_{0.999}$, whenever detecting changes in the mean value (μ) or the scale parameter (δ) of the process ($n = 5$).

μ	-3.0	-2.0	-1.5	-1.0	0.0	1.0	1.5	2.0	3.0	LCL	UCL
					$Logistic(\mu,1)$						
$\overline{X}$	1.000	0.883	0.523	0.140	0.002	0.150	0.542	0.892	0.999	-1.475	1.454
TMd	0.999	0.897	0.545	0.147	0.002	0.156	0.562	0.905	1.000	-1.452	1.434
					$LN_2(\mu,1,0.5)$						
$\overline{X}$	0.999	0.975	0.876	0.563	0.002	0.032	0.152	0.501	1.000	-0.985	1.946
TMd	1.000	0.983	0.901	0.603	0.002	0.045	0.219	0.654	1.000	-1.028	1.709
					$LN_2(\mu,1,1)$						
$\overline{X}$	0.997	0.985	0.958	0.858	0.002	0.004	0.009	0.024	0.268	-0.616	3.140
TMd	1.000	0.997	0.986	0.927	0.002	0.019	0.101	0.538	1.000	-0.631	1.775
δ	0.1	0.3	0.5	0.7	1.0	1.5	2.0	3.0	5.0	LCL	UCL
					$Exp(0,\delta)$						
$\overline{X}$	0.854	0.101	0.017	0.004	0.002	0.033	0.142	0.457	0.825	-0.854	1.945
TMd	0.793	0.090	0.015	0.004	0.002	0.028	0.116	0.393	0.770	-0.881	1.781
					$Gamma(0,\delta,2)$						
$\overline{X}$	1.000	0.521	0.077	0.011	0.002	0.068	0.310	0.773	0.983	-0.997	1.780
TMd	1.000	0.457	0.067	0.010	0.002	0.058	0.268	0.722	0.972	-1.045	1.707
					$Weibull(0,\delta,2)$						
$\overline{X}$	1.000	0.881	0.162	0.018	0.002	0.193	0.650	0.966	1.000	-1.177	1.531
TMd	1.000	0.772	0.120	0.015	0.002	0.166	0.585	0.944	0.999	-1.250	1.571

4 Some overall conclusions

We here summarize the main conclusions:

1. The total median TMd is a fully non-parametric location estimator, highly efficient and robust for small-to-moderate rational subgroup sizes, the most usual ones in SQC.

2. The sample mean $\overline{X}$ is an efficient estimator of the mean value of a symmetric distribution with moderate tails, although it is not at all robust; for symmetric parents the sample median turns out to be the most robust estimator (among the ones considered) for sample sizes smaller than 10, being the total median the most robust one for sample sizes larger than 10. If the parent model has not too heavy tails, i.e. if we compute the "degree of robustness" only on the basis of a smaller set of symmetric distributions, choosing a threshold in the tail-weight, the total median works better than the sample median, from a point of view of robustness, for sample sizes larger than a value n which decreases as the threshold decreases.

3. For the estimation of either the mean value or the median of an asymmetric distribution we suggest the total median as a robust and efficient location

estimator for small-to-moderate samples. Whenever estimating the mean value, the sample median exhibits the best performance for very small sample sizes; for large sample sizes and not too heavy tail models the sample mean may exhibit the best performance. For the estimation of the median, the TMd clearly overpasses all the other estimators for sample sizes $n \geq 5$.

4. The TMd statistic must however be carefully chosen, because in non-normal situations the alarm rate of such a control chart can be much smaller than expected, particularly if n is large. Together with the consideration of the TMd control statistic, the use of a rational subgroup size $n = 5$ is highly advisable in practice.

5. As expected, for normal data the TMd chart is less efficient than the $\overline{X}$-chart to monitor the location of the process; however, for some magnitudes of shift both charts are approximately equivalent, even for normal data. For other distributions, like for instance the lognormal, the TMd chart performs better, being thus advisable in practice due to its robustness comparatively to the common $\overline{X}$ control chart.

6. The bootstrap median, which might appear as a serious alternative to the sample median shows the worst performance among the location estimators herewith considered.

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