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# Growth without scale effects due to entropy

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U.PORTO FEP ECONOMICS AND MANAGEMENT

# Growth without scale effects due to entropy

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**Abstract**

We eliminate scale effects in the Balanced Growth Path of an expanding-variety endogenous growth model using the concept of entropy as a complexity effect. This allows us to gradually diminish scale-effects as the economy develops along the transitional dynamics, which conciliates evidence of the existence of scale effects long ago in history with evidence for no scale effects in today's economies. We show that empirical evidence supports entropy as a stylized form of the complexity effect. Then we show that the model can replicate well the take-off after the industrial revolution. Finally, we show that a model with both network effects (as spillovers in R&D) and entropy (as complexity effects) can replicate the main facts of the very long-run evolution of the economy since A.D. 1. Future scenarios may help to explain (part of) the growth crises affecting the current generation.

**Keywords:** endogenous economic growth, network effects, complexity effects, entropy.

**JEL Classification:** O10, O30, O40, E22

## 1 Introduction

Endogenous growth theorists have debated the existence of a scale effect of the population on the *per capita* GDP growth rate since its emergence. In fact, in the first examples of R&D-led endogenous growth models (Grossmann and Helpman, 1990; Romer, 1990; Aghion and Howitt, 1992), GDP *per capita* growth depended on the population level, implying that more populated countries would grow more than less populated countries, which is clearly rejected by current empirical cross-country evidence. However, historical examples long before the industrial revolution indicated that the level of population and connections between civilizations may determine the economic growth of those civilizations (e.g. Kremer, 1993). On the theoretical ground, the evolution of endogenous growth theory had made efforts to eliminate the so-called scale effects (Jones, 1999). Until now, the empirical contradiction on the existence of the scale effects between the today's evidence and historical evidence has not been completely incorporated into the economic growth literature. We fill this gap. As a first step, we infer a time-varying state-dependent complexity effect from the data. The next step is to find and calibrate a function

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that approximates the time series inferred from the data reasonably well. We motivate a closed-form specification for the complexity effect by deriving it from the concept of generalized entropy, as a measure of disorder, redundancy or diversity.

In the context of an endogenous growth model of horizontal R&D, complexity rises as the economy develops and increases the number of varieties of capital (technological) goods. Indeed, complexity may be understood as a process. Economies at earlier stages of development are characterized by relatively simple production methods and a limited availability of specialized inputs, in a context in which the diversity of human activities and of produced goods and services is low (Ciccone and Matsuyama, 1996). We argue that economies at these stages may display distinct market sizes, namely due to different population sizes, but are characterized altogether by similar low degrees of complexity. As such, innovation (or adoption) costs due to complexity effects are small and market size exerts no significant influence on them.

Complexity rises as the economy develops and increases the diversity of its activities. In developed economies production industries make extensive use of highly specialized inputs and have access to a wide variety of producer services, such as equipment repair and maintenance, transportation and communication services, engineering and legal supports, accounting, advertising, and financial services (see Ciccone and Matsuyama, 1996, and the extensive references therein). At this stage we expect the innovation costs due to complexity effects to be significant and become roughly proportional to market size. Thus, formally, we consider that complexity, though time-varying, is not a direct function of time but is state dependent, where the state of the economy is given by the number of varieties. This allows us to obtain scale-effects during the transition path which gradually decrease through time and eventually vanish in the steady state.

We contribute to the literature in two main issues: (1) bring the concept of entropy to model the complexity effect in the endogenous growth theory; (2) use the model with a complexity effect modelled through entropy to replicate both the take-off from the industrial revolution and the very long-run evolution of the economy since A.D. 1.

This paper is organized as follows. Section 2 reviews the literature related to the removal of the scale effects in endogenous growth theory and motivates the use of entropy functions to approximate the time-varying state-dependent complexity effect. Section 3 presents empirical motivation that allows us to infer the complexity effect from the data and calibrate the time-varying complexity effect. Then, Section 4 presents the model, its steady-state and transitional dynamics. Section 5 presents the quantitative simulation of the model for the take-off since the Industrial Revolution. Section 6 incorporates time-varying spillovers in the R&D production function and simulates the evolution of the economy since A.D. 1. In Sections 5 and 6 we provide a discussion of the results in the light of historical evidence. Finally, Section 7 draws conclusions and also highlights possible lines for further research.

# 2 Literature Review

## 2.1 R&D market complexity effect and the removal of scale effects on growth

A strand of the endogenous-growth literature has considered a market complexity effect in R&D activities as a mechanism for the elimination of the scale effects on growth along the Balanced Growth Path – BGP (e.g., Dinopoulos and Thompson, 1999, 2000; Barro and Sala-i-Martin 2004; Etro, 2008; Gil et al., 2013). Scale effects on growth are removed by considering a complexity effect in R&D such that the difficulty of introducing new products and replacing old ones is proportional to the market size measured by the absolute level of output attributable to the product targeted by R&D: the larger the latter, the greater the costs necessary to discover, develop and market the associated technology e.g., costs pertaining to the construction of prototypes and samples, new assembly lines and training of workers, and generic coordination, organizational and transportation costs.

These complexity costs offset the positive effect of scale on the (expected) profits of the successful innovator. In this way, scale variables like the level of the population do not influence the BGP growth rate because the probability of research success is also independent of those variables. This assumption guarantees that spending on R&D increases at the same rate as output and, thereby, allows us to establish a (BGP) relationship between a constant R&D to output ratio and a constant economic growth rate, as seems to be the case empirically.

As shown by Dinopoulos and Thompson (1999), this approach can be interpreted as a reduced form of the mechanism considered in the models by Dinopoulos and Thompson (1998), Peretto (1998), Howitt (1999) and Li (2000), among others, in which the removal of scale effects explicitly relies on the combination of vertical and horizontal R&D: the scale effect of rising population is offset by the growth of the number of varieties of products (through horizontal innovation) that fragments the growing demand for these products, thus preventing the reward to any specific product innovation from rising with population. This mechanism allows for the removal of scale effects while preserving the fully-endogenous growth result of the first-generation endogenous growth models (e.g., Romer, 1990; Aghion and Howitt, 1992) and is in contrast to the semi-endogenous growth result obtained by considering decreasing returns to the accumulated knowledge stock, as in, e.g., Jones (1995) and Segerstrom (1998). Over time the empirical literature has presented evidence more supportive of the fully-endogenous growth models than of the semi-endogenous approach (e.g., Dinopoulos and Thompson, 2000; Zachariadis, 2003; Laincz and Peretto, 2006; Ha and Howitt, 2007; Madsen, 2008; Sedgley and Elmslie, 2010; Ang and Madsen, 2015).

However, in spite of the modern evidence against scale effects on growth, there are compelling arguments for scale effects from historical evidence (e.g., Kremer, 1993). The dichotomy between historical and modern evidence on scale effects may be addressed from the point of view of deviations from the BGP. Observe that even in the already cited non-scale models, scale effects may still be present along the transition path. Transitional dynamics for the semi-endogenous and the fully-endogenous specifications have been analyzed by, e.g., Jones (1995) and Dinopoulos and Thompson (1998), respectively. But a key question concerns the persistence of deviations from the BGP and, specifically, whether the speed of adjustment back to the BGP is sufficiently slow to reveal scale effects over long periods of time.

Alternatively, it may be recognized that the complexity effect in R&D activities may be large enough to eliminate the scale effects in modern fully-developed economies, but may be small or non-existent in transition economies. We motivate the existence of a time-varying state-dependent complexity effect that increases with the economies' level of development by deriving it from the concept of generalized entropy.

## 2.2 A theoretic foundation of a time-varying complexity effect

Although the concept of entropy originated in Thermodynamics – commonly understood as a measure of molecular disorder within a macroscopic system – and its statistical definition was developed in Statistical Mechanics, it has been adapted and extended by other fields of study, including Information Theory, Biology, and Economics. As such, several indices have been created to measure entropy: examples are the Boltzmann-Gibbs entropy index and the Tsallis entropy index (a generalization of Boltzmann-Gibbs) as a measure of uncertainty in Statistical Mechanics (e.g., Tsallis, 1988); the Shannon index as a measure of redundancy in Information Theory (Shannon, 1948); the Patil and Taillie index (equivalent to the Tsallis index) as a measure of diversity in Biology (Patil and Taillie, 1982); and the Atkinson index (a transformation of a generalized entropy index) as a measure of income inequality in Economics (Atkinson, 1970).

A widely used family of entropy indices based on the contributions by Tsallis, and Patil and Taillie, is usually written in the form

$$S_q = \begin{cases} \frac{1-\sum_{i=1}^{W} p_i^q}{q-1} & , q \neq 1 \\ -\sum_{i=1}^{W} p_i \ln p_i & , q = 1 \end{cases}, \tag{1}$$

where $p_i$ is the probability of state $i$, $W$ is the number of states, and $q$ is a non-negative parameter. $S_q$ represents a parametric family of indices labeled by $q$, with some limiting values representing well-known indices that measure biological diversity or informational redundancy. For instance, with $q = 1$, one recovers the Boltzmann-Gibbs entropy and the Shannon entropy indices as a special case of the Tsallis index. In general, each specific application of the index $S_q$ requires the determination of a particular value of $q$.

If the distribution is uniform, such that, e.g., all the messages in the message space or the biological varieties (species) in the variety space are equiprobable, $p = 1/W$, the entropy is maximized and $S_q$ becomes

$$S_q = \begin{cases} \frac{1-W^{1-q}}{q-1} & , q \neq 1 \\ \ln W & , q = 1 \end{cases}, \tag{2}$$

The entropy indices may also be computed up to a constant of proportionality (in Thermodynamics this is known as the Boltzmann constant).

Adapting to an R&D production function, entropy as a measure of disorder, redundancy, or diversity captures the essence of the complexity effect in R&D put forward by the fully-endogenous growth literature. In the context of our model the number of states is represented by the number of varieties of technological goods, $A$, such that

$$(1+\psi)(A_{t+1} - A_t) = \delta \cdot (A_t) \cdot \frac{L_t^A}{L_t^{\chi(A_t)}}, \tag{3}$$

using $\psi$ as a measure of creative destruction as in Jones and Williams (2000), $L_t^A$ is the amount of labor allocated to R&D activities, $L_t$ is total amount of labor in the economy, and $\chi(A)$ is the time-varying state-dependent complexity effect. There are two sectors in the economy and consequently labor may be allocated to the R&D sector or to the final-good sector, which will be described below.

Considering $p = 1/A$, we obtain

$$S_q(A_t) = \begin{cases} b\frac{1-A_t^{1-q}}{q-1} & , q \neq 1 \\ b\ln A_t & , q = 1 \end{cases}, \tag{4}$$

with $b$ being the equivalent to the Boltzmann constant. We will consider $\chi(A_t)$ as

$$\chi(A_t) = max\left\{0, S_q(A_t)\right\}. \tag{5}$$

Equation (5) means that the definition of entropy given by the Tsallis index may be used to write the complexity effect $\chi(A_t)$ as a positive and concave function of the technological level $A_t$. Figure 1 shows a graphical representation of equation (5).

The following lemma shows that there is a specific combination of the parameters of the entropy function in equation (5) for which $\chi(A_t)$ converges to 1.

**Lemma 1.** *With $q > 1$, then $\chi(A_t)$ converges to $\frac{b}{q-1}$. Thus for $b = q - 1$, $\chi(A_t)$ converges to 1 as A goes to infinity. With $q \leq 1$ then $\chi(A_t)$ converges to $+\infty$.*

*Proof.* Calculate $\lim_{A\to+\infty}\chi(A_t)$ for $q > 1$ and $q \leq 1$ in equation (4), respectively. □

In the next section we will show that this theoretical foundation for the complexity effect can reasonably

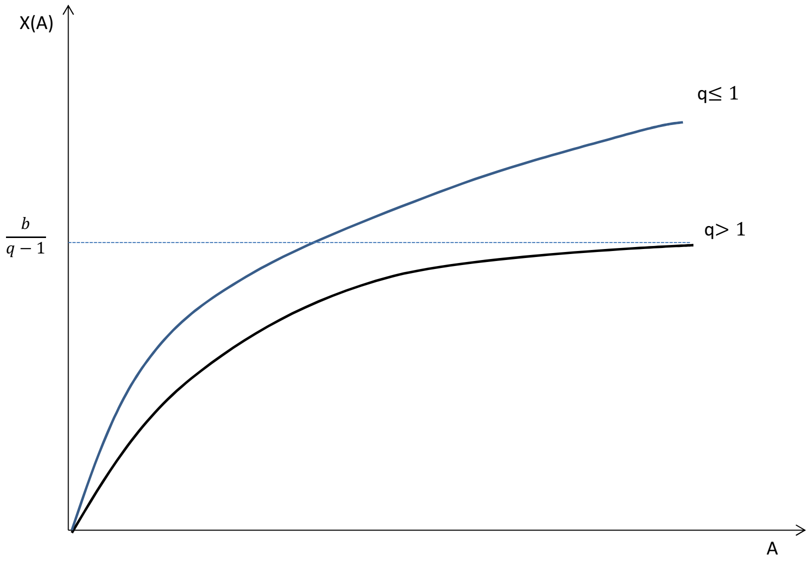



Figure 1: $\chi(A)$ function for different intervals of $q$

match the empirical series, given the available data.[1]

# 3 Empirical Motivation for complexity effect using entropy

Let $\Delta A_{t+1} = A_{t+1} - A_t$ in (3). By applying logs and solving for $\chi$, we have the recursive equation:

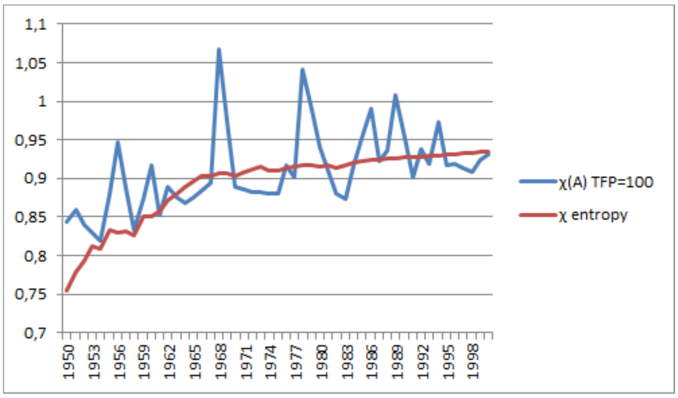
$$\chi = \frac{ln\delta - ln(1+\psi) + lnA_t + lnL_t^A - ln(\Delta A_{t+1})}{lnL_t}$$

To obtain estimates for $\chi$ over time, we consider the calibrated values of $\delta$ and the time series data for $A$, $L^A$ and $L$. We have used total labor force for $L$ between 1950 and 2000, from the Penn World Tables (PWT) 8.1. For the number of workers employed in R&D ($L^A$) we used the Number of Full-Time-Equivalent (FTE) R&D scientists and engineers in R&D-performing companies from the National Science Foundation. Finally for $A$, we use a TFP index (with the 2005 level equalized to 100) inferred from the (*per capita*) production function $y_t = (A_t)^\sigma (l_t^Y)^{1-\alpha} k_t^\alpha$, using output per worker for $y_t$ and physical capital per worker for $k_t$ (from the Penn World Tables 8.1), with parameters $\alpha = 0.36$ (share of physical capital in GDP) and $\sigma$ (returns to specialization, i.e. the measure of the social benefit from innovations) being 0.196 or 0.64 in different exercises and $\psi = 1.648$ (creative destruction) (values taken from Jones and Williams, 2000: Table 2). Finally note that in the model workers are allocated either to the final-good production or to the R&D labs, so we infer the ratio of workers in the production sector to total labor

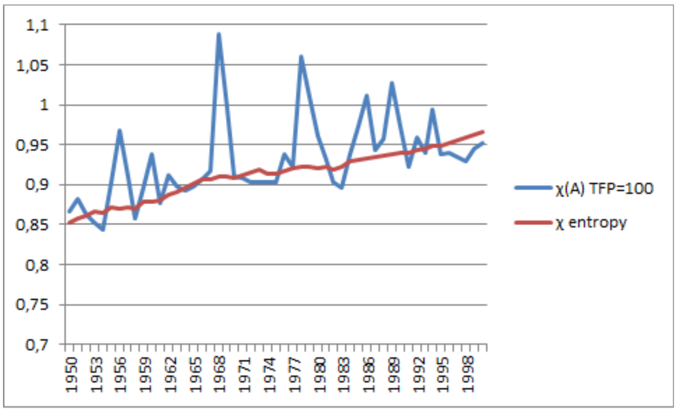
[1] We have tested other functions to adjust to the complexity effect. However, due to the matching between the concept of entropy and the difficulty or complexity effect in R&D, as well as due to the flexibility of the entropy function, we introduce this concept into the endogenous growth theory. Moreover, we show that it not only matches empirical series for the complexity effect, but also allows for historical plausible evolution of the economy in the long run.

force $l_t^Y$ using the data for $L^A$ and $L$ already mentioned above. The parameter $\delta$ is adjusted such that we obtain a steady-state growth rate in the model of 1.87%.[2] The resulting series are plotted in Figure 2.[3]

Also in the same Figure 2, we plot the theoretical adjustment to the empirical function using the definition of entropy in (5). We adjust $b$ and $q$ such that we obtain a reasonable adjustment between the theoretical and the empirical series. We can observe that we obtain very reasonable adjustment between the empirical series (for the lengthier time series available) and our theoretical formulation of the complexity effect.




(a) TFP index 2005=100, $\sigma = 0.196$, $q = 2.15$ and $b = 1.08$




(b) TFP index 2005=100, $\sigma = 0.64$, $q = 1.29$ and $b = 0.38$

Figure 2: Comparison between empirical series for $\chi$ (blue series) and entropy series for $\chi(A)$ (red series)

# 4 Model Setup

## 4.1 Households

We consider a model of overlapping generations (OLG). The young generation members supply one unit of labor from which they earn wages $w_t$ and smooth their consumption, dividing their income between the consumption in the current period $c_t^1$ and in the second period $c_{t+1}^1$. Old generation members do not work and make a living from their savings. Young individuals born in period 1 maximize utility $u_t = log(c_t^1) + \beta log(c_{t+1}^1)$, where $\beta$ is the discount factor, subject to the following constraints: $c_t^1 = w_t - s_t$, where $s_t$ are savings, and $c_{t+1}^2 = R_{t+1}s_t$, where $R_{t+1}$ is the expected gross interest rate. This standard OLG setup provides a well-known solution for *per capita* savings:

[2] This is the average annual growth rate of GDP per worker in the USA between 1950 and 2000, from the PWT 8.1.

[3] Note that we make a correspondence between "model" TFP and "data" TFP, without considering any further adjustment.

$$s_t = \frac{\beta}{1-\beta} w_t. \tag{6}$$

Population has dimension $L$ and grows at rate $n$. An exogenous growth rate may be appropriate to include forces that enlarge the market while proving convenient in deriving analytical results and in focusing the paper on the evolution of the technology side of economic activity (for a similar approach concerning population dynamics, see Peretto, 2015).

## 4.2 Firms

A continuum of size 1 of competitive firms produces a homogeneous output using a Cobb-Douglas technology and employing physical capital and labor. In period $t$ each firm $i$ uses labor $L^Y{}_t(i)$ and intermediate capital goods $K_t(i)$ to produce output. Aggregating over all firms, we have aggregated output as:

$$Y_t = A_t^{\sigma} K_t^{\alpha} L_t^{Y\,1-\alpha}, \tag{7}$$

where $0 < \alpha < 1$ is the share of physical capital in national income, $1 - \alpha$ the share of labor in the national income (as usual in the Cobb-Douglas settings) and $\sigma$ is a parameter that governs the returns to specialization. This allows us to proceed as Jones and Williams (2000) and Alvarez-Pelaez and Groth (2005) and disentangle the effect of returns to specialization from the share of physical capital in the final good production. For simplicity and without any loss of generality we assume that capital depreciates fully within one generation. Profit maximization yields the following first-order conditions:

$$w_t = (1-\alpha)\frac{Y_t}{L_t^Y}, \tag{8}$$

$$p_t = \alpha \frac{Y_t}{K_t}, \tag{9}$$

where $p_t$ is the price of aggregated capital good. The composite capital good $K$ is a CES aggregate of quantities, $x_i$, of specialized capital goods:

$$K_t = A_t \left[ \frac{1}{A_t} \sum_{0}^{A_t} x_i^{\alpha} \right]^{\frac{1}{\alpha}} \tag{10}$$

Using equations (9) and (10), we obtain the demand for individual varieties:

$$x_i = \frac{1}{A_t} \left( \frac{\alpha Y}{K^{\alpha} p_i} \right)^{\frac{1}{1-\alpha}} \tag{11}$$

With all varieties being produced in the same quantities, $K_t = A_t x_{it}$.

In the intermediate-goods sector (in which there is monopoly power) the firm maximizes profits $\pi = p_i x_i - r x_i$, from which we obtain the usual profit maximizing price, after substituting $x_i$ from equation (11), as $p_i = \frac{r}{\alpha}$. Using the profits equation from the intermediate-goods sector, the profits maximizing price and equation (9) for the wage, we obtain the following expression for profits in the intermediate-goods firm $\pi_t = (1-\alpha)\alpha Y_t / A_t$.

The number of varieties $A_t$ is produced according to equation (3). The free-entry condition into the R&D sector (which employs labor) is $w_t L_A = \pi_t (1+\psi)\Delta\mathrm{A}$ , which equates the costs and the profits of inventing $(1+\psi)\Delta A$ new units. Using equation (3), this yields $w_t \frac{L_t^{\chi(A_t)}}{\delta A_t} = \pi_t$. We equate both equations for profits. Then we use equation (8) and the labor market clearing condition $L_t = L_t^A + L_t^Y$ to obtain the shares of labor employed in the R&D sector and in the final-goods sector as follows ($l_t^a = \frac{L_t^a}{L_t}$ , where

$a = A, Y$ ):

$$l_t^Y = \min\left\{1, \frac{1}{\alpha\delta}\frac{1}{L_t^{1-\chi}}\right\} \qquad ; l_t^A = \max\left\{0, 1 - \frac{1}{\alpha\delta}\frac{1}{L_t^{1-\chi}}\right\} \tag{12}$$

# 5 Equilibrium dynamics: transitional dynamics and steady state

Using equations (6), (8), (7) and the *per capita* versions of the variables as small caps of the same letters, such that $y_t = \frac{Y_t}{L_t}$ is *per capita* income, $k_t = \frac{K_t}{L_t}$ is physical capital *per capita* and $c_t = \frac{C_t}{L_t}$ is consumption *per capita*, the model can be summarized by the following equations:

$$s_t = \frac{\beta}{1+\beta} w_t \tag{13}$$

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t}{L_{t+1}} s_t \tag{14}$$

$$w_t = (1-\alpha) y_t / l_t^Y \tag{15}$$

$$y_t = (A_t)^\sigma (l_t^Y)^{1-\alpha} k_t^\alpha = c_t + k_{t+1} \tag{16}$$

$$L_{t+1} = (1+n) L_t \tag{17}$$

Inserting (13) into (14), then substituting $w_t$ from expression (15) and finally using (12) and (16), we obtain the difference equation for physical capital *per capita* as follows:

$$k_{t+1} = \bar{a} \frac{(A_t)^\sigma L_t^{\alpha(1-\chi(A_t))} k_t^\alpha}{1+n} \tag{18}$$

where $\bar{a} = \frac{\beta(\alpha\delta)^\alpha (1-\alpha)}{1+\beta}$ . Note that when the complexity effect reaches $\chi(A) = 1$ (one), then equation (18) becomes free of scale effects. When $\chi(A) < 1$ scale effects are present but decreasing as $\chi(A)$ increases.

Using equations (3), (12), and (18), we drive the two dynamic equations that describe growth dynamics in this model:

$$\Delta k_t = \bar{a} \frac{(A_t)^\sigma L_t^{\alpha(1-\chi(A_t))} k_t^\alpha}{1+n} - k_t \tag{19}$$

and

$$(1+\psi)\Delta A_t = \delta \cdot (A_t) \left( L_t^{1-\chi(A_t)} - \frac{1}{\alpha\delta} \right). \tag{20}$$

Proposition 1 characterizes and states the conditions for a feasible steady state with endogenous growth. Note that this model evolves to fully endogenous economic growth, depending only on the parameters of the model, if $\chi(A_t)$ converges to a constant and population is constant. The scale effect vanishes if $\chi(A_t) = 1$ and becomes negative if $\chi(A_t) > 1$. However, with an increasing population, the unique steady state occurs with $\chi(A_t) = 1$.

**Proposition 1.** *In the steady state with increasing population, it must be that* $\chi(A_t) = 1$*; consequently:* $\frac{\Delta A_t}{A_t} = \frac{1}{1+\psi}(\delta - 1/\alpha)$, $\frac{\Delta k_t}{k_t} = \frac{\sigma}{1-\alpha}\frac{1}{1+\psi}(\delta - 1/\alpha)$ *and there is a feasible steady state with endogenous growth if and only if* $\delta\alpha > 1$.

*Proof.* Substitute $\chi(A_t) = 1$ in equation (20) and in equation (19). □

Using equations (19) and (20) and Lemma 1 we can conclude that for $q > 1$, the complexity effect converges to a constant, allowing for a steady state with constant population and some (positive if $b < q-1$ or negative if $b > q-1$) scale effects or to a steady state with growing population without scale effects (if $b = q-1$). Also, it indicates that for $q \leq 1$, in the limit growth vanishes to zero. This means that the model is flexible enough to replicate different outcomes, depending on the quantitative calibration of the R&D function. In the next sections we calibrate the model and evaluate quantitatively the plausible behavior of the economy governed by this model.

# 6 Calibration and the evolution of this economy

We mostly followed Jones and Williams (2000) to calibrate the model. So we used $\alpha = 0.36$, $\sigma = 0.196$ and $\psi = 1.648$, as did they. We used $\beta = 0.25$, which replicates a gross domestic savings rate (as a percentage of GDP) in the USA of 21% (average between 1974 and 2013). We replicate an annual average growth of GDP per worker in the USA of 1.87% (average between 1950 and 2000) by setting the value of $\delta$ to 9. The values that shape the entropy function for the complexity effect – equation (4) – come from the empirical exercise shown in Section 3, thus $q = 2.15$ and $b = 1.08$. We also test the differences of the results to the consideration of an alternative value of $\sigma = 0.64$, a value also considered in Jones and Williams (2000) and for which we also show empirical results in Section 3. In this case, we consider $\delta = 3.965$, $q = 1.29$, and $b = 0.38$. We include a series for the growth rate of population $n_t$ from Maddison (2008), taking the average growth rate for each generation of 20 years, considering the population values for 12 countries in western Europe (Austria, Belgium, Denmark, Finland, France, Germany, Italy, Netherlands, Norway, Sweden, Switzerland and the United Kingdom), the USA, and Australia. In the simulation exercise we show results for the path of the economy in the post-industrial revolution, beginning in 1880. In this case we want to replicate the main facts of the evolution of the economy in this period and will also compare the evolution of the "model" series with "data" series. Note that as we do not make any assumption on the values that govern the entropy R&D function $\chi(A)$ our objective is also to replicate the empirical value for $\chi(A)$, in 2000, which is tending to one.

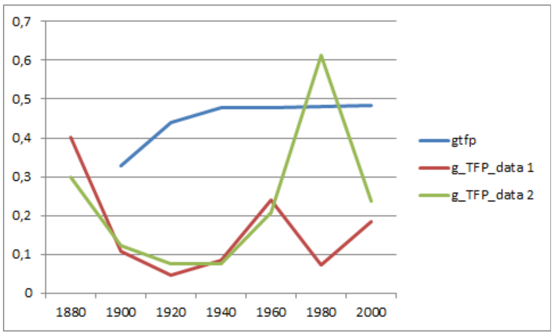
## 6.1 From the Industrial Revolution to the present days

In this subsection we will present the results of our simulations in our benchmark model. We compare the series for TFP growth, GDP *per capita* growth and GDP *per capita* levels, and the complexity (entropy) effect that emerges from the model with the series found in data. For GDP *per capita* growth and GDP *per capita* levels in data,[4] we use series for the USA and the UK (from the Penn World Tables 8.1) and series for the UK from Clark (2009). For TFP growth we use series for the USA from Baier *et. al.* (2000) and for the UK from Clark (2009). Finally, we compare the model complexity effect with the available observations we have for the same effect (also shown in Section 3).

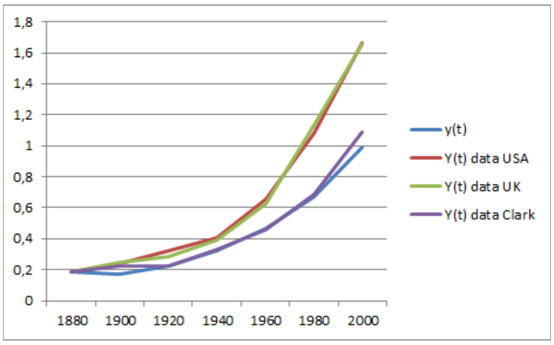
Figure 3 shows our main results. The model replicates an acceleration of TFP growth which tends to stabilize over the XX$^{th}$ century. It seems that the model tends to show an acceleration of TFP growth that happens too soon relative to data. Especially when comparing TFP model series with data series, it is worth noting that the model has not the sufficient ingredients to replicate the empirical effect of the Great Depression between the 1920 and 1940 generations.[5] Despite that, the model replicates quite well the evolution of the growth rate of *per capita* output, showing clearly the acceleration due to the industrial revolution. Regarding the level of *per capita* GDP, the model almost mimics the evolution of GDP shown by Clark (2009) for the UK being slightly below the estimates for the USA. Figure 3d shows that the evolution of the complexity effects closely replicates the data we obtained for this effect in the last half of the XX$^{th}$ century. It is interesting to note that a lower complexity effect has two reinforcing

[4]In the case of the level of GDP *per capita* we normalize the data value to match the model value in 1880. Growth rates are gross growth rates for one generation, 20 years.

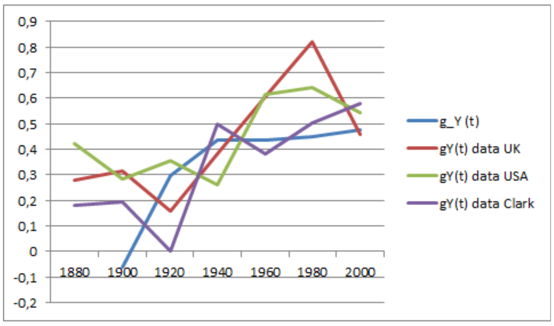
[5]A stylized model of endogenous growth as this one intends to replicate the long-run stylized facts (also those affecting the transitional dynamics) and not cyclical phenomena. In fact, most countries have returned to their long-run trend of long-run growth after the great depression.



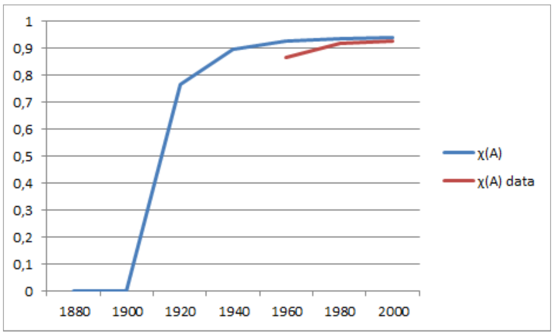

(a) TFP growth; g_TFP_1 is from Baier *et. al.* (2000) and g_TFP_2 is from Clark (2009).




(b) GDP *per capita* levels; Y(t) data UK and Y(t) data USA are from PWT 8.1 and Y(t) data Clark is data for the UK from Clark (2009)




(c) GDP *per capita* growth rates; gY(t) data UK and gY(t) data USA are from PWT 8.1 and gY(t) data Clark is data for the UK from Clark (2009)




(d) Complexity Effect $\chi(A)$

Figure 3: Evolution of the main "model" series (blue line series) and comparison with "data" series ($\sigma = 0.196$, $q = 2.15$ and $b = 1.08$, $k_0 = 0.073$, $L_0 = 0.8$, $A_0 = 0.5$).

effects: while it increases TFP growth (at higher than plausible rates), it also increases capital growth due to a stronger scale effect, as the lower the complexity effect the higher the scale effect on growth.

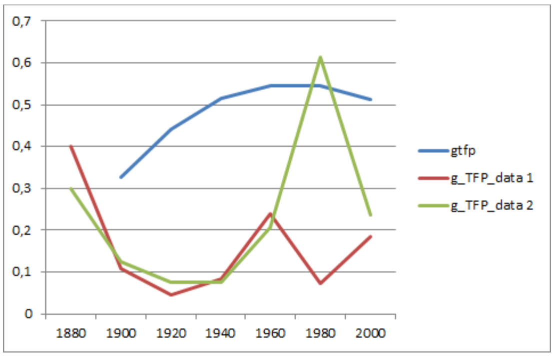
It is worth noting that with some adjustment in the parameters of the complexity function we can replicate the productivity slowdown after the 1980's and the main pattern related to the evolution of the other variables. In Figure 4 we show the evolution of the main variables in the economy for $q = 1.1$ and $b = 0.206$. In this figure the pattern of the evolution of TFP growth is closer to the data series (which now seem to replicate closer the long-run effect of the Great Depression), while the evolution of GDP *per capita* growth rates, and GDP *per capita* levels are again close to the data. The complexity effect $\chi(A)$ approaches the value of 1 in 2000, also close to the available data.

Next we present an exercise with the alternative value for $\sigma = 0.64$. In this case there is a slowdown in TFP growth rate in the late XX$^{th}$ century (but not a decrease in TFP growth as some of the data series show). Overall, TFP growth, GDP *per capita* growth and GDP *per capita* levels predicted by the model are very close to the data series. The poorest prediction of the model in this case is that of the complexity effect, although increasing considerablyt from the early XX$^{th}$ century on, is far from the 0.9 value that we found in the estimates based on the data.

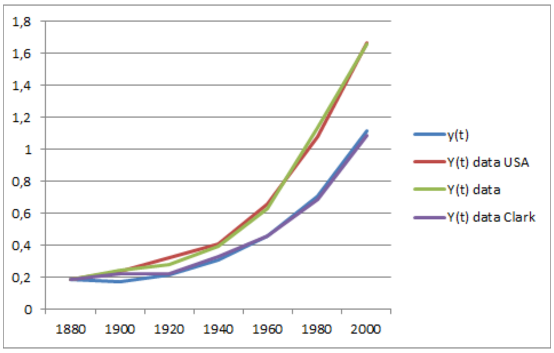
In the next section we will present counterfactual exercises to evaluate the importance of considering a varying state-dependent complexity effect to approximate the transitional dynamics of the model with the data series.

### 6.1.1 Counterfactual exercises with no varying complexity effect

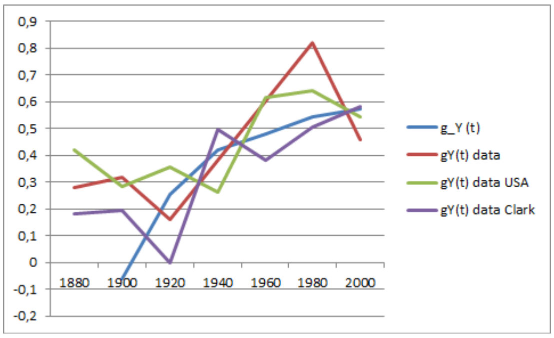
In this section we wish to show the relevance of considering a time-varying complexity effect in the R&D technology, which implies that the scale effects gradually vanish. First we assume that the complexity



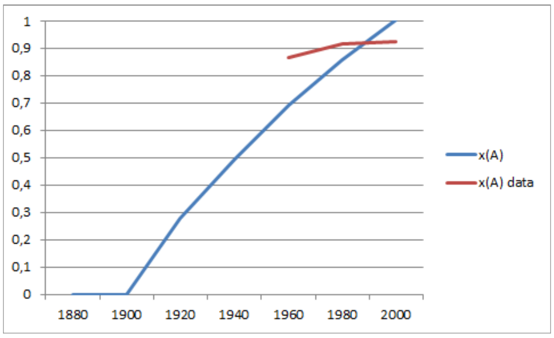

(a) TFP growth; g_TFP_1 is from Baier *et. al.* (2000) and g_TFP_2 is from Clark (2009).




(b) GDP *per capita* levels; Y(t) data UK and Y(t) data USA are from PWT 8.1 and Y(t) data Clark is data for the UK from Clark (2009)




(c) GDP *per capita* growth rates; gY(t) data UK and gY(t) data USA are from PWT 8.1 and gY(t) data Clark is data for the UK from Clark (2009)




(d) Complexity Effect $\chi(A)$

Figure 4: Evolution of the main "model" series (blue line series) and comparison with "data" series ($\sigma = 0.196$, $q = 1.1$ and $b = 0.206$, $k_0 = 0.073$, $L_0 = 0.8$, $A_0 = 0.5$).

effect $\chi(A)$ is always one, which implies that there are no scale effects throughout history. Second, we consider that $\chi(A) = 0$ always, which implies maximum scale effects throughout history. Both exercises show us the plausibility of considering a time-varying and increasing complexity effect in R&D. In the first case (Figure 6a), the main implausible prediction of the model is that the growth of TFP would always be constant from 1880 to 2000. In the second exercise (Figure 6b) the growth of TFP is always much higher than the data show, and instead of slowing down after 1980, it increases a great deal. In 2000 the growth rate of TFP would be implausibly high (near an annualized rate of 3.53%, while in data it is close to an annualized rate of 0.9%).[6] Additionally, by assumption we are imposing a complexity effect of zero (0), which is counterfactual with the evidence we showed in Section 3.

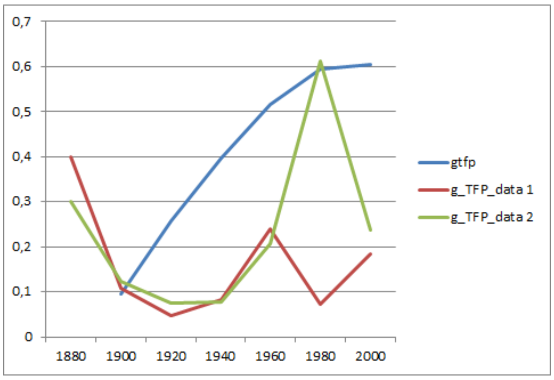
In the next subsection we introduce the change that R&D may benefit from knowledge diffusion across borders (as e.g. in Bottazzi and Peri, 2007). Our aim is to test the robustness of our results to this change in the setup, as the use of foreign ideas could diminish the importance of the complexity effect and eventually overcome it.

#### 6.1.2 International knowledge diffusion

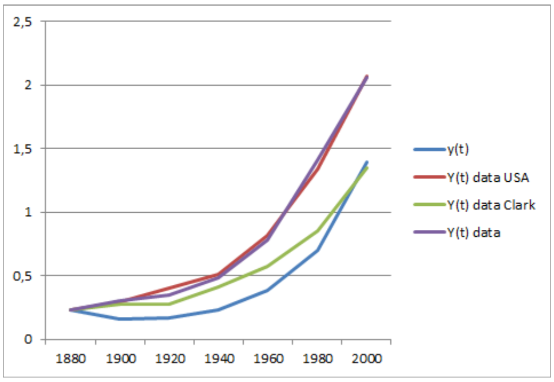
In this subsection we change the R&D production function to

$$(1+\psi)(A_{t+1} - A_t) = \delta \cdot (A_t)^g \left(A_t^w\right)^\mu \cdot \frac{L_t^A}{L_t^{\chi(A_t)}}, \tag{21}$$

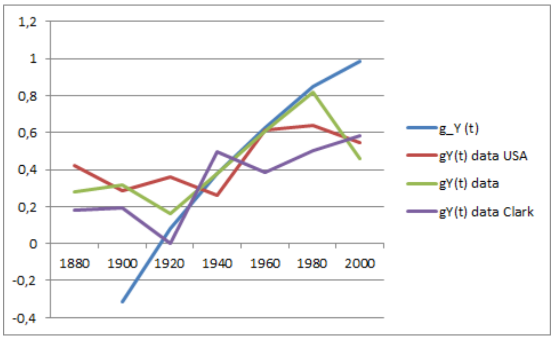
[6] As the main implausible features of considering a non-varying complexity effect come from the behavior of the TFP growth, we only include figures for this series. However, other figures are available upon request. We should note that the changes in the assumptions about the complexity effect have no effect in the steady-state growth rate of 1.87% that we replicate.



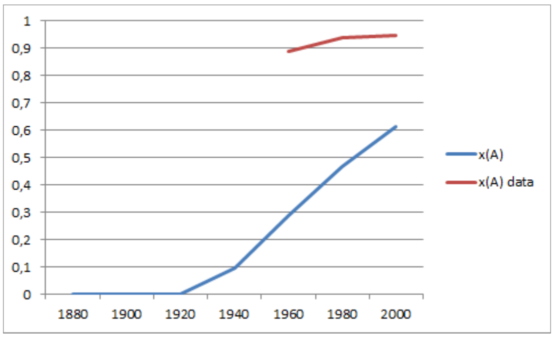

(a) TFP growth; g_TFP_1 is from Baier *et. al.* (2000) and g_TFP_2 is from Clark (2009).




(b) GDP *per capita* levels; Y(t) data UK and Y(t) data USA are from PWT 8.1 and Y(t) data Clark is data for the UK from Clark (2009)

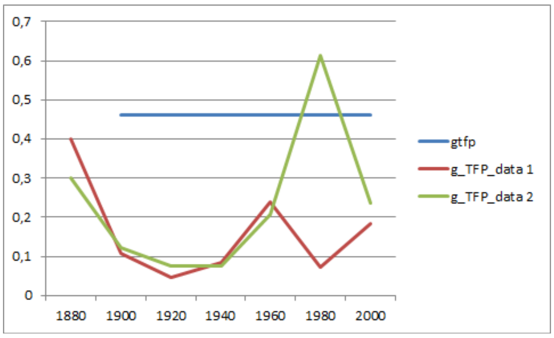



(c) GDP *per capita* growth rates; gY(t) data UK and gY(t) data USA are from PWT 8.1 and gY(t) data Clark is data for the UK from Clark (2009)

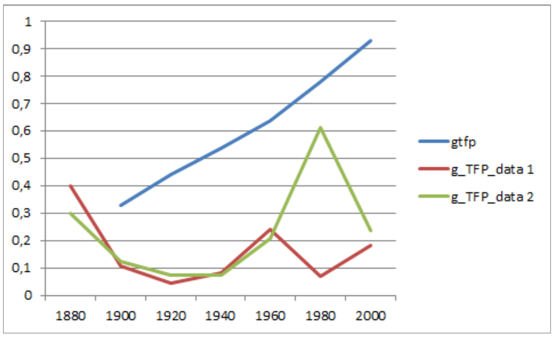



(d) Complexity Effect $\chi(A)$

Figure 5: Evolution of the main "model" series (blue line series) and comparison with "data" series ($\sigma = 0.64$, $q = 1.29$ and $b = 0.38$, $k_0 = 0.073$, $L_0 = 0.8$, $A_0 = 0.5$).




(a) TFP growth with $\chi(A) = 1$; g_TFP_1 is from Baier *et. al.* (2000) and g_TFP_2 is from Clark (2009).

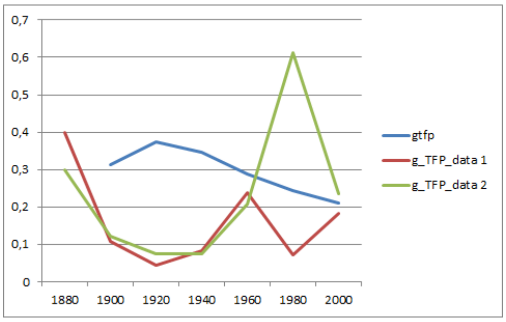



(b) TFP growth with $\chi(A) = 0$; g_TFP_1 is from Baier *et. al.* (2000) and g_TFP_2 is from Clark (2009).

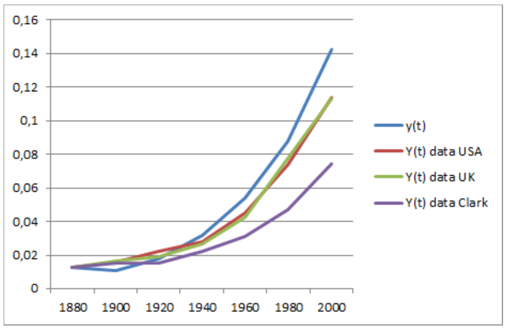
Figure 6: Evolution of the main "model" series (blue line series) and comparison with "data" TFP series ($\sigma = 0.196$, imposed $\chi(A)$, $k_0 = 0.073$, $L_0 = 0.8$, $A_0 = 0.5$).

where in addition to the specification in equation (3) we include domestic spillovers $g$, following Jones (1995), and $A_t^w$, following Bottazzi and Peri (2007), representing an exogenous and foreign stock of ideas influencing the production of domestic new technologies.[7] There is also a degree of international spillovers measured by $\mu$. In order to measure $A_t^w$ we use the stock of patents issued to non-residents in the USA (registered in the US patent office). First we set $A_0^w$ equal to the data ratio between patents issued to non-residents in the USA and patents issued to residents in the USA in 1880 (the first observation in the simulations), which was 2.3875%. Then, we calculate the growth rate of $A_t^w$ as the growth rate of patents issued to non-residents in the USA in the US patent office. We set $g = 0.864$, which is the highest value in Jones and Williams (2000) – a value that is also consistent with empirical estimates in Pessoa (2005) and Porter and Stern (2000) – and use an approximate value to the most robust estimates in Bottazzi and Peri (2007: Table 4) – a coefficient of 0.2 – to set $\mu = 0.0272$.[8] Figure 7 shows the results.

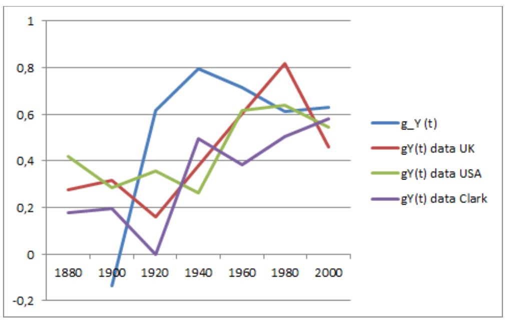
We can observe that this extension allows us to improve the replication of data time-series, namely those of GDP *per capita* (now more similar to the evolution of the USA data) and growth rates of GDP *per capita*. In fact, the simulation mimics quite well the final growth rates (both for GDP *per capita* and TFP) in 2000 and plots a slight recovery between 1980 and 2000. Again it also mimics the final values (for the three generations for which we have data between 1960 and 2000) of the complexity effect. Thus, we conclude that the model modification to account for foreign knowledge diffusion does not diminish the accuracy of the model in replicating the complexity effect, while allowing for a better replication of the final growth rates and approximating the evolution of GDP *per capita* to the USA data series. However, as a drawback, TFP slowdown begins too early in history (after 1920) and the model tends to show too high GDP *per capita* growth rates in the middle of the XX$^{th}$ century (between 1920 and 1960).



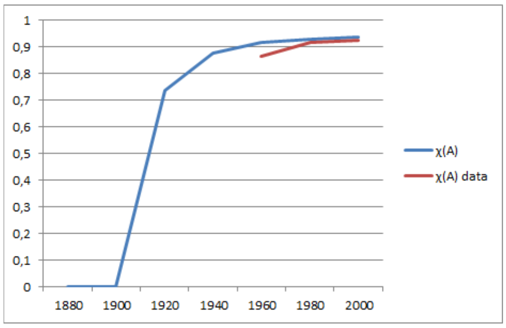

(a) TFP growth; g_TFP_1 is from Baier *et. al.* (2000) and g_TFP_2 is from Clark (2009).




(b) GDP *per capita* levels; Y(t) data UK and Y(t) data USA are from PWT 8.2 and Y(t) data Clark is data for the UK from Clark (2009)




(c) GDP *per capita* growth rates; gY(t) data UK and gY(t) data USA are from PWT 8.1 and gY(t) data Clark is data for the UK from Clark (2009)




(d) Complexity Effect $\chi(A)$

Figure 7: Evolution of the main "model" series (blue line series) and comparison with "data" series ($\sigma = 0.196$, $q = 2.15$ and $b = 1.08$, $\phi = 0.864$, $\mu = 0.0272$, $k_0 = 0.00004$, $L_0 = 0.8$, $A_0 = 0.5$).

[7] The deduction of the model with this change in the R&D production function is presented in the Appendix.

[8] $\mu = (1 - \phi) \times coefficient$.

# 7 An extended model to the very long run

The literature on Unified Growth Theory offers an insightful approach to economic growth in the very long run and its gradual take-off, by looking into the historical context of the pre-take-off and take-off phases of today's most developed countries, from the 18th century to the early 20th century. Galor and other authors' (e.g., Galor and Weil, 2000; Galor and Moav, 2002; see Galor, 2005, for a review) analytical results address the transition between distinct growth regimes – Malthusian, post-Malthusian and modern growth regimes – exploring the interaction between population growth and fertility rates (demographic transition), human-capital accumulation and output and *per capita* output growth. A recent strand of the literature shows that the gradual take-off of the economic growth rate can be complementarily explained by a mechanism centered on technological change, knowledge diffusion and the role of firms and entrepreneurs. For instance, Strulik (2009, 2014) focuses on the interaction of physical-capital accumulation and knowledge diffusion, while shutting down the interaction between demographic and economic forces and their impact on fertility and human capital. It is assumed that capital-embodied (learning-by-doing) technological change improves the transmission of knowledge by reducing the effective distance between firms. In this way capital accumulation improves the appropriation of knowledge through increased network or globalization effects. Then, more knowledge spillovers lead to higher factor productivity, which in turn, triggers more investment and higher growth.

Peretto (2015) explores a Schumpeterian model with two types of innovation activity (vertical and horizontal) in order to stress the role of forces at the industry level, the firms' and entrepreneurs' incentives to innovate and how they respond to changes in aggregate market size. Peretto addresses, in particular, the historical evidence concerning the transition from the first to the second Industrial Revolution by analyzing the conditions under which innovation is initially carried out by individual entrepreneurs who market it by starting up a new firm, while, later, innovation is mainly carried out by incumbent firms performing in-house R&D activities. Strulik (2009, 2014) does not remove scale effects in the steady state, while Peretto (2015) sterilizes scale effects in the steady state provided that both types of innovation occur, as in, e.g., Peretto (1998) and Dinopoulos and Thompson (1998), thereby generating non-scale fully-endogenous growth.

In this section we follow the second strand of the literature and extend our model to replicate the economy in the very long run, from A.D. 1 to the present days, including the Industrial Revolution and the subsequent take-off, the emergence of R&D-based economic activities and intensive learning activities, all of which will contribute to economic growth.

In this model there will be a learning mechanism as in Strulik (2014), as well as spillovers in R&D that are positively related to network effects related to physical capital, also present in Strulik (2009). These mechanisms, as in Strulik (2009, 2014), intend to represent the effect of globalization.[9] To this framework we also add our time-varying state-dependent complexity mechanism linked to entropy in the R&D technology. This more complete model should allow us to replicate the very long-run evolution of the economy, while incorporating a mechanism that tends to dissolve scale effects in modern times but allows for them in more ancient historical periods.

## 7.1 The Model

The main model is enlarged to consider a more complete production function such that *per capita* output can be written as:

$$y_t = {A_L}_t^{\zeta} {A_R}_t^{\sigma} k_t^{\alpha} l_t^{Y\,1-\alpha}, \tag{22}$$

[9] Because of this we do not include knowledge diffusion incorporating foreign ideas into the R&D production function, as we did in Section 6.1.2. However, adding that mechanism of knowledge diffusion does not change our main results.

where $A_L$ represents knowledge accumulated by experience or learning-by-doing, such that:

$$A_{Lt} = \bar{A} k_t^{g(k_t)}, \tag{23}$$

where $\bar{A}$ is the level of learning that is independent on the physical capital, $g(k)$ is a time-varying state-dependent network or globalization effect, based on the small world theory (e.g. in Strulik, 2014), and is given by:

$$g(k_t) = 1 - \left( \frac{\omega}{\omega + exp(k_t)} \right)^3 .^{10} \tag{24}$$

Furthermore, R&D is also enlarged to include spillovers that are modelled as network effects also given by equation (24):

$$(1+\psi)(A_{Rt+1} - A_{Rt}) = \delta \cdot (A_{Rt})^{g(k_t)} \cdot \frac{L_t^A}{L_t^{\chi(A_{Rt})}}. \tag{25}$$

Spillovers in R&D that are less than one are common in the endogenous growth literature since Jones (1995). These effects had been introduced with the motivation of nullifying the scale effects that would arise if spillovers were complete (i.e. $g = 1$). However, the way Strulik (2009) introduced the R&D spillovers in $g(k)$ in the model implies that scale effects may be absent in the transitional dynamics but should increase and be maximum when $g(k_t) = 1$, exactly in the steady state (or in the fully developed economy). This is exactly in opposition to historical evidence according to which scale effects seem to have been present in certain historical periods, but tend to be null (or very small) in the fully developed economies of modern times. Thus our complexity effect modelled as entropy is again essential to decrease and eventually nullify the scale effects in the fully developed economy, while allowing for their existence in the past.[11] Using (22), (23), (25) and the associated free-entry condition $w_t \frac{L_t^{\chi(A_t)}}{\delta A_t^{g(k_t)}} = \pi_t$ and following the same steps as in Section 5 (using again (13), (14), (15) and (17), and then using the identity in (16)), we obtain the dynamic equation for physical capital *per capita*:

$$\Delta k_t = \bar{a} \frac{(A_t)^{\sigma - \alpha(1-g(k_t))} L_t^{\alpha(1-\chi(A_t))} k_t^{\alpha + g(k_t)\zeta}}{1+n} - k_t \tag{26}$$

where $\bar{a} = \frac{\beta(\alpha\delta)^\alpha(1-\alpha)}{1+\beta} \bar{A}^\zeta$. Note that (26) highlights again that a steady state with a growing population can be attained when $\chi(A) = 1$.

## 7.2 Calibration

Following the same steps as in the benchmark model (Section 6), we use both literature values and data to calibrate the parameters in the model. Thus, we continue to use the same values for the gains of specialization ($\sigma$=0.196), the share of capital ($\alpha$=0.36), the creative destruction ($\psi = 1.648$), and $\beta = 0.25$, which replicates a gross domestic savings rate (as a percentage of GDP) in the USA of 21% (average between 1974 and 2013) . We infer the data for TFP level as before (using output and capital per worker from PWT) with the difference that now the TFP level is ${A_L}_t^\zeta {A_R}_t^\sigma$. We obtain $A_{Rt}$ from the US Patent Office series for patents issued in the NBER classifications 1 to 5, which are supposedly the most intensive in intentional research (chemical, computers and communications, Drugs and Medical, Electrical and Electronics, and Mechanical)[12] – see Marco *et. al.,* 2015. These data are used to disentangle the series for $A_{Rt}$ and $A_{Lt}$. We use the growth rate of capital per worker and the growth rate of patents (both calculated in the time interval from 1950 and 2000) to obtain $\zeta = 0.389$, using also the calibrated values

[10] We rely on Strulik (2014) for detailed foundation for this functional form. In particular, it can be derived from first principles under the assumption that the economy is a small world network in which the number of long-distance links depends positively on the capital stock. It tends to unity as $k$ goes to infinity and leaves only one parameter $\omega$ to calibrate.

[11] Note that when $g(k) = \chi(A) = 1$, scale effects are absent.

[12] From the total number of patents issued, this excludes classifications 7 (others), 8 (not classified) and 9 (missing).

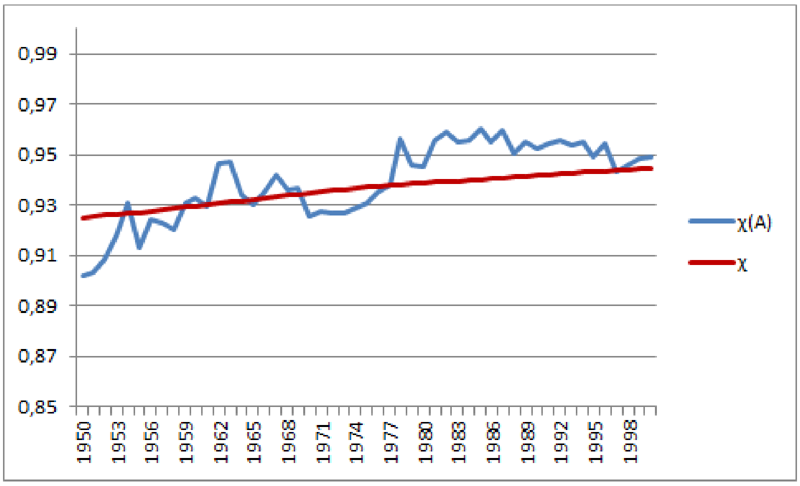
for $\alpha$ and $\sigma$. We adjust $\delta = 4.39$ such that the model replicates a steady-state growth rate of 1.87% (the same procedure as before). Population growth rates are again the 20 year average population growth from A.D. 1 to A.D. 2000 from Maddison (2008).

Finally, transforming equations (23) and (25) to their log version, we use the following recursive system to obtain the state-dependent functions $g$ and $\chi$:

$$g = \frac{lnA_t^L - ln\bar{A}}{lnk_t}$$

$$\chi = \frac{ln\delta + g \cdot lnA_t^R + lnL_t^A - ln(\Delta A_{t+1}^R)}{lnL_t}$$

To obtaint empirical estimates for $g$ and $\chi$ over time, we consider the calibrated values of $\delta$ and $\bar{A}$ and the time series data for $A^L$, $A^R$, $k$, $L^A$ and $L$. We follow Strulik (2014) in adjusting for the scale in which the learning variable is measured in the data up to a constant $\mu$, such that the learning variable in the model should be the learning variable in data multiplied by $\mu$. As before, we consider indices of learning $A^L$ and technological knowledge $A^R$ to be indices with the value 100 in 2000. Then, we proceed as before and adjust $q$ and $b$ such that the theoretical series for $\chi$ approximate the empirical series. We plot the series in Figure 8, obtained with $b = 0.919$ and $q = 1.961$, which we will use in the baseline simulations in this Section. Also as in Strulik (2014), $\omega$ is adjusted such that we obtain the best possible historical adjustment.




(a) TFP index (2005=100), $\gamma = 0.196$, $q = 1.961$ and $b = 0.919$

Figure 8: Comparison between empirical series for $\chi$ (blue series) and entropy series for $\chi(A)$ (red series)

### 7.3 From A.D. 1 to present days

In this section we present simulations of the extended model to replicate the main facts of the economic evolution from A.D. 1 to A.D. 2000 Figures 9 and 10 show the evolution of the main variables. In Figure 9a the globalization effect is plotted. Interestingly, the globalization effect is smoothly increasing until the middle age, doubling between A.D. 1 and near A.D. 1000 Historically, this period corresponds to the christianization of Europe, the decadence of the eastern Roman Empire, the rise of the Arab Empire, which flourished from Iberia to China, and the Vikings expeditions. From near 1000 to the middle of XV$^{th}$ century, globalization stagnated on a level near 0.4. This corresponds to the Middle-Ages, when barriers to trade and to free thinking took place all over Europe. After a slight increase in globalization due to the first exploration voyages (e.g., those of Columbus and Vasco da Gama) between 1400 and

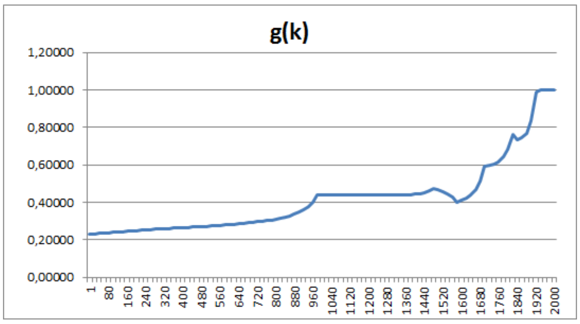
1460, there is a drop in globalization. Although this continues to be an epoch of maritime explorations (e.g., Magellan's voyages), at the same time, reformation and counter-reformation shaped Europe, and the Turkish hegemony on the east decreased trade through the eastern route to Asia, while the new maritime routes had not reached their maximum capacity yet.

Interestingly, it is after 1580 that globalization begins to expand rapidly. This corresponds to the entrance of the Dutch and the English in the world trade maritime routes, defying and eventually defeating the Iberian countries.[13] Between 1820 and 1840, there is a period of falling globalization, which historically corresponds to the collapse of the Iberian empires in the South and Central America, the consequent independence of several countries in that region of the globe, and protectionist practices in the USA and Canada. After 1840, a new age of growing globalization emerges and which reaches the maximum in 1940. This epoch corresponds to the maximum extension of the British empire, the emergence of the USA as a global power, and to the first and second Industrial Revolutions. Figures 9b, 9d and 9f show that the take-off for modern growth is situated around 1840, with the emergence of R&D as an intentional activity and which is highly intensified after 1920 (see historical evidence e.g. in Mokyr, 2005). The emergence of learning-driven knowledge began earlier than intentional R&D knowledge, with a positive evolution since the early XVI$^{th}$ century, but with a great intensification on the eve of the XIX$^{th}$ century. The evolution of labor allocated to R&D activities (Figure 9e) confirms that allocation to intentional R&D began in 1820, but by a very small amount. Figure 9d confirms that this effort only implied a significant change in technological knowledge, led by intentional R&D, after 1860. Figure 9e shows the evolution of the complexity effect, conducted by our entropy formulation. The complexity effect is positive only in 1840 and increases steadily until 1920, after which it grows slowly and eventually stabilizes around one in 2000. Figure 10 shows the evolution of the economy in growth rates. All the changes in growth rates before the emergence of the Industrial Revolution are driven by changes in population growth, implying that the model does well in replicating a large period of stagnation with Malthusian effects due to population growth. In fact, sustained growth begins only after 1840. The model replicates a slight decrease in the growth of technology (Figure 10c) in the transition between the first and the second Industrial Revolution at the end of the XIX$^{th}$ century and beginning of the XX$^{th}$ and a productivity growth slowdown after 1960, roughly corresponding to the productivity slowdown usually verified after the 1970s. It is worth noting that the productivity slowdown shown in Figure 10c is consistent with an ever increasing labor in R&D activities (see also Figure 9e). The growth rate of physical capital (Figure 10a) also slows down between 1940 and 1960 (roughly corresponding to the war period), while the growth rate of learning drops (Figure 10d) are more pronounced after 1920 and 1960 (including the historical periods of the Great Depression and World War II, respectively). These different growth rates implied an always increasing *per capita* output growth rate from 1840 and 1940, which then tends to slow down until 2000 (Figure 10b).

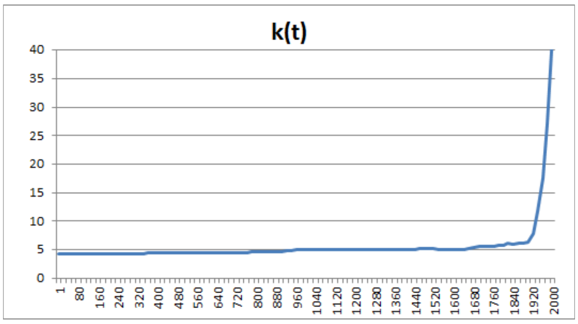
When comparing the model series with data series, which we show in Figure 11 for output *per capita* and growth rate of output *per capita*, we conclude that the model behaves quite well in predicting the timing of the take-off to modern growth and also the levels of output *per capita*. In fact, Figures 11a and 11c show that the evolution of output *per capita* in the model is in between the UK data and the US data from the Maddison Project and almost mimics the evolution of the UK data from Clark (2009). Also in growth rates (Figures 11b and 11d), the model replicates the evolution in the data quite well, with the take-off in the middle of the XIX$^{th}$ century and the slowdown after the 1960s. It is worth noting that the smoothing in growth happens close to the period in which empirical series also tend to stabilize. Note that this economy displays a very small scale effect, as the limit of $\chi(A)$ is 0.956296. This also means that the economy will not stabilize *per capita* growth if population remains growing. A slight change in the value of $q$ to 1.919 (or $b$ to 0.969) would imply that the limit of $\chi(A)$ would be one and no scale effect would be present in the current economy. Thus, growth would stabilize even with population growth. As we wish

[13] The defeat of the invincible Armada occurred in 1588.

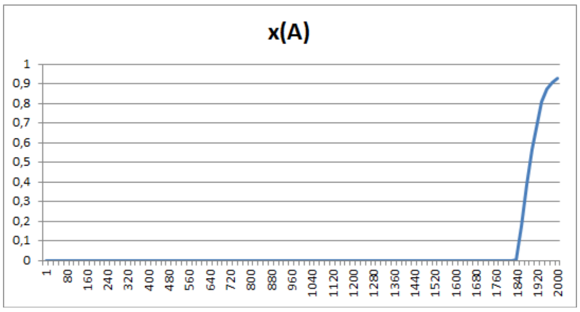
to remain tied to the empirical estimates of the complexity effect parameters, we present the results with the precise values that come from the calibration exercise. However, note that a (slight) change in these parameters that would yield a fully-endogenous steady-state would not change our transitional dynamics results. Additionally, this slight change in the parameter $q$ would emphasize the productivity slowdown after the second half of the XX$^{th}$ century.[14]



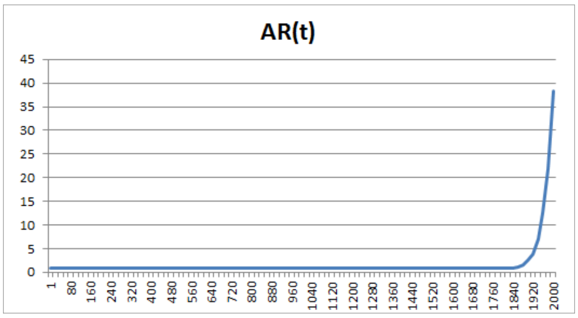

(a) Globalization Effect $g(k)$



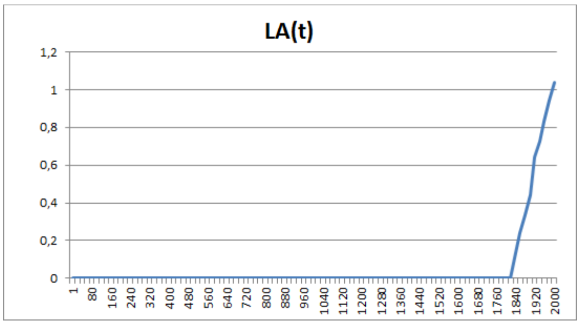

(b) Physical Capital *per capita*



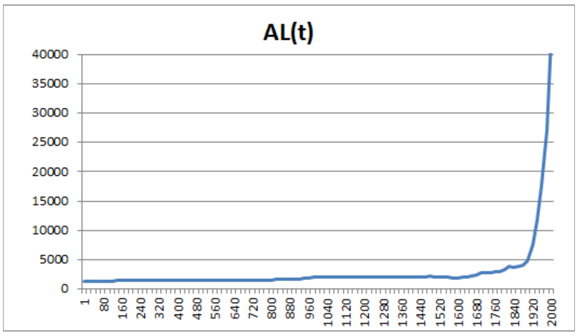

(c) Complexity Effect $\chi(A)$




(d) Technological knowledge



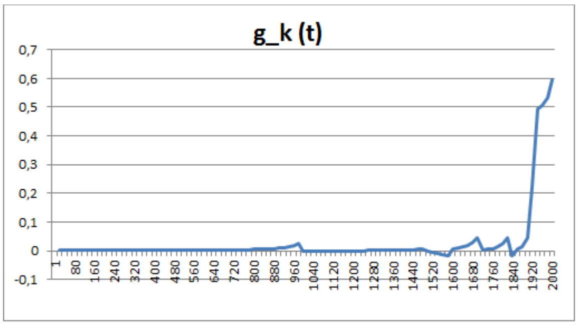

(e) Labor in R&D Activities



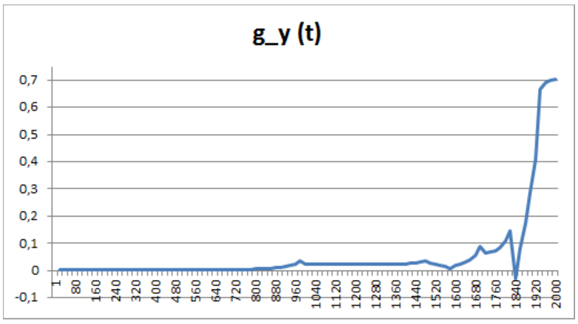

(f) Learning-by-doing knowledge

Figure 9: Evolution of the main "model" series ($\sigma = 0.169$, $q = 1.961$ and $b = 0.919$, $k_0 = 4.2$, $L_0 = 0.1$, $A_{R0} = 1$, $\bar{A} = 1000$, $\omega = 742.734390035$).

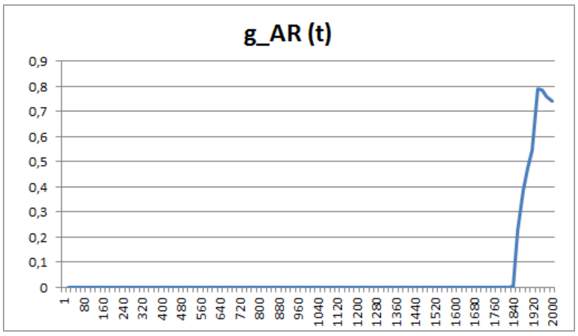
[14]Results are available upon request. In this slightly different exercise both the level and the growth rates of output *per capita* would be as close to the data as in the Figure 10.



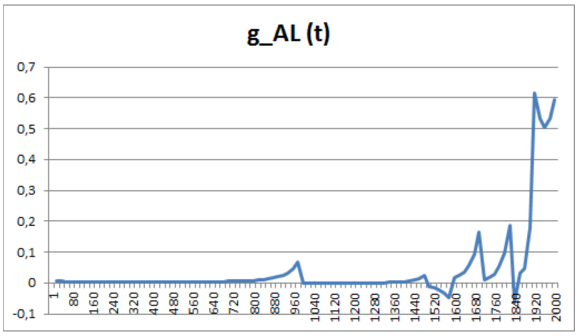

(a) Growth rate of physical capital *per capita*




(b) Growth rate of output *per capita*

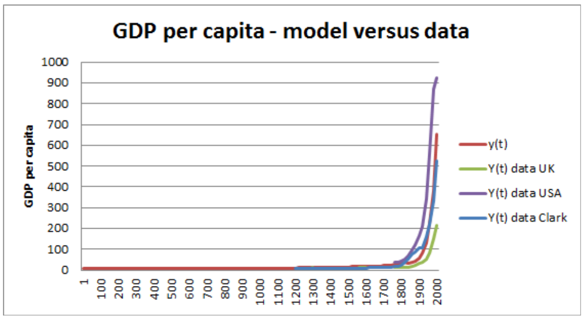



(c) Growth rate of technological knowledge

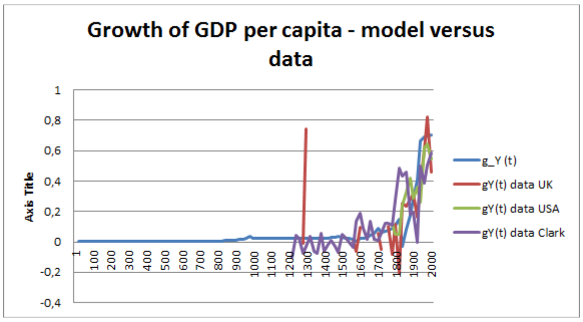



(d) Growth rate of learning-by-doing knowledge

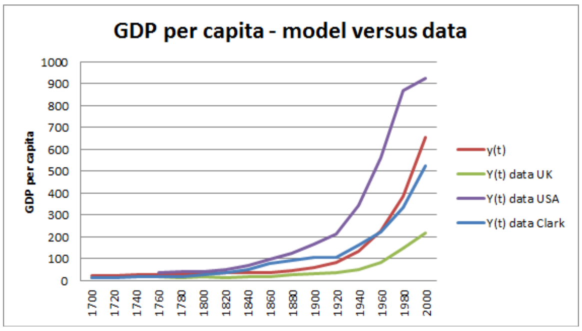
Figure 10: Evolution of the main "model" series ($\sigma = 0.169$, $q = 1.961$ and $b = 0.919$, $k_0 = 4.2$, $L_0 = 0.1$, $A_{R0} = 1$, $\bar{A} = 1000$, $\omega = 742.734390035$).



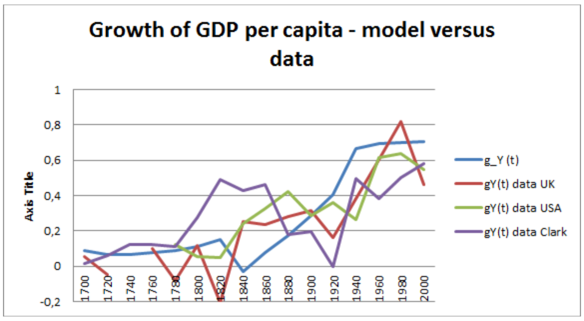

(a) Output *per capita* (A.D. 1 - A.D. 2000)




(b) growth rate of output *per capita* (A.D. 1 - A.D. 2000




(c) Output *per capita* (A.D. 1700 - A.D. 2000)




(d) growth rate of output *per capita* (A.D. 1700 - A.D. 2000)

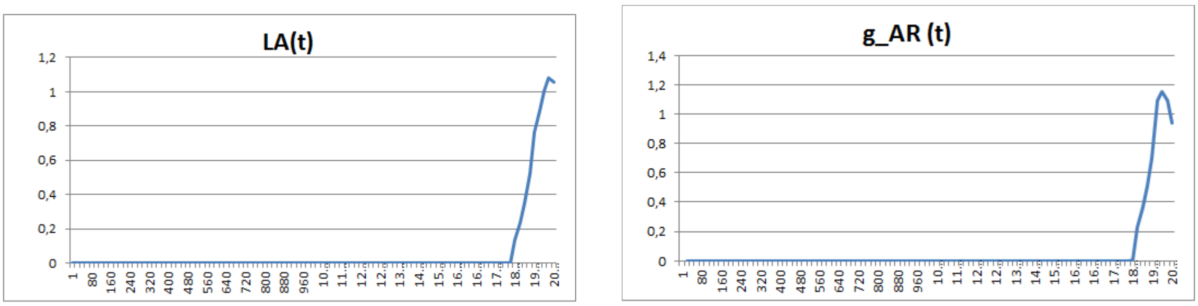
Figure 11: Evolution of the main "model" series (blue line series) (same as in Figure 10) and comparison with "data" series for the USA (source: Maddison Project, Bolt and Zaden, 2014, green line series), for the UK (source: Maddison Project, Bolt and Zaden, 2014, red line series) and for the UK (source: Clark, 2009, purple line series); output *per capita* in the data is set equal to the value in the model for the first observation available in the data (for the UK, Maddison Project the first observation is A.D. 1; for the UK, Clark data the first observation is A.D. 1200; for the USA, Maddison Project the first observation is A.D. 1640).

In the following subsection, we replicate a more pronounced productivity slowdown and simultaneously approximate the predicted series for GDP *per capita* to the data series for the US.

### 7.3.1 The Productivity Slowdown

In order to replicate a more pronounced productivity slowdown in the post-1970 decade, we change the calibration of the complexity effect. This is intended also to show the flexibility of our formulation to replicate different historical scenarios, especially those linked with productivity evolution and the take-off to the modern growth regime based on physical capital accumulation and R&D. In fact, Figure 12 shows that both labor allocated to R&D and productivity growth linked with technological knowledge decrease after 1960. Figure 12b shows that the growth rate of technological knowledge in 2000 would be as low as the 1940s' values. As in the benchmark model, the adjustment in the complexity function introduced in this paper is sufficient to mimic the productivity slowdown in the last quarter of the XX$^{th}$ century. This phenomenon is more pronounced in this extension than in the previous one because in this case the complexity effect $\chi(A)$ approaches one rapidly and does not tend to become constant, as occurred in the previous simulation. The evolution of other variables and the take-off timing are similar to those presented in Figures 9 and 10, so we do not reproduce the time-series for the other variables.

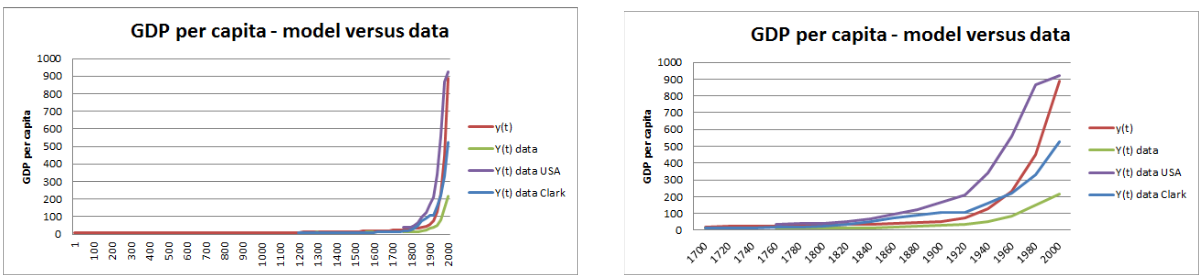
Figure 13 compares the evolution of output *per capita* in the data and in the model. We can thus observe that the model is quite precise in mimicking the evolution of GDP *per capita* both when looking at the larger time span from A.D. 1 to A.D. 2000 and when we focus on the period post-1700.




(a) labor allocated to R&D.

(b) growth rate of technological knowledge.

Figure 12: The productivity slowdown ($\sigma = 0.169$, $q = 1$ and $b = 0.21$, $k_0 = 4.2$, $L_0 = 0.1$, $A_{R0} = 1$, $\bar{A} = 1000$, $\omega = 742.734390035$).




(a) Output *per capita* (A.D. 1- A.D. 2000).

(b) Output *per capita* (A.D. 1700 - A.D. 2000).

Figure 13: The productivity slowdown: Evolution of the main "model" series (blue line series) (same as in Figure 10) and comparison with "data" series for the USA (source: Maddison Project, Bolt and Zaden, 2014, green line series), for the UK (source: Maddison Project, Bolt and Zaden, 2014, red line series) and for the UK (source: Clark, 2009, purple line series); output *per capita* in the model is made equal to the value in data for the first observation available in data (for the UK, Maddison Project the first observation is A.D. 1; for the UK, Clark data the first observation is A.D. 1200; for the USA, Maddison Project the first observation is A.D. 1640).

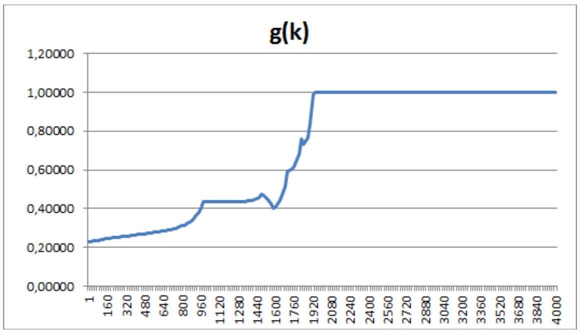
### 7.4 Futuristic Scenarios

In this section we extend the previous scenarios to the future and conclude that the model predicts a significant slowdown of the economy (in *per capita* output) after 2000. In fact, growth rates will only tend to increase again in around 2200. According to demographic projections, population growth rates will tend to stagnate as fertility rates decrease and tend to stabilize at around 2.1 children per woman. We use Maddison's (2008) projections for population until 2030 (using population for the same countries as before) and then steadily decrease population growth by 0.01 percentage points in each 20-year period until it reaches zero in 3020. Figure 14 shows the results, plotting the main variables and emphasizing the growth rates time path. The pattern for the globalization effect $g(k)$ almost mimics that presented in Figure 9a with the level of globalization remaining constant at one after 1940. Research effort will increase until the year 3000, when it will stabilize. The complexity effect $\chi(A)$ stabilizes around 2000, as already mentioned when describing Figure 9. The most interesting features of this simulation are related to the evolution of growth rates. In fact, after the physical-capital growth rate recovery after World War II, it will reach the maximum in 2020 and then decrease steadily until 2160 (this corresponds to a drop of the annual growth rate from 2.36% to 2.20%). After 2160 and until 3000, physical capital will increase steadily from an annual average of 2.20% to 2.52%, after which this growth rate will decrease marginally until it stabilizes (see Figure 14b). After a decrease in technological knowledge ($A_R$) growth after 1960 (annual growth rate of 2.94%), that will continue until 2120, there will be a recovery until the year 3000, from an annual growth rate of 2.74% to 3.21% (see Figure 14d). Due to the growth pattern of physical capital (Figure 14b) and learning-by-doing (not shown) which are very similar, the *per capita* output growth rate (Figure 14f), after lowering down between 1960 and 2000, it will decrease from 2.70% (in 2000) to 2.46% (in 2220) and then recover to 2.56% around the year 2660, and then will again slightly decrease to 2.52% around the year 3000.

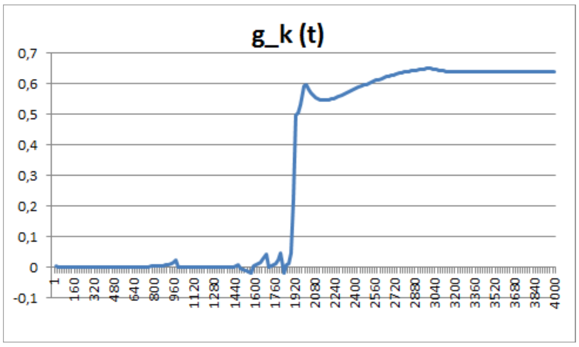
However, for an economy that attained the best growth rates in 2000 (for instance the value for the UK in the 1980s generation, reaching an annual average of 3% according to the Maddison Project data), the economy should behave more like the simulation in Section 7.3.1, which also replicates well the productivity slowdown after the 1970s. If we extend this simulated economy to the future, assuming that when $\chi(A)$ reaches one it will remain there, we obtain more pronounced oscillations in the growth rates. In this case, the output *per capita* growth rate will experience a huge and continuous drop from 2000 to around 2900 from an annual growth rate of near 3.47% to 1.88% (tending then to 1.87%, the steady-state value) – see Figure 15f. This drop is explained by a productivity slowdown due to R&D from 1960 to 2040 (Figure 15d), which stabilizes afterwards, and a slower fall in the physical capital and learning growth rates, lasting from the year 2000 to nearly the year 3000. Figure 15 shows the evolution of these variables in this alternative exercise.

Finally, if we are willing to allow for negative scale effects in the future, we may relax the assumption according to which when $\chi(A)$ reaches one, it remains there indefinetly, and instead let $\chi(A)$ grow to levels higher than 1, as it happens if $b > 1 - q$ or $q = 1$. In this case, the evolution of the economy will be much worse, with the growth rate dropping to zero as the population growth also goes to zero (near the year 3000). Moreover, R&D will end in the year 2160 and as a consequence technological knowledge will stabilize. Thus, the economy would return to a regime in which growth is driven solely by physical-capital accumulation and knowledge driven by learning. This should occur between 2160 and 3000, after which the economy would stagnate. This scenario is presented in Figure 16. Such a scenario seems to be excluded by our empirical estimates of the parameters of the complexity effect function, which, on the contrary, seems to point to a value of $\chi(A)$ that tends to one or less.

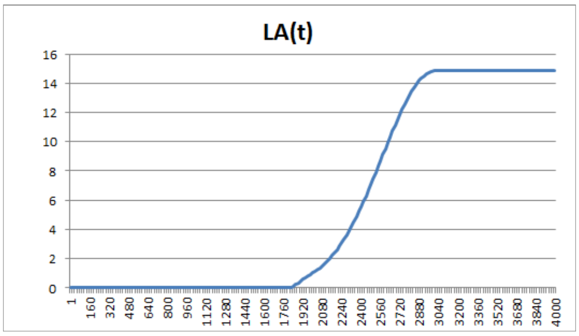
Interestingly, whatever the scenario, the model may explain the decreasing growth of the world economy after 2000, which is due entirely to the complexity effect (note that the simulations in Strulik (2014) did not show this effect). According to the model, after the decrease in technological knowledge growth



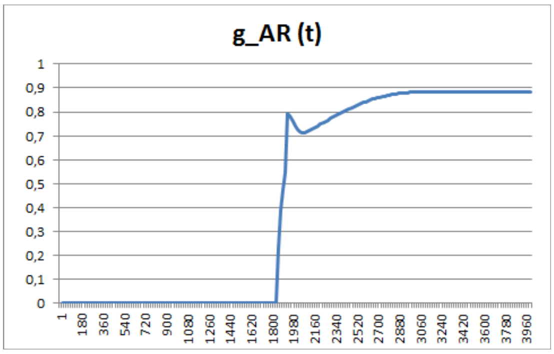

(a) Globalization Effect $g(k)$



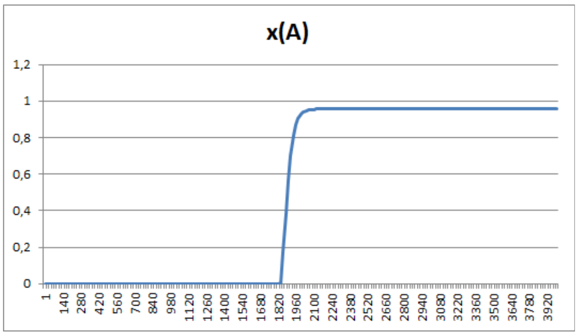

(b) Growth rate of Physical Capital per capita



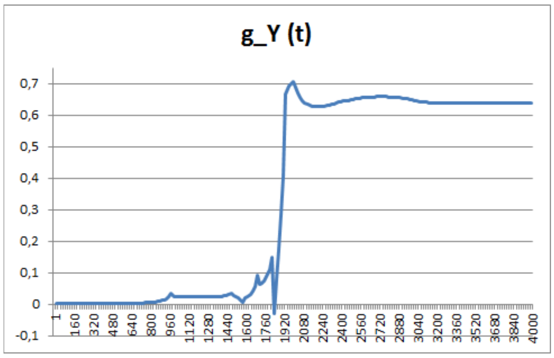

(c) Labor in R&D Activities




(d) Growth rate of Technological knowledge



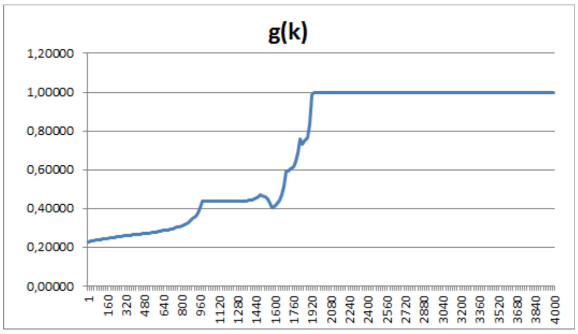

(e) Complexity Effect $\chi(A)$



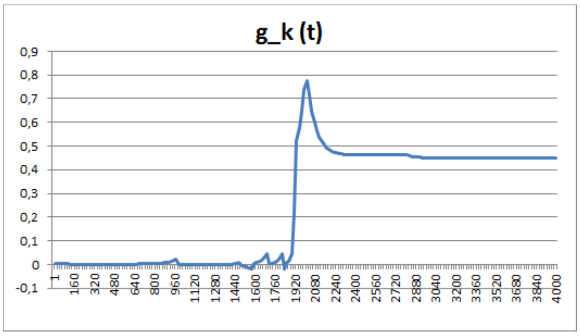

(f) Growth rate of Output *per capita*

Figure 14: Evolution of the main "model" series ($\sigma = 0.169$, $q = 1.961$ and $b = 0.919$, $k_0 = 4.2$, $L_0 = 0.1$, $A_{R0} = 1$, $\bar{A} = 1000$, $\omega = 742.734390035$.)

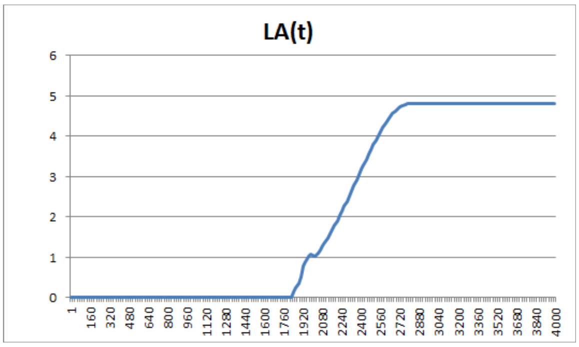
after the 1960s, physical capital and learning-by-doing, and eventually output *per capita* experience a subsequent fall in growth rates after the year 2000. In the model the process of learning-by-doing is almost entirely determined by the evolution of physical capital, due to the network effect, thus resulting in very similar time paths for the two variables. The R&D productivity slowdown is a consequence of the rise in the complexity effect and happens in the generation that follows the maximum value for the complexity effect. This slowdown affects income growth, which thus affects savings and physical-capital accumulation, eventually leading to its slowdown and drop after the year 2000. As the network effect $g(k)$ stabilizes before this period, this will no longer affect R&D and thus R&D productivity will stabilize. Although the slowdown of the 2000s has been attributed mainly to financial frictions and inefficiencies (which our stylized model does not incorporate), some authors began to question the sole financial roots of the crisis. For instance, in Kasparov *et. al.* (2012), the authors point out that the collapse of advanced-country growth is not merely a result of the financial crisis. At its root, they argue, these countries' weakness reflects secular stagnation in technology and innovation. Most recently, Brinca *et. al.* (2016) – in a forthcoming chapter in the Handbook of Macroeconomics – had pointed to the efficiency wedge (measured by $A$ in the final good production function) as the main source of the great recession at the



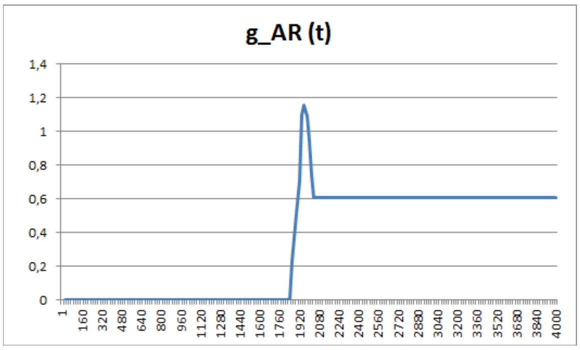

(a) Globalization Effect $g(k)$



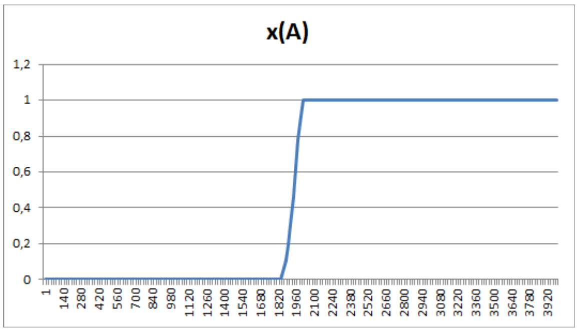

(b) Growth rate of Physical Capital *per capita*



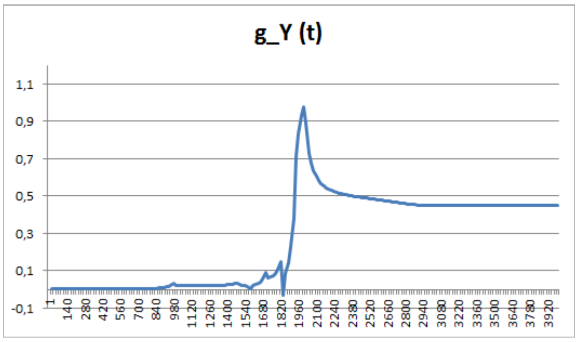

(c) Labor in R&D Activities




(d) Growth rate of Technological knowledge




(e) Complexity Effect $\chi(A)$



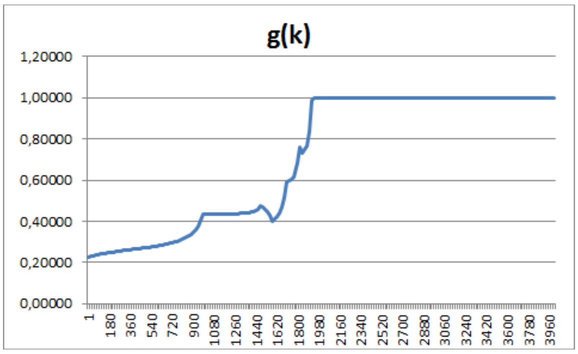

(f) Growth rate of Output *per capita*

Figure 15: Evolution of the main "model" series ($\sigma = 0.169$, $q = 1$ and $b = 0.21$, $k_0 = 4.2$, $L_0 = 0.1$, $A_{R0} = 1$, $\bar{A} = 1000$, $\omega = 742.734390035$, $max(\chi(A)) = 1$).

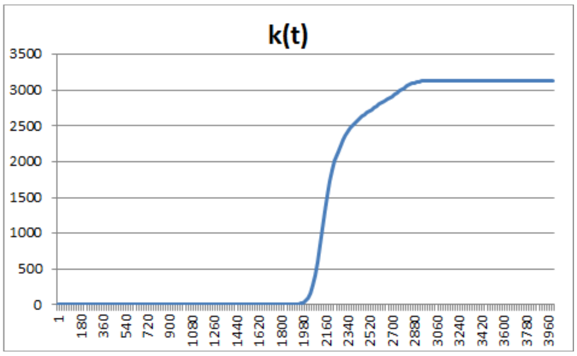
end of the 2000s' decade.

# 8 Conclusions

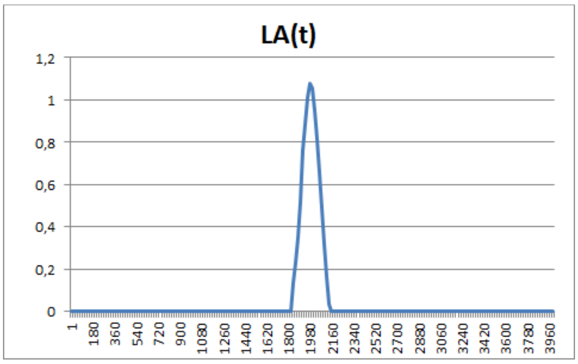
We introduce the concept of entropy into the endogenous growth theory to describe the complexity effect that tends to dilute the market-size effect in fully developed economies. Firms in fully developed economies face the intensive use of producer services, such as equipment repair and maintenance, transportation and communication services, engineering and legal supports, accounting, advertising, and financial services. At this stage, complexity in dealing with such a variety of services may exert as much pressure as the positive effect of the market size. The concept of entropy fits well into a complexity effect that is state-dependent and time-varying. We relate the complexity function to the number of varieties that firms in the industrialized world have to deal with. Thus, as it seems to be suggested by empirical evidence, this allows for the market-size or scale effects to be present in historical periods and to slowly vanish until the present day economies. We test this hypothesis against the available data and show that the entropy function, as a complexity effect, adjusts well to the complexity effect estimated by taking the R&D functional forms



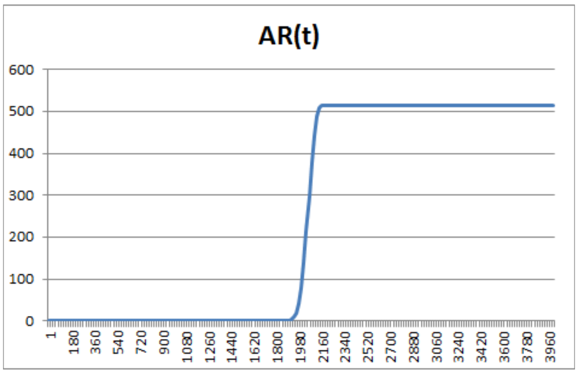

(a) Globalization Effect $g(k)$



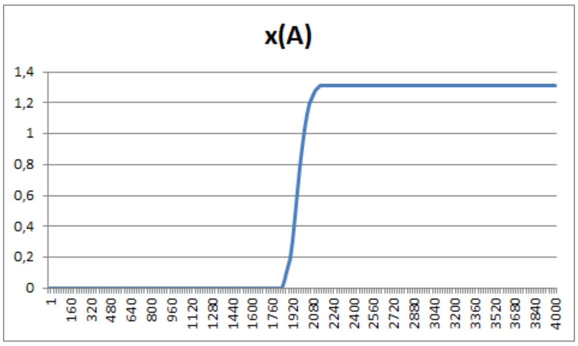

(b) Physical Capital per capita



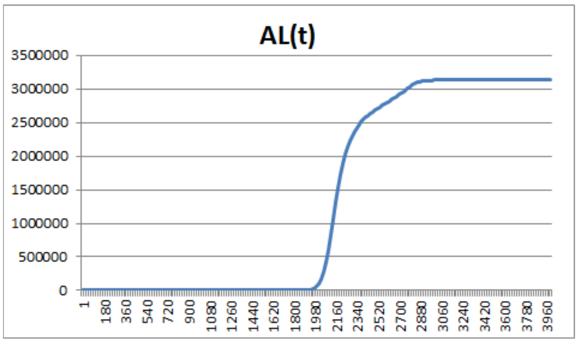

(c) Labor in R&D Activities




(d) Technological knowledge




(e) Complexity Effect $\chi(A)$




(f) Learning-by-doing knowledge

Figure 16: Evolution of the main "model" series ($\sigma = 0.169$, $q = 1$ and $b = 0.21$, $k_0 = 4.2$, $L_0 = 0.1$, $A_{R0} = 1$, $\bar{A} = 1000$, $\omega = 742.734390035$).

of the model to the data. The estimated complexity effect is increasing and reaches values close to one near the year 2000, which tends to support the "fully-endogenous growth theory"without (or with very small) scale effects, as a good description of the fully developed economy.

We devise an R&D-based endogenous growth model with a state-dependent time-varying complexity effect and formally obtain its transitional dynamics. Then, we use the adjustment of the complexity effect to the data to calibrate the parameters that shape that function. Together with literature references and the data, we fully calibrate the model and simulate its transitional dynamics for the period after the Industrial Revolution. We note that the state-dependent time-varying effect linked with the concept of entropy is essential to replicate the post-Industrial Revolution evolution of the economy, both in levels and in growth rates. In particular, it replicates well the evolution of GDP *per capita* and growth rates in the UK, as well as a varying TFP growth, which can mimic the productivity slowdown in the last part of the XX$^{th}$ century. Furthermore, the simulated model replicates the final values of the complexity effect obtained from the empirical estimates.

Finally, we devise a model that intends to replicate the very long-run evolution of the economy from A.D. 1 to A.D. 2000. We consider a spillover or network effect that is state-dependent (borrowed from Strulik, 2014) together with our complexity effect based on the entropy concept. The model's transitional dynamics are derived. Then, we calibrate it to simulate the evolution of the economy. We discover that the model does well in predicting a take-off from the "Malthusian" stage to the post-industrial revolution stage. It is important to note that the globalization effect simulation fits well on historical evolution, with a stagnation during the middle-ages, a first push with the European expansion and empires in the XVII$^{th}$ centuries and a final rise during the final part of the XIX$^{th}$ century and the eve of the XX$^{th}$ century. The complexity effect rises with the intentional R&D activity that emerged in the XIX$^{th}$ century. We show that the state-dependent time-varying effect led by entropy is again essential to replicate a productivity slowdown in the final part of the last century. Additionally, the model predicts well the evolution of the levels of GDP *per capita* and their growth rates. Some futuristic scenarios show that the model accommodates very different evolutions of the future economy. If the model replicates a mild growth rate in 2000 (similar to those in the UK), then the evolution should be positive, after a very small adjustment following 2000, with a slow and gradual increase in growth rates in the future centuries. However, if the model replicates a high current growth rate (similar to those of the USA), then the post-2000 adjustment is more dramatic in growth rates. Finally, in a more implausible scenario in which the complexity effect rises above one, the economy may stagnate in the very long future with R&D activity tending to vanish within the next two centuries. This would imply that after that time, the economy would return to a long period in which physical capital and learning-by-doing would be the only sources of growth, such as what occurred before the intentional R&D emerged in the XIX$^{th}$ century. Whatever the scenario in the future, the state-dependent time-varying complexity effect seems to contribute to the explanation of the growth slowdown that happened in the current 2000-2020 generation.

# References


Aghion, P., and P. Howitt (1992): "A Model of Growth Through Creative Destruction," *Econometrica*, 60(2), 323–351.

Alvarez-Pelaez, M., and C. Groth (2005): 'Too little or too much R&D?," *European Economic Review*, 49, 437-456.

Ang, J. B., and J. B. Madsen (2015): "What Drives Ideas Production Across the World?," *Macroeconomic Dynamics*, 19, 79–115.

Atkinson, A. B. (1970): "On the Measurement of Inequality," *Journal of Economic Theory*, 2(3), 244–263.

Baier, S. L., G. Dwyer, and R. Tamura (2006): "How Important are Capital and Total Factor Productivity for Economic Growth?," *Economic Inquiry*, 44 (1), 23–49.

Barro, R., and X. Sala-i-Martin (2004): *Economic Growth.* Cambridge, Massachusetts: MIT Press, second edn.

Bolt, J., and J. van Zaden (2014): 'The Maddison Project: collaborative research on historical national accounts," *The Economic History Review*, 67(3), 627-651. Data available at http://www.ggdc.net/maddison/maddison-project/orihome.htm (accessed in December, 26, 2015).

Bottazzi, L., and G. Peri (2007): "The international dynamics of RD and innovation in the long-run and in the short-run," *The Economic Journal*, 117, 586-511.

Brinca, P., V. Chari, P. Kehoe, and E. McGratten (2016): "Accounting for Business Cycles with Wedges," *Research Department Staff Report 528*, March, Federal Reserve Bank of Minneapolis, forthcoming in the *Handbook of Macroeconomics*, second edn (Editors: Harald Uhlig and John Taylor).

Ciccone, A., and K. Matsuyama (1996): "Start-up costs and pecuniary externalities as barriers to economic development," *Journal of Development Economics*, 49, 33–59.

Clark, G. (2009): "The Macroeconomic Aggregates for England, 1209-2008 ," *Working Papers, University of California, Department of Economics*, 09, 19. Available at http://hdl.handle.net/10419/58383 (accessed December 26, 2015).

Dinopoulos, E., and P. Thompson (1998): "Schumpeterian Growth Without Scale Effects," *Journal of Economic Growth*, 3 (December), 313–335.

——— (1999): "Scale Effects in Schumpeterian Models of Economic Growth," *Journal of Evolutionary Economics*, 9, 157–185.

——— (2000): "Endogenous Growth in a Cross-Section of Countries," *Journal of International Economics*, 51, 335–362.

Etro, F. (2008): "Growth Leaders," *Journal of Macroeconomics*, 30, 1148–1172.

Feenstra, R. C., R. Inklaar, and P. Timmer (2015): "The Next Generation of the Penn World Table," *American Economic Review*, 105(10): 3150-82. Data available at www.ggdc.net/pwt (accessed in December 2015, 26).

Galor, O. (2005): "From Stagnation to Growth: Unified Growth Theory," in *Handbook of Economic Growth, Volume 1*, ed. by P. Aghion, and S. Durlauf, pp. 171–293. Amsterdam: Elsevier.

Galor, O., and O. Moav (2002): "Natural Selection and the Origin of Economic Growth," *Quartely Journal of Economics*, 117, 1133–1191.

Galor, O., and D. Weil (2000): "Population, Technology, and Growth: From the Malthusian Regime to the Demographic Transition and Beyond," *American Economic Review*, 90, 806–828.

Gil, P. M., P. Brito, and O. Afonso (2013): "Growth and Firm Dynamics with Horizontal and Vertical R&D," *Macroeconomic Dynamics*, 17 (7), 1438–1466.

Ha, J., and P. Howitt (2007): "Accounting for Trends in Productivity and R&D: A Schumpeterian Critique of Semi-Endogenous Growth Theory," *Journal of Money Credit and Banking*, 39, 733–774.

Howitt, P. (1999): "Steady Endogenous Growth with Population and R&D Inputs Growing," *Journal of Political Economy*, 107(4), 715–730.

Jones, C. (1995): "R&D Based Models of Economic Growth," *Journal of Political Economy*, 103, 759–784.

Jones, C., and J. Williams (2000): "Too Much of a Good Thing? The Economics of Investment in R&D," *Journal of Economic Growth*, 5, 65–85.

Kasparov, G., and M. Levchin and P. Thiel (2012): *The Blueprint: Reviving Innovation, Rediscovering Risk, and Rescuing the Free Market* . W. W. Norton  Company.

Kremer, M. (1993): "Population and Growth: One Million BC to 1990," *Quarterly Journal of Economics*, (108), 681–716.

Laincz, C. A., and P. F. Peretto (2006): "Scale Effects in Endogenous Growth Theory: An Error of Aggregation Not Specification," *Journal of Economic Growth*, 11, 263–288.

Li, C.-W. (2000): "Endogenous vs. Semi-Endogenous Growth in a Two-R&D-Sector Model," *Economic Journal*, 110, No. 462, Conference Papers (March), C109–C122.

Maddison (2007): *Contours of the World Economy 1-2030 AD*. Oxford University Press. Data available in http://www.ggdc.net/MADDISON/oriindex.htm, (accessed in December, 26, 2015).

Madsen, J. B. (2008): "Semi-Endogenous versus Schumpeterian Growth Models: Testing the Knowledge Production Function Using International Data," *Journal of Economic Growth*, 13, 1–26.

Marco, A. C., M. Carley , S. Jackson , and A. Myers (2015): "The USPTO Historical Patent Data Files," *U.S. Patent and Trademark Office Working Paper*, 2015-1. Available at http://www.uspto.gov/economics. Data available at. http://www.uspto.gov/learning-and-resources/electronic-data-products/historical-patent-data-files. (Accessed January 15, 2016).

Patil, G. P., and C. Taillie (1982): "Diversity as a Concept and its Measurement," *Journal of the American Statistical Association*, 77(379), 548–561.

Mokyr, J. (2005): "Long-term Economic Growth and the history of technology," *Handbook of Economic Growth*, Volume 1B: chapter 17. Elsevier, Editors: Philippe Aghion and Steven N. Durlauf.

Peretto, P. (1998): "Technological Change and Population Growth," *Journal of Economic Growth*, 3 (December), 283–311.

——— (2015): "From Smith to Schumpeter: A Theory of Take-Off and Convergence to Sustained Growth," *European Economic Review*, 78, 1–26.

Pessoa, A. (2005): " "Ideas" driven growth: the OECD evidence," *Portuguese Economic Journal*, 4 (1), 46–67.

Porter, M., and S. Stern (2000): 'Measuring the "Ideas" Production Function: Evidence from International Patent Output," *NBER Working-Paper*, 7891.

Romer, P. M. (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98(5), 71–102.

Sedgley, N., and B. Elmslie (2010): "Reinterpreting the Jones Critique: A Time Series Approach to Testing and Understanding Idea Driven Growth Models with Transitional Dynamics," *Journal of Macroeconomics*, 32, 103–117.

Segerstrom, P. (1998): "Endogenous Growth Without Scale Effects," *American Economic Review*, 88 (5), 1290–1310.

Shannon, C. E. (1948): "A Mathematical Theory of Communication," *Bell System Technical Journal*, 27, 379–423 & 623–656.

Strulik, H. (2009): "Knowledge and Growth in the Very Long Run," *Hannover University Discussion Paper*, 414, 1–32.

——— (2014): "Knowledge and Growth in the Very Long Run," *International Economic Review*, 55(2), 459–482.

Tsallis, C. (1988): "Possible Generalization of Boltzmann-Gibbs Statistics," *Journal of Statistical Physics*, 52(1/2), 479–487.

Zachariadis, M. (2003): "R&D, Innovation, and Technological Progress: A Test of the Schumpeterian Framework Without Scale Effects," *Canadian Journal of Economics*, 36, 566–586.

# A Appendix: the model with international knowledge diffusion

Using (16), (21) and the associated free-entry condition $w_t \frac{L_t^{\chi(A_t)}}{\delta A_t^g (A_t^w)^\mu} = \pi_t$, and following the steps as in Section 5 (using again equations (15)), we first obtain the share of labor allocated to R&D and the final good production as:

$$l_t^Y = \min\left\{1, \frac{1}{\alpha\delta}\frac{1}{L_t^{1-\chi}}\frac{A_t^{1-g}}{(A_t^w)^\mu}\right\} \quad ; l_t^A = \max\left\{0, 1 - \frac{1}{\alpha\delta}\frac{1}{L_t^{1-\chi}}\frac{A_t^{1-g}}{(A_t^w)^\mu}\right\} \tag{A.1}$$

Inserting (13) into (14), then replacing $w_t$ with expression (15) and finally using (A.1) and (16), we obtain the difference equation for physical capital as follows:

$$\Delta k_t = \bar{a}\frac{(A_t)^{\sigma-\alpha(1-g)}(A_t^w)^{\alpha\mu}L_t^{\alpha(1-\chi(A_t))}k_t^\alpha}{1+n} - k_t \tag{A.2}$$

where $\bar{a} = \frac{\beta(\alpha\delta)^\alpha(1-\alpha)}{1+\beta}$ .

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