On uniformly continuous functions for some profinite topologies*

Dedicated to Antonio Restivo for his 70th birthday

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Abstract

Given a variety of finite monoids $\mathbf{V}$, a subset of a monoid is a $\mathbf{V}$-subset if its syntactic monoid belongs to $\mathbf{V}$. A function between two monoids is $\mathbf{V}$-preserving if it preserves $\mathbf{V}$-subsets under preimages and it is hereditary $\mathbf{V}$-preserving if it is $\mathbf{W}$-preserving for every subvariety $\mathbf{W}$ of $\mathbf{V}$. The aim of this paper is to study hereditary $\mathbf{V}$-preserving functions when $\mathbf{V}$ is one of the following varieties of finite monoids: groups, p-groups, aperiodic monoids, commutative monoids and all monoids.

1 Introduction

This article is a follow-up of [12], where the authors started the study of $\mathbf{V}$-preserving functions. Let us first remind the definition. Let M be a monoid and let $\mathbf{V}$ be a variety of finite monoids. A recognizable subset S of M is said to be a $\mathbf{V}$-*subset* if its syntactic monoid belongs to $\mathbf{V}$. A function $f : M \to N$ is called $\mathbf{V}$-*preserving* if, for each $\mathbf{V}$-subset of N, $f^{-1}(L)$ is a $\mathbf{V}$-subset of M. A function is *hereditary* $\mathbf{V}$-*preserving* if it is $\mathbf{W}$-preserving for every subvariety $\mathbf{W}$ of $\mathbf{V}$.

Let us first consider the case where f is a function from A^* to B^*, where A and B are finite alphabets. If $\mathbf{V}$ is the variety $\mathbf{M}$ of all finite monoids, a $\mathbf{V}$-preserving function is also called *regularity-preserving*, according to the terminology used in [5, 16, 18]. The characterization of regularity-preserving functions is a long-term objective, but in spite of intensive research (see [10] for a detailed bibliography), it it is still out of reach. For the variety $\mathbf{G}_p$ of finite p-groups, the situation is more advanced. Indeed, the authors gave in [13] a characterization of $\mathbf{G}_p$-preserving functions when B is a one-letter alphabet

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and a preliminary step towards a general solution can be found in [10]. For the variety **G** of finite groups and for the variety **A** of finite aperiodic monoids, the only known contribution to the study of **V**-preserving functions seems to be the article of Reutenauer and Schützenberger on rational functions [14].

This paper focuses on hereditary **V**-preserving functions when **V** is one of the varieties **M**, **G**, $\mathbf{G}_p$ and **A**. We consider functions from a free monoid or a free commutative monoid to $\mathbb{N}$ and, in the case of the varieties **G** and $\mathbf{G}_p$, we also study functions from A^* to $\mathbb{Z}$ or from $\mathbb{Z}^k$ to $\mathbb{Z}$. The case of a one-letter alphabet was also discussed in [3]. Our results are summarized in the table below.

V	$\mathbf{G}_p$	**G**	**A**	**M**
$A^* \to \mathbb{Z}$	Theorem 3.19	Theorem 4.3	Open	Open
$\mathbb{Z}^k \to \mathbb{Z}$	Theorem 3.5	Theorem 4.1	Irrelevant	Corollary 7.5
$\mathbb{N}^k \to \mathbb{Z}$	Theorem 3.12	Theorem 4.2	Irrelevant	Open
$\mathbb{N}^k \to \mathbb{N}$	Theorem 3.12	Theorem 4.2	Theorem 5.4	Theorem 7.2

Characterization of hereditary V-preserving functions.

2 Preliminaries

In this section, we review the basic notions used in this paper.

2.1 Varieties

A *variety of finite monoids* is a class of finite monoids closed under taking submonoids, quotients and finite direct products. In the sequel, we shall use freely the term *variety* instead of *variety of finite monoids*.

We denote by **M** (respectively **Com**, **G**, **Ab**, **A**) the variety of all finite monoids (respectively finite commutative monoids, finite groups, finite abelian groups, finite aperiodic monoids). Given a prime number p, we denote by $\mathbf{G}_p$ the variety of all finite p-groups and by $\mathbf{Ab}_p$ the variety of all finite abelian p-groups. Each finite monoid M generates a variety, denoted by (M). The join of a family of varieties $(\mathbf{V}_i)_{i\in I}$ is the least variety containing all the varieties $\mathbf{V}_i$, for $i \in I$.

For $n > 0$, C_n denotes the cyclic group of order n. Throughout the paper, we shall use the well-known structure theorem for finite abelian groups [15], which shows that **Ab** is the variety generated by the finite cyclic groups.

Proposition 2.1 *Every finite abelian group is isomorphic to a direct product of finite cyclic groups.*

2.2 Ultrametrics and pseudo-ultrametrics

A *pseudo-ultrametric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties, for all $x, y, z \in X$:

(P_1) $d(x, y) \geqslant 0$,

(P$_2$) $d(x,x) = 0$,
(P$_3$) $d(x,y) = d(y,x)$,
(P$_4$) $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$.

An *ultrametric* satisfies a stronger version of (P$_2$):

(P$_5$) $d(x,y) = 0$ if and only if $x = y$.

2.3 Uniformly continuous functions

Given two pseudometric spaces (X_1, d_1) and (X_2, d_2), a function $f : X_1 \to X_2$ is *uniformly continuous* if, for every positive real number ε there exists a positive real number $\delta > 0$ such that for all $(x,y) \in X^2$,

$$d_1(x,y) < \delta \text{ implies } d_2(f(x), f(y)) < \varepsilon. \tag{2.1}$$

It follows in particular that if $d_1(x,y) = 0$, then $d_2(f(x), f(y)) = 0$. Moreover this condition is sufficient if 0 is an isolated point in the range of d_1 and d_2. We shall only need a weaker version of this result.

Proposition 2.2 *If d_1 and d_2 have finite range, a function $f : (X_1, d_1) \to (X_2, d_2)$ is uniformly continuous if and only if*

$$d_1(x,y) = 0 \textit{ implies } d_2(f(x), f(y)) = 0. \tag{2.2}$$

Proof. Since d_2 has finite range, there exists a positive real number ε such that $d_2(u,v) < \varepsilon$ implies $d_2(u,v) = 0$. If f is uniformly continuous, there exists δ such that $d_1(x,y) < \delta$ implies $d_2(f(x), f(y)) < \varepsilon$. By the choice of ε, this actually implies $d_2(f(x), f(y)) = 0$ and thus (2.2) holds.

Since d_1 has finite range, there exists a positive real number δ such that $d_1(u,v) < \delta$ implies $d_1(u,v) = 0$. Suppose that (2.2) holds and let ε be a positive integer. If $d_1(u,v) < \delta$ then $d_1(u,v) = 0$ and by (2.2), $d_2(f(x), f(y)) = 0$. It follows in particular that $d_2(f(x), f(y)) < \varepsilon$ and thus f is uniformly continuous. □

Nonexpansive functions form an interesting subclass of the class of uniformly continuous functions. A function $f : (X_1, d_1) \to (X_2, d_2)$ is *nonexpansive* if, for all $(x,y) \in X_1 \times X_1$,

$$d_2(f(x), f(y)) \leqslant d_1(x,y)$$

We shall use nonexpansive functions in Section 3.

2.4 Pro-V metrics

For the remainder of this section, let $\mathbf{V}$ denote a variety of finite monoids. Let M be a monoid and let $u, v \in M$. We say that a monoid N *separates* u and v if there exists a monoid morphism $\varphi : M \to N$ such that $\varphi(u) \neq \varphi(v)$. A monoid M is *residually* $\mathbf{V}$ if any two distinct elements of M can be separated by a monoid in $\mathbf{V}$.

We shall use the conventions $\min \emptyset = \infty$ and $2^{-\infty} = 0$. For all $u, v \in M$, let

$$r_{\mathbf{V}}(u,v) = \min \Big\{ |N| \;\Big|\; N \text{ is in } \mathbf{V} \text{ and separates } u \text{ and } v \Big\}$$

and $d_{\mathbf{V}}(u,v) = 2^{-r_{\mathbf{V}}(u,v)}$. Then $d_{\mathbf{V}}$ is a pseudo-ultrametric, called the *pro-*$\mathbf{V}$ metric on M (see [12]). If the monoid is residually $\mathbf{V}$, then $d_{\mathbf{V}}$ is an ultrametric.

In this paper, we consider free monoids, free commutative monoids and free abelian groups of finite rank: they are all finitely generated and residually $\mathbf{V}$ for the main varieties considered in this paper: monoids, (abelian) groups, abelian p-groups, (commutative) aperiodic monoids.

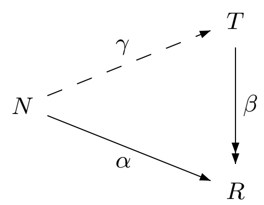
2.5 V-uniform continuity and V-hereditary continuity

Let M and N be monoids. A function $f : M \to N$ is said to be $\mathbf{V}$*-uniformly continuous* if it is uniformly continuous for the pro-$\mathbf{V}$ pseudometric on M and N. The following result was proved in [12, Theorem 4.1].

Proposition 2.3 *A function $f : M \to N$ is $\mathbf{V}$-preserving if and only if it is $\mathbf{V}$-uniformly continuous.*

We say that f is $\mathbf{V}$*-hereditarily continuous* if it is $\mathbf{W}$-uniformly continuous for each subvariety $\mathbf{W}$ of $\mathbf{V}$. Closure properties of this notion under various operators are analysed in [12, Subsection 4.3].

A monoid N is called $\mathbf{V}$*-projective* if the following property holds: if $\alpha : N \to R$ is a morphism and if $\beta : T \to R$ is a surjective morphism, where T (and hence R) is a monoid of $\mathbf{V}$, then there exists a morphism $\gamma : N \to T$ such that $\alpha = \beta \circ \gamma$.



For example, any free monoid (in particular $\mathbb{N}$) is $\mathbf{V}$-projective for every variety of finite monoids. Similarly, any free group (in particular $\mathbb{Z}$) is $\mathbf{V}$-projective for every variety of finite groups. Note that a $\mathbf{V}$-projective monoid is $\mathbf{W}$-projective for every subvariety $\mathbf{W}$ of $\mathbf{V}$.

The following results were proved in [12]:

Proposition 2.4 [12, Proposition 5.7] *Let $\mathbf{V}$ be the join of a family $(\mathbf{V}_i)_{i\in I}$ of varieties of finite commutative monoids. A function from a monoid to a $\mathbf{V}$-projective monoid is $\mathbf{V}$-hereditarily continuous if and only if it is $\mathbf{V}_i$-hereditarily continuous for all $i \in I$.*

Proposition 2.5 [12, Proposition 5.4] *A function from a monoid to a commutative monoid is $\mathbf{V}$-hereditarily continuous if and only if it is $(\mathbf{V} \cap \mathbf{Com})$-hereditarily continuous.*

In constrast, note that a $\mathbf{V}$-uniformly continuous function from a monoid to a commutative monoid is not necessarily $(\mathbf{V} \cap \mathbf{Com})$-hereditarily continuous. For instance, the function f from $\{a,b\}^*$ to $\mathbb{N}$ defined by $f(ab) = 1$ and $f(u) = 0$ if $u \neq ab$ is $\mathbf{M}$-uniformly continuous but is not $\mathbf{Com}$-uniformly continuous.

2.6 p-adic valuations

Let p be a prime number. If n is a non-zero integer, the *p-adic valuation* of n is the integer

$$v_p(n) = \max\left\{k \in \mathbb{N} \mid p^k \text{ divides } n\right\}$$

By convention, $v_p(0) = +\infty$. Note that the equality $v_p(nm) = v_p(n) + v_p(m)$ holds for all integers n, m.

The *p-adic norm* of n is the real number

$$|n|_p = p^{-v_p(n)}.$$

The p-adic norm satisfies the following properties, for all $n, m \in \mathbb{Z}$:

(N_1) $|n|_p \geqslant 0$,
(N_2) $|n|_p = 0$ if and only if $n = 0$,
(N_3) $|mn|_p = |m|_p|n|_p$,
(N_4) $|m+n|_p \leqslant \max\{|m|_p, |n|_p\}$.

The p-adic valuation and the p-adic norm can be extended to $\mathbb{Z}^k$ as follows. Given $n = (n_1, \ldots, n_k) \in \mathbb{Z}^k$, we set

$$v_p(n) = \min_{1\leqslant j\leqslant k}\{v_p(n_j)\} \quad \text{and} \quad |n|_p = p^{-v_p(n)} = \max_{1\leqslant j\leqslant k}\{|n_j|_p\}.$$

The p-adic norm on $\mathbb{Z}^k$ still satisfies (N_1), (N_2) and (N_4), as well as the following weaker version of (N_3):

(N_5) for all $n, m \in \mathbb{Z}^k$, $|mn|_p \leqslant |m|_p|n|_p$.

The p-adic norm on $\mathbb{Z}^k$ induces the *p-adic ultrametric* d_p on $\mathbb{Z}^k$, defined by $d_p(u, v) = |u - v|_p$. Note that the pro-$\mathbf{Ab}_p$ metric $d_{\mathbf{Ab}_p}$ and d_p are strongly equivalent metrics.

2.7 Binomial coefficients

Let A be a finite alphabet. We denote by A^* the free monoid on A. Note that if $|A| = 1$, then A^* is isomorphic to the additive monoid $\mathbb{N}$.

Let u and v be two words of A^*. Let $u = a_1 \cdots a_n$, with $a_1, \ldots, a_n \in A$. Then u is a *subword* of v if there exist $v_0, \ldots, v_n \in A^*$ such that $v = v_0a_1v_1 \ldots a_nv_n$. Set

$$\binom{v}{u} = |\{(v_0, \ldots, v_n) \mid v = v_0a_1v_1 \cdots a_nv_n\}|.$$

Note that if $A = \{a\}$, $u = a^n$ and $v = a^m$, then $\binom{v}{u} = \binom{m}{n}$ and hence these numbers constitute a generalization of the classical binomial coefficients. See [7, Chapter 6] for more details. Sometimes, it will be useful to use the convention $\binom{m}{n} = 0$ for $m \geqslant 0$ and $n \in \mathbb{Z} \setminus \{0, \ldots, m\}$, which is compatible with the usual properties of binomial coefficients.

2.8 Mahler expansions

For a fixed $v \in A^*$, we can view the generalized binomial coefficient $\binom{-}{v}$ as a function from A^* to $\mathbb{N}$. The functions $\left\{\binom{-}{v} \mid v \in A^*\right\}$ constitute a *locally finite* family of functions in the sense that, for each $u \in A^*$, the image of u is 0 for all but finitely many elements of the family.

It is clear that the sum of a locally finite family of functions is well defined. In particular, if $(g_v)_{v \in A^*}$ is a family of elements of an abelian group G, then there is a well-defined function f from A^* into G defined by the formula (in additive notation)

$$f(u) = \sum_{v \in A^*} g_v \binom{u}{v}$$

The generalized binomial coefficients provide a unique decomposition of the functions from A^* into G, which will be referred as *Mahler expansion*:

Proposition 2.6 (Lothaire [7]) *Let G be an abelian group and let $f : A^* \to G$ be an arbitrary function. Then there exists a unique family $\langle f, v \rangle_{v \in A^*}$ of elements of G such that, for all $u \in A^*$, $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$. This family is given by the inversion formula*

$$\langle f, v \rangle = \sum_{w \in A^*} (-1)^{|v|+|w|} \binom{v}{w} f(w) \tag{2.3}$$

A similar result holds for functions from $\mathbb{N}^k$ to an abelian group G. If r is an element of $\mathbb{N}^k$ (or more generally of $\mathbb{Z}^k$), we denote by r_i its i-th component, so that $r = (r_1, \ldots, r_k)$. First observe that the family

$$\left\{ \binom{-}{r_1} \cdots \binom{-}{r_k} \mid r \in \mathbb{N}^k \right\}$$

is a locally finite family of functions from $\mathbb{N}^k$ into $\mathbb{N}$. Thus, given a family $(g_r)_{r \in \mathbb{N}^k}$, the formula

$$f(n) = \sum_{r \in \mathbb{N}^k} g_r \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

defines a function $f : \mathbb{N}^k \to G$. Conversely, each function from $\mathbb{N}^k$ to G admits a unique *Mahler expansion*, a result proved in a more general setting in [2, 1].

Proposition 2.7 *Let G be an abelian group and let $f : \mathbb{N}^k \to G$ be an arbitrary function. Then there exists a unique family $\langle f, r \rangle_{r \in \mathbb{N}^k}$ of elements of G such that, for all $n \in \mathbb{N}^k$,*

$$f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

The coefficients $\langle f, r \rangle$ are given by

$$\langle f, r \rangle = \sum_{i_1=0}^{r_1} \ldots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \binom{r_1}{i_1} \cdots \binom{r_k}{i_k} f(i).$$

3 $\mathbf{G}_p$-hereditary continuity

Let p be a prime number. We proved in [11, 13] that $\mathbf{G}_p$-uniformly continuous functions from A^* to $\mathbb{Z}$ can be characterized by properties of their Mahler expansions. The case where A is a one-letter alphabet corresponds to the classical Mahler's Theorem from p-adic number theory [8, 9].

Theorem 3.1 *Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then the following conditions are equivalent:*

(1) f *is* $\mathbf{G}_p$*-uniformly continuous;*
(2) $\lim_{|v|\to\infty}|\langle f, v\rangle|_p = 0$.

A similar result (Amice, [2]) holds when A^* is replaced by $\mathbb{Z}^k$ (see also [13, Corollary 6.3] for an alternative proof). In this section, we obtain analogous results for $\mathbf{G}_p$-hereditary continuity. A first step is to reduce $\mathbf{G}_p$-hereditary continuity to a simpler property.

Lemma 3.2 *A function from a monoid to a* $\mathbf{G}_p$*-projective commutative monoid is* $\mathbf{G}_p$*-hereditarily continuous if and only if it is* (C_{p^n})*-uniformly continuous for all* $n > 0$.

Proof. By Proposition 2.5, f is $\mathbf{G}_p$-hereditarily continuous if and only if it is $(\mathbf{G}_p \cap \mathbf{Com})$-hereditarily continuous. Since

$$\mathbf{G}_p \cap \mathbf{Com} = \mathbf{G}_p \cap \mathbf{Ab} = \mathbf{Ab}_p = \bigvee_{n>0} (C_{p^n})$$

by Proposition 2.1, Proposition 2.4 implies that f is $\mathbf{G}_p$-hereditarily continuous if and only if f is (C_{p^n})-hereditarily continuous for every $n \in \mathbb{N}$. Since the only subvarieties of (C_{p^n}) are those of the form (C_{p^i}) with $i \leqslant n$, the lemma follows. □

Let $\mathbf{V}$ be a variety of groups. Since any morphism from $\mathbb{N}^k$ to a finite group extends uniquely to a morphism from $\mathbb{Z}^k$ to that same group, the pro-$\mathbf{V}$ pseudo-metric on $\mathbb{N}^k$ is the restriction of the pro-$\mathbf{V}$ pseudo-metric on $\mathbb{Z}^k$. Therefore the forthcoming results hold for $\mathbb{N}^k$ even though they are stated and proved for $\mathbb{Z}^k$.

We denote by $e_1, \ldots, e_k$ the canonical generators of both $\mathbb{N}^k$ and $\mathbb{Z}^k$. Thus $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 occurs in position j.

Lemma 3.3 *Let* $n \in \mathbb{N}$ *and let* d *be the pro-*(C_{p^n}) *pseudo-metric on* $\mathbb{Z}^k$*. For* $r, s \in \mathbb{Z}^k$*, one has* $d(r, s) = 2^{-p^m}$ *where*

$$m = \min\left\{ i \leqslant n \mid \text{there exists } j \in \{1, \ldots, k\} \text{ such that } r_j \not\equiv s_j \bmod p^i \right\}.$$

Proof. Suppose that $r_j \not\equiv s_j \pmod{p^i}$ for some $i \leqslant n$ and $j \in \{1, \ldots, k\}$. Let $f : \mathbb{Z}^k \to C_{p^i}$ be defined by $f(n) = n_j$. Clearly, $C_{p^i} \in (C_{p^n})$ and f separates r and s, hence $d(r, s) \geqslant 2^{-p^i}$ and so $d(r, s) \geqslant 2^{-p^m}$. Note that this last inequality holds trivially if $m = \infty$.

If $d(r, s) = 0$, equality follows. Otherwise, we may assume that $f : \mathbb{Z}^k \to G \in (C_{p^n})$ is a morphism that separates r and s with $|G|$ minimum. By Proposition 2.1, G is a direct product of cyclic groups. Since their order must divide $|G|$ which is a power of p, each one of these factor groups is of the form C_{p^i}. Since any group in (C_{p^n}) must satisfy the identity $x^{p^n} = 1$, we conclude that $i \leqslant n$ in each case. If G were a nontrivial direct product, we could decompose f into its components and contradict the minimality of G, thus $G = C_{p^i}$ with $i \leqslant n$.

Suppose that $r_j \equiv s_j \pmod{p^i}$ for every $j \in \{1, \ldots, k\}$. Then $r_j = s_j$ in C_{p^i} for every j and so

$$f(r) = \sum_{j=1}^{k} r_j f(e_j) = \sum_{j=1}^{k} s_j f(e_j) = f(s),$$

a contradiction. Thus $r_j \not\equiv s_j \pmod{p^i}$ for some $j \in \{1, \ldots, k\}$ and so $i \geqslant m$. It follows that $d(r, s) = 2^{-p^i} \leqslant 2^{-p^m}$ and so $d(r, s) = 2^{-p^m}$ as required. □

The next corollary shows how the pro-(C_{p^n}) pseudo-metric relates to the p-adic norm:

Corollary 3.4 *Let $n \in \mathbb{N}$ and let d denote the pro-(C_{p^n}) pseudo-metric on $\mathbb{Z}^k$. For all $r, s \in \mathbb{Z}^k$, we have*

$$d(r,s) = \begin{cases} 2^{-\frac{p}{|r-s|_p}} & \text{if } |r-s|_p > p^{-n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let

$$m = \min \left\{ i \leqslant n \mid \text{ exists } j \in \{1, \ldots, k\} \text{ such that } r_j \not\equiv s_j \bmod p^i \right\}.$$

It is easy to check that

$$m = \begin{cases} v_p(r-s) + 1 & \text{if } v_p(r-s) < n \\ \infty & \text{otherwise.} \end{cases}$$

Clearly, $v_p(r-s) < n$ if and only if $|r-s|_p > p^{-n}$. In this case,

$$p^m = p^{v_p(r-s)+1} = \frac{p}{|r-s|_p}$$

and the claim follows from Lemma 3.3. □

We arrive to our characterization of $\mathbf{G}_p$-hereditarily continuous functions.

Theorem 3.5 *A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{G}_p$-hereditarily continuous if and only if it is nonexpansive for the p-adic norm.*

Proof. Let d_n denote the pro-(C_{p^n}) pseudo-metric. By Lemma 3.2, f is hereditarily $\mathbf{G}_p$-uniformly continuous if and only if, for all $n > 0$, it is uniformly continuous for d_n. By Proposition 2.2, this holds if and only if, for all $r, s \in \mathbb{Z}^k$,

$$d_n(r,s) = 0 \text{ implies } d_n(f(r), f(s)) = 0. \tag{3.4}$$

By Corollary 3.4, $d_n(r, s) = 0$ if and only if $|r-s|_p \leqslant p^{-n}$, thus (3.4) is equivalent to stating that for all $r, s \in \mathbb{Z}^k$,

$$|r-s|_p \leqslant p^{-n} \text{ implies } |f(r) - f(s)|_p \leqslant p^{-n}. \tag{3.5}$$

Clearly, (3.5) holds for every n if and only if $|f(r) - f(s)|_p \leqslant |r - s|_p$, which proves the result. □

It follows easily from Theorem 3.5 that all polynomial functions from $\mathbb{Z}^k$ to $\mathbb{Z}$ are $\mathbf{G}_p$-hereditarily continuous. We shall use the Mahler expansion of functions given by Proposition 2.7 to characterize all the $\mathbf{G}_p$-hereditarily continuous functions from $\mathbb{N}^k$ to $\mathbb{Z}$. Polynomial functions will appear then as the finitely generated case. We shall need a few lemmas:

Lemma 3.6 *The sum of a locally finite family of* $\mathbf{G}_p$*-hereditarily continuous functions from* $\mathbb{N}^k$ *to* $\mathbb{Z}$ *is* $\mathbf{G}_p$*-hereditarily continuous.*

Proof. Let $\{f_i : \mathbb{N}^k \to \mathbb{Z} \mid i \in I\}$ be a locally finite family of $\mathbf{G}_p$-hereditarily continuous functions and let $f = \sum_{i \in I} f_i$. By Theorem 3.5, each f_i is nonexpansive for the p-adic norm, and since the p-adic norm satisfies (N_4), f is also nonexpansive. □

The following result is due to Kummer [6]. See also [17, 4].

Proposition 3.7 *Let* $n, r \in \mathbb{N}$ *with* $0 \leqslant r \leqslant n$*. Then* $v_p(\binom{n}{r})$ *is equal to the number of carries it takes to add* r *and* $n - r$ *in base* p*.*

Taking $n = p^s$ yields the following corollary

Lemma 3.8 *Let* $r, s \in \mathbb{N}$ *with* $0 < r \leqslant p^s$*. Then* $v_p\left(\binom{p^s}{r}\right) = s - v_p(r)$*.*

We also need a result stated in [3, Lemma 2.8], for which we give a shorter proof.

Lemma 3.9 *Let* $n, r, s \in \mathbb{N}$*. Then*

$$p^s \text{ divides } (\operatorname*{lcm}_{1 \leqslant j \leqslant r} j)\left(\binom{n+p^s}{r} - \binom{n}{r}\right) \tag{3.6}$$

or equivalently,

$$s \leqslant \max_{1 \leqslant j \leqslant r} v_p(j) + \nu_p\left(\binom{n+p^s}{r} - \binom{n}{r}\right). \tag{3.7}$$

Proof. Since $\binom{n+p^s}{r} - \binom{n}{r} = \sum_{j=1}^{r} \binom{p^s}{j}\binom{n}{r-j}$, one gets by Lemma 3.8 the relation

$$\left|\binom{n+p^s}{r} - \binom{n}{r}\right|_p \leqslant \max_{1 \leqslant j \leqslant r} \left|\binom{p^s}{j}\right|_p \left|\binom{n}{r-j}\right|_p \leqslant \max_{1 \leqslant j \leqslant r} \left|\binom{p^s}{j}\right|_p = \max_{1 \leqslant j \leqslant r} p^{v_p(j)-s}$$

or equivalently,

$$v_p\left(\binom{n+p^s}{r} - \binom{n}{r}\right) \geqslant \min_{1 \leqslant j \leqslant r} (s - v_p(j)) = s - \max_{1 \leqslant j \leqslant r} v_p(j)$$

which gives (3.7). □

We shall need two elementary results on nonexpansive functions.

Lemma 3.10 *Let* $f : \mathbb{N} \to \mathbb{Z}$ *be a nonexpansive function for the p-adic norm and let* $s \in \mathbb{N}$*. Then for* $0 \leqslant i \leqslant p^s$*,* p^s *divides* $\binom{p^s}{i}(f(i) - f(0))$*, or equivalently,* $s \leqslant v_p\left(\binom{p^s}{i}\right) + v_p(f(i) - f(0))$*.*

Proof. Since f is nonexpansive, one has $|f(i) - f(0)|_p \leqslant |i - 0|_p$ and thus $v_p(f(i) - f(0)) \geqslant v_p(i)$. Since $v_p\left(\binom{p^s}{i}\right) = s - v_p(i)$ by Lemma 3.8, the relation $s \leqslant v_p\left(\binom{p^s}{i}\right) + v_p(f(i) - f(0))$ follows immediately. □

Corollary 3.11 *Let $f : \mathbb{N} \to \mathbb{Z}$ be a nonexpansive function for the p-adic norm and let $s \in \mathbb{N}$. Then p^s divides $\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i)$.*

Proof. Newton's binomial formula yields

$$0 = (1-1)^{p^s} = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i},$$

hence

$$\sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} f(i) = \sum_{i=0}^{p^s} (-1)^i \binom{p^s}{i} (f(i) - f(0)).$$

The result now follows from Lemma 3.10. □

Theorem 3.12 *Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \ldots \binom{n_k}{r_k}$ be the Mahler expansion of a function $f : \mathbb{N}^k \to \mathbb{Z}$. Then the following conditions are equivalent:*

(1) *f is $\mathbf{G}_p$-hereditarily continuous,*

(2) *$v_p(j) \leqslant v_p(\langle f, r \rangle)$ holds for all j, r such that $1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\}$.*

Proof. (1) ⇒ (2). For all $r, t \in \mathbb{N}^k$, let us set

$$m_r(t) = \sum_{i_1=0}^{r_1} \ldots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \ldots \tbinom{r_k}{i_k} f(i+t).$$

By Proposition 2.7, we have $m_r(0, \ldots, 0) = \langle f, r \rangle$. We next show that

$$\min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\} \leqslant \min_{t \in \mathbb{N}^k} \{v_p(m_{r+s}(t))\} \tag{3.8}$$

for all $r, s \in \mathbb{N}^k$. By transitivity, we may assume that $s_1 + \ldots + s_k = 1$. By symmetry, we may assume that $s = (1, 0, \ldots, 0)$. Let $\ell = \min_{t \in \mathbb{N}^k} \{v_p(m_r(t))\}$. For all $t \in \mathbb{N}^k$, we have

$$\begin{aligned}
m_{r+s}(t) &= \sum_{i_1=0}^{r_1+1} \sum_{i_2=0}^{r_2} \ldots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{1+r_1}{i_1} \tbinom{r_2}{i_2} \ldots \tbinom{r_k}{i_k} f(i+t) \\
&= \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \ldots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \tbinom{r_2}{i_2} \ldots \tbinom{r_k}{i_k} f(i+t) \\
&+ \sum_{i_1=1}^{r_1+1} \sum_{i_2=0}^{r_2} \ldots \sum_{i_k=0}^{r_k} (-1)^{1+r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1-1} \tbinom{r_2}{i_2} \ldots \tbinom{r_k}{i_k} f(i+t) \\
&= -\sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \ldots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \tbinom{r_2}{i_2} \ldots \tbinom{r_k}{i_k} f(i+t) \\
&+ \sum_{i_1=0}^{r_1} \sum_{i_2=0}^{r_2} \ldots \sum_{i_k=0}^{r_k} (-1)^{r_1+\ldots+r_k+i_1+\ldots+i_k} \tbinom{r_1}{i_1} \tbinom{r_2}{i_2} \ldots \tbinom{r_k}{i_k} f(i+t+s) \\
&= -m_r(t) + m_r(t+s).
\end{aligned}$$

Since $p^\ell \mid m_r(t)$ and $p^\ell \mid m_r(t+s)$, it follows that $p^\ell \mid m_{r+s}(t)$ and so (3.8) holds.

Now we show that

$$s \leqslant v_p\big(m_{p^s e_j}(t)\big) \tag{3.9}$$

for all $s \in \mathbb{N}$, $t \in \mathbb{N}^k$ and $j = 1, \ldots, k$.

By symmetry, we may assume that $j = 1$, so that (3.9) becomes

$$p^s \mid \sum_{i=0}^{p^s} (-1)^{p^s+i} \binom{p^s}{i} f(i + t_1, t_2, \ldots, t_k). \tag{3.10}$$

Fix $t \in \mathbb{N}^k$ and let $g : \mathbb{N} \to \mathbb{Z}$ be the function defined by

$$g(n) = f(n + t_1, t_2, \ldots, t_k).$$

By Theorem 3.5, g is $\mathbf{G}_p$-hereditarily continuous and thus (3.10) follows from Corollary 3.11. Therefore (3.10) holds and so does (3.9).

We now show that

$$1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\} \Rightarrow v_p(j) \leqslant v_p(m_r(t)) \tag{3.11}$$

holds for all $j \in \mathbb{N}$ and $r, t \in \mathbb{N}^k$.

We use induction on $q = r_1 + \ldots + r_k$. The claim holds trivially for $q = 0$, hence we assume that $q > 0$ and (3.11) holds for smaller values of q. By symmetry, we may assume that $r_1 > 0$.

Assume first that $1 \leqslant j \leqslant \max\{r_1 - 1, \ldots, r_k\}$. By the induction hypothesis on q, we have $v_p(j) \leqslant v_p(m_{r_1-1,r_2\ldots r_k}(t))$ for all $t \in \mathbb{N}^k$. Thus $v_p(j) \leqslant v_p(m_r(t))$ by (3.8).

The remaining case corresponds to $j = r_1 > \max\{r_1 - 1, \ldots, r_k\}$. If j is not a power of p, then we may write $j = j_1 j_2$ with $j_1 < j$ and $v_p(j_1) = v_p(j)$, falling into the previous case. Thus we may assume that $j = p^i$ for some $i \in \mathbb{N}$. By (3.9), we have $i \leqslant v_p\big(m_{p^i,0,\ldots,0}(t)\big)$ for all $t \in \mathbb{N}^k$. Since $r_1 = j = p^i$, it follows from (3.8) that $v_p(j) = i \leqslant v_p(m_r(t))$ and (3.11) holds.

Considering now the particular case $t = 0$, we obtain Condition (2).

(2) $\Rightarrow$ (1). By Lemma 3.6, it is enough to show that the function

$$g(n) = \langle f, r \rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

is $\mathbf{G}_p$-hereditarily continuous for a fixed $r \in \mathbb{N}^k$. Write $m = \langle f, r \rangle$. Let $x, y \in \mathbb{N}^k$ and assume that $p^s \mid x - y$. By Theorem 3.5, it suffices to show that

$$p^s \mid m\Big(\binom{x_1}{r_1} \cdots \binom{x_k}{r_k} - \binom{y_1}{r_1} \cdots \binom{y_k}{r_k}\Big). \tag{3.12}$$

We have $p^s \mid x - y$ if and only $y = x + p^s z$ for some $z \in \mathbb{Z}^k$. Clearly, we can obtain y from x by successively adding or subtracting $p^s e_i$ $(i = 1, \ldots, k)$. Since $p^s \mid \ell$ and $p^s \mid \ell'$ together imply $p^s \mid \ell - \ell'$, we may assume without loss of generality that $x = y + p^s e_i$. By symmetry, we may also assume that $i = 1$. Therefore (3.12) will follow from

$$p^s \mid m\Big(\binom{y_1 + p^s}{r_1} - \binom{y_1}{r_1}\Big). \tag{3.13}$$

By condition (2), we have $v_p(j) \leqslant v_p(m)$ if $1 \leqslant j \leqslant r_1$, hence Lemma 3.9 yields

$$s \leqslant \max_{1 \leqslant j \leqslant r_1} v_p(j) + \nu_p\Big(\binom{y_1+p^s}{r_1} - \binom{y_1}{r_1}\Big) \leqslant v_p(m) + \nu_p\Big(\binom{y_1+p^s}{r_1} - \binom{y_1}{r_1}\Big)$$

and (3.13) holds as required. $\square$

It followed from Theorem 3.5 that all polynomial functions $f : \mathbb{N}^k \to \mathbb{Z}$ with integer coefficients are $\mathbf{G}_p$-hereditarily continuous. There are of course only countably many such functions. Theorem 3.12 implies the existence of uncountably many $\mathbf{G}_p$-hereditarily continuous functions:

Corollary 3.13 *There are uncountably many $\mathbf{G}_p$-hereditarily continuous functions $f : \mathbb{N}^k \to \mathbb{Z}$.*

Proof. For every $r \in \mathbb{N}^k$, let

$$\ell_r = \max\{v_p(j) \mid 1 \leqslant j \leqslant \max\{r_1, \ldots, r_k\}\}.$$

By Theorem 3.12 and Proposition 2.7, the map

$$(n_r)_{r\in\mathbb{N}^k} \mapsto \sum_{r\in\mathbb{N}^k} p^{\ell_r} n_r \binom{-}{r_1} \cdots \binom{-}{r_k}$$

is a bijection between $\mathbb{Z}^{(\mathbb{N}^k)}$ and the set of all $\mathbf{G}_p$-hereditarily continuous functions from $\mathbb{N}^k$ to $\mathbb{Z}$. $\square$

We now consider functions from a free monoid A^* to $\mathbb{Z}$. Let $h : A^* \to \mathbb{N}^A$ be the canonical morphism defined by $h(u) = (|u|_a)_{a\in A}$, where $|u|_a$ denotes as usual the number of occurrences of the letter a in u. Let $\sim$ be the *commutative equivalence*, formally defined by $u \sim v$ if and only if $h(u) = h(v)$.

Lemma 3.14 *Let $f : A^* \to \mathbb{Z}$ be a $\mathbf{G}_p$-hereditarily continuous function and let $u, v \in A^*$ be commutatively equivalent. Then $f(u) = f(v)$.*

Proof. Let us choose s such that $p^s > |f(u)-f(v)|$ and let d (respectively d') be the pro-(C_{p^s}) pseudo-metric on A^* (respectively $\mathbb{Z}$). Since f is hereditarily $\mathbf{G}_p$-uniformly continuous, it is in particular (C_{p^s})-uniformly continuous. Now, if u and v are commutatively equivalent, then $d(x, y) = 0$ and hence $d'(f(x), f(y)) = 0$, which means that $f(x) \equiv f(y) \bmod p^s$. Since $p^s > |f(u) - f(v)|$, this finally implies that $f(u) = f(v)$. $\square$

Lemma 3.15 *Let $u \in A^*$ and $r = (r_a)_{a\in A} \in \mathbb{N}^A$. Then $\sum_{v\in h^{-1}(r)} \binom{u}{v} = \prod_{a\in A} \binom{|u|_a}{r_a}$.*

Proof. Let $\mathbb{Z}\langle A\rangle$ be the ring of polynomials in noncommutative variables in A with integer coefficients. The monoid morphism μ from A^* to the multiplicative monoid $\mathbb{Z}\langle A\rangle$ defined, for each letter $a \in A$, by $\mu(a) = 1 + a$, is called the *Magnus transformation*. By [7, Proposition 6.3.6], the following formula holds for all $u \in A^*$:

$$\mu(u) = \sum_{v\in A^*} \binom{u}{v} v \tag{3.14}$$

Let $\mathbb{Z}[A]$ be the ring of polynomials in commutative variables in A with integer coefficients. The commutative version of the Magnus transformation is the monoid morphism $\underline{\mu}$ from A^* to the multiplicative monoid $\mathbb{Z}[A]$ defined, for each letter $a \in A$, by $\underline{\mu}(a) = 1 + a$. Thus by definition, one has, for each word $v \in A^*$,

$$\underline{\mu}(u) = \prod_{a\in A}(1+a)^{|u|_a} = \prod_{a\in A}\left(\sum_{0\leqslant r_a\leqslant|u|_a}\binom{|u|_a}{r_a}a^{r_a}\right) = \sum_{0\leqslant r_a\leqslant|u|_a}\left(\prod_{a\in A}\binom{|u|_a}{r_a}\right)\prod_{a\in A}a^{r_a} \tag{3.15}$$

and on the other hand, (3.14) shows that

$$\underline{\mu}(u) = \sum_{v\in A^*}\binom{u}{v}\prod_{a\in A}a^{|v|_a} = \sum_{r\in\mathbb{N}^A}\left(\sum_{v\in h^{-1}(r)}\binom{u}{v}\right)\prod_{a\in A}a^{r_a} \tag{3.16}$$

Comparing (3.15) and (3.16) now gives the formula $\sum_{v\in h^{-1}(r)}\binom{u}{v} = \prod_{a\in A}\binom{|u|_a}{r_a}$. $\square$

Lemma 3.16 *Let $f : A^* \to G$ be a function from A^* to some abelian group with Mahler expansion $f(\_) = \sum_{w\in A^*}\langle f, w\rangle\binom{\_}{w}$. Then the following conditions are equivalent:*

(1) *for any two commutatively equivalent words u and v, $\langle f, u\rangle = \langle f, v\rangle$,*
(2) *for any two commutatively equivalent words u and v, $f(u) = f(v)$.*

Proof. (1) implies (2). Suppose that (2) holds. For each $r \in \mathbb{N}^k$, let $\langle k, r\rangle$ be the common value of $\langle f, v\rangle$ for all $v \in h^{-1}(r)$. With the help of Lemma 3.15, we now obtain

$$\begin{aligned} f(u) &= \sum_{v\in A^*}\langle f, v\rangle\binom{u}{v} = \sum_{r\in\mathbb{N}^k}\sum_{v\in h^{-1}(r)}\langle k, r\rangle\binom{u}{v} \\ &= \sum_{r\in\mathbb{N}^k}\langle k, r\rangle\sum_{v\in h^{-1}(r)}\binom{u}{v} = \sum_{r\in\mathbb{N}^k}\langle k, r\rangle\prod_{a\in A}\binom{|u|_a}{r_a} \end{aligned}$$

It follows immediately that if u and v are commutatively equivalent, then $f(u) = f(v)$.

(2) implies (1). Let $g : A^* \to G$ be the function defined by $g(u) = (-1)^{|u|}\langle f, u\rangle$. It follows from the inversion formula (2.3) that $\langle g, x\rangle = (-1)^{|x|}f(x)$. Thus if (2) holds, then for any two commutatively equivalent words u and v, $\langle g, u\rangle = \langle g, v\rangle$. By the first part of the proof applied to g, it follows that $g(u) = g(v)$ and thus $\langle f, u\rangle = \langle f, v\rangle$. $\square$

Lemma 3.17 *Let $g : \mathbb{N}^k \to \mathbb{Z}$ be a function and let $\mathbf{V}$ be a variety of finite groups. Then g is $\mathbf{V}$-hereditarily continuous if and only if $g \circ h$ is $\mathbf{V}$-hereditarily continuous.*

Proof. By Proposition 2.5, g or $g \circ h$ are $\mathbf{V}$-hereditarily continuous if and only if they are $(\mathbf{V} \cap \mathbf{Ab})$-hereditarily continuous. Let $\mathbf{W}$ be a subvariety of $\mathbf{V} \cap \mathbf{Ab}$ and let d denote the pro-$\mathbf{W}$ pseudo-metric. Since h is surjective, every element of $\mathbb{N}^k$ can be written in the form $h(u)$ for some $u \in A^*$. Therefore g is $\mathbf{W}$-uniformly continuous if and only if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $u, v \in A^*$,

$$d\left(h(u), h(v)\right) < \delta \text{ implies } d\left(g \circ h(u), g \circ h(v)\right) < \varepsilon \tag{3.17}$$

Since any morphism from A^* to an abelian group factors through $\mathbb{N}^k$, one has $d(u,v) = d(h(u), h(v))$ for all $u, v \in A^*$. Therefore (3.18) can be rewritten as

$$d(u,v) < \delta \text{ implies } d\,(g \circ h(u), g \circ h(v)) < \varepsilon \tag{3.18}$$

and thus g is **W**-uniformly continuous if and only if $g \circ h$ is **W**-uniformly continuous. $\square$

Lemma 3.18 *Let $g : \mathbb{N}^k \to \mathbb{Z}$ be a function and let*

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r\rangle \binom{n_1}{r_1} \dots \binom{n_k}{r_k} \text{ and } g \circ h(u) = \sum_{v \in A^*} \langle g \circ h, v\rangle \binom{u}{v}$$

be the Mahler expansions of g and $g \circ h$. Then $\langle g, r\rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k}\rangle$ for every $r \in \mathbb{N}^k$.

Proof. We have

$$\begin{aligned} g(n) &= g \circ h(a_1^{n_1} \dots a_k^{n_k}) = \sum_{v \in A^*} \langle g \circ h, v\rangle \binom{a_1^{n_1} \dots a_k^{n_k}}{v} \\ &= \sum_{v \in a_1^* \dots a_k^*} \langle g \circ h, v\rangle \binom{a_1^{n_1} \dots a_k^{n_k}}{v} = \sum_{r_1=0}^{n_1} \dots \sum_{r_k=0}^{n_k} \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k}\rangle \binom{n_1}{r_1} \dots \binom{n_k}{r_k}. \end{aligned}$$

By the uniqueness of the Mahler expansion in Proposition 2.7, we conclude that $\langle g, r\rangle = \langle g \circ h, a_1^{r_1} \cdots a_k^{r_k}\rangle$ for every $r \in \mathbb{N}^k$. $\square$

Theorem 3.19 *Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v\rangle \binom{u}{v}$ be its Mahler expansion. Then f is $\mathbf{G}_p$-hereditarily continuous if and only if it satisfies the following conditions:*

1. *for any two commutatively equivalent words u and v, $\langle f, u\rangle = \langle f, v\rangle$,*
2. *$v_p(j) \leqslant v_p(\langle f, v\rangle)$ holds for all $v \in A^*$ and $1 \leqslant j \leqslant \max_{a \in A} |v|_a$.*

Proof. Assume that f is $\mathbf{G}_p$-hereditarily continuous. By Lemmas 3.14 and 3.16, condition (1) holds. Moreover, by Lemma 3.14, we may write $f = g \circ h$, where $h : A^* \to \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \to \mathbb{Z}$ is defined by

$$g(n) = f(a_1^{n_1} \dots a_k^{n_k})$$

By Lemma 3.18, the Mahler expansion

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r\rangle \binom{n_1}{r_1} \cdots \binom{n_k}{r_k}$$

of g is defined by $\langle g, r\rangle = \langle f, a_1^{r_1} \dots a_k^{r_k}\rangle$.

Assume that $v \in A^*$ and $j \in \mathbb{N}$ are such that $1 \leqslant j \leqslant |v|_{a_i}$ for every $i \in \{1, \dots, k\}$. Let $r = (|v|_{a_1}, \dots, |v|_{a_k})$. By Lemma 3.17, g is $\mathbf{G}_p$-hereditarily continuous and so we get $v_p(j) \leqslant v_p(\langle g, r\rangle) = v_p(\langle f, a_1^{r_1} \dots a_k^{r_k}\rangle)$ by Theorem 3.12. Since $v \sim a_1^{r_1} \dots a_k^{r_k}$, we get $\langle f, v\rangle = \langle f, a_1^{r_1} \dots a_k^{r_k}\rangle$ by Lemma 3.16 and so $v_p(j) \leqslant v_p(\langle f, v\rangle)$. Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, $f(u) = f(v)$ whenever $u \sim v$ and so there exists a function $g : \mathbb{N}^k \to \mathbb{Z}$ such that $f = g \circ h$. By Lemma 3.17, it suffices to show that g is $\mathbf{G}_p$-hereditarily continuous. Let

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \dots \binom{n_k}{r_k}$$

be the Mahler expansion of g and suppose that $1 \leqslant j \leqslant \max\{r_1, \dots, r_k\}$. By Theorem 3.12, we only need to show that

$$v_p(j) \leqslant v_p(\langle g, r \rangle). \tag{3.19}$$

By Lemma 3.18, we have $\langle g, r \rangle = \langle f, a_1^{r_1} \dots a_k^{r_k} \rangle$. Since

$$1 \leqslant j \leqslant \max\{r_1, \dots, r_k\} = \max\{|a_1^{r_1} \dots a_k^{r_k}|_{a_1}, \dots, |a_1^{r_1} \dots a_k^{r_k}|_{a_k}\},$$

it follows from condition (2) that $v_p(j) \leqslant v_p(\langle f, a_1^{r_1} \dots a_k^{r_k} \rangle)$ and so (3.19) holds as required. □

4 G-hereditary continuity

Let $\mathbb{P}$ denote the set of all positive primes.

Theorem 4.1 *A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{G}$-hereditarily continuous if and only if, for each prime p, it is nonexpansive for the p-adic norm.*

Proof. Since $\mathbf{G} \cap \mathbf{Com} = \bigvee_{p \in \mathbb{P}} (\mathbf{G}_p \cap \mathbf{Com})$, it follows from Propositions 2.4 and 2.5 that a function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is $\mathbf{G}$-hereditarily continuous if and only if it is $\mathbf{G}_p$-hereditarily continuous for every $p \in \mathbb{P}$. It now remains to apply Theorem 3.5 to conclude. □

Theorem 3.12 yields:

Theorem 4.2 *Let $f(n) = \sum_{r \in \mathbb{N}^k} \langle f, r \rangle \binom{n_1}{r_1} \dots \binom{n_k}{r_k}$ be the Mahler expansion of a function $f : \mathbb{N}^k \to \mathbb{Z}$. Then the following conditions are equivalent:*

(1) *f is $\mathbf{G}$-hereditarily continuous;*

(2) *j divides $\langle f, r \rangle$ for all $j \in \mathbb{N}$ and $r \in \mathbb{N}^k$ such that $1 \leqslant j \leqslant \max\{r_1, \dots, r_k\}$.*

We present now the analogue of Theorem 3.19 through an adaptation of its proof. We keep the notation introduced in Section 3.

Theorem 4.3 *Let $f : A^* \to \mathbb{Z}$ be a function and let $f(u) = \sum_{v \in A^*} \langle f, v \rangle \binom{u}{v}$ be its Mahler expansion. Then f is $\mathbf{G}$-hereditarily continuous if and only if it satisfies the following conditions:*

(1) *if u and v are commutatively equivalent, then $\langle f, u \rangle = \langle f, v \rangle$,*

(2) *j divides $\langle f, v \rangle$ for all $v \in A^*$ and $1 \leqslant j \leqslant \max_{a \in A} |v|_a$.*

Proof. Assume that f is $\mathbf{G}$-hereditarily continuous. Since $\mathbf{G}$-hereditarily continuous implies $\mathbf{G}_p$-hereditarily continuous, Lemma 3.14 remains valid for $\mathbf{G}$. Together with Lemma 3.16, this yields condition (1). Moreover, by Lemma 3.14, we may write $f = gh$, where $h : A^* \to \mathbb{N}^k$ is the canonical morphism and $g : \mathbb{N}^k \to \mathbb{Z}$ is defined by

$$g(n) = f(a_1^{n_1} \dots a_k^{n_k})$$

By Lemma 3.18, the Mahler expansion

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \dots \binom{n_k}{r_k}$$

of g is defined by $\langle g, r \rangle = \langle f, a_1^{r_1} \dots a_k^{r_k} \rangle$.

Assume that $1 \leqslant j \leqslant |v|_{a_i}$ for some $v \in A^*$ and $i \in \{1, \dots, k\}$. Let $r = (|v|_{a_1}, \dots, |v|_{a_k})$. By Lemma 3.17, g is $\mathbf{G}$-hereditarily continuous and so we get $j \mid \langle g, r \rangle = \langle f, a_1^{r_1} \dots a_k^{r_k} \rangle$ by Theorem 4.2. Since $v \sim a_1^{r_1} \dots a_k^{r_k}$, we get $\langle f, v \rangle = \langle f, a_1^{r_1} \dots a_k^{r_k} \rangle$ by Lemma 3.16 and so $j \mid \langle f, v \rangle$. Thus condition (2) holds.

Conversely, assume that conditions (1) and (2) hold. By Lemma 3.16, $f(u) = f(v)$ whenever $u \sim v$ and so there exists a function $g : \mathbb{N}^k \to \mathbb{Z}$ such that $f = gh$. By Lemma 3.17, it suffices to show that g is $\mathbf{G}$-hereditarily continuous. Let

$$g(n) = \sum_{r \in \mathbb{N}^k} \langle g, r \rangle \binom{n_1}{r_1} \dots \binom{n_k}{r_k}$$

be the Mahler expansion of g and suppose that $1 \leqslant j \leqslant \max\{r_1, \dots, r_k\}$. By Theorem 4.2, we only need to show that

$$j \mid \langle g, r \rangle. \tag{4.20}$$

By Lemma 3.18, we have $\langle g, r \rangle = \langle f, a_1^{r_1} \dots a_k^{r_k} \rangle$. Since

$$1 \leqslant j \leqslant \max\{r_1, \dots, r_k\} = \max\{|a_1^{r_1} \dots a_k^{r_k}|_{a_1}, \dots, |a_1^{r_1} \dots a_k^{r_k}|_{a_k}\},$$

it follows from condition (2) that $j \mid \langle f, a_1^{r_1} \dots a_k^{r_k} \rangle$ and so (4.20) holds as required. □

5 A-uniform continuity

Given a variety $\mathbf{V}$, let $\mathbf{CV} = \mathbf{Com} \cap \mathbf{V}$. In particular $\mathbf{CA}$ is the variety of commutative and aperiodic monoids. For each $t \in \mathbb{N}$, let $\mathbf{A}_t = [\![x^{t+1} = x^t]\!]$ and $\mathbf{CA}_t = \mathbf{Com} \cap \mathbf{A}_t$ be the variety of commutative aperiodic monoids of exponent t.

Let also N_t denote the monogenic monoid presented by $\langle x \mid x^t = x^{t+1} \rangle$. We usually view N_t as a quotient of $\mathbb{N}$ in order to represent its elements by natural numbers. The following results are folklore.

Proposition 5.1 *Every variety of commutative monoids is generated by its monogenic monoids. In particular* $\mathbf{CA}_t = (N_t)$ *for every* $t \in \mathbb{N}$*. Moreover, if* $\mathbf{V} \subseteq \mathbf{CA}$*, then* $\mathbf{V} = \mathbf{CA}_t$ *for some* $t \in \mathbb{N}$*.*

Given $m, n \in \mathbb{N}$, let us set

$$(m \wedge n) = \begin{cases} \min\{m,n\} & \text{if } m \neq n \\ \infty & \text{if } m = n \end{cases}$$

More generally, for $u, v \in \mathbb{N}^k$, we set write

$$(u \wedge v) = \min\{u_1 \wedge v_1, \ldots, u_k \wedge v_k\}.$$

Lemma 5.2 *Let $u, v \in \mathbb{N}^k$ and $t \in \mathbb{N}$. Then:*

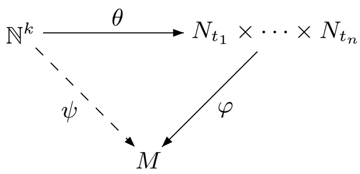
(1) $r_{\mathbf{A}}(u,v) = r_{\mathbf{CA}}(u,v) = (u \wedge v) + 2$;

(2) $r_{\mathbf{A}_t}(u,v) = r_{\mathbf{CA}_t}(u,v) = \begin{cases} (u \wedge v) + 2 & \text{if } (u \wedge v) < t \\ \infty & \text{otherwise} \end{cases}$

Proof. We may assume that $u \neq v$. Let $\mathbf{V} \subseteq \mathbf{A}$. Since $\mathbf{CV} \subseteq \mathbf{V}$ and every quotient of $\mathbb{N}^k$ in $\mathbf{V}$ is necessarily in $\mathbf{CV}$, we have $r_{\mathbf{V}}(u,v) = r_{\mathbf{CV}}(u,v)$. We show next that

$$r_{\mathbf{CV}}(u,v) = \min\{|N_t| \mid N_t \in \mathbf{CV} \text{ and separates } u \text{ and } v\}. \tag{5.21}$$

Indeed, if $M \in \mathbf{CV}$ separates u and v through $\psi : \mathbb{N}^k \to M$, it follows from the proof of Proposition 5.1 that there exists an onto homomorphism $\varphi : N_{t_1} \times \cdots \times N_{t_n} \to M$, where each N_{t_i} may be assumed to be a submonoid of M. Since $\mathbb{N}^k$ is a free commutative monoid, we may factor ψ through θ:



Since $\psi(u) \neq \psi(v)$, one of the component morphisms $\theta_i : \mathbb{N}^k \to N_{t_i}$ must separate u and v. Therefore the smallest $M \in \mathbf{CV}$ separating u and v must be of the form N_t and so (5.21) holds.

(1) By (5.21), we have

$$r_{\mathbf{CA}}(u,v) = \min\{|N_t| \mid N_t \text{ separates } u \text{ and } v\}. \tag{5.22}$$

If $u \wedge v = u_i \wedge v_i$, it is immediate that the projection on the i-th component induces a morphism from $\mathbb{N}^k$ to $N_{(u \wedge v)+1}$ separating u and v.

Suppose now that $\eta : \mathbb{N}^k \to N_t$ separates u and v with $t \leqslant (u \wedge v)$. Since

$$\sum_{i=1}^{k} \eta(u_i e_i) = \eta(u) \neq \eta(v) = \sum_{i=1}^{k} \eta(v_i e_i),$$

we have $\eta(u_i e_i) \neq \eta(v_i e_i)$ for some $i \in \{1, \ldots, k\}$. Hence $\eta(e_i) \geqslant 1$. Since $u_i, v_i \geqslant t$, it follows that $\eta(u_i e_i) = t = \eta(v_i e_i)$, a contradiction.

Thus $N_{(u \wedge v)+1}$ is the smallest N_t separating u and v. In view of (5.21), it follows that

$$r_{\mathbf{CA}}(u,v) = |N_{(u \wedge v)+1}| = (u \wedge v) + 2.$$

(2) By (5.21) and Proposition 5.1, we have

$$r_{\mathbf{CA}_t}(u,v) = \min\{|N_s| \mid s \leqslant t \text{ and } N_s \text{ separates } u \text{ and } v\}.$$

In view of (5.22), it follows that

$$r_{\mathbf{CA}_t}(u,v) = \begin{cases} r_{\mathbf{CA}}(u,v) & \text{if } r_{\mathbf{CA}}(u,v) \leqslant t+1 = |N_t| \\ \infty & \text{otherwise} \end{cases}$$

By (1), $r_{\mathbf{CA}}(u,v) \leqslant t+1$ is equivalent to $(u \wedge v) < t$ and the claim follows. □

Theorem 5.3 *Let $f : \mathbb{N} \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is **A**-uniformly continuous,*

(2) *for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u, v \in \mathbb{N}$, $u \wedge v \geqslant s$ implies $f(u) \wedge f(v) \geqslant n$,*

(3) *for every $n \in \mathbb{N}$, $f^{-1}(n)$ is either finite or cofinite.*

Proof. (1) ⇔ (2). It follows from the definition that f is **A**-uniformly continuous if and only if for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that, for all $u, v \in \mathbb{N}$,

$$r_{\mathbf{A}}(u,v) \geqslant s \text{ implies } r_{\mathbf{A}}(f(u), f(v)) \geqslant n,$$

that is equivalent to (2) by Lemma 5.2.

(2) ⇒ (3). Suppose that $f^{-1}(m)$ is neither finite nor cofinite. Let $s \in \mathbb{N}$ be arbitrary. Take $u_s \in f^{-1}(m)$ and $v_s \in \mathbb{N} \setminus f^{-1}(m)$ such that $u_s, v_s \geqslant s$. Thus the relation

$$u_s \wedge v_s \geqslant s \text{ and } f(u_s) \wedge f(v_s) \leqslant m$$

holds for all $s \in \mathbb{N}$, and so (2) fails.

(3) ⇒ (2). Let $n \in \mathbb{N}$. Suppose first that $f^{-1}(m)$ is cofinite for some $m \in \mathbb{N}$. Let $s = \max(\mathbb{N} \setminus f^{-1}(m))$. If $u \wedge v \geqslant s+1$, then $u \neq v$ implies $u, v \geqslant s+1$ and so $f(u) = m = f(v)$, hence we have $f(u) = f(v)$ in any case and $f(u) \wedge f(v) > n$ trivially.

Assume now that $f^{-1}(i)$ is finite for every $i \in \mathbb{N}$. Let $s = \max \cup_{i=0}^{n-1} f^{-1}(i)$. If $u \wedge v \geqslant s+1$ and $u \neq v$, then $u, v \geqslant s+1$ and so $u, v \notin \cup_{i=0}^{n} f^{-1}(i)$. Hence $f(u), f(v) \geqslant n$ and so $f(u) \wedge f(v) \geqslant n$. Therefore (ii) holds. □

Similarly, we get

Theorem 5.4 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is **A**-uniformly continuous;*

(2) *for all $n \in \mathbb{N}$, there exists $s \in \mathbb{N}$ such that for all $u, v \in \mathbb{N}^k$, $u \wedge v \geqslant s$ implies $f(u) \wedge f(v) \geqslant n$.*

However, there is no analogue of condition (3) of Theorem 5.3 in this case: if we define $f : \mathbb{N}^2 \to \mathbb{N}$ by $f(m,n) = m$, it is immediate that f is **A**-uniformly continuous and $f^{-1}(m)$ is infinite for every $m \in \mathbb{N}$.

Theorem 5.5 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping and $t \in \mathbb{N}$. Then the following conditions are equivalent:*

(1) *f is $\mathbf{A}_t$-uniformly continuous,*

(2) *for all $u, v \in \mathbb{N}^k$, $u \wedge v \geqslant t$ implies $f(u) \wedge f(v) \geqslant t$.*

Proof. Since $\mathbb{N}^k$ and $\mathbb{N}$ are commutative, the pseudo-metrics $d_{\mathbf{A}_t}$ and $d_{\mathbf{CA}_t}$ coincide in both monoids. Hence f is $\mathbf{A}_t$-uniformly continuous if and only if it is $\mathbf{CA}_t$-uniformly continuous.

Since $\operatorname{Im} r_{\mathbf{CA}_t} = \{2, \ldots, t+1, \infty\}$ by Lemma 5.2 (2), it follows from Proposition 2.2 that f is $\mathbf{CA}_t$-uniformly continuous if and only if for all $u, v \in \mathbb{N}^k$, $r_{\mathbf{CA}_t}(u, v) = \infty$ implies $r_{\mathbf{CA}_t}(f(u), f(v)) = \infty$. Now the claim follows from the same Lemma 5.2 (2). □

6 A-hereditary continuity

Lemma 6.1 *A function from a monoid M to $\mathbb{N}$ is $\mathbf{A}$-hereditarily continuous if and only if it is $\mathbf{CA}_t$-uniformly continuous for every $t \in \mathbb{N}$.*

Proof. By Proposition 2.5, a function is **A**-hereditarily continuous if and only if it is **CA**-hereditarily continuous. The lemma now follows from [12, Proposition 5.9]. □

Theorem 6.2 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is $\mathbf{A}$-hereditarily continuous,*

(2) *for all $u, v \in \mathbb{N}^k$, $u \wedge v \leqslant f(u) \wedge f(v)$,*

(3) *f is $\mathbf{A}$-nonexpansive.*

Proof. (1) is equivalent to (2). By Lemma 6.1, f is **A**-hereditarily continuous if and only it is $\mathbf{CA}_t$-uniformly continuous for every $t \in \mathbb{N}$. In view of Lemma 5.2, this amounts to stating that, for all $t \in \mathbb{N}$ and for all $u, v \in \mathbb{N}^k$, $u \wedge v \geqslant t$ implies $f(u) \wedge f(v) \geqslant t$.

(2) is equivalent to (3). By Lemma (5.2) (1), an equivalent formulation of (2) is that, for all $u, v \in \mathbb{N}^k$, $r_{\mathbf{A}}(u, v) \leqslant r_{\mathbf{A}}(f(u), f(v))$, which is equivalent to (3). □

We now look for a more explicit characterization of **A**-hereditary continuity. Given a function $f : \mathbb{N}^k \to \mathbb{N}$, we say that $g : \mathbb{N} \to \mathbb{N}$ is a *slice function* of f if there exists some $j \in \{1, \ldots, k\}$ and $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k \in \mathbb{N}$ such that $g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k)$ for every $x \in \mathbb{N}$.

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be *extensive* if $x \leqslant f(x)$ for every $x \in \mathbb{N}$ and *truncated* if there exists some $m \in \mathbb{N}$ such that $x \leqslant f(x)$ for $x \leqslant m$ and $f(x) = m$ for $x > m$. Functions that are either extensive or truncated can be described by the following single property:

(C) if $b = \min\{x \in \mathbb{N} \mid f(x) < x\}$, then $f(x) = b - 1$ for every $x \geqslant b$.

Indeed, the case $b = \infty$ corresponds to extensive functions and the case b finite corresponds to truncated functions.

Lemma 6.3 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping satisfing condition (C) and assume that $f(a_1, \ldots, a_k) < \min\{a_1, \ldots, a_k\}$. Then:*

(1) *$f(x_1, \ldots, x_k) = f(a_1, \ldots, a_k)$ for all $x_1 \geqslant a_1, \ldots, x_k \geqslant a_k$;*

(2) *there exists some $c \leqslant \min\{a_1, \ldots, a_k\}$ such that $f(a_1, \ldots, a_k) = f(c, \ldots, c) = c - 1$.*

Proof. (1) We use induction on k. For $k = 1$, assume that $f(a) < a$ and $x \geqslant a$. Then there exists $b = \min\{y \in \mathbb{N} \mid f(y) < y\}$ and so, by condition (C), $x \geqslant a \geqslant b$ implies $f(x) = f(a) = b - 1$.

Assume now that $k > 1$ and (i) holds for smaller values of k. Let $x_1 \geqslant a_1, \ldots, x_k \geqslant a_k$. By condition (C), we have $f(a_1, \ldots, a_{k-1}, x_k) = f(a_1, \ldots, a_k)$: indeed, if we take $b = \min\{x \in \mathbb{N} \mid f(a_1, \ldots, a_{k-1}, x) < x\}$, then $b \leqslant a_k \leqslant x_k$ and so $f(a_1, \ldots, a_{k-1}, x) = b - 1 = f(a_1, \ldots, a_k)$.

Define now $g : \mathbb{N}^{k-1} \to \mathbb{N}$ by $g(y_1, \ldots, y_{k-1}) = f(y_1, \ldots, y_{k-1}, x_k)$. Since f satisfies (C), so does g. Moreover,

$$g(a_1, \ldots, a_{k-1}) = f(a_1, \ldots, a_{k-1}, x) = b-1 = f(a_1, \ldots, a_k) < \min\{a_1, \ldots, a_{k-1}\}.$$

By the induction hypothesis, we get $g(x_1, \ldots, x_{k-1}) = g(a_1, \ldots, a_{k-1})$ since $x_1 \geqslant a_1, \ldots, x_{k-1} \geqslant a_{k-1}$. Thus

$$\begin{aligned} f(x_1, \ldots, x_k) &= g(x_1, \ldots, x_{k-1}) = g(a_1, \ldots, a_{k-1}) \\ &= f(a_1, \ldots, a_{k-1}, x_k) = f(a_1, \ldots, a_k) \end{aligned}$$

as required.

(2) We use induction on k. For $k = 1$, assume that $f(a) < a$. Then there exists $c = \min\{y \in \mathbb{N} \mid f(y) < y\}$ and so, by condition (C), $a \geqslant c$ implies $f(a) = f(c) = c - 1$.

Assume now that $k > 1$ and (ii) holds for smaller values of k. Let

$$b = \min\{y \in \mathbb{N} \mid f(a_1, \ldots, a_{k-1}, y) < y\}.$$

Then $b \leqslant a_k$. Define g as above. We have

$$g(a_1, \ldots, a_{k-1}) = f(a_1, \ldots, a_{k-1}, b) = b - 1 = f(a_1, \ldots, a_k)$$

by condition (C). By the induction hypothesis, there exists $c \leqslant a_1, \ldots, a_{k-1}$ such that $g(a_1, \ldots, a_{k-1}) = g(c, \ldots, c) = c - 1$. Thus $c - 1 = g(a_1, \ldots, a_{k-1}) = b - 1$ and so $b = c$. Since $b \leqslant a_k$, we get $c \leqslant a_1, \ldots, a_k$. Thus $f(a_1, \ldots, a_k) = c - 1 = g(c, \ldots, c) = f(c, \ldots, c)$ as required. □

Theorem 6.4 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then the following conditions are equivalent:*

(1) *f is* **A***-hereditarily continuous;*

(2) *every slice function of f is either extensive or truncated.*

Proof. (1) ⇒ (2). Let g be the slice function of f defined by

$$g(x) = f(a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k)$$

and let $a_j = \min\{x \in \mathbb{N} \mid g(x) < x\}$. If $x > a_j$, then one gets by Theorem 6.2

$$a_j = (a_1, \ldots, a_k) \wedge (a_1, \ldots, a_{j-1}, x, a_{j+1}, \ldots, a_k) \leqslant g(a_j) \wedge g(x),$$

and since $g(a_j) < a_j$, it follows that $g(a_j) = g(x)$.

Let $z = g(a_j)$. It remains to prove that $z = a_j - 1$. Since $z < a_j$, we have $z + 1 \leqslant a_j$. Suppose that $z + 1 < a_j$. By Theorem 6.2, one has

$$z+1 = (a_1, \ldots, a_k) \wedge (a_1, \ldots, a_{j-1}, z+1, a_{j+1}, \ldots, a_k) \leqslant g(a_j) \wedge g(z+1) = z \wedge g(z+1),$$

hence $g(z+1) = z < z+1$. Since $z + 1 < a_j$, this contradicts the minimality of a_j. Thus $z + 1 = a_j$ and (C) holds.

(2) $\Rightarrow$ (1). We use induction on k. For $k = 1$, assume that f satisfies condition (C) and let $u, v \in \mathbb{N}$ be distinct. By Theorem 6.2, we must prove that $u \wedge v \leqslant f(u) \wedge f(v)$. Let $Y = \{y \in \mathbb{N} \mid f(y) < y\}$. If $u, v \notin Y$, the claim follows. Hence we may assume that $Y \neq \emptyset$ and $b = \min Y$. If $u, v \in Y$, then $f(u) = b - 1 = f(v)$ by condition (C) and so $u \wedge v \leqslant \infty = f(u) \wedge f(v)$. Finally, assume that $u \in Y$ and $v \notin Y$. Then $f(u) = b - 1$. Without loss of generality, we may assume that $f(v) \neq b - 1$. Hence $v < b$ and so $v \leqslant (f(u) \wedge f(v))$. Thus $u \wedge v \leqslant f(u) \wedge f(v)$ and the result holds for $k = 1$.

Assume now that $k > 1$ and the theorem holds for smaller values of k. Assume that $f : \mathbb{N}^k \to \mathbb{N}$ satisfies condition (C) and let $u, v \in \mathbb{N}^k$ be distinct.

Assume first that $u_i = v_i$ for some $i \in \{1, \ldots, k\}$. Without loss of generality, we may assume that $i = k$. Define $g : \mathbb{N}^{k-1} \to \mathbb{N}$ by $g(y_1, \ldots, y_{k-1}) = f(y_1, \ldots, y_{k-1}, u_k)$. Since f satisfies (C), so does g. By the induction hypothesis and Theorem 6.2, we get

$$\begin{aligned} u \wedge v &= (u_1, \ldots, u_{k-1}) \wedge (v_1, \ldots, v_{k-1}) \\ &\leqslant g(u_1, \ldots, u_{k-1}) \wedge g(v_1, \ldots, v_{k-1}) = f(u) \wedge f(v) \end{aligned}$$

as required.

Hence we may assume that $u_i \neq v_i$ for every $i \in \{1, \ldots, k\}$. Without loss of generality, we may also assume that $u \wedge v = u_1$. Suppose first that $f(v) < u_1$. Since $u_1 \leqslant v_1, \ldots, v_k$, we may apply Lemma 6.3(ii) and get some $c \leqslant v_1, \ldots, v_k$ such that $f(v) = f(c, \ldots, c) = c - 1$. Thus $c - 1 < u_1$ and so $c < u_1 \leqslant u_2, \ldots, u_k$. By Lemma 6.3(i), it follows that $f(u) = f(c, \ldots, c) = f(v)$. Therefore $u \wedge v \leqslant f(u) \wedge f(v)$.

Next we assume that $f(u) < u_1$. Since $u_1 \leqslant u_2, \ldots, u_k$, we may apply Lemma 6.3(ii) and get some $c \leqslant u_1$ such that $f(u) = f(c, \ldots, c) = c - 1$. Since $v_1, \ldots, v_k \geqslant u_1 \geqslant c$, it follows from Lemma 6.3(i) that $f(v) = f(c, \ldots, c) = f(u)$. Therefore $u \wedge v \leqslant f(u) \wedge f(v)$ also in this case.

The final case $f(u), f(v) \geqslant u_1 = u \wedge v$ is trivial. □

7 M-hereditary continuity

Proposition 7.1 *Let M be a monoid and $f : M \to \mathbb{N}$ a mapping. Then the following conditions are equivalent:*

(1) *f is* **M***-hereditarily continuous;*

(2) *f is both* **G***- and* **A***-hereditarily continuous;*

(3) *f is both* **Ab**- *and* **CA**-*hereditarily continuous.*

Proof. The equivalence of (1) and (3) follows from [12, Proposition 5.8] and that of (2) and (3) from Proposition 2.5. □

Theorem 7.2 *Let $f : \mathbb{N}^k \to \mathbb{N}$ be a mapping. Then f is* **M**-*hereditarily continuous if and only if:*

(1) $\gcd\{u_i - v_i \mid i = 1, \dots, k\}$ *divides $f(u) - f(v)$ for all $u, v \in \mathbb{N}^k$,*

(2) *every slice function of f is either extensive or constant.*

Proof. By Proposition 7.1, f is **M**-hereditarily continuous if and only if it is both **G**- and **A**-hereditarily continuous. Now condition (1) is equivalent to **G**-hereditary continuity by Theorem 4.1. By Theorem 6.4, **A**-hereditary continuity is equivalent to every slice function of f being either extensive or truncated. Clearly, every constant function $f : \mathbb{N} \to \mathbb{N}$ is necessarily truncated. It remains to prove that every truncated slice function must be indeed constant in these circumstances.

Suppose that $g : \mathbb{N} \to \mathbb{N}$ defined by $g(x) = f(a_1, \dots, a_{j-1}, x, a_{j+1}, \dots, a_k)$ is truncated with $f(x) = m$ for every $x > m$. Let $M = \max(\operatorname{Im} f)$ and take $s < m$ arbitrary. We consider

$$u = (a_1, \dots, a_{j-1}, s, a_{j+1}, \dots, a_k), \quad v = (a_1, \dots, a_{j-1}, M + s + 1, a_{j+1}, \dots, a_k).$$

Since

$$M + 1 = \gcd_{1 \leqslant i \leqslant k} (u_i - v_i) \mid f(u) - f(v)$$

and $|f(u) - f(v)| \leqslant M$, it follows that $f(u) = f(v)$, hence $g(s) = g(M + s + 1) = m$. Therefore g is constant as claimed. □

Corollary 7.3 *Let $f : \mathbb{N} \to \mathbb{N}$ be a mapping. Then f is* **M**-*hereditarily continuous if and only if f is extensive or constant, and $u - v$ divides $f(u) - f(v)$ for all $u, v \in \mathbb{N}$.*

We can now adapt the proof of Corollary 3.13 to strengthen it:

Theorem 7.4 *There are uncountably many* **M**-*hereditarily continuous functions from $\mathbb{N}$ to $\mathbb{N}$.*

Proof. By the uniqueness of Mahler expansions, the mapping

$$(n_r)_{r \in \mathbb{N}} \mapsto \sum_{r \in \mathbb{N}} n_r \operatorname{lcm}(1, \dots, r) \binom{-}{r}$$

induces an injection θ from $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$. Let $f \in \operatorname{Im} \theta$. By Theorem 4.2, f is **G**-hereditarily continuous. Since $n_1 \geqslant 1$, we have

$$\sum_{r \in \mathbb{N}} n_r \operatorname{lcm}(1, \dots, r) \binom{x}{r} \geqslant n_1 \binom{x}{1} \geqslant x$$

for every $x \in \mathbb{N}$ and so f is extensive and thus **A**-hereditarily continuous by Theorem 6.4. Therefore f is **M**-hereditarily continuous by Proposition 7.1.

Since $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ is uncountable and θ is one-to-one, $\operatorname{Im}\theta$ is an uncountable set of **M**-hereditarily continuous functions from $\mathbb{N}$ to $\mathbb{N}$. □

We can also settle the case of functions from $\mathbb{Z}^k$ to $\mathbb{Z}$.

Corollary 7.5 *A function from $\mathbb{Z}^k$ to $\mathbb{Z}$ is* **M***-hereditarily continuous if and only if, for each prime p, it is nonexpansive for the p-adic norm.*

Proof. Let $f : \mathbb{Z}^k \to \mathbb{Z}$ be a function and let **V** denote a subvariety of **M**. Since every quotient of $\mathbb{Z}^k$ is necessarily a group, the pseudo-metrics $d_{\mathbf{V}}$ and $d_{\mathbf{V}\cap\mathbf{G}}$ coincide in $\mathbb{Z}^k$ (and in particular in $\mathbb{Z}$). It follows that f is **V**-uniformly continuous if and only if it is $\mathbf{V}\cap\mathbf{G}$-uniformly continuous. Since $\mathbf{V}\cap\mathbf{G}$ takes all possible values among the subvarieties of **G**, it follows that f is **M**-hereditarily continuous if and only if it is **G**-hereditarily continuous. One can now apply Theorem 4.1 to conclude. □

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