A surface with canonical map of degree 24

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Abstract

We construct a complex algebraic surface with geometric genus $p_g = 3$, irregularity $q = 0$, self-intersection of the canonical divisor $K^2 = 24$ and canonical map of degree 24 onto $\mathbb{P}^2$.
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1 Introduction

Let S be a smooth minimal surface of general type with geometric genus $p_g \geq 3$. Denote by $\phi : S \dashrightarrow \mathbb{P}^{p_g-1}$ the canonical map and let $d := \deg(\phi)$. The following Beauville's result is well-known.

Theorem 1 ([Be]). *If the canonical image $\Sigma := \phi(S)$ is a surface, then either:*

(i) $p_g(\Sigma) = 0$, *or*

(ii) Σ *is a canonical surface (in particular* $p_g(\Sigma) = p_g(S)$*).*

Moreover, in case (i) $d \leq 36$ *and in case* (ii) $d \leq 9$.

Beauville has also constructed families of examples with $\chi(\mathcal{O}_S)$ arbitrarily large for $d = 2, 4, 6, 8$ and $p_g(\Sigma) = 0$. Despite being a classical problem, for $d > 8$ the number of known examples drops drastically. Tan's example [Ta, §5] with $d = 9$ and Persson's example [Pe] with $d = 16$, $q = 0$ are well known. Du and Gao [DG] show that if the canonical map is an abelian cover of $\mathbb{P}^2$, then these are the only possibilities for $d > 8$. More recently the author has given examples with $d = 12$ [Ri2] and $d = 16$, $q = 2$ [Ri3].

In this paper we construct a surface S with $p_g = 3$, $q = 0$ and $d = 24$, obtained as a $\mathbb{Z}_2^4$-covering of $\mathbb{P}^2$. The canonical map of S factors through a $\mathbb{Z}_2^2$-covering of a surface with $p_g = 3$, $q = 0$ and $K^2 = 6$ having 24 nodes, which in turn is a double covering of a Kummer surface.

Notation

We work over the complex numbers. All varieties are assumed to be projective algebraic. A $(-n)$-curve on a surface is a curve isomorphic to $\mathbb{P}^1$ with self-intersection $-n$. Linear equivalence of divisors is denoted by $\equiv$. The rest of the notation is standard in Algebraic Geometry.

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2 $\mathbb{Z}_2^n$-coverings

The following is taken from [Ca], the standard reference is [Pa].

Proposition 2. *A normal finite $G \cong \mathbb{Z}_2^r$-covering $Y \to X$ of a smooth variety X is completely determined by the datum of*

1. *reduced effective divisors D_σ, $\forall \sigma \in G$, with no common components;*
2. *divisor classes $L_1, \dots, L_r$, for $\chi_1, \dots, \chi_r$ a basis of the dual group of characters $G^\vee$, such that*
$$2L_i \equiv \sum_{\chi_i(\sigma)=-1} D_\sigma.$$

Conversely, given 1. and 2., one obtains a normal scheme Y with a finite $G \cong \mathbb{Z}_2^r$-covering $Y \to X$, with branch curves the divisors D_σ.

The covering $Y \to X$ is embedded in the total space of the direct sum of the line bundles whose sheaves of sections are the $\mathcal{O}_X(L_i)$, and is there defined by equations

$$u_{\chi_i} u_{\chi_j} = u_{\chi_i \chi_j} \prod_{\chi_i(\sigma)=\chi_j(\sigma)=-1} x_\sigma,$$

where x_σ is a section such that $\operatorname{div}(x_\sigma) = D_\sigma$.

The scheme Y can be seen as the normalization of the Galois covering given by the equations

$$u_{\chi_i}^2 = \prod_{\chi_i(\sigma)=-1} x_\sigma,$$

and Y is irreducible if $\{\sigma | D_\sigma > 0\}$ generates G.

For a covering $\pi : Y \to X$ with ramification divisor R, the Hurwitz formula $K_Y = \pi^*(K_X) + R$ holds. Let us describe the canonical system for the case where π is a $\mathbb{Z}_2^2$-covering with smooth branch divisor. We have branch curves D_1, D_2, D_3 and relations $2L_i \equiv D_j + D_k$, for all permutations (i, j, k) of $\{1, 2, 3\}$. The covering π factors as

$$\phi : Y \to W_i, \quad \varphi : W_i \to X,$$

where φ is the double covering corresponding to L_i. Let R_i be the ramification divisor of ϕ. One has

$$K_Y \equiv \phi^*(K_{W_i}) + R_i \quad \text{and} \quad K_{W_i} \equiv \varphi^*(K_X + L_i),$$

which gives

$$K_Y \equiv \pi^*(K_X + L_i) + \frac{1}{2}\pi^*(D_i), \quad i = 1, 2, 3.$$

Finally we notice that taking the quotient by a subgroup H of the Galois group of the covering corresponds to considering the subalgebra generated by the line bundles L_χ^{-1}, where χ ranges over the characters orthogonal to H.

3 The construction

We show in the Appendix the existence of reduced plane curves C_6 of degree 6 and C_7 of degree 7 through points $p_0, \dots, p_5$ such that:

· C_7 has a triple point at p_0 and tacnodes at $p_1, \dots, p_5$;

· C_6 is smooth at p_5, has a node at p_0 and tacnodes at $p_1, \dots, p_4$;

· the branches of the tacnode of C_j at p_i are tangent to the line T_i through p_0, p_i, $j = 1, 2$, $i = 1, \dots, 4$;

· the branches of the tacnode of C_7 at p_5 are tangent to C_6;

· the singularities of $C_6 + C_7$ are resolved via one blow-up at p_0 and two blow-ups at each of $p_1, \dots, p_5$.

Step 1 (Construction)
Consider the map

$$\mu : X \longrightarrow \mathbb{P}^2$$

which resolves the singularities of the curve C_7. Then μ is given by blow-ups at

$$p_0, p_1, p_1', \dots, p_5, p_5',$$

where p_i' is infinitely near to p_i. Let $E_0, E_1, E_1', \dots, E_5, E_5'$ be the corresponding exceptional divisors (with self-intersection -1).

Let x, y, z, w be generators of the group $\mathbb{Z}_2^4$ and

$$\psi : Y \longrightarrow X$$

be the $\mathbb{Z}_2^4$-covering defined by

$$D_x := \widetilde{T}_1 - E_0 - 2E_1',$$

$$D_y := \widetilde{T}_2 - E_0 - 2E_2',$$

$$D_z := \widetilde{C}_6 - 2E_0 - \sum_1^4 (2E_i + 2E_i') - 2E_5',$$

$$D_w := \widetilde{C}_7 + \widetilde{T}_4 - 4E_0 - \sum_1^3 (2E_i + 2E_i') - (2E_4 + 4E_4') - (2E_5 + 2E_5'),$$

$$D_{xy} := \widetilde{T}_3 - E_0 - 2E_3',$$

$$D_{xz} := \cdots := D_{zw} := 0,$$

where the notation $\widetilde{\cdot}$ stands for the total transform $\mu^*(\cdot)$.

We note that each of the divisors D_x, D_y, D_{xy} and $\widetilde{T}_4 - E_0 - 2E_4'$ (contained in D_w) is a disjoint union of two (-2)-curves.

For $i, j, k, l \in \{-1, 1\}$, let χ_{ijkl} denote the character which takes the value i, j, k, l on x, y, z, w, respectively. There exist divisors L_{ijkl} such that

$$2L_{ijkl} \equiv \sum_{\chi_{ijkl}(\sigma)=-1} D_\sigma, \qquad (1)$$

thus the covering ψ is well defined. Since there is no 2-torsion in the Picard group of X, then ψ is uniquely determined. The surface Y is smooth because the curves $D_x, \ldots, D_{xy}$ are smooth and disjoint. Division of the equations (1) by 2 gives that the L_{ijkl} are according to the following table. For instance $L_{-1111} \equiv \widetilde{T} - E_0 - E_1' - E_3'$.

	$\widetilde{T}$	E_0	E_1	E_1'	E_2	E_2'	E_3	E_3'	E_4	E_4'	E_5	E_5'
L_{-1111}	1	−1	0	−1	0	0	0	−1	0	0	0	0
L_{1-111}	1	−1	0	0	0	−1	0	−1	0	0	0	0
L_{-1-111}	1	−1	0	−1	0	−1	0	0	0	0	0	0
L_{11-11}	3	−1	−1	−1	−1	−1	−1	−1	−1	−1	0	−1
L_{-11-11}	4	−2	−1	−2	−1	−1	−1	−2	−1	−1	0	−1
L_{1-1-11}	4	−2	−1	−1	−1	−2	−1	−2	−1	−1	0	−1
$L_{-1-1-11}$	4	−2	−1	−2	−1	−2	−1	−1	−1	−1	0	−1
L_{111-1}	4	−2	−1	−1	−1	−1	−1	−1	−1	−2	−1	−1
L_{-111-1}	5	−3	−1	−2	−1	−1	−1	−2	−1	−2	−1	−1
L_{1-11-1}	5	−3	−1	−1	−1	−2	−1	−2	−1	−2	−1	−1
$L_{-1-11-1}$	5	−3	−1	−2	−1	−2	−1	−1	−1	−2	−1	−1
L_{11-1-1}	7	−3	−2	−2	−2	−2	−2	−2	−2	−3	−1	−2
$L_{-11-1-1}$	8	−4	−2	−3	−2	−2	−2	−3	−2	−3	−1	−2
$L_{1-1-1-1}$	8	−4	−2	−2	−2	−3	−2	−3	−2	−3	−1	−2
$L_{-1-1-1-1}$	8	−4	−2	−3	−2	−3	−2	−2	−2	−3	−1	−2

Step 2 (Invariants)
Since

$$K_X \equiv -3\widetilde{T} + E_0 + \sum_1^5 (E_i + E_i'),$$

then

$$\chi(\mathcal{O}_Y) = 16\chi(\mathcal{O}_X) + \frac{1}{2}\sum \left(L_{ijkl}^2 + K_X L_{ijkl}\right) =$$
$$= 16 - 1 - 1 - 1 + 0 - 1 - 1 - 1 + 0 - 1 - 1 - 1 + 0 - 1 - 1 - 1 = 4.$$

For the computation of

$$p_g(Y) = p_g(X) + \sum h^0(X, \mathcal{O}_X(K_X + L_{ijkl})),$$

let

$$\mathcal{T}_1 := \left(\widetilde{T}_4 - E_0 - 2E_4' + E_5 - E_5'\right),$$

$$\mathcal{T}_2 := \left(\widetilde{T}_2 + \widetilde{T}_3 + \widetilde{T}_4 - 3E_0 - \sum_2^4 2E_i' + E_5 - E_5'\right),$$

$$\mathcal{L}_1 := \left|3\widetilde{T} - E_0 - \sum_1^3 (E_i + E_i') - E_4 - E_5\right|$$

and

$$\mathcal{L}_2 := \left|2\widetilde{T} - (E_1 + E_1') - E_2 - E_3 - E_4 - E_5\right|.$$

Each of $\mathcal{T}_1$, $\mathcal{T}_2$ is a disjoint union of (-2)-curves intersecting negatively $K_X + L_{11-1-1}$, $K_X + L_{1-1-1-1}$, respectively, thus we have

$$|K_X + L_{11-1-1}| = \mathcal{T}_1 + \mathcal{L}_1$$

and

$$|K_X + L_{1-1-1-1}| = \mathcal{T}_2 + \mathcal{L}_2.$$

We show in the Appendix that $\mathcal{L}_1$ has only one element and $\mathcal{L}_2$ is empty. Hence

$$h^0(X, \mathcal{O}_X(K_X + L_{11-1-1})) = 1$$

and

$$h^0(X, \mathcal{O}_X(K_X + L_{1-1-1-1})) = 0.$$

Analogously

$$h^0(X, \mathcal{O}_X(K_X + L_{-11-1-1})) = h^0(X, \mathcal{O}_X(K_X + L_{-1-1-1-1})) = 0.$$

It is easy to see that

$$h^0(X, \mathcal{O}_X(K_X + L_{11-11})) = h^0(X, \mathcal{O}_X(K_X + L_{111-1})) = 1$$

and

$$h^0(X, \mathcal{O}_X(K_X + L_{ijkl})) = 0$$

for the remaining cases. We conclude that

$$p_g(Y) = 0 + 1 + 1 + 1 = 3.$$

Now we compute the self-intersection of the canonical divisor for the minimal model S of Y. The divisor

$$\xi_1 := \frac{1}{2}\psi^* \left(\sum_1^3 \left(\widetilde{T}_i - E_0 - 2E'_i \right) \right)$$

is a disjoint union of $8 \times 6 = 48$ (-1)-curves and the divisor

$$\xi_2 := \frac{1}{2}\psi^* \left(\widetilde{T}_4 - E_0 - 2E'_4 + E_5 - E'_5 \right)$$

is a disjoint union of $8 \times 3 = 24$ (-1)-curves.

The covering ψ factors through the double covering $\varphi : W \to X$ with branch locus $D_z + D_w$. We have $K_W \equiv \varphi^*(K_X + L_{11-1-1})$, hence the Hurwitz formula gives

$$K_Y \equiv \xi_1 + \psi^*(K_X + L_{11-1-1}).$$

Thus one of the canonical curves of Y is

$$\xi_1 + 2\xi_2 + \psi^*(\mathcal{C}),$$

where $\mathcal{C}$ is the unique element in the linear system $\mathcal{L}_1$ defined above. From $\xi_1\xi_2 = \xi_1\psi^*(\mathcal{C}) = \psi^*(\mathcal{C})^2 = 0$ and $\xi_2\psi^*(\mathcal{C}) = 24$, we get $K_Y^2 = -48$. We show in the Appendix that the curve $\mathcal{C}$ is irreducible, therefore $\psi^*(\mathcal{C})$ is nef and then $K_S^2 = 24$.

Step 3 (The canonical map)
The divisors

$$D_z,\ D_w,\ D_{zw}$$

define a $\mathbb{Z}_2^2$-covering

$$\rho : U \to X.$$

We have

$$\chi(\mathcal{O}_U) = 4\chi(\mathcal{O}_X) + \frac{1}{2}\sum \left(L_{11kl}^2 + K_X L_{11kl}\right) = 4 + 0 + 0 + 0 = 4$$

and

$$p_g(U) = p_g(X) + \sum h^0(X, \mathcal{O}_X(K_X + L_{11kl})) = 0 + 1 + 1 + 1 = 3.$$

The surface U is the quotient of Y by the subgroup H generated by x, y. The group H acts on the minimal model S of Y with only isolated fixed points, so S/H is the canonical model $\bar{U}$ of U and then

$$K_{\bar{U}}^2 = 6.$$

Finally we want to show that the canonical map of U is of degree 6 onto $\mathbb{P}^2$. It suffices to verify that the canonical system has no base component nor base points. The canonical system of U is generated by the divisors

$$\begin{aligned} K_1 &:= \frac{1}{2}\rho^*(D_z) + \rho^*(K_X + L_{111-1}), \\ K_2 &:= \frac{1}{2}\rho^*(D_w) + \rho^*(K_X + L_{11-11}), \\ K_3 &:= \frac{1}{2}\rho^*(D_{zw}) + \rho^*(K_X + L_{11-1-1}). \end{aligned}$$

Denote by $\vartheta_1, \ldots, \vartheta_4$ the four (-1)-curves

$$\frac{1}{2}\rho^*(\widetilde{T}_4 - E_0 - 2E_4')$$

and by ϑ_5, ϑ_6 the two (-1)-curves

$$\frac{1}{2}\rho^*(E_5 - E_5').$$

Let

$$\pi : U \to U'$$

be the contraction to the minimal model and $q_1, \ldots, q_6 \in U'$ be the points obtained by contraction of $\vartheta_1, \ldots, \vartheta_6$. If κ is an effective canonical divisor of U', then

$$H := \pi^*(\kappa) + \vartheta_1 + \cdots + \vartheta_6$$

is a canonical curve of U. So, the multiplicity of a curve ϑ_i in H is 1 if and only if the curve κ does not contain the point q_i.

Since the multiplicity of $\vartheta_5 + \vartheta_6$ in K_1 is 1, the points q_5, q_6 are not base points of the canonical system of U'. The multiplicity of $\vartheta_1 + \cdots + \vartheta_4$ in K_2 is 1, so also the points $q_1, \ldots, q_4$ are not base points of the canonical system of U'. Now to conclude the non-existence of other base points, it suffices to show that the plane curves

$$\mu \circ \rho(K_i), \ i = 1, 2, 3,$$

have common intersection $\{p_0, p_1, \ldots, p_5\}$ and their singularities are no worse than stated. This is done in the Appendix. Here we just note that these curves are

$$T_4 + C_6, \quad C_7, \quad T_4 + C_3,$$

where C_3 is the plane cubic corresponding to the unique element in the linear system $\mathcal{L}_1$, defined in Step 2 above.

Step 4 (Conclusion)
The $\mathbb{Z}_2^4$-covering $\psi : Y \to X$ factors as

$$Y \xrightarrow{4:1} U \xrightarrow{4:1} X.$$

Since $p_g(Y) = p_g(U) = 3$ and the canonical map of U is of degree 6, then the canonical map of Y is of degree 24.

Remark 3. *Consider the intermediate double covering $\epsilon : Q \to X$ of ρ with branch locus D_z. Then Q is a Kummer surface: each divisor $\epsilon^* \left(\widetilde{T} - E_0 - 2E_i'\right)$ is a disjoint union of four (-2)-curves. The surface U contains 24 disjoint (-2)-curves $A_1, \ldots, A_{24}$, the pullback of $\sum_1^3 \epsilon^* \left(\widetilde{T}_i - E_0 - 2E_i'\right)$, such that the covering $Y \to U$ is a $\mathbb{Z}_2^2$-Galois covering ramified over the divisors*

$$A_1 + \cdots + A_8, \; A_9 + \cdots + A_{16}, \; A_{17} + \cdots + A_{24}.$$

Appendix

The following code is implemented with the Computational Algebra System Magma [BCP], version V2.21-8.

First we compute the curves C_6 and C_7 referred in Section 3. We choose the points $p_0, \ldots, p_5$ with a symmetry axis and compute the curves using the Magma function *LinSys* given in [Ri1].

```
A<x,y>:=AffineSpace(Rationals(),2);
P:=[A![0,0],A![2,2],A![-2,2],A![3,1],A![-3,1],A![0,5]];
M1:=[[2],[2,2],[2,2],[2,2],[2,2],[1,1]];
M2:=[[3],[2,2],[2,2],[2,2],[2,2],[2,2]];
T:=[[],[[1,1]],[[-1,1]],[[3,1]],[[-3,1]],[[1,0]]];
J6:=LinSys(LinearSystem(A,6),P,M1,T);
J7:=LinSys(LinearSystem(A,7),P,M2,T);
C6:=Curve(A,Sections(J6)[1]);
C7:=Curve(A,Sections(J7)[1]);
```

We consider the projective closure of the curves and verify that they are irreducible and the singularities are exactly as stated.

```
P2<x,y,z>:=ProjectiveClosure(A);
C6:=ProjectiveClosure(C6);
C7:=ProjectiveClosure(C7);
IsAbsolutelyIrreducible(C6);
IsAbsolutelyIrreducible(C7);
```

```
SingularPoints(C6 join C7);
HasSingularPointsOverExtension(C6 join C7);
[ResolutionGraph(C6,P[i]):i in [1..#P-1]];
[ResolutionGraph(C7,P[i]):i in [1..#P]];
[ResolutionGraph(C6 join C7,P[i]):i in [1..#P]];
```

To clarify the situation at the origin, we use:

```
d:=DefiningEquation(TangentCone(C7,A![0,0]));
d eq y*(x^2 + 40585383/1587545*y^2);
```

thus the singularity is ordinary.

The defining polynomials of C_6 and C_7 are

```
289*x^6+754326*x^4*y^2+2610657*x^2*y^4+1906344*y^6-2013848*x^4*y*z
-17946576*x^2*y^3*z-22212504*y^5*z+1336400*x^4*z^2
+35856160*x^2*y^2*z^2+89326224*y^4*z^2-22270208*x^2*y*z^3
-146421504*y^3*z^3+295936*x^2*z^4+84049920*y^2*z^4
```

and

```
8683464*x^6*y-494984955*x^4*y^3-1064093674*x^2*y^5-558251235*y^7
-11358312*x^6*z+1253331746*x^4*y^2*z+8340957732*x^2*y^4*z
+7286240034*y^6*z-920312219*x^4*y*z^2-17394911410*x^2*y^3*z^2
-32292289971*y^5*z^2+179839940*x^4*z^3+11716330200*x^2*y^2*z^3
+55580514660*y^4*z^3-1270036000*x^2*y*z^4-32468306400*y^3*z^4
```

Now we show that the linear system $\mathcal{L}_1$, defined in Step 2 above, has exactly one element. Let L_1 be the corresponding linear system of plane cubics. By parameter counting, $\dim(L_1) \geq 0$. If $\dim(L_1) \geq 1$, then one of its curves contains the line T_3, because

$$\left(\widetilde{T}_3 - E_0 - E_3 - E_3'\right)\left(3\widetilde{T} - E_0 - \sum_1^3 (E_i + E_i') - E_4 - E_5\right) = 0.$$

The other component of this curve is a conic, but one can verify that the conic through p_4 tangent to the lines T_1, T_2 at p_1, p_2, which is given by the equation

$$x^2 - 9y^2 + 32y - 32 = 0,$$

does not contain the point p_5. We compute the unique plane cubic C_3 in L_1 and show that it is irreducible:

```
M:=[[1],[1,1],[1,1],[1,1],[1,0],[1,0]];
J3:=LinSys(LinearSystem(A,3),P,M,T);
#Sections(J3) eq 1;
C3:=ProjectiveClosure(Curve(A,Sections(J3)[1]));
IsAbsolutelyIrreducible(C3);
```

The defining polynomial of C_3 is

```
17*x^3-924*x^2*y-153*x*y^2-996*y^3+1164*x^2*z
+544*x*y*z+6516*y^2*z-544*x*z^2-7680*y*z^2
```

To conclude that the linear system $\mathcal{L}_2$, defined in Step 2, is empty, it suffices to note that the conic C through $p_1, \ldots, p_5$ is not tangent to the line T_1 at the point p_1. An equation for C is

$$-12x^2 + 11y^2 - 93y + 190 = 0.$$

Finally we verify that the curves

$$T_4 + C_6, \quad C_7, \quad T_4 + C_3,$$

referred in the end of Section 3, have intersection $\{p_0, p_1, \ldots, p_5\}$:

```
T4:=Curve(P2,x+3*y);
PointsOverSplittingField((T4 join C6) meet C7 meet (T4 join C3));
```

and the singularities are no worse than stated:

```
[ResolutionGraph(T4 join C3 join C6 join C7,p):p in P];
```

To clarify the situation at the origin, we use:

```
TC:=TangentCone(T4 join C3 join C6 join C7,P2![0,0,1]);
DefiningEquation(TC) eq y*(x+3*y)*(x + 240/17*y)
*(x^2 + 82080/289*y^2)*(x^2 + 40585383/1587545*y^2);
```

thus the singularity is ordinary.

References

[Be] A. Beauville, *L'application canonique pour les surfaces de type général*, Invent. Math., **55** (1979), no. 2, 121–140.

[BCP] W. Bosma, J. Cannon and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput., **24** (1997), no. 3-4, 235–265.

[Ca] F. Catanese, *Differentiable and deformation type of algebraic surfaces, real and symplectic structures*, *Symplectic 4-manifolds and algebraic surfaces*, vol. 1938 of *Lecture Notes in Math.*, Springer, Berlin (2008), 55–167.

[DG] R. Du and Y. Gao, *Canonical maps of surfaces defined by abelian covers*, Asian J. Math., **18** (2014), no. 2, 219–228.

[Pa] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math., **417** (1991), 191–213.

[Pe] U. Persson, *Double coverings and surfaces of general type*, *Algebraic geometry (Proc. Sympos., Univ. Tromsø, Tromsø, 1977)*, vol. 687 of *Lecture Notes in Math.*, Springer, Berlin (1978), 168–195.

[Ri1] C. Rito, *On the computation of singular plane curves and quartic surfaces*, arXiv:0906.3480 [math.AG] (2009).

[Ri2] C. Rito, *New canonical triple covers of surfaces*, P. Am. Math. Soc., **143** (2015), no. 11, 4647–4653.

[Ri3] C. Rito, *A surface with $q = 2$ and canonical map of degree 16*, Michigan Math. J., **66** (2017), no. 1, 99–105.

[Ta] S.-L. Tan, *Surfaces whose canonical maps are of odd degrees*, Math. Ann., **292** (1992), no. 1, 13–29.

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