A Partially Reduced-Bias Class of Value-at-Risk Estimators

4 authors, including:

Maria Ivette Gomes
University of Lisbon

Frederico Caeiro
New University of Lisbon

Fernanda Otila Sousa Figueiredo
University of Porto

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M. Ivette Gomes*
CEAUL and DEIO, Universidade de Lisboa, Lisboa, Portugal – ivette.gomes@fc.ul.pt

Frederico Caeiro
CMA and FCT, Universidade Nova de Lisboa, Caparica, Portugal – fac@fct.unl.pt

Fernanda Figueiredo
CEAUL, and Faculdade de Economia, Universidade do Porto, Porto, Portugal – otilia@fep.up.pt

Dinis Pestana
CEAUL and DEIO, Universidade de Lisboa, Lisboa, Portugal – dinis.pestana@fc.ul.pt

For any level q, $0 < q < 1$, and on the basis of a sample $(X_1, \ldots, X_n)$ of either independent, identically distributed or possibly weakly dependent and stationary random variables from an unknown model F with a heavy right-tail function, the value-at-risk at the level q, denoted by VaR_q, the size of the loss that occurred with a small probability q, is estimated by a recent semi-parametric procedure based on a partially reduced-bias extreme value index (EVI) class of estimators, a generalization of the classical Hill EVI-estimator, related to the mean-of-order-p of an adequate set of statistics. Such an estimator depends on two tuning parameters p and k, with $p \geq 0$ and $1 \leq k < n$ the number of top order statistics involved in the semi-parametric estimation, and outperforms previous estimation procedures. The adequate choice of k and p can be done through the use of either a computer-intensive double-bootstrap method or through reliable heuristic procedures. An application in the field of finance is also provided.

Keywords: extreme value theory; semi-parametric estimation; statistics of extremes; value-at-risk.

1 Introduction and scope of the article

Let $(X_1, \ldots, X_n)$ be a sample of independent, identically distributed or possibly weakly dependent and stationary *random variables* (RVs), from an underlying *cumulative distribution function* (CDF) F. Let us denote by $(X_{1:n} \leq \cdots \leq X_{n:n})$ the sample of associated ascending order statistics. If there exist sequences of real numbers, (a_n, b_n), with $a_n > 0$ and $b_n \in \mathbb{R}$, such that the sequence of linearly normalized maxima, $\{(X_{n:n} - b_n)/a_n\}_{n\geq 1}$, converges to a non-degenerate RV, then (Gnedenko, 1943) such a RV is of the type of a general *extreme value* (EV) CDF,

$$\mathrm{EV}_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{-1/\xi}), 1+\xi x > 0, & \text{if} \quad \xi \neq 0, \\ \exp(-\exp(-x)),\ x > 0, & \text{if} \quad \xi = 0. \end{cases} \tag{1}$$

We then say that F is in the max-domain of attraction of EV_ξ, use the notation $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_\xi)$, (a_n, b_n) are the so-called attraction coefficients of F to the limiting law EV_ξ, and the parameter ξ is the *extreme value index* (EVI), one of the most relevant parameters in the field of statistics of extremes.

We shall here consider heavy right tails, i.e. $\xi > 0$ in (1), and we are interested in dealing with the semi-parametric estimation of the *value-at-risk* (VaR_q) at the level q, the size of the loss that occurs with a small probability q. We are thus dealing with the high quantile

$$\chi_{1-q} \equiv \mathrm{VaR}_q := F^{\leftarrow}(1-q),$$

of the unknown CDF F, with $F^{\leftarrow}(y) = \inf\{x : F(x) \geq y\}$ denoting the generalized inverse function of F. As usual, let us denote by $U(t)$ the *tail quantile function* (TQF), i.e. $U(t) := F^{\leftarrow}(1-1/t), \quad t \geq 1$, the generalized inverse function of $1/(1-F)$. For small q, we thus want to estimate the parameter $\mathrm{VaR}_q = U(1/q)$, $q = q_n \to 0$, $nq_n \le 1$, extrapolating beyond the sample, possibly working in the whole $\mathcal{D}_{\mathcal{M}}(\mathrm{EV}_{\xi>0}) =: \mathcal{D}^+_{\mathcal{M}}$, assuming thus that $U(t) \sim Ct^{\xi}$, as $t \to \infty$, where the notation $a(t) \sim b(t)$ means that $a(t)/b(t) \to 1$, as $t \to \infty$.

Weissman (1978) proposed the semi-parametric VaR_q-estimator,

$$\mathrm{Q}^{(q)}_{\hat{\xi}}(k) := X_{n-k:n}\,(k/(nq))^{\hat{\xi}}, \tag{2}$$

where $\hat{\xi}$ can be any consistent estimator for ξ and Q stands for quantile. For $\xi > 0$, the classical EVI-estimator, usually the one which is used in (2), for a semi-parametric quantile estimation, is the Hill estimator $\hat{\xi} = \hat{\xi}(k) =: \mathrm{H}(k)$ (Hill, 1975),

$$\mathrm{H}(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \quad V_{ik} = \ln\frac{X_{n-i+1:n}}{X_{n-k:n}},\ 1 \le i \le k. \tag{3}$$

If we plug in (2) the Hill estimator, $\mathrm{H}(k)$, we get the so-called Weissman-Hill quantile or VaR_q-estimator, with the obvious notation, $\mathrm{Q}^{(q)}_{\mathrm{H}}(k)$.

Noticing that we can write

$$\mathrm{H}(k) = \sum_{i=1}^{k} \ln\left(\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k} = \ln\left(\prod_{i=1}^{k}\frac{X_{n-i+1:n}}{X_{n-k:n}}\right)^{1/k}, \quad 1 \le i \le k < n,$$

the Hill estimator is thus the logarithm of the geometric mean (or mean-of-order-0) of

$$\underline{\mathbb{U}} := \{U_{ik} := X_{n-i+1:n}/X_{n-k:n},\ 1 \le i \le k < n\}. \tag{4}$$

More generally, Brilhante *et al.* (2013) considered as basic statistics the *mean-of-order-p* (MOP) of $\underline{\mathbb{U}}$, in (4), with $p \ge 0$, and the associated class of EVI-estimators,

$$\mathrm{H}_p(k) := \begin{cases} \frac{1}{p}\left(1 - \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^p\right)^{-1}\right), & \text{if} \quad p > 0, \\ \mathrm{H}(k), & \text{if} \quad p = 0, \end{cases} \tag{5}$$

with $\mathrm{H}_0(k) \equiv \mathrm{H}(k)$, given in (3). The class of MOP EVI-estimators in (5) depends now on this tuning parameter $p \ge 0$, and was shown to be valid for $0 \le p < 1/\xi$, whenever $k = k_n$ is an intermediate sequence, i.e. a sequence of integers $k = k_n$, $1 \le k < n$, such that $k = k_n \to \infty$ and $k_n = o(n)$, as $n \to \infty$. If we plug in (2) the MOP EVI-estimator, $\mathrm{H}_p(k)$, we get the so-called MOP quantile or VaR_q-estimator, with the obvious notation, $\mathrm{Q}^{(q)}_{\mathrm{H}_p}(k)$, studied asymptotically and for finite samples in Gomes *et al.* (2015b).

The MOP EVI-estimators in (5) can often have a high asymptotic bias, and bias reduction has recently been a vivid topic of research in the area of statistics of extremes. Working just for technical simplicity in the particular class of Hall-Welsh models in (Hall and Welsh, 1986), with a TQF $U(t) = Ct^{\xi}\left(1 + \xi\beta t^{\rho}/\rho + o(t^{\rho})\right)$, as $t \to \infty$, dependent on a vector (β, ρ) of unknown second-order parameters, the asymptotic distributional representation of the Hill EVI-estimator, given in (3), or equivalently, of $\mathrm{H}_p(k)$, given in (5), for $p = 0$, led Caeiro *et al.* (2005) to directly remove the dominant component of the bias of the Hill EVI-estimator, given by $\xi\beta(n/k)^{\rho}/(1-\rho)$, considering the *corrected-Hill* (CH) EVI-estimators,

$$\mathrm{CH}(k) \equiv \mathrm{CH}_{\hat{\beta},\hat{\rho}}(k) := \mathrm{H}(k)\Big(1 - \frac{\hat{\beta}}{1-\hat{\rho}}\left(\frac{n}{k}\right)^{\hat{\rho}}\Big), \tag{6}$$

a *minimum-variance reduced-bias* (MVRB) class of EVI-estimators for suitable second-order parameter estimators, $(\hat{\beta}, \hat{\rho})$. Estimators of ρ can be found in a large variety of articles, including Fraga Alves *et al.* (2003). Regarding the β-estimation, we refer to Gomes and Martins (2002), also among others. Gomes and Pestana

(2007) have used the EVI-estimator in (6) to build classes of MVRB VaR_q-estimators, that we obviously denote by $\mathrm{Q}_{\mathrm{CH}}^{(q)}(k)$. Recent overviews including the topic of reduced-bias estimation can be seen in Beirlant *et al.* (2012) and Gomes and Guillou (2014).

Working with values of p such that the asymptotic normality of the estimators in (5) holds, i.e. more specifically with $0 \le p < 1/(2\xi)$, Brilhante *et al.* (2014) noticed that there is an optimal value $p \equiv p_{_M} = \varphi_\rho/\xi$, with

$$\varphi_\rho = 1 - \rho/2 - \sqrt{(1-\rho/2)^2 - 1/2}, \tag{7}$$

which maximises the asymptotic efficiency of the class of estimators in (5). Then, they considered the optimal RV $\mathrm{H}_{p_M}(k)$, with $\mathrm{H}_p(k)$ given in (5), deriving its asymptotic behaviour. Such a behaviour has led Gomes *et al.* (2015a) to introduce a *partially reduced-bias* (PRB) class of MOP EVI-estimators based on $\mathrm{H}_p(k)$, in (5), with the functional expression

$$\mathrm{PRB}_p(k;\hat\beta,\hat\rho) := \mathrm{H}_p(k)\Big(1 - \frac{\hat\beta(1-\varphi_{\hat\rho})}{1-\hat\rho-\varphi_{\hat\rho}}\Big(\frac{n}{k}\Big)^{\hat\rho}\Big), \tag{8}$$

still dependent on a tuning parameter p and with φ_ρ defined in (7). It is thus sensible to use the class of EVI-estimators given in (8), and to consider the associated VaR_q-estimators, that we obviously denote by $\mathrm{Q}_{\mathrm{PRB}_p}^{(q)}(k)$.

In this article, apart from the description of a small-scale Monte-Carlo simulation, in Section 2, to illustrate the comparative behavior of the different VaR-estimators under consideration, an application in the field of finance is provided in Section 3. Finally, Section 4 sketches some conclusions of this study.

2 A Monte-Carlo illustration

We have implemented multi-sample Monte-Carlo simulation experiments of size, 5000×20, essentially for the class of VaR-estimators, $\mathrm{Q}_{\mathrm{PRB}_p}^{(q)}(k)$, and for a few values of n and p, in comparison with the H and CH VaR-estimators. Further details on multi-sample simulation can be found in Gomes and Oliveira (2001).

In Figure 1 an illustration of the obtained results is given for the VaR-estimators under consideration and for an $\mathrm{EV}_{0.1}$ parent. In this figure, we show, for $n = 1000$, $q = 1/n$, and on the basis of the first $N = 5000$ runs, the simulated patterns of mean value, $\mathrm{E}_Q[\cdot]$, and root mean squared error, $\mathrm{RMSE}_Q[\cdot]$, of the standardized PRB MOP VaR-estimators, for $p = p_\ell = \ell/(8\xi)$, $\ell = 1(1)7$, representing only the best two among the considered ℓ-values, the classical H VaR-estimators and the MVRB VaR-estimators.

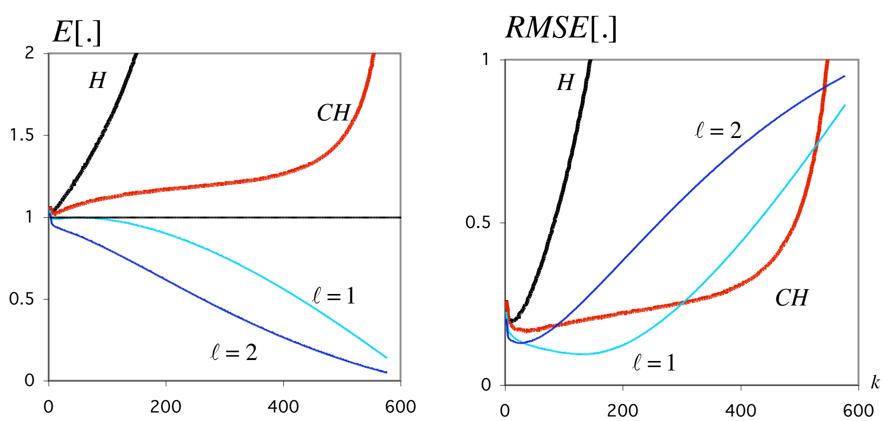

Figure 1: Mean values of $\mathrm{Q}_\bullet^{(1/n)}(k)/\mathrm{VaR}_q$ (*left*) and RMSE of $\mathrm{Q}_\bullet^{(1/n)}(k)/\mathrm{VaR}_q$ (*right*), for underlying EV parent with $\xi = 0.1$, for a sample size $n = 1000$

We have further computed the Weissman-Hill VaR-estimator $\mathrm{Q}_{\mathrm{H}}^{(q)}(k)$ at the simulated value of $k_{0|\mathrm{H}}^{(q)} := \arg\min_k \mathrm{RMSE}\big(\mathrm{Q}_{\mathrm{H}}^{(q)}(k)\big)$, the simulated optimal k in the sense of minimum RMSE. Such a value is not highly relevant in practice, but provides an indication of the best possible performance of the Weissman-Hill VaR-estimator. Such an estimator is denoted by $Q_{00} := Q_{H|0}$. We have also computed $Q_{0p} := Q_{PRB_p|0}$ at simulated optimal levels, for a few values of p, and the simulated indicators,

$$\mathrm{REFF}_{0|p} := \mathrm{RMSE}\,(Q_{00})/\mathrm{RMSE}\,(Q_{p0}).$$

A similar REFF-indicator, $\mathrm{REFF}_{CH|0}$ has also been computed for the MVRB VaR-estimator. For a visualisation of the obtained results, we represent Figure 2, again related to an $EV_{0.1}$ parent CDF.

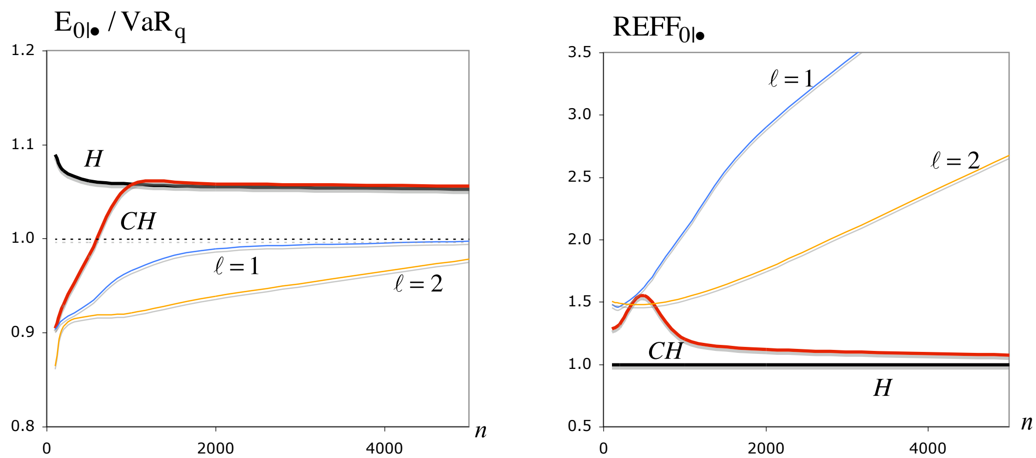

Figure 2: Normalized mean values (*left*) and REFF-indicators (*right*) of the VaR_q-estimators under study, at optimal levels, for $q = 1/n$, $EV_{0.1}$ parents and $100 \le n \le 5000$

3 A case-study in the field of finance

We shall here consider the performance of the above mentioned estimators in the analysis of Euro-UK Pound daily exchange rates from January 4, 1999 till December 14, 2004, the data already analyzed in Gomes and Pestana (2007). We have worked with the $n_0 = 725$ positive log-returns:

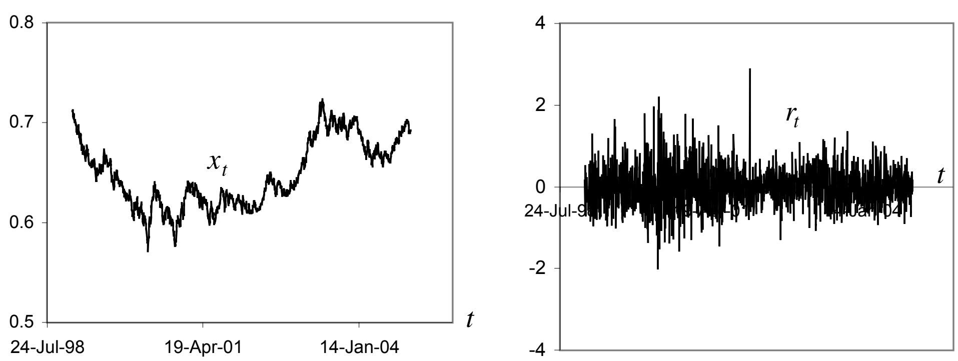

Figure 3: Euro-UK Pound daily exchange rates from January 4, 1999 till December 14, 2004 (*left*) and associated log-returns (*right*)

The sample paths of the VaR–estimators under study, for $q = 0.001$, are pictured in Figure 4, where PRB* represents the PRB_p VaR-estimator associated with an heuristic choice of p, performed in the lines of Gomes *et al.* (2013) and Neves *et al.* (2015).

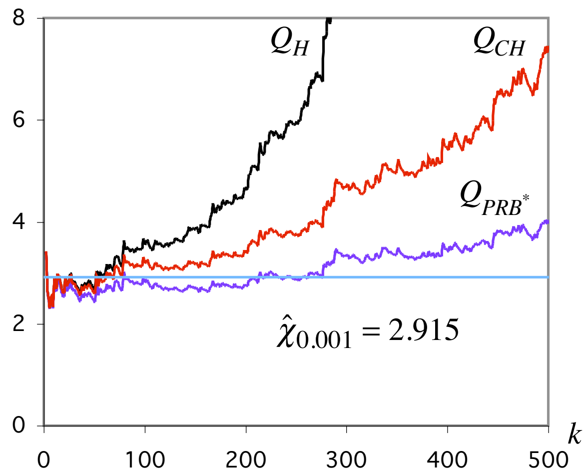

Figure 4: VaR_q-estimates provided through the different classes of VaR-estimators, for the Daily Log-Returns on the Euro-UK Pound and $q = 0.001$

For $q = 0.001$, any of the usual stability criterion for moderate values of k led us to the choice of the estimator Q_{PRB^*} and to the estimate 2.915 for $VaR_{0.001}$.

4 Concluding remarks

- It is clear that Weissman-Hill VaR-estimation leads to a strong over-estimation of VaR and the RB MOP, or even the MOP methodology can provide a more adequate VaR-estimation, being even able to beat the MVRB VaR-estimators in Gomes and Pestana (2007) in a large variety of situations.
- The obtained results lead us to strongly advise the use of the quantile estimator $Q_{PRB_p}(k)$, for a suitable choice of the tuning parameters p and k, provided by an algorithm like for instance the bootstrap algorithm of the type devised for an RB EVI-estimation in Gomes *et al.* (2012), among others, or heuristic algorithms of the type of the ones in Gomes *et al.* (2013) and Neves *et al.* (2015).
- For small values of $|\rho|$ the use of Q_{PRB_p}, with a suitable value of p, always enables a reduction in RMSE regarding the Weissman-Hill estimator and even the CH VaR_q-estimator. Moreover, the bias is also reduced comparatively with the bias of the Weissman-Hill VaR-estimator, resulting in estimates closer to the target value VaR_q, for small values of q comparatively to n.

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