An automata-theoretic approach to the study of fixed points of endomorphisms

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Contents

1 Languages and automata **3**
- 1.1 Words and languages 3
- 1.2 Automata 5
- 1.3 Transducers 9
- 1.4 Rewriting systems 11

2 Automata in group theory **12**
- 2.1 Free groups 12
- 2.2 Virtually free groups 14
- 2.3 Hyperbolic groups 16
- 2.4 Automatic groups 18
- 2.5 Self-similar groups 20

3 Fixed points of endomorphisms **22**
- 3.1 A brief introduction 22
- 3.2 Fixed points of transductions 23
- 3.3 Virtually free group endomorphisms 27

4 Fixed points in the boundary **29**
- 4.1 A brief introduction 29
- 4.2 A model for the boundary of virtually free groups 30
- 4.3 Uniformly continuous endomorphisms 35

4.4 Fixed points in the boundary of virtually free groups 36
4.5 Classification of the infinite fixed points 43

5 Open problems **46**

References **48**

This chapter contains an extended version of the contents of the 3-hour course *Fixed points of virtually free group endomorphisms* of the *Summer School on Automorphisms of Free Groups*, held at the CRM (Bellaterra, Barcelona) from the 25th to the 29th September 2012.

We present basic definitions and results on automata and make a short review of their role in group theory. We compile a brief history of the study of (finite and infinite) fixed points of group endomorphisms and discuss the case of virtually free groups with the help of automata.

1 Languages and automata

We introduce in this section basic facts from automata theory (see also Section 2 of Chapter 2). For broader perspectives, the interested reader is referred to [9, 50].

1.1 Words and languages

In this context, an *alphabet* is a set and its elements are called *letters*. Usually, alphabets are finite and denoted by capital letters such as A or Σ. A finite sequence of letters is appropriately called a *word*. This includes the empty word, conventionally denoted by 1. Nonempty words (on the alphabet A) are usually written in the form $a_1 \dots a_n$ with $a_1, \dots, a_n \in A$.

The set of all words on A is denoted by A^* and turns out to be the *free monoid on A* when it is endowed with the *concatenation product*, defined by

$$(a_1 \dots a_n)(b_1 \dots b_m) = a_1 \dots a_n b_1 \dots b_m$$

for nonempty words and taking 1 as the identity element. Therefore, if $\varphi : A \to M$ is a mapping from A into some monoid M, there exists a unique monoid homomorphism $\Phi : A^* \to M$ extending φ.

We denote by A^ω the set of all (right) infinite words $a_1a_2a_3\dots$ on the alphabet A, and write also $A^\infty = A^* \cup A^\omega$. Infinite words will play a major role in Section 4.

Free monoids lead us naturally to free groups. Given an alphabet A, we denote by $\overline{A}$ a set of *formal inverses* of A (i.e. $a \mapsto \overline{a}$ defines a bijection of A onto some set $\overline{A}$ disjoint from A). We use the notation $\widetilde{A} = A \cup \overline{A}$ and extend the mapping $a \mapsto \overline{a}$ to an involution of the free monoid $\widetilde{A}^*$ by

$$\overline{\overline{a}} = a \quad (a \in A),$$

$$\overline{uv} = \overline{v}\,\overline{u} \quad (u, v \in \widetilde{A}^*).$$

The *free group on* A, denoted by F_A, is the quotient of $\widetilde{A}^*$ by the congruence generated by the relation

$$\mathcal{R}_A = \{(a\overline{a}, 1) \mid a \in \widetilde{A}\}. \tag{1}$$

Thus two words $u, v \in \widetilde{A}^*$ are equivalent in F_A if and only if one can be transformed into the other by successively inserting/deleting factors of the form $a\overline{a}$ ($a \in \widetilde{A}$).

We denote by $\theta : \widetilde{A}^* \to F_A$ the canonical morphism. Note that θ is *matched* in the sense that $\theta(\overline{a}) = (\theta(a))^{-1}$ for every $a \in A$. In fact, $\theta(\overline{u}) = (\theta(u))^{-1}$ holds for every $u \in \widetilde{A}^*$. Matched homomorphisms will be ubiquitous in this chapter.

A subset of A^* is called an A-*language*, or just language when the alphabet is implicit or irrelevant. We remark that *language theory* is an important branch of theoretical computer science which aims at classifying languages and exploring the algorithmic potential of various subclasses. The pioneering work of Noam Chomsky in the fifties [15] is at its origin, hence language theory developped initially within linguistics rather than within computer science or mathematics.

The most intensively studied class of languages is the class of *rational languages*, also known as regular or recognizable languages (according to the definition used, see also Section 2 of Chapter 2). In order to define them, we need to introduce the *rational operators* on languages: *union, product* and *star*. Union is just the set-theoretic operation on subsets. Given $K, L \subseteq A^*$, we define

$$KL = \{uv \mid u \in K,\ v \in L\},$$
$$L^* = \bigcup_{n \geq 0} L^n,$$

using the convention $L^0 = \{1\}$. Note that L^* is the submonoid of A^* generated by L.

Now we can define the family of rational A-languages, denoted by $\mathrm{RAT}(A^*)$, as the smallest family of A-languages containing the finite A-languages and closed under the rational operators. Equivalently, an A-language L is rational if and only if it can be obtained from finite A-languages through finitely many applications of the rational operators.

Rational languages satisfy many important closure and algorithmic properties, such as closure under boolean operators.

Example 1.1 *If $A = \{a, b\}$ and L denotes the set of words on A containing precisely two a's, then $L = b^*ab^*ab^*$ and is therefore rational.*

We remark that, given any monoid M, we can replace languages on the alphabet A by subsets of M in the definition of $\mathrm{RAT}(A^*)$ to obtain $\mathrm{RAT}(M)$, the family of rational subsets of M. Since the rational operators commute with homomorphisms, it is easy to see that rational subsets are preserved by monoid homomorphisms, i.e. whenever $\varphi : M \to N$ is a monoid homomorphism and $L \in \mathrm{RAT}(M)$, then $\varphi(L) \in \mathrm{RAT}(N)$. Moreover, if φ is onto, then every $K \in \mathrm{RAT}(N)$ is of the form $K = \varphi(L)$ for some $L \in \mathrm{RAT}(M)$ [9, Proposition III.2.2].

1.2 Automata

We assume from now on that A is a finite alphabet. We say that $\mathcal{A} = (Q, A, E, q_0, F)$ is an *automaton* if:

- Q is a nonempty set;
- A is a finite alphabet;
- $E \subseteq Q \times A \times Q$;
- $q_0 \in Q$ and $F \subseteq Q$.

The set Q is said to be the set of *vertices* (or *states*), q_0 is the *initial vertex*, F is the set of *final vertices* and E is the set of *edges* (or *transitions*). The automaton is *finite* if Q is finite.

A *finite nontrivial path* in $\mathcal{A}$ is a sequence

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} \dots \xrightarrow{a_n} p_n$$

such that $(p_{i-1}, a, p_i) \in E$ for $i = 1, \dots, n$. Its *label* is the word $a_1 \dots a_n \in A^*$. It is said to be a *successful* path if $p_0 = q_0$ and $p_n \in F$. We consider also the *trivial path* $p \xrightarrow{1} p$ for every $p \in Q$, which is successful if $p = q_0 \in F$. We denote by $p \xrightarrow{u} q$ any path with label u connecting p to q.

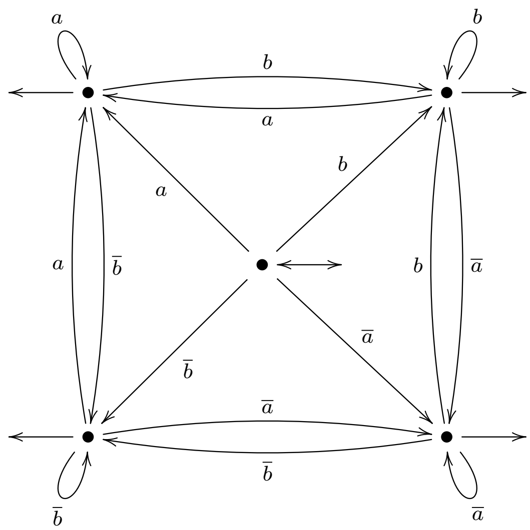
The *language* $L(\mathcal{A})$ *recognized by* $\mathcal{A}$ is the set of all labels of successful paths in $\mathcal{A}$. If $(p_{i-1}, a_i, p_i) \in E$ for every $i \geq 1$, we may consider also the *infinite path*

$$p_0 \xrightarrow{a_1} p_1 \xrightarrow{a_2} p_2 \xrightarrow{a_3} \dots$$

Its label is the infinite word $a_1a_2a_3 \dots \in A^\omega$. We denote by $L_\omega(\mathcal{A})$ the set of labels of all infinite paths $q_0 \longrightarrow \dots$ in $\mathcal{A}$.

Finite automata admit a natural combinatorial description as finite directed labelled graphs. The initial and terminal vertices may be conventionally identified through unlabelled incoming and outcoming arrows (respectively):

Example 1.2 *Let $A = \{a, b\}$ and let $\mathcal{A}$ be the automaton depicted by*



Then

$$L(\mathcal{A}) = \widetilde{A}^* \setminus \widetilde{A}^*\{a\overline{a}, \overline{a}a, b\overline{b}, \overline{b}b\}\widetilde{A}^*$$

is the set of all free group reduced words *on the alphabet $\widetilde{A}$.*

The automaton $\mathcal{A} = (Q, A, E, q_0, F)$ is said to be *deterministic* if

$$(p, a, q), (p, a, r) \in E \;\Rightarrow\; q = r$$

for all $p, q, r \in Q$ and $a \in A$. Then E can be described by means of a partial mapping $Q \times A \to Q$ which extends to a partial mapping $Q \times A^* \to Q$: $(p, u) \mapsto pu$ as follows: for all $p \in Q$ and $u \in A^*$, pu is defined if and only there exists a path in $\mathcal{A}$ of the form $p \xrightarrow{u} q$, and in that case $pu = q$.

In theoretical computer science, deterministic automata are models for computing devices admitting *bounded memory*: the finitely many vertices of the automata (usually called *states* in that context) represent the finitely many memory configurations, the alphabet represents the set of possible elementary actions, and the edges encode the changes in the memory configurations induced by each possible elementary action.

It turns out that the deterministic and nondeterministic versions of finite automata have the same expressive power: the rational languages. The first equivalence in the following theorem uses the classical *subset construction* and is due to Rabin and Scott [48]; the second is Kleene's Theorem [40] (see Theorem 2.1 of Chapter 2 for proofs):

Theorem 1.3 *Let $L \subseteq A^*$. Then the following conditions are equivalent:*

(i) $L = L(\mathcal{A})$ *for some finite automaton* $\mathcal{A}$*;*

(ii) $L = L(\mathcal{A})$ *for some finite deterministic automaton* $\mathcal{A}$*;*

(iii) $L \in \mathrm{RAT}(A^*)$.

Beyond determinism, there are other important properties of automata that we care to define. An automaton $\mathcal{A} = (Q, A, E, q_0, F)$ is said to be:

- *trim* if every vertex of Q lies in some successful path;
- *complete* if, for all $p \in Q$ and $a \in A$, there exists some edge $(p, a, q) \in E$ for some $q \in Q$.

Automata appear in relation with groups since the pioneering work of Benois in the sixties [8]. Indeed, a fundamental role is played by *Benois Theorem* (closure under free group reduction). Given a word $u \in \widetilde{A}^*$, we denote by $\widehat{u}$ the (unique) reduced word obtained by successively cancelling from u factors of the form $a\overline{a}, \overline{a}a$ $(a \in A)$. Given $L \subseteq \widetilde{A}^*$, write $\widehat{L} = \{\widehat{u} \mid u \in L\}$. The set $R_A = \widehat{\widetilde{A}^*}$ of all reduced words constitutes a well-known set of *normal forms* for F_A.

Theorem 1.4 [8] *If $L \in \mathrm{RAT}(\widetilde{A}^*)$, then also $\widehat{L} \in \mathrm{RAT}(\widetilde{A}^*)$.*

It follows easily that $\mathrm{RAT}(F_A)$ is closed under the boolean operations.

On the other hand, the notion of rational subset of a group constitutes a very useful generalization to subsets of the notion of finitely generated subgroup. Evidence is provided by Anisimov and Seifert's Theorem:

Theorem 1.5 [3] *Let H be a subgroup of a group G. Then $H \in \mathrm{RAT}(G)$ if and only if H is finitely generated.*

Proof. Let H be a rational subgroup of G and let $\pi : \widetilde{B}^* \to G$ be a matched epimorphism. Then $H = \pi(L)$ for some $L \in \mathrm{RAT}(\widetilde{B}^*)$. Hence $L \in \mathrm{RAT}(\widetilde{A}^*)$ for some finite subset A of B. It follows that $L = L(\mathcal{A})$ for some finite automaton $\mathcal{A} = (Q, \widetilde{A}, E, q_0, F)$. Let $m = |Q|$ and let

$$X = \{u \in \pi^{-1}(H) : |u| < 2m\}.$$

Since A is finite, so is X. We claim that $H = \langle \pi(X) \rangle$. To prove it, it suffices to show that

$$u \in L \Rightarrow \pi(u) \in \langle \pi(X) \rangle \tag{2}$$

holds for every $u \in \widetilde{A}^*$. We use induction on $|u|$. By definition of X, (2) holds for words of length $< 2m$. Assume now that $|u| \geq 2m$ and (2) holds for shorter words. Write $u = vw$ with $|w| = m$. Then there exists a path

$$\to q_0 \xrightarrow{v} q \xrightarrow{z} t \to$$

in $\mathcal{A}$ with $|z| < m$. Thus $vz \in L(\mathcal{A}) = L$ and by the induction hypothesis $\pi(vz) \in \langle \pi(X) \rangle$. On the other hand, $|z^{-1}w| < 2m$ and $\pi(z^{-1}w) = \pi(z^{-1}v^{-1})\pi(vw) \in H$, hence $z^{-1}w \in X$ and so $\pi(u) = \pi(vz)\pi(z^{-1}w) \in \langle \pi(X) \rangle$, proving (2) as required.

The converse implication follows from the equality $\langle X \rangle = (X \cup X^{-1})^*$. □

The existence of inverses in groups leads naturally to the concept of inverse automaton. Let $\mathcal{A} = (Q, \widetilde{A}, E, q_0, F)$ be an automaton. We say that $\mathcal{A}$ is:

- *involutive* if it satisfies

$$(p, a, q) \in E \iff (q, \overline{a}, p) \in E$$

  for every $a \in A$;

- *inverse* if it is deterministic, trim and involutive.

Cayley graphs of groups (see Subsection 5.1 of Chapter 2) provide natural examples of inverse automata. Let G be a group generated by A, i.e. consider a matched epimorphism $\pi : \widetilde{A}^* \to G$. The *Cayley graph* $\Gamma_A(G)$ has the elements of G as vertices and edges of the form

$$g \xrightarrow{a} ga \quad (g \in G,\ a \in \widetilde{A}),$$

where $ga = g(\pi(a))$. Fixing the identity 1 as the initial and unique final vertex (we call such a vertex a *basepoint*), we obtain an inverse automaton which recognizes the language $\pi^{-1}(1)$.

As we shall see in Subsection 2.1, finite inverse automata play a major role in the study of finitely generated subgroups of free groups. They are also an essential ingredient of the geometric theory of inverse monoids (see Stephen [59]).

1.3 Transducers

The concept of automaton can be given extra structure by considering an output function. We present a restricted definition, sufficient for our purposes.

We say that $\mathcal{T} = (Q, A, E, q_0, F)$ is a finite *transducer* if:

- Q is a finite set;
- A is a finite alphabet;
- $q_0 \in Q$ and $F \subseteq Q$;
- $E \subseteq Q \times A \times A^* \times Q$ is finite.

The concept of path is defined as in the automaton case, ignoring the third components in E.

A finite transducer $\mathcal{T} = (Q, A, E, q_0, F)$ is said to be *deterministic* if

$$(p, a, u, q), (p, a, v, r) \in E \;\Rightarrow\; (q = r \text{ and } u = v)$$

for all $p, q, r \in Q$, $a \in A$ and $u, v \in A^*$. Similarly to the case of deterministic automata, E can then be described by means of two partial mappings $Q \times A \to Q$ and $Q \times A \to A^*$ (with the same domain!) which extend to partial mappings $Q \times A^* \to Q : (p, w) \mapsto pw$ (the transition mapping) and $\lambda : Q \times A^* \to A^*$ (the output mapping). More precisely, given $p \in Q$ and $w = a_1 \dots a_n$ ($a_i \in A$), there exists at most one path in $\mathcal{T}$ of the form

$$p = p_0 \xrightarrow{a_1|u_1} p_1 \xrightarrow{a_2|u_2} \dots \xrightarrow{a_n|u_n} p_n \tag{3}$$

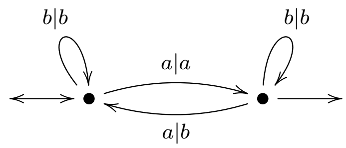
with $(p_{i-1}, a_i, u_i, p_i) \in E$ for $i = 1, \dots, n$. In that case, we set $pw = p_n$ and $\lambda(p, w) = u_1 \dots u_n$. If no such path exists, pw and $\lambda(p, w)$ remain undefined. If $u = u_1 \dots u_n$, we may denote by $p \xrightarrow{w|u} p_n$ the (unique) path (3). Finally, we define a partial mapping $\widehat{\mathcal{T}} : A^* \to A^*$ by

$$\widehat{\mathcal{T}}(w) = \begin{cases} \lambda(q_0, w) & \text{if } q_0 w \text{ is defined and } q_0 w \in F \\ \text{undefined} & \text{otherwise} \end{cases}$$

We say that $\widehat{\mathcal{T}}$ is the *transduction* defined by $\mathcal{T}$.

A finite transducer $\mathcal{T} = (Q, A, E, q_0, F)$ is said to be *complete* if, for all $p \in Q$ and $a \in A$, there exists some edge $(p, a, u, q) \in E$ for some $u \in A^*$ and $q \in Q$.

Example 1.6 *Let $A = \{a, b\}$ and let $\mathcal{T}$ be the finite transducer depicted by*



Then:

(i) $\mathcal{T}$ is deterministic and complete;

(ii) the domain of $\widehat{\mathcal{T}}$ is $L = (b^*ab^*a)^*(b^* \cup b^*ab^*)$;

(iii) for every $w \in L$, $\widehat{\mathcal{T}}(w)$ is the word obtained by replacing each occurrence of even order of a in w by b.

Similarly to the case of inverse automata, it is convenient to introduce inverse transducers. Let $\mathcal{T} = (Q, \widetilde{A}, E, q_0, F)$ be a finite transducer. We say that $\mathcal{T}$ is:

- *involutive* if it satisfies

$$(p, a, u, q) \in E \iff (q, \overline{a}, \overline{u}, p) \in E$$

  for all $a \in A$ and $u \in \widetilde{A}^*$;

- *inverse* if it is deterministic, complete and involutive.

Note that, with respect to the case of inverse automata, completeness replaces trimness.

The next result shows that inverse transducers are appropriate to work in the context of free groups. This is essentially [53, Proposition 3.1] with the adaptations which follow from using a slightly different definition of inverse transducer.

Given a homomorphism $\varphi : M \to N$, we use the notation

$$\ker(\varphi) = \{(u, v) \in M \times M \mid \varphi(u) = \varphi(v)\}$$

for the *kernel congruence* of φ.

Proposition 1.7 [53, Proposition 3.1] *Let $\mathcal{T} = (Q, \widetilde{A}, E, q_0, F)$ be a finite inverse transducer. Then:*

(i) the transition mapping $Q \times \widetilde{A}^* \to Q$ induces a mapping $Q \times F_A \to Q$ through $q(\theta(u)) = qu$;

(ii) the partial mapping $\widehat{\mathcal{T}} : \widetilde{A}^* \to \widetilde{A}^*$ induces a partial mapping $\widetilde{\mathcal{T}} : F_A \to F_A$ through $\widetilde{\mathcal{T}}(\theta(u)) = \theta(\widehat{\mathcal{T}}(u))$.

Proof. (i) Since $\ker(\theta)$ is generated by $\mathcal{R}_A$, it suffices to show that $qva\overline{a}w = qvw$ for all $q \in Q$; $v, w \in \widetilde{A}^*$ and $a \in \widetilde{A}$. Since $\mathcal{T}$ is complete, we have a path

$$q \xrightarrow{v|v'} q_1 \xrightarrow{a|u} q_2 \xrightarrow{\overline{a}|u'} q_3 \xrightarrow{w|w'} q_4 \qquad (4)$$

in $\mathcal{T}$. Since $\mathcal{T}$ is involutive and deterministic, we must have $u' = \overline{u}$ and $q_3 = q_1$, hence we also have a path

$$q \xrightarrow{v|v'} q_1 \xrightarrow{w|w'} q_4$$

and so $qva\overline{a}w = q_4 = qvw$ as required.

(ii) We proceed similarly to part (i), noting that by (i) the domain of $\widetilde{\mathcal{T}}$ is necessarily a union of θ-classes. □

We say that $\widetilde{\mathcal{T}}$ is the transduction of F_A defined by the inverse transducer $\mathcal{T}$. A partial transformation τ of F_A is called a *transduction* if $\tau = \widetilde{\mathcal{T}}$ for some inverse transducer $\mathcal{T}$.

1.4 Rewriting systems

Rewriting systems play a very important role in both computer science and combinatorial group theory. We present here the very few notions needed to understand this chapter, but the reader can find more details in Subsection 1.1 of Chapter 2.

Let A be a finite alphabet and let $\mathcal{R} \subseteq A^* \times A^*$. The relation $\mathcal{R}$ determines a relation $\underset{\mathcal{R}}{\Longrightarrow}$ on A^* by: $u \underset{\mathcal{R}}{\Longrightarrow} v$ if there exist $(r, s) \in \mathcal{R}$ and $x, y \in A^*$ such that $u = xry$ and $v = xsy$. We say that $\underset{\mathcal{R}}{\Longrightarrow}$ is the *rewriting system* over A^* determined by $\mathcal{R}$. The reflexive and transitive closure of $\underset{\mathcal{R}}{\Longrightarrow}$ is denoted by $\overset{*}{\underset{\mathcal{R}}{\Longrightarrow}}$. When the rewriting system is clear from the context, we omit the subscript $\mathcal{R}$. For details on rewriting systems, the reader is referred to [12].

We say that $\underset{\mathcal{R}}{\Longrightarrow}$ is:

- *length-reducing* if $|r| > |s|$ for every $(r, s) \in \mathcal{R}$;
- *noetherian* if there is no infinite chain of the form

$$v_0 \underset{\mathcal{R}}{\Longrightarrow} v_1 \underset{\mathcal{R}}{\Longrightarrow} v_2 \underset{\mathcal{R}}{\Longrightarrow} \dots$$

- *confluent* if, whenever $u \xRightarrow[\mathcal{R}]{*} v$ and $u \xRightarrow[\mathcal{R}]{*} w$, there exists some $z \in A^*$ such that $v \xRightarrow[\mathcal{R}]{*} z$ and $w \xRightarrow[\mathcal{R}]{*} z$;
- *convergent* if it is confluent and noetherian.

Clearly, every length-reducing rewriting system is noetherian. A word $u \in A^*$ is an *irreducible* if no $v \in A^*$ satisfies $u \xRightarrow[\mathcal{R}]{} v$. We denote by $\mathrm{IRR}(\mathcal{R})$ the set of all irreducible words in A^* with respect to $\xRightarrow[\mathcal{R}]{}$.

We denote by $\xLeftrightarrow[\mathcal{R}]{*}$ the symmetric closure of $\xRightarrow[\mathcal{R}]{*}$. Equivalently, $\xLeftrightarrow[\mathcal{R}]{*}$ is the congruence on A^* generated by $\mathcal{R}$. Given $u \in A^*$, let $[u]$ denote the congruence class of u and let $M = A^*/\xLeftrightarrow[\mathcal{R}]{*} = \{[u] \mid u \in A^*\}$. If $\xRightarrow[\mathcal{R}]{}$ is convergent, it turns out that each congruence class $[u]$ contains a unique irreducible word, denoted by $\widehat{u}$ (see Subsection 1.2 of Chapter 2 for a proof). Hence the equivalence

$$u \xLeftrightarrow[\mathcal{R}]{*} v \qquad \text{if and only if} \qquad \widehat{u} = \widehat{v} \tag{5}$$

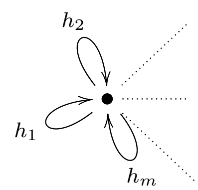
solves the word problem for M in such a case. This is precisely what happens with free groups, since F_A is the quotient of $\widetilde{A}^*$ by the congruence generated by the relation $\mathcal{R}_A$ from (1), and $\mathcal{R}_A$ constitutes a convergent rewriting system. We have of course $\mathrm{IRR}(\mathcal{R}_A) = R_A$, hence we are consistent in our use of the notation $\widehat{u}$.

2 Automata in group theory

We present in this section a very brief account of the role played by automata in various classes of infinite groups, very much inspired by a survey talk given at DCFS'12 [55]. A deeper and more extended survey on the interactions groups/automata can be found in two chapters written in collaboration with Bartholdi for a handbook [5, 6].

2.1 Free groups

Finite automata provide today the most efficient representation of finitely generated subgroups H of a free group F_A. The algorithm known as *Stallings construction* builds an automaton $\S(H)$ which can be used for solving the membership problem for H within F_A and many other applications. Many features of $\S(H)$, which admits a geometric interpretation (the core of the Schreier graph of H), were (re)discovered over the years and were known to Reidemeister, Schreier, and particularly Serre [52]. One of the greatest contributions of Stallings [58] is certainly the algorithm to construct $\S(H)$: taking a finite set of generators $h_1, \ldots, h_m$ of H in reduced form, we start with the so-called flower automaton $\mathcal{F}(H)$, where *petals* labelled by the words h_i (and their inverse edges) are glued to a basepoint q_0:



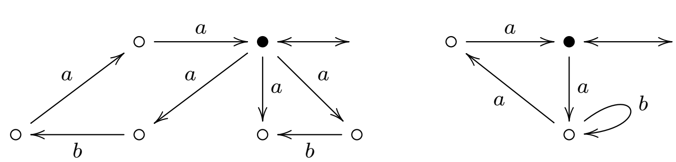
Then we proceed by successively folding pairs of edges of the form $q \xleftarrow{a} p \xrightarrow{a} r$ until reaching a deterministic automaton. And we will have just built $\S(H)$. For details and applications of the Stallings construction, see [6, 37, 44].

The geometric interpretation of $\S(H)$ shows that its construction is independent of the finite set of generators of H chosen at the beginning, and of the particular sequence of foldings followed. And the membership problem is a consequence of the following result:

Theorem 2.1 [58] *Let H be a finitely generated subgroup of F_A and let $u \in R_A$. Then $\theta(u) \in H$ if and only if $u \in L(\S(H))$.*

The main reason for this is that any irreducible word representing an element of H can be obtained by successively cancelling factors $a\overline{a}$ in a word accepted by the flower automaton of H, and folding edges provides a geometric realization of such cancellations.

Example 2.2 *Let $A = \{a, b\}$ and $H = \langle aba^{-1}, aba^2 \rangle \leq F_A$. We get*



$\mathcal{F}(H)$ $\S(H)$

thus a^3 represents an element of H but a^4 does not.

The applications of Stallings automata to the algorithmics of finitely generated subgroups of a free group are manifold. One of the most important is the construction of a *basis* for H (a free group itself by Nielsen's Theorem) using a *spanning tree* of $\S(H)$.

The following result illustrates how automata-theoretic properties of $\S(H)$ can determine group-theoretic properties of H:

Proposition 2.3 [58] *Let H be a finitely generated subgroup of F_A. Then H is a finite index subgroup of F_A if and only if $\S(H)$ is a complete automaton.*

In particular, the subgroup H of Example 2.2 has infinite index.

2.2 Virtually free groups

A group G is said to be *virtually free* if it has a free subgroup F of finite index. Since $\cap_{g \in G}\, gFg^{-1}$ is then a finite index normal subgroup of G (and free by Nielsen's Theorem, being a subgroup of F), we may assume that F is a normal subgroup of G.

It is well-known that a finite index subgroup of a finitely generated group is always finitely generated, hence a finitely generated group is virtually free if and only if it has a finitely generated free (normal) subgroup of finite index.

Virtually free groups admit various characterizations of different types, and the reader will find proofs of such equivalences in Chapter 2. Nevertheless, we shall briefly describe a few, since virtually free groups are also the main object of study in this chapter.

Before introducing the first, we recall the inverse automaton built from the Cayley graph $\Gamma_A(G)$ in Subsection 1.2, recognizing the language $\pi^{-1}(1)$, where $\pi : \widetilde{A}^* \to G$ is the corresponding matched homomorphism. Clearly, $\pi^{-1}(1)$ determines the structure of G, and it is easy to show that $\pi^{-1}(1)$ is rational if and only if G is finite (see the proof of Corollary 2.4 in Chapter 2).

What happens beyond the rational level? The next level in the classical Chomsky's hierarchy is the class of *context-free* languages, discussed in detail in Section 2 of Chapter 2. The celebrated theorem proved by Muller and Schupp (with a contribution from Dunwoody) states the following:

Theorem 2.4 [45] *Let A be a finite alphabet and let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a group G. Then $\pi^{-1}(1)$ is a context-free language if and only if G is virtually free.*

A proof of this theorem is given in Chapter 2.

In the next characterization, rewriting systems are central. We say that a path $p \xrightarrow{u} q$ in $\Gamma_A(G)$ is a *geodesic* if it has shortest length among all the paths connecting p to q in $\Gamma_A(G)$. We denote by $\mathrm{Geo}_A(G)$ the set of labels of all geodesics in $\Gamma_A(G)$. Note that, since $\Gamma_A(G)$ is vertex-transitive (the left action of G on itself produces enough automorphisms of $\Gamma_A(G)$ to make it completely symmetric), it is irrelevant whether or not we fix a basepoint for this purpose.

The next result, due to Gilman, Hermiller, Holt and Rees, shows that geodesics behave rather nicely in the case of virtually free groups.

Theorem 2.5 [25, Theorem 1] *Let G be a finitely generated group. Then the following conditions are equivalent:*

(i) there exists a finite alphabet A, a matched epimorphism $\pi : \widetilde{A}^* \to G$ and a finite $\mathcal{R} \subseteq \mathrm{Ker}\, \pi$ such that $\underset{\mathcal{R}}{\Longrightarrow}$ is length-reducing and $\mathrm{Geo}_A G = \mathrm{IRR}(\mathcal{R})$;

(ii) G is virtually free.

The proof uses Bass-Serre theory, which leads to yet another characterization of virtually free groups. We mention it briefly, avoiding introducing all the required definitions:

It is known that finitely generated virtually free groups are, up to isomorphism, the fundamental groups of graphs of groups where the graph, the vertex groups and the edge groups are all finite. Moreover, they can be obtained from finite groups by finitely many successive applications of free products with amalgamation over finite groups and HNN extensions over finite groups [39].

Once again, the reader may find proofs for all these equivalences in Chapter 2.

We end this account with a very recent characterization. The Stallings construction naturally invites generalizations to broader classes of groups. For instance, an elegant geometric construction of Stallings type automata was achieved for amalgams of finite groups by Markus-Epstein [42]. On the other hand, the most general results were obtained by Kapovich, Weidmann and Miasnikov [38], but the complex algorithms were designed essentially to solve the generalized word problem, and it seems very hard to extend other features of the free group case, either geometric or algorithmic. In joint work with Soler-Escrivà and Ventura [56], we developed a new idea: restricting the type of irreducible words used to represent elements (leading to the concept of *Stallings section*), and find out which groups admit a representation of finitely generated subgroups by finite automata obtained through edge folding from some sort of flower automaton. It turns out that the groups admitting a Stallings section are precisely the virtually free groups! And many of the geometric/algorithmic features of the classical free group case can then be generalized to the virtually free case.

2.3 Hyperbolic groups

Automata also play an important role in the beautiful geometric theory of hyperbolic groups, introduced by Gromov in the eighties [33]. For details on this class of groups, the reader is referred to [23].

Given a group G (finitely) generated by A, the general philosophy is to consider geometric conditions on the structure of $\Gamma_A(G)$ that can lead to a global understanding of the Cayley graph through the local structure (organizing a chart system based on finite subgraphs of $\Gamma_A(G)$). But which conditions? The answer came in the form of *hyperbolic geometry*.

The *geodesic distance* d_A on G is defined by taking $d_A(g,h)$ to be the length of a geodesic from g to h in $\Gamma_A(G)$. Given $X \subseteq G$ nonempty and $g \in G$, we define

$$d_A(g,X) = \min\{d_A(g,x) \mid x \in X\}.$$

A *geodesic triangle* in $\Gamma_A(G)$ is a collection of three geodesics

$$P_1 : g_1 \longrightarrow g_2, \quad P_2 : g_2 \longrightarrow g_3, \quad P_3 : g_3 \longrightarrow g_1$$

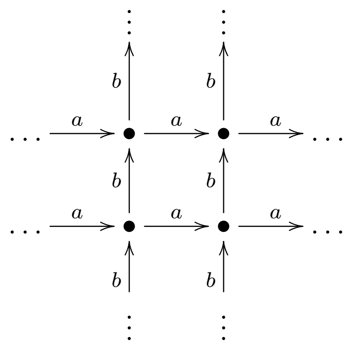
connecting three vertices $g_1, g_2, g_3 \in G$. Let $V(P_i)$ denote the set of vertices occurring in the path P_i. Given $\delta \geq 0$, we say that $\Gamma_A(G)$ is *δ-hyperbolic* if

$$\forall g \in V(P_1) \qquad d_A(g, V(P_2) \cup V(P_3)) \leq \delta$$

holds for every geodesic triangle $\{P_1, P_2, P_3\}$ in $\Gamma_A(G)$. If this happens for some $\delta \geq 0$, we say that G is *hyperbolic*. It is well known that the concept is independent from both alphabet and matched epimorphism, but the hyperbolicity constant δ may change.

Fundamental groups of compact riemannian manifolds with negative (not necessarily constant) sectional curvature are among the most important examples of hyperbolic groups. So are virtually free groups: in fact, they can be characterized by strengthening the geometric condition in the definition of hyperbolicity, replacing geodesic triangles by geodesic polygons.

However, the free Abelian group $\mathbb{Z} \times \mathbb{Z}$, whose Cayley graph (for canonical generators a, b) is the infinite grid



is not hyperbolic. However, there exist plenty of hyperbolic groups: Gromov remarked that, under some reasonable assumptions, the probability of a finitely presented group being hyperbolic is 1.

One of the extraordinary geometric properties of hyperbolic groups is closure under quasi-isometry, being thus one of the few examples where algebra gets away with the concept of *deformation*.

From an algorithmic viewpoint, hyperbolic groups enjoy excellent properties: they have solvable word problem, solvable conjugacy problem and many other positive features. The next two results involve rational languages and rewriting systems:

Theorem 2.6 [20, Theorem 3.4.5] *Let A be a finite alphabet and let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a hyperbolic group G. Then the set of geodesics $\mathrm{Geo}_A(G)$ is a rational language.*

Theorem 2.7 [2] *Let A be a finite alphabet and let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a group G. Then the following conditions are equivalent:*

(i) G is hyperbolic;

(ii) there exists a finite $\mathcal{R} \subseteq \widetilde{A}^* \times \widetilde{A}^*$ such that $\underset{\mathcal{R}}{\Longrightarrow}$ is length-reducing and

$$u \in \pi^{-1}(1) \qquad \text{if and only if} \qquad u \overset{*}{\underset{\mathcal{R}}{\Longrightarrow}} 1$$

holds for every $u \in \widetilde{A}^*$.

It follows easily from this result that $\pi^{-1}(1)$ is a *context-sensitive* language (the third level of Chomsky's hierarchy) if G is hyperbolic. However, the converse fails, $\mathbb{Z} \times \mathbb{Z}$ being a counter-example.

Other connections between automata and hyperbolic groups will be unveiled in the next subsection, which features the wider class of automatic groups.

2.4 Automatic groups

Also in the eighties, another very interesting idea germinated in geometric group theory, and automata were to play the leading role. The new concept was due to Cannon, Epstein, Holt, Levy, Paterson and Thurston [20] (see also [7]).

In view of Theorem 2.6, it is easy to see that every hyperbolic group admits a rational set of normal forms. But this is by no means an exclusive of hyperbolic groups, and rational normal forms are not enough to understand the structure of a group. We need to understand the product, or at least the action of generators on the set of normal forms. Can automata help?

There are different ways of encoding mappings as languages, synchronously or asynchronously. We shall describe the most popular synchronous method, through *convolution*.

Given a finite alphabet A, we assume that \$ is a new symbol (called the *padding symbol*) and define a new alphabet

$$A_{\$} = (A \times A) \cup (A \times \{\$\}) \cup (\{\$\} \times A).$$

For all $u, v \in A^*$, $u \diamond v$ is the unique word in $A_{\$}^*$ whose projection to the first (respectively second) components yields a word in $u\$^*$ (respectively $v\$^*$). For instance, $a \diamond ba = (a, b)(\$, a)$.

Let $\pi : A^* \to G$ be a homomorphism onto a group G. We say that $L \in \mathrm{RAT}(A^*)$ is a *section* for π if $\pi(L) = G$. For every $u \in A^*$, write

$$L_u = \{v \diamond w \mid v, w \in L,\ \pi(vu) = \pi(w)\}.$$

We say that $L \in \mathrm{RAT}(A^*)$ is an *automatic structure* for π if:

- L is a section for π;
- $L_a \in \mathrm{RAT}(A_{\$}^*)$ for every $a \in A \cup \{1\}$.

It can be shown that the existence of an automatic structure is independent from the alphabet A or the homomorphism π, and implies the existence of an *automatic structure with uniqueness* (i.e. $\pi|_L$ is injective). A group is said to be *automatic* if it admits an automatic structure.

The class of automatic groups contains all hyperbolic groups (in fact, $\mathrm{Geo}_A(G)$ is then an automatic structure! [20, Theorem 3.4.5]) and is closed under such operators as free products, finite extensions or direct products. As a consequence, it contains all free abelian groups of finite rank and so automatic groups need not be hyperbolic. By the following result of Gilman, hyperbolic groups can be characterized within automatic groups by a language-theoretic criterion:

Theorem 2.8 [24] *Let A be a finite alphabet and let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism onto a group G. Then the following conditions are equivalent:*

(i) G is hyperbolic;

(ii) G admits an automatic structure with uniqueness L for π such that the language

$$\{u\$v\$w \mid u, v, w \in L,\ \pi(uvw) = 1\}$$

is context-free.

Among many other good algorithmic properties, automatic groups are finitely presented and have decidable word problem (in quadratic time). The reader is referred to [7, 20] for details.

Geometry also plays an important part in the theory of automatic groups, through the *fellow traveller property.* Given a word $u \in A^*$, let $u^{[n]}$ denote the prefix of u of length n (or u itself if $n > |u|$). Let $\pi : \widetilde{A}^* \to G$ be a matched homomorphism and recall the geodesic distance d on G introduced in Subsection 2.3 in connection with the Cayley graph $\Gamma_A(G)$. We say that a section L for π satisfies the fellow traveller property if there exists some constant $K > 0$ such that

$$\forall u, v \in L \quad (d_A(\pi(u), \pi(v)) \leq 1 \quad \Rightarrow \quad \forall n \in \mathbb{N} \quad d_A(\pi(u^{[n]}), \pi(v^{[n]})) \leq K).$$

Intuitively, this expresses the fact that two paths in $\Gamma_A(G)$ labelled by words $u, v \in L$ which start at the same vertex and end up in neighbouring (or equal) vertices *stay close all the way through.*

This geometric property provides an alternative characterization of automatic groups which avoids convolution:

Theorem 2.9 [20, Theorem 2.3.5] *Let* $\pi : \widetilde{A}^* \to G$ *be a matched homomorphism onto a group* G *and let* L *be a rational section for* π*. Then the following conditions are equivalent:*

(i) L *is an automatic structure for* π*;*

(ii) L *satisfies the fellow traveller property.*

The combination of automata-theoretic and geometric techniques is typical of the theory of automatic groups.

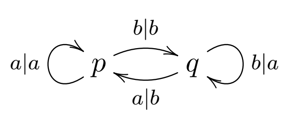
2.5 Self-similar groups

Self-similar groups, also known as *automaton groups*, were introduced in the sixties by Glushkov [26] (see also [1]) but it was through the leading work of Grigorchuk in the eighties [29, 30] that they became a main research subject in geometric group theory. Here automata play a very different role compared with previous subsections.

We can view a free monoid A^* as a rooted tree T with edges u — ua for all $u \in A^*$, $a \in A$ and root 1. The automorphism group of T, which is uncountable if $|A| > 1$, is self-similar in the following sense: if we restrict an automorphism φ of T to a cone uA^*, we get a mapping of the form $uA^* \to (\varphi(u))A^* : uv \mapsto \varphi(u)\psi(v)$ for some automorphism ψ of T. This leads to wreath product decompositions (see [46]) and the possibility of recursion.

But $\mathrm{Aut}(T)$ is huge and non finitely generated except in trivial cases, hence it is a natural idea to study subgroups G of T generated by a finite set of self-similar generators (in the above sense) to keep all the chances of effective recursion methods within a finitely generated context. It turns out that this is equivalent to define G through a finite invertible *Mealy automaton*.

A Mealy automaton on the alphabet A is a finite complete deterministic transducer where edges are labelled by pairs of letters of A, no initial/terminal vertices being assigned. It is said to be *invertible* if the local transformations of A (induced by the labels of the edges leaving a given vertex) are permutations. Here is a famous example of an invertible Mealy automaton:

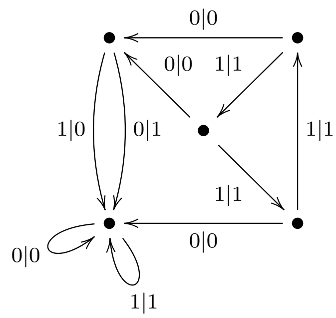


The transformations of $A = \{a, b\}$ induced by the vertices p and q are the identity mapping and the transposition (ab), respectively.

Each vertex q of a Mealy automaton $\mathcal{A}$ defines an endomorphism φ_q of the tree T through the paths $q \xrightarrow{u|\varphi_q(u)} \ldots$ $(u \in A^*)$. If the automaton is invertible, each φ_q is indeed an automorphism and the set of all φ_q, for all vertices q of $\mathcal{A}$, satisfies the desired self-similarity condition. The (finitely generated) subgroup of $\mathrm{Aut}(T)$ generated by the φ_q is the self-similar group $\mathcal{G}(\mathcal{A})$ generated by $\mathcal{A}$.

For instance, the self-similar group generated by the Mealy automaton in the above example is the famous *lamplighter group* [32].

Self-similar groups have decidable word problem. Moreover, the recursion potential offered by their wreath product decompositions allowed successful computations which were hard to foresee with more traditional techniques and turned self-similar groups into the most rich source of counterexamples in infinite group theory ever. The Grigorchuk group [30], generated by the Mealy automaton



is the most famous of the lot, but their exist many others exhibiting fascinating exotic properties [34, 31].

An interesting infinite family of Mealy automata has been studied by the author in joint papers with Steinberg [57] and Kambites and Steinberg [36]: *Cayley machines* of finite groups G (the Cayley graph is adapted by taking edges

$$g \xrightarrow{a|ga} ga$$

and all the elements of the group as generators). If G is abelian, these Cayley machines generate the wreath product $G \wr \mathbb{Z}$, and the lamplighter group corresponds to the case $G = \mathbb{Z}_2$.

Surprising connections with fractals were established in recent years. We shall briefly describe one instance. Given a matched homomorphism $\pi : \widetilde{A}^* \to G$ and a subgroup P of G, the *Schreier graph* $\Gamma_A(G, P)$ has the cosets Pg as vertices and edges $Pg \xrightarrow{a} Pga$ for all $g \in G$ and $a \in \widetilde{A}$. Note that $P = \{1\}$ yields the familiar Cayley graph $\Gamma_A(G)$. It turns out that classical fractals can be obtained as limits of the sequence of graphs $(\Gamma_A(G, P_n))_n$ for some adequate self-similar group G, where P_n denotes the stabilizer of the nth level of the tree T [4, 46]. Note that P_n has finite index and so the Schreier graphs $\Gamma_A(G, P_n)$ are finite.

3 Fixed points of endomorphisms

We discuss in this section fixed point subgroups of group endomorphisms, and we even go a little beyond that in the free group case.

3.1 A brief introduction

We start with a very summarized account of the research on this subject. We have no ambition of being exhaustive, concentrating on results directly related with the material of the course. The reader is referred to the survey of Ventura [63] for a more detailed exposition of this subject.

Gersten proved in the eighties that the fixed point subgroup of a free group automorphism φ is finitely generated [22], with a proof that can be considered as automata-theoretic. Using a different approach, Cooper gave in [17] an alternative topological proof.

Gersten's result was generalized to further classes of groups and endomorphisms in subsequent years. Goldstein and Turner extended it to monomorphisms of free groups [28], and later to arbitrary endomorphisms [27], using what can also be considered an automata-theoretic approach. With respect to automorphisms, the widest generalization is to hyperbolic groups and is due to Paulin [47].

Bestvina and Handel achieved in the late eighties major progress through their innovative train track techniques, bounding the rank of the fixed point subgroup [10]. Their results were subsequently generalized to arbitrary endomorphisms by Imrich and Turner [35] and automorphisms of free products of freely indecomposable groups by Collins and Turner [16]. In [60, 61], Sykiotis generalized these rank bounds to free products using the concepts of *symmetric endomorphism* and *Kurosh rank*. As a consequence, he generalized Gersten's theorem to arbitrary endomorphisms of virtually free groups.

Graph groups (also known as *right angled Artin groups*) are defined through commutation relations between (some) generators. In a joint paper of the author with Rodaro and Sykiotis [49], it was proved that the fixed point subgroup is finitely generated for every endomorphism of a graph group G if and only if G is a free product of abelian groups.

Train-tracks were also used by Maslakova in 2003, who considered the problem of effectively computing a basis for the fixed point subgroup of a free group automorphism. However, the paper [43] turned out to contain some errors. In 2012, a new paper by Bogopolski and Maslakova [11] was posted in arXiv with the purpose of correcting the aforementioned problems.

3.2 Fixed points of transductions

Let End(G) (respectively Aut(G)) denote the monoid of endomorphisms (respectively group of automorphisms) of a group G. Given a (partial) transformation τ of G, write

$$\mathrm{Fix}(\tau) = \{u \in G \mid \tau(u) = u\}.$$

It is easy to see that every $\varphi \in \mathrm{End}(F_A)$ is a transduction of F_A: we have $\varphi = \widetilde{\mathcal{T}}$ for $\mathcal{T} = (\{q\}, \widetilde{A}, E, q, \{q\})$, where

$$E = \{(q, a, \varphi(a), q) \mid a \in \widetilde{A}\}.$$

However, in general, neither transductions of F_A are endomorphisms, nor their fixed points constitute a subgroup. We present in this subsection a generalization to transductions of Goldstein and Turner's proof [27] for fixed points of free group endomorphisms. This result will be applied in Subsection 3.3 to fixed points of virtually free group endomorphisms.

Theorem 3.1 [53, Theorem 3.2] *Let τ be a transduction of F_A and let $z \in F_A$. Then*

$$X_\tau^z = \{g \in F_A \mid \tau(g) = gz\}$$

is rational.

Proof. Write $\tau = \widetilde{\mathcal{T}}$ for some finite inverse transducer $\mathcal{T} = (Q, \widetilde{A}, E, q_0, F)$ with output function λ. Let $\mathcal{T}' = (Q, \widetilde{A}, E, q_0, Q)$ and $\tau' = \widetilde{\mathcal{T}'}$. For every $g \in F_A$, let

$$P_1(g) = g^{-1}\tau'(g) \in F_A, \qquad P(g) = (P_1(g), q_0 g).$$

Note that

$$g \in X_{\tau'}^z \quad \text{if and only if} \quad P_1(g) = z,$$

$$g \in X_\tau^z \quad \text{if and only if} \quad P_1(g) = z \text{ and } q_0 g \in F.$$

We define a deterministic automaton $\mathcal{A}_\mathcal{T} = (P, \widetilde{A}, E', (1, q_0), F')$ by

$P = \{P(g) \mid g \in F_A\}$;

$F' = P \cap (\{z\} \times F)$;

$E' = \{(P(g), a, P(ga)) \mid g \in F_A,\ a \in \widetilde{A}\}$.

Clearly, $\mathcal{A}_\mathcal{T}$ is a possibly infinite automaton. Note that, since $\mathcal{T}$ is inverse, we have $qa\overline{a} = q$ for all $q \in Q$ and $a \in \widetilde{A}$. It follows that, whenever $(p, a, p') \in E$, then also $(p', \overline{a}, p) \in E$. Hence $\mathcal{A}_\mathcal{T}$ is inverse if it is trim. Since every vertex $P(g)$ lies in the path $P(1)\xrightarrow{\widehat{g}}P(g)$, this happens if and only if $F' \neq \emptyset$.

Since every $w \in \widetilde{A}^*$ labels a unique path $P(1)\xrightarrow{w}P(\theta(w))$, it follows that

$$\begin{array}{rcl} L(\mathcal{A}_\mathcal{T}) & = & \{w \in \widetilde{A}^* \mid P(\theta(w)) \in F'\} = \{w \in \widetilde{A}^* \mid P_1(\theta(w)) = z,\ q_0w \in F\} \\ & = & \theta^{-1}(X^z_\tau). \end{array}$$

We claim that to prove that X^z_τ is rational, it suffices to construct a finite subautomaton $\mathcal{B}_\mathcal{T}$ of $\mathcal{A}_\mathcal{T}$ such that

$$\widehat{X^z_\tau} \subseteq L(\mathcal{B}_\mathcal{T}). \tag{6}$$

Indeed, if this holds then

$$\theta(L(\mathcal{A}_\mathcal{T})) = X^z_\tau = \theta(\widehat{X^z_\tau}) \subseteq \theta(L(\mathcal{B}_\mathcal{T})) \subseteq \theta(L(\mathcal{A}_\mathcal{T}))$$

yields $X^z_\tau = \theta(L(\mathcal{B}_\mathcal{T}))$. Since rational subsets are preserved under homomorphic images, the claim follows from $\mathcal{B}_\mathcal{T}$ being finite.

To construct $\mathcal{B}_\mathcal{T}$, we fix

$$M = \max\{|\lambda(q, a)| : q \in Q,\ a \in \widetilde{A}\}, \quad N = \max\{2M + 1, |z|\}$$

and

$$P' = \{P(g) \in P : |P_1(g)| \leq N\}.$$

Since A and $\mathcal{T}$ are finite, so is P'. However, infinitely many $g \in F_A$ may yield the same state $P(g)$.

We say that an edge $(p_1, a, p_2) \in E'$ is:

- *central* if $p_1, p_2 \in P'$;
- *compatible* if it is not central and $\widehat{p_1}$ starts with a.

The proof of the following lemma is straightforward and can be found in [53]:

Lemma 3.2 [53, Lemma 3.3]

(i) There are only finitely many central edges in $\mathcal{A}_\mathcal{T}$.

(ii) If $(p_1, a, p_2) \in E'$ is not central, then either (p_1, a, p_2) or $(p_2, \overline{a}, p_1)$ is compatible.

(iii) For every $p \in P$, there is at most one compatible edge leaving p.

A (possibly infinite) path $q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \ldots$ in $\mathcal{A}_\mathcal{T}$ is:

- *central* if all the vertices in it are in P';
- *compatible* if all the edges in it are compatible and no intermediate vertex is in P'.

We also omit the straightforward proof of the following lemma:

Lemma 3.3 [53, Lemma 3.4] *Let $u \in \widehat{X^z_\mathcal{T}}$. Then there exists a path*

$$(1, q_0) = p'_0 \xrightarrow{u_0} p''_0 \xrightarrow{v_1} p_1 \xrightarrow{\overline{w_1}} p'_1 \xrightarrow{u_1} \ldots \xrightarrow{v_n} p_n \xrightarrow{\overline{w_n}} p'_n \xrightarrow{u_n} p''_n \in T'$$

in $\mathcal{A}_\mathcal{T}$ such that:

(i) $u = u_0 v_1 \overline{w_1} u_1 \ldots v_n \overline{w_n} u_n$;

(ii) the paths $p'_j \xrightarrow{u_j} p''_j$ are central;

(iii) the paths $p''_{j-1} \xrightarrow{v_j} p_j$ and $p'_j \xrightarrow{w_j} p_j$ are compatible;

(iv) $p_j \notin P'$ if both v_j and w_j are nonempty.

We say that a compatible path is *maximal* if it is infinite or cannot be extended (to the right) to produce another compatible path.

Lemma 3.4 *For every $p \in P'$, there exists in $\mathcal{A}_\mathcal{T}$ a unique maximal compatible path M_p starting at p.*

Indeed, every compatible path can be extended to a maximal compatible path, and uniqueness follows from Lemma 3.2(iii).

We define now

$$P'_1 = \{p \in P' \mid M_p \text{ has finitely many distinct edges }\}$$

and $P'_2 = P' \setminus P'_1$. Hence M_p contains no cycles if $p \in P'_2$. By Lemma 3.4, if M_p and $M_{p'}$ intersect at vertex $r_{pp'}$, then they coincide from $r_{pp'}$ onwards.

In particular, if M_p and $M_{p'}$ intersect, then $p \in P'_1$ if and only if $p' \in P'_1$. Let

$$Y = \{(p,p') \in P'_2 \times P'_2 \mid \ M_p \text{ intersects } M_{p'}\}.$$

For every $(p,p') \in Y$, let $M_p \setminus M_{p'}$ denote the (finite) subpath $p \longrightarrow r_{pp'}$ of M_p. In particular, if $p' = p$, then $M_p \setminus M_{p'}$ is the trivial path at p.

Let $\mathcal{B}_{\mathcal{T}}$ be the subautomaton of $\mathcal{A}_{\mathcal{T}}$ containing:

- all vertices in P' and all central edges;
- all vertices and edges in the paths M_p ($p \in P'_1$) and their inverses;
- all vertices and edges in the paths $M_p \setminus M_{p'}$ ($(p,p') \in Y$) and their inverses.

It follows easily from Lemma 3.2(i) and the definitions of P'_1 and $M_p \setminus M_{p'}$ that $\mathcal{B}_{\mathcal{T}}$ is a finite subautomaton of $\mathcal{A}_{\mathcal{T}}$. We show that $\widehat{X^z_\tau} \subseteq L(\mathcal{B}_{\mathcal{T}})$.

Let $u \in \widehat{X^z_\tau}$. Since $\mathcal{B}_{\mathcal{T}}$ contains all the central edges of $\mathcal{A}_{\mathcal{T}}$, it suffices to show that all subpaths

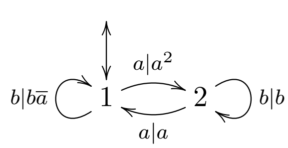
$$p''_{j-1} \xrightarrow{v_j} p_j \xrightarrow{\overline{w_j}} p'_j$$

appearing in the factorization provided by Lemma 3.3 are paths in $\mathcal{B}_{\mathcal{T}}$.

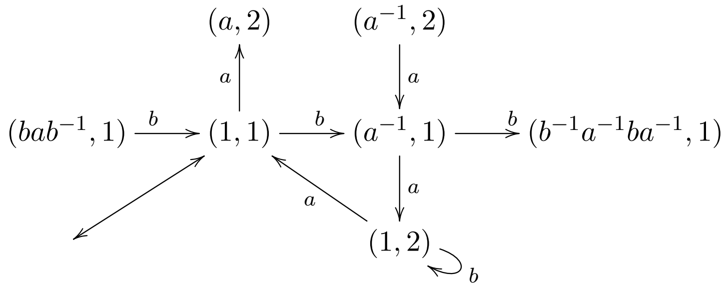
Without loss of generality, we may assume that $v_j \neq 1$. If $w_j = 1$, then $p''_{j-1} \in P'_1$ and we are done, hence we may assume that also $w_j \neq 1$. Now, if one of the vertices p''_{j-1}, p'_j is in P'_1, so is the other and the claim holds since $\mathcal{B}_{\mathcal{T}}$ contains all the edges in the paths M_p ($p \in P'_1$) and their inverses. Hence we may assume that $p''_{j-1}, p'_j \in P'_2$. It follows that $p_j = r_{p''_{j-1},p'_j}$ (since $v_j\overline{w_j} \in R_A$, the paths $M_{p''_{j-1}}$ and $M_{p'_j}$ cannot meet before p_j). Thus $p''_{j-1} \xrightarrow{v_j} p_j$ is $M_{p''_{j-1}} \setminus M_{p'_j}$ and $p'_j \xrightarrow{\overline{w_j}^{-1}} p_j$ is $M_{p'_j} \setminus M_{p''_{j-1}}$, and so these are also paths in $\mathcal{B}_{\mathcal{T}}$ as required.

Therefore (6) holds and so X^z_τ is rational. $\square$

Example 3.5 *Let $A = \{a,b\}$ and let $\mathcal{T}$ be the inverse transducer depicted by*



Let $\tau = \widetilde{\mathcal{T}}$ and let $\mathcal{A}_{\mathcal{T}}$ recognize $X_\tau^1 = \mathrm{Fix}(\tau)$. We can compute any finite subautomaton of $\mathcal{A}_{\mathcal{T}}$ such as



In general, we have no algorithm to decide if our finite subautomaton contains all the reduced cycles of $\mathcal{A}_{\mathcal{T}}$. In this particular example, we can argue that claim: indeed, if

$$i \xrightarrow{x|u} j \xrightarrow{y|v} k$$

are consecutive edges in $\mathcal{T}$ with uv not reduced, then we have either

$$1 \xrightarrow{b|b\overline{a}} 1 \xrightarrow{a|a^2} 2 \quad \text{or} \quad 2 \xrightarrow{\overline{a}|\overline{a}^2} 1 \xrightarrow{\overline{b}|a\overline{b}} 1.$$

It is easy to see that this implies that these edges must always occur consecutively in any fixed point of τ, hence we can take the above subautomaton of $\mathcal{A}_{\mathcal{T}}$ as $\mathcal{B}_{\mathcal{T}}$.

3.3 Virtually free group endomorphisms

We can use Theorem 3.1 to prove Sykiotis' theorem:

Theorem 3.6 [60, Proposition 3.4] *Let φ be an endomorphism of a finitely generated virtually free group. Then* $\mathrm{Fix}(\varphi)$ *is finitely generated.*

Proof. We consider a decomposition of G as a disjoint union

$$G = Fb_0 \cup Fb_1 \cup \ldots \cup Fb_m, \tag{7}$$

where $F = F_A \trianglelefteq G$ is a free group of finite rank and $b_0, \ldots, b_m \in G$ with $b_0 = 1$.

Let $\tau : F_A \to F_A$ and $\eta : F_A \to \{0, \ldots, m\}$ be defined by

$$\varphi(g) = \tau(g) b_{\eta(g)} \quad (g \in F_A).$$

Since the union in the decomposition (7) is disjoint, $\tau(g)$ and $\eta(g)$ are both uniquely determined by $\varphi(g)$, and so both mappings are well defined. We show next that τ is a transduction.

Write $Q = \{0, \ldots, m\}$. For all $p \in Q$ and $a \in \widetilde{A}$, we have $b_p(\varphi(a)) = h_{p,a} b_{\delta(p,a)}$ for some (unique) $h_{p,a} \in F_A$ and $\delta(p,a) \in Q$. We define a finite deterministic complete transducer $\mathcal{T} = (Q, \widetilde{A}, E, 0, Q)$ by taking

$$E = \{(p, a, \widehat{h_{p,a}}, \delta(p,a)) \mid p \in Q, a \in \widetilde{A}\}.$$

Assume that

$$p \xrightarrow{a|\widehat{h_{p,a}}} \delta(p,a) = q$$

is an edge of $\mathcal{T}$. Then $b_p(\varphi(a)) = h_{p,a} b_q$ and so also

$$b_p = b_p(\varphi(a))(\varphi(a^{-1})) = h_{p,a} b_q(\varphi(a^{-1})) = h_{p,a} h_{q,\overline{a}} b_{\delta(q,\overline{a})}.$$

This yields $h_{p,a} h_{q,\overline{a}} = 1$ and $\delta(q, \overline{a}) = p$, thus there is an edge $q \xrightarrow{\overline{a}|\widehat{\overline{h_{p,a}}}} \delta(q, \overline{a}) = p$ in $\mathcal{T}$ and so $\mathcal{T}$ is an inverse transducer.

We claim that $\widetilde{\mathcal{T}} = \tau$. Indeed, let $g = a_1 \ldots a_n \in F_A$ ($a_i \in \widetilde{A}$). Then there exists a (unique) successful path in $\mathcal{T}$ of the form

$$0 = p_0 \xrightarrow{a_1|\widehat{h_{p_0,a_1}}} p_1 \xrightarrow{a_2|\widehat{h_{p_1,a_2}}} \ldots \xrightarrow{a_n|\widehat{h_{p_{n-1},a_n}}} p_n.$$

Moreover, $p_i = \delta(p_{i-1}, a_i)$ for $i = 1, \ldots, n$. It follows that

$$\begin{aligned} \varphi(g) &= b_{p_0}(\varphi(a_1)) \ldots (\varphi(a_n)) = h_{p_0,a_1} b_{p_1}(\varphi(a_2)) \ldots (\varphi(a_n)) \\ &= h_{p_0,a_1} h_{p_1,a_2} b_{p_2}(\varphi(a_3)) \ldots (\varphi(a_n)) = \ldots = h_{p_0,a_1} \ldots h_{p_{n-1},a_n} b_{p_n} \end{aligned}$$

and so, since all the vertices in $\mathcal{T}$ are terminal, we get

$$\tau(g) = h_{p_0,a_1} \ldots h_{p_{n-1},a_n} = \theta(\widehat{h_{p_0,a_1}} \ldots \widehat{h_{p_{n-1},a_n}}) = \widetilde{\mathcal{T}}(g).$$

Thus $\widetilde{\mathcal{T}} = \tau$.

It follows also from the preceding computation that $0g = \eta(g)$ holds in $\mathcal{T}$ for every $g \in F_A$. For every $q \in Q$, let $\mathcal{T}_q = (Q, \widetilde{A}, E, 0, \{q\})$ and $\tau_q = \widetilde{\mathcal{T}_q}$. It follows that

$$\tau_q(g) = \begin{cases} \tau(g) & \text{if } \eta(g) = q \\ \text{undefined} & \text{otherwise} \end{cases}$$

Next let

$$Y = \{(p,q) \in Q \times Q \mid b_q(\varphi(b_p)) \in F_A b_p\}.$$

For every $(p,q) \in Y$, let $z_{p,q} \in F_A$ be such that $b_q(\varphi(b_p)) = z_{p,q}^{-1} b_p$. We show that

$$\mathrm{Fix}(\varphi) = \bigcup_{(p,q) \in Y} X_{\tau_q}^{z_{p,q}} b_p. \tag{8}$$

Indeed, if $(p, q) \in Y$ and $g \in X_{\tau_q}^{z_{p,q}}$, then $gz_{p,q} = \tau_q(g) = \tau(g)$ and $\eta(g) = q$, hence

$$\varphi(gb_p) = \varphi(g)\varphi(b_p) = \tau(g)b_{\eta(g)}(\varphi(b_p)) = gz_{p,q}b_q(\varphi(b_p)) = gz_{p,q}z_{p,q}^{-1}b_p = gb_p$$

and so $gb_p \in \mathrm{Fix}(\varphi)$.

Conversely, let $gb_p \in \mathrm{Fix}(\varphi)$ with $g \in F_A$ and $p \in Q$. Then

$$gb_p = \varphi(gb_p) = \varphi(g)\varphi(b_p) = \tau(g)b_{g\eta}(\varphi(b_p))$$

and so $b_{\eta(g)}(\varphi(b_p)) \in F_A b_p$. Writing $q = \eta(g)$, it follows that $(p, q) \in Y$ and so

$$gb_p = \tau(g)b_q(\varphi(b_p)) = \tau(g)z_{p,q}^{-1}b_p.$$

Hence $g = \tau(g)z_{p,q}^{-1} = \tau_q(g)z_{p,q}^{-1} \in X_{\tau_q}^{z_{p,q}}$ and so (8) holds.

Now $X_{\tau_q}^{z_{p,q}} \in \mathrm{RAT}(F_A)$ for every $(p, q) \in Y$ by Theorem 3.1. Since $F_A \subseteq G$, we get $X_{\tau_q}^{z_{p,q}} \in \mathrm{RAT}(G)$ and so (8) yields $\mathrm{Fix}(\varphi) \in \mathrm{RAT}(G)$. Since $\mathrm{Fix}(\varphi)$ is a subgroup of G, it follows from Theorem 1.5 that $\mathrm{Fix}(\varphi)$ is finitely generated. □

Unfortunately, our approach does not lead directly to an algorithm to compute a basis of $\mathrm{Fix}(\varphi)$ (see [11]) because it is not clear how to decide in Subsection 3.2 whether $p \in P'$ belongs to P'_1 or P'_2 and how to compute the paths M_p and $M_p \setminus M_{p'}$.

4 Fixed points in the boundary

4.1 A brief introduction

One of the remarkable features of hyperbolic groups is the role played by the *boundary*. Let $G = \langle A \rangle$ be a hyperbolic group. The elements of the boundary $\partial_A(G)$ can be described as equivalence classes of *rays* (infinite words on $\widetilde{A}$ whose finite factors are geodesics in $\Gamma_A(G)$). Two rays are equivalent if the Hausdorff (geodesic) distance between them is finite [23, Section 7.1], i.e. there exists some $N > 0$ such that every point from one of the rays is at geodesic distance $\leq N$ from the other ray. In the classical case $G = F_A$, the boundary consists of all reduced words of $\widetilde{A}^\omega$.

The boundary has very rich properties of geometrical, topological and dynamical nature. Its topological structure can be defined with the help of the *Gromov product*: the Gromov product (with respect to A and the basepoint 1) induces a distance d' on G, and the completion of the metric space (G, d') turns out to be $\Omega(G) = G \cup \partial_A(G)$. This defines univocally (up to homeomorphism) $\partial_A(G)$ and its topology. Both $\partial_A(G)$ and $\Omega(G)$ are compact, which immediately unveils part of the advantages procured by both boundary and hyperbolic completion. Since alternative (finite) generating sets induce a quasi-isometry of the respective Cayley graphs, $\partial_A(G)$ is (up to homeomorphism) independent from the choice of A, and the same holds for the completion.

Clearly, every uniformly continuous $\varphi \in \mathrm{End}(G)$ (with respect to the metric d') induces a unique continuous extension $\Phi : \Omega(G) \to \Omega(G)$. The elements of $\mathrm{Fix}(\Phi) \setminus \mathrm{Fix}(\varphi)$ are called the *infinite fixed points* of φ. It is well known that monomorphisms admit continuous extensions.

It is easy to see that $\mathrm{Fix}(\Phi)$ is a closed subspace of $\Omega(G)$. An infinite fixed point $\alpha \in \mathrm{Fix}(\Phi) \cap \partial_A(G)$ is said to be *singular* if α belongs to the topological closure $(\mathrm{Fix}(\varphi))^c$ of $\mathrm{Fix}(\varphi)$. Otherwise, α is said to be *regular*. We denote by $\mathrm{Sing}(\Phi)$ (respectively $\mathrm{Reg}(\Phi)$) the set of all singular (respectively regular) infinite fixed points of Φ. Clearly, there exists a natural action of $\mathrm{Fix}(\varphi)$ on the left of $\mathrm{Fix}(\Phi)$, hence we have the concept of $(\mathrm{Fix}(\varphi))$-*orbit*.

The first results on infinite fixed points of free group automorphisms are due to Cooper [17], which showed that $\mathrm{Reg}(\Phi)$ has finitely many $(\mathrm{Fix}(\varphi))$-orbits. A major breakthrough is done by Gaboriau, Jaeger, Levitt and Lustig in [21], where Bestvina and Handel's results are extended to consider also orbits of regular infinite fixed points, and a classification of infinite fixed points is provided. They also remark that some of the results would hold for virtually free groups with some adaptations.

In [54], we discussed infinite fixed points for monomorphisms of free products of cyclic groups, the group case of a more general setting based on the concept of special confluent rewriting system. This was a follow-up of previous work with Cassaigne [13, 14] on endomorphisms over these systems and their periodic points (finite and infinite).

In what follows, we shall present the results from [53], where we consider virtually injective endomorphisms of virtually free groups (which are precisely the uniformly continuous endomorphisms for the metric d'), and we discuss the dynamical nature of the regular fixed points in the automorphism case, generalizing the results of [21] on free groups.

4.2 A model for the boundary of virtually free groups

Given two words $\alpha, \beta \in A^\infty$, we denote by $\alpha \wedge \beta$ their longest common prefix.

We can use Theorem 2.5 to prove the following result:

Lemma 4.1 [53, Lemma 5.1] *Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5. Then there exists a positive integer N_0 such that, for all $u \in \mathrm{Geo}_A(G)$ and $v \in \widetilde{A}^*$:*

(i) there exists some $w \in \mathrm{Geo}_A(G)$ such that $\pi(w) = \pi(uv)$ and $|u \wedge w| \geq |u| - N_0|v|$;

(ii) there exists some $z \in \mathrm{Geo}_A(G)$ such that $\pi(z) = \pi(vu)$ and $|\overline{u} \wedge \overline{z}| \geq |u| - N_0|v|$.

In fact, we take N_0 to be the twice the length of the longest relator in the finite length-reducing rewriting system arising from Theorem 2.5.

We assume for the remainder of the chapter that G is a finitely generated virtually free group, $\pi : \widetilde{A}^* \to G$ a matched epimorphism and N_0 a positive integer satisfying the conditions of Lemma 4.1. Since G is hyperbolic, it follows from [20, Theorem 3.4.5] that $\mathrm{Geo}_A(G)$ is an automatic structure for G with respect to π and so the fellow traveller property holds for some constant $K_0 > 0$ (which can be taken as $2(\delta + 1)$, if δ is the hyperbolicity constant). This amounts to say that

$$d_A(\pi(u), \pi(v)) \leq 1 \quad \Rightarrow \quad \forall n \in \mathbb{N} \quad d_A(\pi(u^{[n]}), \pi(v^{[n]})) \leq K_0$$

holds for all $u, v \in \mathrm{Geo}_A(G)$.

We fix a total ordering of $\widetilde{A}$. The *shortlex ordering* of $\widetilde{A}^*$ is defined by

$$u \leq_{sl} v \text{ if } \begin{cases} |u| < |v| \\ \text{or} \\ |u| = |v| \text{ and } u = wau', v = wbv' \text{ with } a < b \text{ in } \widetilde{A} \\ \text{or} \\ u = v \end{cases}$$

This is a well-known well-ordering of $\widetilde{A}^*$, compatible with multiplication on the left and on the right. Let

$$M_A(G) = \{u \in \mathrm{Geo}_A(G) \mid u \leq_{sl} v \text{ for every } v \in \pi^{-1}(\pi(u))\}, \qquad (9)$$

i.e. the set of all the shortlex minimal representatives of the elements of G. By [20, Theorem 2.5.1], $M_A(G)$ is also an automatic structure for G with respect to π, and therefore rational. We note that, since $\leq_{sl}$ is compatible with multiplication, $M_A(G)$ is *factorial* (a factor of a word in $M_A(G)$ is still in $M_A(G)$).

Given $g \in G$, let $\widehat{g}$ denote the unique word of $M_A(G)$ representing g. This corresponds precisely to free group reduction if $G = F_A$ and $\pi = \theta$. Since we shall not be using free group reduction from now on, we write also $\widehat{u} = \widehat{\pi(u)}$ for every $u \in \widetilde{A}^*$ to simplify notation.

It is easy to see that, for every finitely generated group G and every matched epimorphism $\pi : \widetilde{A}^* \to G$, the rewriting system over A^* defined by

$$\mathcal{R} = \{(u, \widehat{u}) : u \in \widetilde{A}^*,\ u \neq \widehat{u}\}$$

is noetherian, confluent and satisfies $\mathrm{IRR}(\mathcal{R}) = M_A(G)$. Since $\ker(\pi)$ is the congruence generated by $\mathcal{R}$, it follows from (5) that the equivalence

$$\pi(u) = \pi(v) \iff \widehat{u} = \widehat{v}$$

solves the word problem for G. The next theorem shows that, within virtually free groups, we can get away with a finite subsystem:

Theorem 4.2 [53, Theorem 5.2] *Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5. Then there exists some constant $M \geq 1$ such that the finite relation*

$$\mathcal{R}' = \{(u, \widehat{u}) : u \in \widetilde{A}^*,\ |u| \leq M,\ u \neq \widehat{u}\}$$

satisfies:

(i) $\underset{\mathcal{R}'}{\Longrightarrow}$ *is noetherian and confluent;*

(ii) $\mathrm{IRR}(\mathcal{R}') = M_A(G)$.

Note that condition (ii) implies that $\mathcal{R}'$ generates $\ker(\pi)$ as well. The constant M can be taken as K_0N_0+1, where N_0 is the constant from Lemma 4.1 and K_0 is a fellow traveller constant for $\mathrm{Geo}_A(G)$ [53, Theorem 5.2].

We remark that, unlike Theorem 2.5, the existence of such a rewriting system is not exclusive of virtually free groups. It is easy to see that $\mathbb{Z} \times \mathbb{Z}$, which is not even hyperbolic, admits such a system.

Another useful consequence of Lemma 4.1 is the following result:

Lemma 4.3 [53, Lemma 6.1(i)] *Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5. Let N_0 be the constant from Lemma 4.1 and let K_0 be a fellow traveller constant for* $\mathrm{Geo}_A(G)$. *Then*

$$|\widehat{g}| \leq |\widehat{g} \wedge \widehat{gh}| + K_0N_0 + N_0|\widehat{h}|$$

holds for all $g, h \in G$.

We can now present a most simplified model for the boundary of a finitely generated virtually free group which will prove itself useful in the study of infinite fixed points. We shall define the already mentioned distance d' on G by means of the *Gromov product* (taking 1 as basepoint).

Given $g, h \in G$, we define

$$(g|h) = \frac{1}{2}(d_A(1,g) + d_A(1,h) - d_A(g,h)).$$

Fix $\varepsilon > 0$ such that $\varepsilon\delta \leq \frac{1}{5}$, where δ is a hyperbolicity constant for $\Gamma_A(G)$. Write $z = e^{\varepsilon}$ and define

$$\rho(g,h) = \begin{cases} z^{-(g|h)} & \text{if } g \neq h \\ 0 & \text{otherwise} \end{cases}$$

for all $g, h \in G$. In general, ρ is not a distance because it fails the triangular inequality. This problem is overcome by defining

$$d'(g,h) = \inf\{\rho(g_0,g_1)+\ldots+\rho(g_{n-1},g_n) \mid g_0 = g,\ g_n = h;\ g_1,\ldots,g_{n-1} \in G\}.$$

By [62, Proposition 5.16] (see also [23, Proposition 7.10]), d' is a distance on G and the inequalities

$$\frac{1}{2}\rho(g,h) \leq d'(g,h) \leq \rho(g,h) \tag{10}$$

hold for all $g, h \in G$. As we mentioned in Subsection 4.1, the completion $\Omega(G) = G \cup \partial_A(G)$ of the metric space (G, d') defines the boundary $\partial_A(G)$ up to homeomorphism. We slightly abuse notation by denoting also by d' the extension of d' to both $\Omega(G)$ and $\partial_A(G)$.

The language $M_A(G)$ introduced in (9) was noted to be rational. Let $\mathcal{A} = (Q, \widetilde{A}, E, q_0, F)$ be a finite trim deterministic automaton recognizing $M_A(G)$ (e.g. the *minimal automaton* of L, see [9]). Since $M_A(G)$ is factorial, we must have $F = Q$. Let

$$\partial M_A(G) = \{\alpha \in \widetilde{A}^{\omega} \mid \alpha^{[n]} \in M_A(G) \text{ for every } n \in \mathbb{N}\}.$$

Equivalently, since $\mathcal{A}$ is trim and deterministic, and $F = Q$, we have $\partial M_A(G) = L_{\omega}(\mathcal{A})$. Write $\Omega M_A(G) = M_A(G) \cup \partial M_A(G)$. We define a mapping $d : \Omega M_A(G) \times \Omega M_A(G) \to \mathbb{R}_0^+$ by

$$d(\alpha,\beta) = \begin{cases} 2^{-|\alpha\wedge\beta|} & \text{if } \alpha \neq \beta \\ 0 & \text{otherwise} \end{cases}$$

It is immediate that d is a distance in $\Omega M_A(G)$, indeed an ultrametric distance since

$$|\alpha \wedge \gamma| \geq \min\{|\alpha \wedge \beta|, |\beta \wedge \gamma|\}$$

holds for all $\alpha, \beta, \gamma \in \Omega M_A(G)$. We slightly abuse notation by denoting also by d the restriction of d to $M_A(G) \times M_A(G)$.

Proposition 4.4 [53, Proposition 6.2]

(i) The mutually inverse mappings $(G, d') \to (M_A(G), d) : g \mapsto \widehat{g}$ *and* $(M_A(G), d) \to (G, d') : u \mapsto \pi(u)$ *are uniformly continuous;*

(ii) $(\Omega M_A(G), d)$ *is the completion of* $(M_A(G), d)$*;*

(iii) $(\partial M_A(G), d)$ *is homeomorphic to* $(\partial_A(G), d')$.

Thus the construction of $\Omega M_A(G)$ constitutes a model for the hyperbolic completion of G. But we must import also to $\Omega M_A(G)$ the algebraic operations of $\Omega(G)$ since we shall be considering homomorphisms soon. Clearly, the binary operation on $M_A(G)$ is defined as

$$M_A(G) \times M_A(G) \to M_A(G) : (u, v) \mapsto \widehat{uv}$$

so that $(G, d') \to (M_A(G), d) : g \mapsto \widehat{g}$ is also a group isomorphism. But there is another important algebraic operation involved. Indeed, for every $g \in G$, the left translation $\psi_g : G \to G : x \mapsto gx$ is uniformly continuous for d' and so admits a continuous extension $\Psi_g : \Omega(G) \to \Omega(G)$. It follows that the left action of G on its boundary, $G \times \partial_A(G) \to \partial_A(G) : (g, \alpha) \mapsto \Psi_g(\alpha)$, is continuous. We can also replicate this operation in $\Omega M_A(G)$ as follows:

Proposition 4.5 [53, Proposition 6.3] *Let* $u \in M_A(G)$*. Then* $\psi_u : M_A(G) \to M_A(G) : v \mapsto \widehat{uv}$ *is uniformly continuous.*

Therefore ψ_u admits a continuous extension $\Psi_u : \Omega M_A(G) \to \Omega M_A(G)$ and the left action $M_A(G) \times \partial M_A(G) \to \partial M_A(G) : (u, \alpha) \mapsto \Psi_u(\alpha)$ is continuous. Write $\widehat{u\alpha} = \Psi_u(\alpha)$. For every $\alpha \in \partial M_A(G)$, and since $\alpha = \lim_{n \to +\infty} \alpha^{[n]}$, we have by continuity

$$\widehat{u\alpha} = \lim_{n \to +\infty} \widehat{u\alpha^{[n]}}$$

in $\Omega M_A(G) \subseteq \widetilde{A}^\infty$, hence $(\Omega M_A(G), d)$ serves as a model for the hyperbolic completion of G, both topologically and algebraically. From now on, we shall pursue our work within $(\Omega M_A(G), d)$.

4.3 Uniformly continuous endomorphisms

Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5 (and therefore of Theorem 4.2). Given $\varphi \in \mathrm{End}(G)$, we denote by $\widehat{\varphi}$ the corresponding endomorphism of $M_A(G)$ for the binary operation induced by the product in G, i.e. $\widehat{\varphi}(u) = \widehat{\varphi(\pi(u))}$. To simplify notation, we shall often write $\varphi(u)$ instead of $\varphi(\pi(u))$ for $u \in \widetilde{A}^*$.

We say that φ satisfies the *bounded cancellation property* if

$$\{|\widehat{\varphi}(u)| - |\widehat{\varphi}(u) \wedge \widehat{\varphi}(uv)| : uv \in M_A(G)\}$$

is bounded (in particular, the product uv must be reduced). In that case, we denote its maximum by B_φ. This property was considered originally for free group automorphisms by Cooper [17].

We recall that a homomorphism with finite kernel is called *virtually injective*.

Theorem 4.6 [53, Theorem 7.1] *Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5. Let φ be a virtually injective endomorphism of G. Then φ satisfies the bounded cancellation property.*

The proof is rather technical and is therefore omitted. This result can be used to identify the uniformly continuous endomorphisms of G, through another technical proof:

Theorem 4.7 [53, Theorem 7.2] *Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5. The following conditions are equivalent for a nontrivial endomorphism φ of G:*

(i) φ is uniformly continuous for d';

(ii) φ is virtually injective.

We show next that the bounded cancellation property extends to the action of G on its boundary.

Given a uniformly continuous endomorphism φ of (G, d'), it follows from Proposition 4.4(i) that $\widehat{\varphi} : M_A(G) \to M_A(G)$ is uniformly continuous for d. Since $\Omega M_A(G)$ is the completion of $(M_A(G), d)$, then $\widehat{\varphi}$ admits a unique continuous extension $\Phi : \Omega M_A(G) \to \Omega M_A(G)$. By continuity, we have

$$\Phi(\alpha) = \Phi(\lim_{n\to+\infty} \alpha^{[n]}) = \lim_{n\to+\infty} \widehat{\varphi}(\alpha^{[n]}). \tag{11}$$

Corollary 4.8 [53, Corollary 7.3] *Let G be a finitely generated virtually free group and let $\pi : \widetilde{A}^* \to G$ be a matched epimorphism satisfying the conditions of Theorem 2.5. Let φ be a uniformly continuous endomorphism of (G, d') and let $u\alpha \in \partial M_A(G)$. Then $|\widehat{\varphi}(u)| - |\widehat{\varphi}(u) \wedge \Phi(u\alpha)| \leq B_\varphi$.*

Proof. We have $\Phi(u\alpha) = \lim_{n\to+\infty} \widehat{\varphi}(u\alpha^{[n]})$ by (11). In view of Theorem 4.7, we have $\lim_{n\to+\infty} |\widehat{\varphi}(u\alpha^{[n]})| = +\infty$, hence $|\widehat{\varphi}(u) \wedge \Phi(u\alpha)| = |\widehat{\varphi}(u) \wedge \widehat{\varphi}(u\alpha^{[m]})|$ for sufficiently large m. Since $u\alpha^{[m]} \in M_A(G)$, the claim follows from the definition of B_φ. □

4.4 Fixed points in the boundary of virtually free groups

Keeping all the notation introduced in the preceding subsections, we assume that G is a finitely generated virtually free group and $\pi : \widetilde{A}^* \to G$ is a matched epimorphism satisfying the conditions of Theorem 2.5. Fix also a virtually injective endomorphism φ of G. We adapt notation introduced in [41] for free groups.

Given $u \in M_A(G)$, let $\sigma(u) = u \wedge \widehat{\varphi}(u)$ and write

$$u = \sigma(u)\tau(u), \quad \widehat{\varphi}(u) = \sigma(u)\rho(u).$$

Define also

$$\sigma'(u) = \wedge\{\sigma(uv) \mid uv \in M_A(G)\}$$

and write $\sigma(u) = \sigma'(u)\sigma''(u)$. Thus $\sigma(u)$ is the longest common prefix of u and $\widehat{\varphi}(u)$, and $\sigma'(u)$ is the longest prefix of u which is also a prefix of $\widehat{\varphi}(uv)$ whenever $uv \in M_A(G)$.

We present the two following technical lemmas without proof:

Lemma 4.9 [53, Lemma 8.1] *Let $uv \in M_A(G)$. Then:*

(i) $|\sigma''(u)| \leq B_\varphi$;

(ii) $|\sigma(u)| - |\sigma(u) \wedge \widehat{\varphi}(uv)| \leq |\sigma''(u)|$;

(iii) $\widehat{\varphi}(uv) = \sigma'(u)(\widehat{\sigma''(u)\rho(u)\widehat{\varphi}(v)})$;

(iv) $\sigma'(uv) = \sigma'(u)(\bigwedge_{uvz\in M_A(G)} (\widehat{\sigma''(u)\rho(u)\widehat{\varphi}(vz)} \wedge \sigma''(u)\tau(u)vz))$.

Recall now the finite trim deterministic automaton $\mathcal{A} = (Q, \widetilde{A}, E, q_0, Q)$ recognizing $M_A(G)$. We want to build an automaton to study both finite and infinite fixed points, and the vertices are going to be quadruples defined with the help of the mappings above.

For every $u \in M_A(G)$, we define

$$\xi(u) = (\sigma''(u), \tau(u), \rho(u), q_0 u).$$

Note that there exists precisely one path of the form $q_0 \xrightarrow{u} q_0 u$ in $\mathcal{A}$.

Lemma 4.10 [53, Lemma 8.2] *Let $u, v \in M_A(G)$ be such that $\xi(u) = \xi(v)$ and let $a \in \widetilde{A}$, $\alpha \in \widetilde{A}^\infty$. Then:*

(i) $ua \in M_A(G)$ if and only if $va \in M_A(G)$;

(ii) if $ua \in M_A(G)$, then $\xi(ua) = \xi(va)$

(iii) $\widehat{u\overline{v}} \in \mathrm{Fix}(\widehat{\varphi})$;

(iv) $u\alpha \in \Omega M_A(G)$ if and only if $v\alpha \in \Omega M_A(G)$;

(v) $u\alpha \in \mathrm{Fix}(\Phi)$ if and only if $v\alpha \in \mathrm{Fix}(\Phi)$;

(vi) if $\alpha \in \partial M_A(G)$, then $\alpha = \lim_{n\to+\infty} \widehat{\alpha^{[n]}u}$.

Given $X \subseteq A^\infty$, write

$$\mathrm{Pref}(X) = \{u \in A^* \mid u\alpha \in X \text{ for some } \alpha \in A^\infty\}.$$

We build a (possibly infinite) automaton $\mathcal{A}'_\varphi = (Q', \widetilde{A}, E', q'_0, F')$ by taking

- $Q' = \{\xi(u) \mid u \in \mathrm{Pref}(\mathrm{Fix}(\Phi))\}$;
- $q'_0 = \xi(1)$;
- $F' = \{\xi(u) \in Q' \mid \tau(u) = \rho(u) = 1\}$;
- $E' = \{(\xi(u), a, \xi(v)) \in Q' \times \widetilde{A} \times Q' \mid v = ua \in \mathrm{Pref}(\mathrm{Fix}(\Phi))\}$.

Note that $\mathcal{A}'_\varphi$ is deterministic by Lemma 4.10(ii) and is also *accessible*: if $u \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$, then there exists a path $q'_0 \xrightarrow{u} \xi(u)$ and so every vertex can be reached from the initial vertex.

Let S denote the set of all vertices $q \in Q'$ such that there exist at least two edges in $\mathcal{A}'_\varphi$ leaving q. The following lemma is an essential ingredient for our finiteness results:

Lemma 4.11 [53, Lemma 8.3] *S is finite.*

Proof. In view of Lemma 4.9(i), the unique components of $\xi(u)$ that may assume infinitely many values are $\tau(u)$ and $\rho(u)$. Moreover, we claim that

$$\tau(u) \neq 1 \Rightarrow |\rho(u)| \leq B_\varphi \tag{12}$$

holds for every $u \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$. Indeed, suppose that $\tau(u) \neq 1$ and $|\rho(u)| > B_\varphi$. Write $\alpha = u\beta$ for some $\alpha \in \mathrm{Fix}(\Phi)$. In view of Corollary 4.8, $|\rho(u)| > B_\varphi$ yields $|\Phi(u\beta) \wedge \widehat{\varphi}(u)| > |\sigma(u)|$ and now $\tau(u) \neq 1$ yields $(\Phi(u\beta) \wedge u\beta) = (\widehat{\varphi}(u) \wedge u) = \sigma(u)$. Since $\beta \neq 1$, this contradicts $\alpha \in \mathrm{Fix}(\Phi)$. Therefore (12) holds.

It is also easy to check that

$$|\rho(u)| > B_\varphi \Rightarrow \xi(u) \notin S \tag{13}$$

holds for every $u \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$. Indeed, if $|\rho(u)| > B_\varphi$ and a is the first letter of $\rho(u)$, then, by definition of B_φ, $\sigma(u)a$ is a prefix of $\Phi(u\alpha)$ whenever $u\alpha \in \mathrm{Fix}(\Phi)$. Therefore any edge leaving $\xi(u)$ in $\mathcal{A}'_\varphi$ must have label a and so (13) holds. Therefore we only need to bound $|\tau(u)|$ for $\xi(u) \in S$.

Write $D_\varphi = \max\{|\widehat{\varphi(a)}| : a \in \widetilde{A}\}$. Consider N_0 and K_0 as before. Since φ is virtually injective, we can define

$$W_0 = \max\{|u| : u \in M_A(G),\ |\widehat{\varphi}(u)| \leq 2(B_\varphi + D_\varphi - 1)\}.$$

Let $Z_0 = B_\varphi + N_0(K_0 + W_0)D_\varphi$. To complete the proof of the lemma, it suffices to prove that

$$|\tau(u)| > Z_0 \Rightarrow \xi(u) \notin S \tag{14}$$

for every $u \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$.

Suppose that $|\tau(u)| > Z_0$ and $(\xi(u), a, \xi(ua)), (\xi(u), b, \xi(ub)) \in E'$ for some $u \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$, where $a, b \in \widetilde{A}$ are distinct. We have $\xi(ua) = \xi(v)$ for some $v \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$. By Lemma 4.10(v), we get $ua\alpha \in \mathrm{Fix}(\Phi)$ for some $\alpha \in \Omega M_A(G)$. By (11), we get $ua\alpha = \lim_{n\to+\infty} \widehat{\varphi}(ua\alpha^{[n]})$ and so $|\widehat{\varphi}(ua\alpha^{[n]})| \geq |u|$ for sufficiently large n. Let

$$p = \min\{n \in \mathbb{N} : |\widehat{\varphi}(ua\alpha^{[n]})| \geq |u|\}.$$

Note that $p > 0$ since $|\tau(u)| > Z_0$ and by (12). Since $|\widehat{\varphi}(ua\alpha^{[p-1]})| < |u|$ by minimality of p, we get

$$|\widehat{\varphi}(ua\alpha^{[p]})| \leq |\widehat{\varphi}(ua\alpha^{[p-1]})| + D_\varphi < |u| + D_\varphi. \tag{15}$$

On the other hand, we claim that

$$|u| - |\widehat{\varphi}(ua\alpha^{[p]}) \wedge u| \leq B_\varphi. \tag{16}$$

Indeed, suppose that $|u| - |\widehat{\varphi}(ua\alpha^{[p]}) \wedge u| > B_\varphi$. Let $c = \widehat{\varphi}(ua\alpha^{[p]}) \wedge u$. Then $c = \sigma(ua\alpha^{[p]})$ and we get

$$|\rho(ua\alpha^{[p]})| = |\widehat{\varphi}(ua\alpha^{[p]})| - |c| > |\widehat{\varphi}(ua\alpha^{[p]})| - |u| + B_\varphi \geq B_\varphi.$$

Since $|\tau(ua\alpha^{[p]})| \geq |u| - |\widehat{\varphi}(ua\alpha^{[p]}) \wedge u| > B_\varphi$ and $ua\alpha^{[p]} \in \mathrm{Pref}(\mathrm{Fix}(\Phi))$, we contradict (12). Thus (16) holds.

Similarly, $ub\beta \in \mathrm{Fix}(\Phi)$ for some $\beta \in \Omega M_A(G)$. Defining

$$q = \min\{n \in \mathbb{N} : |\widehat{\varphi}(ub\beta^{[n]})| \geq |u|\},$$

we get

$$|\widehat{\varphi}(ub\beta^{[q]})| < |u| + D_\varphi \tag{17}$$

and

$$|u| - |\widehat{\varphi}(ub\beta^{[q]}) \wedge u| \leq B_\varphi. \tag{18}$$

Since $|u| > Z_0 > B_\varphi$, we may write $u = u_1u_2$ with $|u_2| = B_\varphi$. Then by (15) and (16) we may write $\widehat{\varphi}(ua\alpha^{[p]}) = u_1x$ for some x such that $|x| < B_\varphi + D_\varphi$. Similarly, (17) and (18) yield $\widehat{\varphi}(ub\beta^{[q]}) = u_1y$ for some y such that $|y| < B_\varphi + D_\varphi$. Writing $w = \widehat{\overline{\beta^{[q]}}\,\overline{b}a\alpha^{[p]}}$, it follows that $\varphi(w) = \pi(\overline{y}x)$ and so $|\widehat{\varphi}(w)| \leq 2(B_\varphi + D_\varphi - 1)$. Hence $|w| \leq W_0$. Applying Lemma 4.3 to $g = \pi(ub\beta^{[q]})$ and $h = \pi(w)$, we get

$$|ub\beta^{[q]}| \leq |ub\beta^{[q]} \wedge ua\alpha^{[p]}| + N_0(K_0 + |w|) \leq |u| + N_0(K_0 + W_0)$$

and so $q < N_0(K_0 + W_0)$. Hence, in view of (12), we get

$$\begin{array}{rcl}
|\tau(u)| & = & |u| - |\sigma(u)| \leq |\widehat{\varphi}(ub\beta^{[q]})| - |\sigma(u)| \\
& \leq & |\widehat{\varphi}(u)| + |\widehat{\varphi}(b\beta^{[q]})| - |\sigma(u)| \leq |\rho(u)| + N_0(K_0 + W_0)D_\varphi \\
& \leq & B_\varphi + N_0(K_0 + W_0)D_\varphi,
\end{array}$$

contradicting $|\tau(u)| > Z_0$. Thus (14) holds and the lemma is proved. □

We use S to build a subautomaton of $\mathcal{A}'_\varphi$ as follows. Let Q'' denote the set of all vertices $q \in Q'$ such that there exists some path $q \longrightarrow p \in S \cup F'$. We define $\mathcal{A}''_\varphi = (Q'', \widetilde{A}, E'', q''_0, F'')$ by taking $q''_0 = q'_0$, $F'' = F'$ and $E'' = E' \cap (Q'' \times \widetilde{A} \times Q'')$.

We can now start to present the main results of this subsection:

Theorem 4.12 [53, Theorem 8.4] *Let φ be a virtually injective endomorphism of a finitely generated virtually free group G. Then:*

(i) the automaton $\mathcal{A}''_\varphi$ is finite;

(ii) $L(\mathcal{A}''_\varphi) = \mathrm{Fix}(\widehat{\varphi})$;

(iii) $L_\omega(\mathcal{A}''_\varphi) = \mathrm{Sing}(\Phi)$.

Proof. (i) The set F' is finite and S is finite by Lemma 4.11. On the other hand, by definition of S, there are only finitely many paths in $\mathcal{A}'_\varphi$ of the form $\nu_j : p' \longrightarrow q'$ with $p', q' \in S \cup F' \cup \{q'_0\}$ and no intermediate vertex in $S \cup F' \cup \{q'_0\}$. Now recall that $\mathcal{A}'_\varphi$ is accessible, hence every path of the form $q \xrightarrow{u} p \in S \cup F'$ can be extended to some path $q'_0 \xrightarrow{v} q \xrightarrow{u} p \in S \cup F'$ which is itself a concatenation of the finitely many paths ν_j. Therefore Q'' is finite and so is $\mathcal{A}''_\varphi$.

(ii) Every $u \in M_A(G)$ labels at most a unique path $q'_0 = \xi(1) \xrightarrow{u} \xi(u)$ out of the initial vertex in $\mathcal{A}'_\varphi$. On the other hand, if $q'_0 = \xi(1) \xrightarrow{u} q'$ is a path in $\mathcal{A}'_\varphi$, then the fourth component of ξ yields a path $q_0 \xrightarrow{u} q$ in $\mathcal{A}$ and so $u \in M_A(G)$. Hence

$$\begin{aligned} L(\mathcal{A}'_\varphi) &= \{u \in M_A(G) \mid \xi(u) \in F'\} = \{u \in M_A(G) \mid \tau(u) = \rho(u) = 1\} \\ &= \mathrm{Fix}(\widehat{\varphi}). \end{aligned}$$

Since $L(\mathcal{A}''_\varphi) = L(\mathcal{A}'_\varphi)$, (ii) holds.

(iii) Let $\alpha \in L_\omega(\mathcal{A}''_\varphi)$. Then there exists some $q'' \in Q''$ and some infinite sequence $(i_n)_n$ such that $q''_0 \xrightarrow{\alpha^{[i_n]}} q''$ is a path in $\mathcal{A}''_\varphi$ for every n. Write $u = \alpha^{[i_1]}$ and let $v_n = \widehat{\alpha^{[i_n]}\overline{u}}$. By Lemma 4.10(iii), we have $v_n \in \mathrm{Fix}(\widehat{\varphi})$ for every n. It follows from Lemma 4.10(vi) that $\alpha = \lim_{n\to+\infty} v_n$, thus $\alpha \in \mathrm{Sing}(\Phi)$.

Conversely, let $\alpha \in \mathrm{Sing}(\Phi)$. Then we may write $\alpha = \lim_{n\to+\infty} v_n$ for some sequence $(v_n)_n$ in $\mathrm{Fix}(\widehat{\varphi})$. Let $k \in \mathbb{N}$. For large enough n, we have $\alpha^{[k]} = v_n^{[k]}$ and so there is some path

$$q''_0 \xrightarrow{\alpha^{[k]}} q''_k \xrightarrow{w} t''_k \in F'',$$

where $\alpha^{[k]}w = v_n$. Thus $\alpha \in L_\omega(\mathcal{A}''_\varphi)$ as required. □

Recall now the continuous extensions $\Psi_u : \Omega M_A(G) \to \Omega M_A(G)$ of the uniformly continuous mappings $\psi_u : M_A(G) \to M_A(G) : v \mapsto \widehat{uv}$ defined for each $u \in M_A(G)$ (see Proposition 4.5). As remarked before, this is equivalent to say that the left action $M_A(G) \times \partial M_A(G) \to \partial M_A(G) : (u, \alpha) \mapsto \widehat{u\alpha}$

is continuous. Identifying $M_A(G)$ with G and $\partial M_A(G)$ with $\partial_A(G)$, we have a continuous action (on the left) of G on $\partial_A(G)$. Clearly, this action restricts to a left action of $\mathrm{Fix}(\varphi)$ on $\mathrm{Fix}(\Phi) \cap \partial_A(G)$: if $g \in \mathrm{Fix}(\varphi)$ and $\alpha \in \mathrm{Fix}(\Phi) \cap \partial_A(G)$, with $\alpha = \lim_{n\to+\infty} g_n$ $(g_n \in G)$, then

$$\begin{aligned}\Phi(g\alpha) &= \Phi(g \lim_{n\to+\infty} g_n) = \Phi(\lim_{n\to+\infty} gg_n) = \lim_{n\to+\infty} \varphi(gg_n)\\ &= \lim_{n\to+\infty} \varphi(g)\varphi(g_n) = \varphi(g) \lim_{n\to+\infty} \varphi(g_n)\\ &= g(\Phi(\lim_{n\to+\infty} g_n)) = g(\Phi(\alpha)) = g\alpha.\end{aligned}$$

Moreover, the $(\mathrm{Fix}(\varphi))$-orbits of $\mathrm{Sing}(\Phi)$ and $\mathrm{Reg}(\Phi)$ are disjoint: if $\alpha \in \mathrm{Sing}(\Phi)$, we can write $\alpha = \lim_{n\to+\infty} g_n$ with the $g_n \in \mathrm{Fix}(\varphi)$ and get $g\alpha = \lim_{n\to+\infty} gg_n$ with $gg_n \in \mathrm{Fix}(\varphi)$ for every n; hence $\alpha \in \mathrm{Sing}(\Phi) \Rightarrow g\alpha \in \mathrm{Sing}(\Phi)$ and the action of g^{-1} yields the converse implication.

We can now prove the following theorem:

Theorem 4.13 [53, Theorem 8.5] *Let φ be a virtually injective endomorphism of a finitely generated virtually free group G. Then $\mathrm{Reg}(\Phi)$ has finitely many $(\mathrm{Fix}(\varphi))$-orbits.*

Proof. Let P be the set of all infinite paths $s'_0 \xrightarrow{a_1} s'_1 \xrightarrow{a_2} \ldots$ in $\mathcal{A}'_\varphi$ such that:

- $s'_0 \in S \cup \{q_0\}$;
- $s'_n \notin S \cup \{q_0\}$ for every $n > 0$;
- $s'_n \neq s'_m$ whenever $n \neq m$.

By Lemma 4.11, there are only finitely many choices for s'_0. Since A is finite and $\mathcal{A}'_\varphi$ is deterministic, there are only finitely many choices for s'_1, and from that vertex onwards, the path is univocally determined due to $s'_n \notin S$ $(n \geq 1)$. Hence P is finite, and we may assume that it consists of paths $p'_i \xrightarrow{\alpha_i} \ldots$ for $i = 1, \ldots, m$. Fix a path $q'_0 \xrightarrow{u_i} p'_i$ for each i and let $X = \{u_1\alpha_1, \ldots, u_m\alpha_m\} \subseteq \partial M_A(G)$. We claim that

$$X \subseteq \mathrm{Reg}(\Phi). \tag{19}$$

Let $i \in \{1, \ldots, m\}$ and write $\beta = u_i\alpha_i$. To show that $\beta \in \mathrm{Fix}(\Phi)$, it suffices to show that $\lim_{n\to+\infty} \widehat{\varphi}(\beta^{[n]}) = \beta$. Let $k \in \mathbb{N}$. We must show that there exists some $r \in \mathbb{N}$ such that

$$n \geq r \Rightarrow |\widehat{\varphi}(\beta^{[n]}) \wedge \beta| > k. \tag{20}$$

Since φ is virtually injective, there exists some $r > k$ such that

$$n \geq r \Rightarrow |\widehat{\varphi}(\beta^{[n]})| > k + B_\varphi.$$

Suppose that $|\widehat{\varphi}(\beta^{[n]}) \wedge \beta| \leq k$ for some $n \geq r$. Then $|\sigma(\beta^{[n]})| \leq k$. Since $k < r \leq n$, it follows that $\tau(\beta^{[n]}) \neq 1$. On the other hand, since $|\widehat{\varphi}(\beta^{[n]})| > k + B_\varphi$, we get $|\rho(\beta^{[n]})| > B_\varphi$. In view of (12), this contradicts $\xi(\beta^{[n]}) \in Q'$. Therefore (20) holds for our choice of r and so $X \subseteq \mathrm{Fix}(\Phi)$. Since the path $q_0' \xrightarrow{\beta} \ldots$ can visit only finitely often a given vertex, then $\beta \notin L_\omega(\mathcal{A}''_\varphi)$ and so (19) follows from Theorem 4.12(iii).

By the previous comments on $(\mathrm{Fix}(\varphi))$-orbits, the $(\mathrm{Fix}(\varphi))$-orbits of the elements of X must be contained in $\mathrm{Reg}(\Phi)$. We complete the proof of the theorem by proving the opposite inclusion.

Let $\beta \in \mathrm{Reg}(\Phi)$. By Theorem 4.12(iii), we have $\beta \notin L_\omega(\mathcal{A}''_\varphi)$ and so there exists a factorization $\beta = u\alpha$ and a path

$$q_0' \xrightarrow{u} p' \xrightarrow{\alpha} \ldots$$

in $\mathcal{A}'_\varphi$ such that p' signals the last occurrence of a vertex from $S \cup \{q_0'\}$. We claim that no vertex is repeated after p'. Otherwise, since no vertex of S appears after p', we would get a factorization of $p' \xrightarrow{\alpha} \ldots$ as

$$p' \xrightarrow{v} q' \xrightarrow{w} q' \xrightarrow{w} q' \xrightarrow{w} \ldots$$

and by Lemma 4.10(iii) and (vi) we would get $\pi(uvw^n\overline{v}\,\overline{u}) \in \mathrm{Fix}(\varphi)$ and

$$\beta = \lim_{n\to+\infty} \widehat{uvw^n\overline{v}\,\overline{u}},$$

contradicting $\beta \in \mathrm{Reg}(\Phi)$. Thus no vertex is repeated after p' and so we must have $p' = p_i'$ and $\alpha = \alpha_i$ for some $i \in \{1, \ldots, m\}$. It follows that $\beta = u\alpha_i$. By Lemma 4.10(iii), we get $\widehat{u\overline{u_i}} \in \mathrm{Fix}(\widehat{\varphi})$ and we are done. $\square$

Theorem 4.13 is somehow a version for infinite fixed points of Theorem 3.6, which we proved before for finite fixed points. Note however that $\mathrm{Sing}(\Phi)$ has *not* in general finitely many $(\mathrm{Fix}(\varphi))$-orbits since $\mathrm{Sing}(\Phi)$ may be uncountable (take for instance the identity automorphism on a free group of rank 2).

Since every finite set is closed in a metric space, we obtain the following corollary from Theorem 4.13:

Corollary 4.14 [53, Corollary 8.6] *Let φ be a virtually injective endomorphism of a finitely generated virtually free group G with* $\mathrm{Fix}(\varphi)$ *finite. Then* $\mathrm{Fix}(\Phi)$ *is finite.*

4.5 Classification of the infinite fixed points

We investigate now the nature of the infinite fixed points of Φ when φ is an automorphism. Since both φ and φ^{-1} are then uniformly continuous by Proposition 4.7, they extend to continuous mappings Φ and Ψ which turn out to be mutually inverse in view of the uniqueness of continuous extensions to the completion. Therefore Φ is a bijection. We say that $\alpha \in \mathrm{Reg}(\Phi)$ is:

- an *attractor* if
$$\exists \varepsilon > 0 \; \forall \beta \in \Omega M_A(G) \; (d(\alpha, \beta) < \varepsilon \Rightarrow \lim_{n \to +\infty} \Phi^n(\beta) = \alpha).$$
- a *repeller* if
$$\exists \varepsilon > 0 \; \forall \beta \in \Omega M_A(G) \; (d(\alpha, \beta) < \varepsilon \Rightarrow \lim_{n \to +\infty} \Phi^{-n}(\beta) = \alpha).$$

The latter amounts to say that α is an attractor for Φ^{-1}. There exist other types but they do not occur in our context as we shall see.

We say that an attractor $\alpha \in \mathrm{Reg}(\Phi)$ is *exponentially stable* if

$$\begin{aligned} &\exists \varepsilon, k, \ell > 0 \; \forall \beta \in \Omega M_A(G) \; \forall n \in \mathbb{N} \\ &\qquad (d(\alpha, \beta) < \varepsilon \Rightarrow d(\alpha, \Phi^n(\beta)) \leq k 2^{-\ell n} d(\alpha, \beta)). \end{aligned}$$

This is equivalent to say that

$$\begin{aligned} &\exists M, N, \ell > 0 \; \forall \beta \in \Omega M_A(G) \; \forall n \in \mathbb{N} \\ &\qquad (|\alpha \wedge \beta| > M \Rightarrow |\alpha \wedge \Phi^n(\beta)| + N \geq \ell n + |\alpha \wedge \beta|). \end{aligned} \tag{21}$$

A repeller $\alpha \in \mathrm{Reg}(\Phi)$ is exponentially stable if it is an exponentially stable attractor for Φ^{-1}.

Theorem 4.15 [53, Theorem 9.1] *Let φ be an automorphism of a finitely generated virtually free group G. Then $\mathrm{Reg}(\Phi)$ contains only exponentially stable attractors and exponentially stable repellers.*

Proof. Let $\alpha \in \mathrm{Reg}(\Phi)$ and write $\alpha = a_1 a_2 \ldots$ with $a_i \in \widetilde{A}$. Then there exists a path

$$\xi(1) \xrightarrow{a_1} \xi(\alpha^{[1]}) \xrightarrow{a_2} \xi(\alpha^{[2]}) \xrightarrow{a_3} \ldots$$

in $\mathcal{A}'_\varphi$. Let $Y_0 = B_\varphi(D_{\varphi^{-1}} + 1) + B_{\varphi^{-1}}(D_\varphi + 1)$ and let

$$V = \{\xi(u) \in Q' : |\tau(u)| > Y_0 \text{ or } |\rho(u)| > Y_0\}.$$

It is easy to see that $Q' \setminus V$ is finite. We saw in the proof of Theorem 4.13 that there are only finitely many repetitions of vertices in a path in $\mathcal{A}'_\varphi$ labelled by a regular fixed point, hence there exists some $n_0 \in \mathbb{N}$ such that

$$\xi(\alpha^{[n]}) \in V \text{ for every } n \geq n_0. \tag{22}$$

Now we consider two cases:

Case I: $\tau(\alpha^{[n_0]}) = 1$.

We claim that

$$\tau(\alpha^{[n]}) = 1 \text{ for every } n \geq n_0. \tag{23}$$

We use induction on n. The case $n = n_0$ holds in Case I, so assume that $\tau(\alpha^{[n]}) = 1$ for some $n \geq n_0$. Then $\xi(\alpha^{[n]}) \in V$ yields $|\rho(\alpha^{[n]})| > Y_0 > 2B_\varphi$. Since $|\widehat{\varphi}(\alpha^{[n+1]})| \geq |\widehat{\varphi}(\alpha^{[n]})| - B_\varphi$ by definition of B_φ, then

$$\begin{array}{rcl} |\rho(\alpha^{[n+1]})| & \geq & |\widehat{\varphi}(\alpha^{[n+1]})| - |\alpha^{[n+1]}| \geq |\widehat{\varphi}(\alpha^{[n]})| - B_\varphi - |\alpha^{[n]}| - 1 \\ & = & |\rho(\alpha^{[n]})| - B_\varphi - 1 > Y_0 - B_\varphi - 1 > B_\varphi. \end{array}$$

By (12), we get $\tau(\alpha^{[n+1]}) = 1$ and so (23) holds.

Next we show that

$$(\Phi(\alpha^{[n]}\gamma))^{[n+1]} = \alpha^{[n+1]} \tag{24}$$

if $n \geq n_0$ and $\alpha^{[n]}\gamma \in \Omega M_A(G)$. Indeed, by (23) we have $\widehat{\varphi}(\alpha^{[n]}) = \alpha^{[n]}(\rho(\alpha^{[n]}))$ and $|\rho(\alpha^{[n]})| > Y_0 > B_\varphi$. By the definition of B_φ and Corollary 4.8, we get $(\Phi(\alpha^{[n]}\gamma))^{[n+1]} = \alpha^{[n]}(\rho(\alpha^{[n]}))^{[1]}$. Considering the particular case $\gamma = a_{n+1}$, we also get

$$(\widehat{\varphi}(\alpha^{[n+1]}))^{[n+1]} = \alpha^{[n]}(\rho(\alpha^{[n]}))^{[1]} = (\Phi(\alpha^{[n]}\gamma))^{[n+1]}.$$

Since $\tau(\alpha^{[n+1]}) = 1$ by (23), we have $(\widehat{\varphi}(\alpha^{[n+1]}))^{[n+1]} = \alpha^{[n+1]}$ and so (24) holds.

Hence we may write $\Phi(\alpha^{[n]}\gamma) = \alpha^{[n+1]}\gamma'$ whenever $\alpha^{[n]}\gamma \in \Omega M_A(G)$. Iterating, it follows that, for all $k \geq n_0$ and $n \in \mathbb{N}$, $\alpha^{[k]}\gamma \in \Omega M_A(G)$ implies $\Phi^n(\alpha^{[k]}\gamma) = \alpha^{[k+n]}\gamma'$ for some γ'. We claim that this implies that

$$|\alpha \wedge \beta| \geq n_0 \Rightarrow |\alpha \wedge \Phi^n(\beta)| \geq n + |\alpha \wedge \beta| \tag{25}$$

holds for all $\beta \in \Omega M_A(G)$ and $n \in \mathbb{N}$. Indeed, write $(\alpha \wedge \beta) = \alpha^{[k]}$ and $\beta = \alpha^{[k]}\gamma$. Then $k \geq n_0$ implies $\Phi^n(\beta) = \Phi^n(\alpha^{[k]}\gamma) = \alpha^{[k+n]}\gamma'$ for some γ' and so $|\alpha \wedge \Phi^n(\beta)| \geq n + k$. Thus (25) holds and so does (21). Therefore α is an exponentially stable attractor in this case.

Now, if $|\alpha^{[t]}\tau| = 1$ for some $t > n_0$, we can always replace n_0 by t and deduce by Case I that α is an exponentially stable attractor. Thus we may assume that:

Case II: $\tau(\alpha^{[n]}) \neq 1$ for every $n \geq n_0$.

By replacing n_0 by a larger integer if necessary, we may assume that (22) is also satisfied when we consider the equivalents of ξ and V for φ^{-1}.

Since φ is injective, there exists some $n_1 \geq n_0$ such that $|\widehat{\varphi}(\alpha^{[n_1]})| \geq n_0 + B_\varphi$. Since $\tau(\alpha^{[n_1]}) \neq 1$, it follows from (12) that $|\rho(\alpha^{[n_1]})| \leq B_\varphi$, hence $\sigma(\alpha^{[n_1]}) = \alpha^{[n_2]}$ for some $n_2 \geq n_0$. Write $x = \rho(\alpha^{[n_1]})$ and $y = \widehat{\varphi^{-1}}(\alpha^{[n_2]})\widehat{\varphi^{-1}}(x)$. Then $\widehat{\varphi}(\alpha^{[n_1]}) = \alpha^{[n_2]}x$ yields $\alpha^{[n_1]} = \widehat{y}$ and so

$$n_1 = |\alpha^{[n_1]}| \leq |\widehat{\varphi^{-1}}(\alpha^{[n_2]})| + |\widehat{\varphi^{-1}}(x)| \leq |\widehat{\varphi^{-1}}(\alpha^{[n_2]})| + B_\varphi D_{\varphi^{-1}}.$$

On the other hand, $|\rho(\alpha^{[n_1]})| \leq B_\varphi < Y_0$ and $\alpha^{[n_1]} \in V$ together yield $Y_0 < |\tau(\alpha^{[n_1]})| = n_1 - n_2$ and so

$$n_2 + B_{\varphi^{-1}} < n_1 - Y_0 + B_{\varphi^{-1}} < n_1 - B_\varphi D_{\varphi^{-1}} \leq |\widehat{\varphi^{-1}}(\alpha^{[n_2]})|.$$

In view of (12), we can apply Case I to φ^{-1}, hence α is an exponentially stable attractor for φ^{-1} and therefore an exponentially stable repeller for φ. $\square$

Example 4.16 *Let $G = \mathbb{Z} \times \mathbb{Z}_2$ and let $\varphi \in \mathrm{End}(G)$ be defined by $\varphi(m, n) = (2m, n)$. We compute and classify the infinite fixed points of φ.*

We take $A = \{a, b, c\}$ as generating set. Note that this is not the canonical set of generators, which would not work. Then the matched homomorphism $\pi : \widetilde{A}^* \to G$ defined by

$$\pi(a) = (1, 0), \quad \pi(b) = (0, 1), \quad \pi(c) = (1, 1)$$

yields

$$\mathrm{Geo}_A(G) = \{a, c\}^* \cup \{\overline{a}, \overline{c}\}^* \cup \{b, \overline{b}\}$$

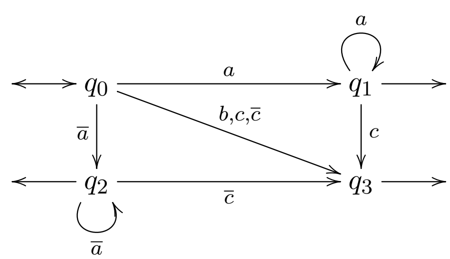
and we can take

$$\begin{aligned} \mathcal{R} &= \{(x\overline{x}, 1) \mid x \in \widetilde{A}\} \cup \{(ab, c), (a\overline{b}, c), (\overline{a}b, \overline{c}), (\overline{a}\,\overline{b}, \overline{c})\} \\ &\cup \{(ba, c), (\overline{b}a, c), (b\overline{a}, \overline{c}), (\overline{b}\,\overline{a}, \overline{c}), (cb, a), (c\overline{b}, a)\} \\ &\cup \{(\overline{c}b, \overline{a}), (\overline{c}\,\overline{b}, \overline{a}), (bc, a), (\overline{b}c, a), (b\overline{c}, \overline{a}), (\overline{b}\,\overline{c}, \overline{a})\} \\ &\cup \{(a\overline{c}, b), (\overline{c}a, b), (\overline{a}c, b), (c\overline{a}, b), (b^2, 1), (\overline{b}^2, 1)\} \end{aligned}$$

to get $\mathrm{Geo}_A(G) = \mathrm{IRR}(\mathcal{R})$. Ordering $\widetilde{A}$ by $a < c < \overline{a} < \overline{c} < b < \overline{b}$, we get

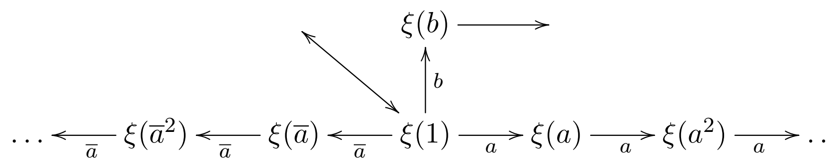
$$M_A(G) = a^*\{1, c\} \cup \overline{a}^*\{1, \overline{c}\} \cup \{b\},$$

recognized by the automaton $\mathcal{A}$ depicted by



Hence $\partial M_A(G) = L_\omega(\mathcal{A}) = \{a^\omega, \overline{a}^\omega\}$.

Now φ is injective and therefore uniformly continuous, admiting a continuous extension Φ to $\Omega M_A(G)$. Since $B_\varphi = 0$, it is easy to check that $\mathcal{A}'_\varphi$ is the automaton



and

$$\xi(1) = (1, 1, 1, q_0), \qquad \xi(b) = (1, 1, 1, q_3),$$
$$\xi(a^n) = (1, 1, a^n, q_1), \qquad \xi(\overline{a}^n) = (1, 1, \overline{a}^n, q_2)$$

for $n \geq 1$. Note that in general we ignore how to compute $\mathcal{A}'_\varphi$, our proofs being far from constructive!

It is immediate that $\mathrm{Fix}(\Phi) = \{1, b, a^\omega, \overline{a}^\omega\}$. Moreover, the regular infinite fixed points a^ω and $\overline{a}^\omega$ are both exponentially stable attractors.

5 Open problems

We end this chapter with some open problems which follow naturally from the results in Sections 3 and 4:

Problem 5.1 *Is it possible to generalize Theorems 3.6, 4.13 and 4.15 to arbitrary finitely generated hyperbolic groups?*

Paulin proved that Theorem 3.6 holds for automorphisms of hyperbolic groups [47].

Problem 5.2 *Is* Fix(φ) *effectively computable when* φ *is an endomorphism of a finitely generated virtually free group?*

For the moment, only the case of free group automorphisms is known (see [11]).

Another natural question to ask in this context is whether similar results hold for *equalizers*. Given homomorphisms $\varphi, \psi : G \to G'$, let

$$\mathrm{Eq}(\varphi, \psi) = \{x \in G \mid \varphi(x) = \psi(x)\}.$$

Problem 5.3 *Given homomorphisms* $\varphi, \psi : G \to G'$ *of finitely generated virtually free groups with* φ *injective, is* Eq(φ, ψ) *finitely generated?*

This question has been solved by Goldstein and Turner for free groups [27]. The restriction to the case where at least one of the homomorphisms is injective is required even in the free group case (see [22] and [63, Section 3] for counterexamples).

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