

R. Chertovskih,^{a,b} A.C.-L. Chian,^{c,d,e} O. Podvigina,^a
E. Rempel,^{d,e} V. Zheligovsky^a

^a*Institute of Earthquake Prediction Theory and Mathematical Geophysics,
Russian Academy of Sciences,*

84/32 Profsoyuznaya St., 117997 Moscow, Russian Federation

^b*Centre for Wind Energy and Atmospheric Flows,
Faculdade de Engenharia da Universidade do Porto,
Rua Dr. Roberto Frias s/n, 4200-465 Porto, Portugal*

^c*Paris Observatory, LESIA, CNRS, 92195 Meudon, France*

^d*Institute of Aeronautical Technology (IEFM/ITA)
and World Institute for Space Environment Research (WISER),
São José dos Campos, São Paulo 12228-900, Brazil*

^e*National Institute for Space Research (INPE),
P.O. Box 515, São José dos Campos, São Paulo 12227-010, Brazil*

Existence, uniqueness and analyticity of space-periodic solutions to the regularised long-wave equation

Abstract

We consider space-periodic evolutionary and travelling-wave solutions to the regularised long-wave equation (RLWE) with damping and forcing. We establish existence, uniqueness and smoothness of the evolutionary solutions for smooth initial conditions, and global in time spatial analyticity of such solutions for analytical initial conditions. The width of the analyticity strip decays at most polynomially. We prove existence of travelling-wave solutions and uniqueness of travelling waves of a sufficiently small norm. The importance of damping is demonstrated by showing that the problem of finding travelling-wave solutions to the undamped RLWE is not well-posed. Finally, we demonstrate the asymptotic convergence of the power series expansion of travelling waves for a weak forcing.

1 Introduction

The regularised long-wave equation (RLWE), also known as the Benjamin–Bona–Mahony (BBM) equation, is a model for the propagation of one-dimensional, unidirectional small-amplitude long waves in nonlinear dispersive me-

dia, being of great interest in the study of propagation of long waves in shallow waters [10] such as tsunami driven by an earthquake [39] and drift waves in a controlled nuclear fusion plasma [17,19]. It was first derived by Peregrine [32], then by Benjamin et al. [3], as an alternative to the Korteweg-de Vries (KdV) equation [10], in response to mathematical difficulties associated with the KdV equation, such as the existence and stability of solutions and other problems related to the dispersion term [3,34]. The RLWE was later derived by He & Salat [17] as a model for nonlinear drift waves in plasmas, with a periodic driving term and a linear damping term introduced ad hoc to study transition to chaos.

The understanding of the evolution of nonlinear physical systems such as the RLWE requires a combined effort of numerical and analytical studies. Numerically simulated nonlinear evolution of a driven-damped RLWE, under the forcing of a periodic wave, has been analyzed in a series of papers. He & Chian [14] discovered a new type of synchronization, the so-called on-off collective imperfect phase synchronization, in the turbulent state of RLWE solutions. In the driver frame, solutions to the RLWE can be represented as a set of coupled oscillators in Fourier space. As the system evolves in time, the oscillators in different spatial scales intermittently adjust themselves to collective imperfect phase synchronization, inducing strong bursts in the wave energy. Rempel & Chian [33] demonstrated that non-attracting chaotic sets known as “chaotic saddles” are responsible for transient and intermittent dynamics in the RLWE. As the driver amplitude is increased, the system undergoes a transition from quasiperiodicity to temporal chaos, then to spatiotemporal chaos. The resulting time series in the spatiotemporal chaos regime display random switching between laminar and bursty phases. Rempel & Chian [33] identified temporally and spatiotemporally chaotic saddles which are responsible for the laminar and bursty phases, respectively. Prior to the transition to permanent spatiotemporal chaos, a spatiotemporally chaotic saddle is responsible for chaotic transients that mimic the dynamics of the post-transition attractor. Chian et al. [7] applied the Fourier-Lyapunov analysis to prove the duality of amplitude and phase synchronization in the RLWE due to multi-scale interactions in chaotic saddles at the onset of permanent spatiotemporal chaos. By computing the power-phase spectral entropy and the time-averaged power-phase spectra, they showed that the laminar/bursty states in the on-off spatiotemporal intermittency correspond, respectively, to the chaotic saddles with higher/lower degrees of amplitude-phase synchronization across spatial scales.

From an analytical perspective, several works have presented studies on the existence, uniqueness and stability of solutions to the RLWE. In the seminal paper [3], Benjamin et al. proved the existence and uniqueness of nonperiodic solutions to the initial-value problem for the RLWE in \mathbb{R}^1 . While for the initial data in Sobolev spaces $H^s(\mathbb{R}^1)$ for $s \geq 0$ this problem for the RLWE

is well-posed [5], it is ill-posed for $s < 0$ [31]. The stability of solitary-wave solutions to the RLWE was shown by Bona [4]; existence and stability of such solutions to the generalized BBM equation is examined in [42] (see also references therein). For the generalised RLWE with an arbitrary nonlinearity and a stronger damping described by the Laplacian, space-periodic solutions have strong finite-dimensional global attractors [40,41] (see also [8,38,37]) in the Sobolev spaces $H^1(\mathbb{T}^1)$ and $H^2(\mathbb{T}^1)$; the attractors consist of real analytical solutions [9]. Jafari et al. [20] (see also [11]) found exact travelling-wave solutions to the RLWE using the simplest equation method [22,23].

All the aforementioned papers examine the RLWE in its original form, without the additional damping term introduced by He and Salat [17]. The goal of the present paper is to present the mathematical theory of space-periodic solutions to the driven-damped RLWE. We begin by proving existence, uniqueness (section 2) and spatial analyticity (section 3) of space-periodic evolutionary solutions to the RLWE. In section 4 we show the existence of travelling-wave solutions to the damped RLWE, as well as uniqueness of solutions whose norm does not exceed a certain threshold (and hence a travelling-wave solution is unique, provided the forcing is sufficiently weak). In section 5 we construct, in the form of infinite power series in the inverse wave speed, a family of fast space-periodic travelling waves that are formal asymptotic solutions to the zero-force RLWE without damping. We do not prove convergence of these asymptotic power series; by construction, upon truncation, the series represent travelling-wave solutions to the undamped RLWE with some weak forcing, whose amplitude can be of the order of any negative power of the wave speed. This shows that in the absence of damping, finding travelling-wave solutions to the RLWE is not a well-posed problem. The amplitude of forcing in numerical investigations [13,14,15,16,17,33,34,39] of the RLWE was small. This has suggested to consider the asymptotic expansions of solutions for a weak forcing; we do this in section 6.

2 Existence and uniqueness of evolutionary solutions

In this section we consider evolutionary solutions to the RLWE:

$$\frac{\partial}{\partial t}(\varphi - a\varphi'') + b\varphi' + c\varphi\varphi' + d\varphi + e(x, t) = 0, \quad (1)$$

where $'$ denotes differentiation in $x \in \mathbb{R}^1$, a, b, c and d are real constants, and $a > 0$. The forcing $e(x, t)$ is prescribed.

Existence and uniqueness of the classical solutions to the forced BBM equation (aka the non-damped RLWE, i.e., (1) for $d = 0$) on the entire line \mathbb{R}^1 (the domain of the x variable) was proved in [3] under the assumption that the

initial “energy” $\int_{-\infty}^{\infty} (\varphi^2 + (\varphi')^2) dx$ is finite, the forcing is continuous and has a finite Lebesgue norm. By contrast, we consider solutions periodic in x (assuming without any loss of generality that the period is 2π); existence and uniqueness of space-periodic solutions to the BBM equation without forcing was proved (using different techniques) in [28,36] (see also [29]).

Theorem 1. Suppose $\varphi_0(x) \in C^\infty(\mathbb{R}^1)$ is 2π -periodic and $e(x, t) \in C^\infty(\mathbb{R}^1 \times \mathbb{R}_+^1)$ is 2π -periodic for any $t \geq 0$. For any constants $a > 0$, b , c and d there exists a unique 2π -periodic solution to the RLWE, $\varphi(x, t) \in C^\infty(\mathbb{R}^1 \times \mathbb{R}_+^1)$, such that $\varphi(x, t)|_{t=0} = \varphi_0(x)$.

Proof exploits the general ideas involved in proofs of similar statements for equations of the hydrodynamic type (see, e.g., [25]).

i. We use the Fourier-Galerkin method and consider an approximation to the solution

$$\varphi^{(N)}(x, t) = \sum_n \hat{\varphi}_n^{(N)}(t) e^{inx},$$

where $\hat{\varphi}_n^{(N)} = 0$ for $|n| > N$. The approximate Fourier coefficients $\hat{\varphi}_n^{(N)}(t)$ satisfy the equations obtained by the orthogonal projection in $L_2([0, 2\pi])$ of the RLWE onto the subspace spanned by the Fourier harmonics e^{inx} for $|n| \leq N$:

$$(1 + an^2) \dot{\hat{\varphi}}_n^{(N)} + (d + ibn) \hat{\varphi}_n^{(N)} + ic \sum_m m \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} + \hat{e}_n(t) = 0, \quad (2)$$

$$\hat{\varphi}_n^{(N)}(t) \Big|_{t=0} = \hat{\varphi}_{0,n},$$

where the dot denotes differentiation in time, and $\hat{e}_n(t)$ and $\hat{\varphi}_{0,n}$ are the Fourier coefficients of $e(x, t)$ and $\varphi_0(x)$, respectively:

$$e(x, t) = \sum_n \hat{e}_n(t) e^{inx}, \quad \varphi_0(x) = \sum_n \hat{\varphi}_{0,n} e^{inx}.$$

We employ the seminorms $\|\cdot\|_s$ defined as follows: for $f(x) = \sum_n \hat{f}_n e^{inx}$,

$$\|f\|_s^2 = \begin{cases} \sum_n |\hat{f}_n|^2 |n|^{2s}, & s > 0; \\ \sum_n |\hat{f}_n|^2 (\max(|n|, 1))^{2s}, & s \leq 0. \end{cases}$$

For $s \geq 0$, $\|\cdot\|_0^2 + \|\cdot\|_s^2$ is the square of the norm in the Sobolev space $H^s(\mathbb{T}^1)$ of 2π -periodic functions.

ii. An energy bound, on which all our constructions are based, is obtained by multiplying (2) by $\hat{\varphi}_{-n}^{(N)} = \overline{\hat{\varphi}_n^{(N)}}$ and summing the results over all n :

$$\frac{1}{2} \frac{d}{dt} \left(\|\varphi^{(N)}\|_0^2 + a \|\varphi^{(N)}\|_1^2 \right) + d \|\varphi^{(N)}\|_0^2 = - \sum_n \hat{\varphi}_{-n}^{(N)} \hat{e}_n(t). \quad (3)$$

The sums involving constants b and c vanish, since by periodicity

$$ib \sum_n n \hat{\varphi}_n^{(N)} \hat{\varphi}_{-n}^{(N)} = \frac{b}{2\pi} \int_0^{2\pi} \varphi^{(N)} \frac{d}{dx} \varphi^{(N)} dx = \frac{b}{4\pi} \int_0^{2\pi} \frac{d}{dx} (\varphi^{(N)})^2 dx = 0$$

and

$$ic \sum_{m,n} m \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} \hat{\varphi}_{-n}^{(N)} = \frac{c}{2\pi} \int_0^{2\pi} (\varphi^{(N)})^2 \frac{d}{dx} \varphi^{(N)} dx = \frac{c}{6\pi} \int_0^{2\pi} \frac{d}{dx} (\varphi^{(N)})^3 dx = 0.$$

By Gronwall's lemma, identity (3) implies the inequality

$$\left(\|\varphi^{(N)}\|_0^2 + a \|\varphi^{(N)}\|_1^2 \right)^{1/2} \leq C_0(t)$$

(recall that $a > 0$), where

$$C_0(t) \equiv \left(\|\varphi_0\|_0^2 + a \|\varphi_0\|_1^2 \right)^{1/2} e^{\tilde{d}t} + \int_0^t \|e(x, \tau)\|_0 e^{\tilde{d}(t-\tau)} d\tau,$$

$$\tilde{d} = \begin{cases} 0, & d \geq 0, \\ |d|, & d < 0. \end{cases}$$

From this inequality we infer bounds, that are uniform in N : $\|\varphi^{(N)}\|_s \leq C_s(t)$ for $s = 0$ and 1 (we can set $C_1(t) = C_0(t)/\sqrt{a}$).

iii We derive now bounds, that are uniform in N , for $\|\varphi^{(N)}\|_s$, where $s > 0$ is arbitrary.

For $s > 1$, multiply (2) by $\hat{\varphi}_{-n}^{(N)} |n|^{2s} (1 + an^2)^{-1}$ and sum the results over n :

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varphi^{(N)}\|_s^2 &= - \sum_n \frac{(d + ibn)|n|^{2s}}{1 + an^2} \hat{\varphi}_n^{(N)} \hat{\varphi}_{-n}^{(N)} - ic \sum_{m,n} \frac{m|n|^{2s}}{1 + an^2} \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} \hat{\varphi}_{-n}^{(N)} \\ &\quad - \sum_n \frac{|n|^{2s}}{1 + an^2} \hat{\varphi}_{-n}^{(N)} \hat{e}_n(t). \end{aligned} \quad (4)$$

We bound each sum in the r.h.s. By the Cauchy-Bunyakovsky-Schwarz inequality,

$$- \sum_n \frac{d|n|^{2s}}{1 + an^2} \hat{\varphi}_n^{(N)} \hat{\varphi}_{-n}^{(N)} \leq \frac{\tilde{d}}{a} \|\varphi^{(N)}\|_s \|\varphi^{(N)}\|_{s-2}.$$

By changing the index of summation $n \rightarrow -n$, we establish

$$\sum_n \frac{ibn|n|^{2s}}{1 + an^2} \hat{\varphi}_n^{(N)} \hat{\varphi}_{-n}^{(N)} = 0.$$

To bound the third sum, note that, by the same inequality,

$$\sum_n \frac{|n|^{s+1}}{1 + an^2} |\hat{\varphi}_{-n}^{(N)}| \leq Q_1 \|\varphi^{(N)}\|_s \quad (5)$$

$$\text{for } Q_1 = \left(2 \sum_{n>0} \left(\frac{n}{1+an^2} \right)^2 \right)^{1/2} \text{ and any } s \geq 0,$$

and

$$|m| |n|^{s-1} \leq Q_{2,s}(|m|^s + |n-m|^s) \quad (6)$$

for all $m, n, s \geq 1$ and some suitable constants $Q_{2,s}$. Therefore,

$$\begin{aligned} & \left| c \sum_{m,n} \frac{m|n|^{2s}}{1+an^2} \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} \hat{\varphi}_{-n}^{(N)} \right| \\ & \leq |c| \sum_n \left(\sum_m Q_{2,s}(|m|^s + |n-m|^s) |\hat{\varphi}_m^{(N)}| |\hat{\varphi}_{n-m}^{(N)}| \right) \frac{|n|^{s+1}}{1+an^2} |\hat{\varphi}_{-n}^{(N)}| \\ & \leq 2|c| Q_{2,s} \|\varphi^{(N)}\|_s \|\varphi^{(N)}\|_0 Q_1 \|\varphi^{(N)}\|_s = Q_{3,s} \|\varphi^{(N)}\|_0 \|\varphi^{(N)}\|_s^2, \end{aligned}$$

where $Q_{3,s} = 2|c|Q_1Q_{2,s}$. Finally,

$$\left| \sum_n \frac{|n|^{2s}}{1+an^2} \hat{\varphi}_{-n}^{(N)} \hat{e}_n(t) \right| \leq \frac{1}{a} \|\varphi^{(N)}\|_s \|e(x, t)\|_{s-2}.$$

Collecting all the bounds, we obtain from (4):

$$\frac{1}{2} \frac{d}{dt} \|\varphi^{(N)}\|_s^2 \leq \frac{\tilde{d}}{a} \|\varphi^{(N)}\|_s \|\varphi^{(N)}\|_{s-2} + Q_{3,s} \|\varphi^{(N)}\|_0 \|\varphi^{(N)}\|_s^2 + \frac{1}{a} \|\varphi^{(N)}\|_s \|e(x, t)\|_{s-2},$$

i.e.

$$\frac{d}{dt} \|\varphi^{(N)}\|_s \leq \frac{\tilde{d}}{a} \|\varphi^{(N)}\|_{s-2} + Q_{3,s} \|\varphi^{(N)}\|_0 \|\varphi^{(N)}\|_s + \frac{1}{a} \|e(x, t)\|_{s-2}.$$

Using Gronwall's lemma, we deduce by induction from this inequality bounds, that are uniform in N :

$$\begin{aligned} \|\varphi^{(N)}(x, t)\|_s & \leq \|\varphi^{(N)}(x, 0)\|_s e^{-Q_{3,s} \int_0^t C_0(\tau) d\tau} \\ & \quad + \frac{1}{a} \int_0^t \left(\tilde{d} C_{s-2}(\tau) + \|e(x, \tau)\|_{s-2} \right) e^{-Q_{3,s} \int_\tau^t C_0(\tau') d\tau'} d\tau \quad (7) \end{aligned}$$

for all even $s \geq 2$. We denote the r.h.s. of (7) by $C_s(t)$. By interpolation, $\|\varphi^{(N)}(x, t)\|_s \leq C_s(t) \equiv C_{S+2}^{1-\mu}(t) C_S^\mu(t)$ holds true for any $s \geq 0$, where $S \geq 0$ is integer, $0 \leq \mu \leq 1$ and $s = (1-\mu)(S+2) + \mu S$. (The specific form of the bounding functions $C_s(t)$ is not important for our purposes.)

iv. We derive now bounds for $\|\dot{\varphi}^{(N)}\|_s$ that are uniform in N .

Multiply (2) by $\dot{\hat{\varphi}}_{-n}^{(N)}|n|^{2s}(1+an^2)^{-1}$ and sum the results over n :

$$\begin{aligned}\|\dot{\varphi}^{(N)}\|_s^2 &= -\sum_n \frac{(d+1bn)|n|^{2s}}{1+an^2} \hat{\varphi}_n^{(N)} \dot{\hat{\varphi}}_{-n}^{(N)} - 1c \sum_{m,n} \frac{m|n|^{2s}}{1+an^2} \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} \dot{\hat{\varphi}}_{-n}^{(N)} \\ &\quad - \sum_n \frac{|n|^{2s}}{1+an^2} \dot{\hat{\varphi}}_{-n}^{(N)} \hat{e}_n(t).\end{aligned}\tag{8}$$

We derive bounds for each sum in the r.h.s. for $s \geq 0$. Clearly,

$$\left| \sum_n \frac{(d+1bn)|n|^{2s}}{1+an^2} \hat{\varphi}_n^{(N)} \dot{\hat{\varphi}}_{-n}^{(N)} \right| \leq Q_4 \|\dot{\varphi}^{(N)}\|_s \|\varphi^{(N)}\|_{s-1},$$

where Q_4 is a constant such that $Q_4(1+an^2) \geq \max(1, |n|)(|d| + |b||n|)$ for all n . For $s = 0$, the second sum can be bounded as follows:

$$\left| c \sum_{m,n} \frac{m}{1+an^2} \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} \dot{\hat{\varphi}}_{-n}^{(N)} \right| \leq Q_{3,0} \|\varphi^{(N)}\|_0 \|\varphi^{(N)}\|_1 \|\dot{\varphi}^{(N)}\|_0,$$

where

$$Q_{3,0} = |c| \left(\sum_n (1+an^2)^{-2} \right)^{1/2}.$$

For $s \geq 1$, we use inequalities (5) applied to $\dot{\varphi}^{(N)}$ instead of $\varphi^{(N)}$ and (6):

$$\begin{aligned}& \left| c \sum_{m,n} \frac{m|n|^{2s}}{1+an^2} \hat{\varphi}_m^{(N)} \hat{\varphi}_{n-m}^{(N)} \dot{\hat{\varphi}}_{-n}^{(N)} \right| \\ & \leq |c| \sum_n \left(\sum_m Q_{2,s} (|m|^s + |n-m|^s) |\hat{\varphi}_m^{(N)}| |\hat{\varphi}_{n-m}^{(N)}| \right) \frac{|n|^{s+1}}{1+an^2} |\dot{\hat{\varphi}}_{-n}^{(N)}| \\ & \leq 2|c| Q_{2,s} \|\varphi^{(N)}\|_s \|\varphi^{(N)}\|_0 Q_1 \|\dot{\varphi}^{(N)}\|_s = Q_{3,s} \|\varphi^{(N)}\|_0 \|\varphi^{(N)}\|_s \|\dot{\varphi}^{(N)}\|_s.\end{aligned}$$

Finally, for $s \geq 0$ the last sum satisfies the inequality

$$\left| \sum_n \frac{|n|^{2s}}{1+an^2} \dot{\hat{\varphi}}_{-n}^{(N)} \hat{e}_n(t) \right| \leq \max(1, a^{-1}) \|\dot{\varphi}^{(N)}\|_s \|e(x, t)\|_{s-2}.$$

Collecting the bounds, we obtain for $s = 0$ and $s \geq 1$ from (8):

$$\|\dot{\varphi}^{(N)}\|_s \leq Q_4 \|\varphi^{(N)}\|_{s-1} + Q_{3,s} \|\varphi^{(N)}\|_0 \|\varphi^{(N)}\|_{\max(s,1)} + \max(1, a^{-1}) \|e(x, t)\|_{s-2}.\tag{9}$$

By induction, (9) yields a bound that is uniform in N , for any integer $s \geq 0$. We denote the r.h.s. of 9 by $D_s(t)$. By interpolation, for any $s \geq 0$

$$\|\dot{\varphi}^{(N)}(x, t)\|_s \leq D_{S+1}^{1-\mu}(t) D_S^\mu(t) \equiv D_s(t)\tag{10}$$

at any time $t \geq 0$, where $S \geq 0$ is integer, $0 \leq \mu \leq 1$ and $s = (1-\mu)(S+1) + \mu S$.

v. Differentiating (2) in time, we find

$$(1 + an^2)\ddot{\hat{\varphi}}_n^{(N)} + (d + \mathbf{i}bn)\dot{\hat{\varphi}}_n^{(N)} + \mathbf{i}c \sum_m m(\dot{\hat{\varphi}}_m^{(N)}\hat{\varphi}_{n-m}^{(N)} + \hat{\varphi}_m^{(N)}\dot{\hat{\varphi}}_{n-m}^{(N)}) + \dot{e}_n(t) = 0.$$

Using this equation and the bounds for $\|\varphi^{(N)}\|_s$ and $\|\dot{\varphi}^{(N)}\|_s$ obtained above for arbitrarily large s , it is easy to show that $|\hat{\varphi}_n^{(N)}|$ are uniformly bounded in N for each n .

Consider a time interval $[0, T]$ for some $T > 0$. We have demonstrated that, for each n , $|\dot{\hat{\varphi}}_n^{(N)}(t)|$ and $|\ddot{\hat{\varphi}}_n^{(N)}(t)|$ are uniformly bounded in N and $t \in [0, T]$, and hence the sets of functions $\hat{\varphi}_n^{(N)}(t)$ and $\dot{\hat{\varphi}}_n^{(N)}(t)$ are equicontinuous. Therefore, applying the Arzelà–Ascoli theorem and using the diagonal process, we can choose a subsequence $N_k \rightarrow \infty$ such that

1) for each n , $\varphi_n^{(N_k)}$ and $\dot{\varphi}_n^{(N_k)}$ uniformly converge to some continuous functions φ_n and δ_n ; furthermore, $\delta_n = \dot{\varphi}_n$, as can be seen by letting $N_k \rightarrow \infty$ in the relation

$$\hat{\varphi}_n^{(N)}(t) = \varphi_n(0) + \int_0^t \dot{\hat{\varphi}}_n^{(N)}(\tau) d\tau;$$

2) the bounds

$$\|\varphi(x, t)\|_s \leq C_s(t) \tag{11}$$

and

$$\|\dot{\varphi}(x, t)\|_s \leq D_s(t) \tag{12}$$

hold true for the limit functions

$$\varphi(x, t) = \sum_n \varphi_n(t) e^{\mathbf{i}nx}, \quad \dot{\varphi}(x, t) = \sum_n \dot{\varphi}_n(t) e^{\mathbf{i}nx}$$

(this can be shown by considering inequalities (7) and (10) for $N_k \rightarrow \infty$).

Thus, at each time t the limit functions $\varphi(x, t)$ and $\dot{\varphi}(x, t)$ are infinitely smooth in x (provided the initial data and the forcing are infinitely smooth). In the limit $N_k \rightarrow \infty$, the Galerkin equation (2) becomes

$$(1 + an^2)\dot{\varphi}_n + (d + \mathbf{i}bn)\varphi_n + \mathbf{i}c \sum_m m\varphi_m\varphi_{n-m} + \hat{e}_n(t) = 0 \tag{13}$$

(passing to the limit in the infinite sum in m is possible, because the sum converges uniformly in N). Relations (13) imply that $\varphi(x, t)$ satisfies the original RLWE in the classical sense.

Differentiating the RLWE $s - 1$ times in t , we incrementally establish (by induction in s) that $\frac{\partial^s}{\partial t^s}(\varphi - a\varphi'')$ and hence $\partial^s \varphi / \partial t^s$ are continuous in time; this proves that $\varphi(x, t) \in C^\infty(\mathbb{R}^1 \times \mathbb{R}_+^1)$.

Finally, if there exist two distinct smooth solutions to the RLWE, application of Gronwall's lemma to the linear equation for the difference between them

establishes that the difference is zero. In particular, the limit functions obtained for different subsequences $N_k \rightarrow \infty$ and/or on different time intervals $[0, T]$ necessarily coincide. Q.E.D.

3 Spatial analyticity of evolutionary solutions

Temporal analyticity of solutions to the zero-force BBM equation was proved in [3]. These authors analysed convergence of Taylor's expansion of the solution in time, employing an integral operator that maps the m -th time derivative of the solution to the time derivative of order $m + 1$. Here we prove the spatial analyticity of φ by the techniques of [43].

For any $\sigma > 0$ we define the Gevrey–Sobolev seminorms of $f(x) = \sum_n \hat{f}_n e^{inx}$ by the relation

$$\|f\|_{\sigma,s}^2 = \begin{cases} \sum_n |\hat{f}_n|^2 |n|^{2s} e^{2\sigma|n|}, & s > 0; \\ \sum_n |\hat{f}_n|^2 (\max(|n|, 1))^{2s} e^{2\sigma|n|}, & s \leq 0. \end{cases}$$

Functions, whose Gevrey–Sobolev norms are finite, are analytic; the first index σ is a lower estimate of the width of the analyticity strip of f around the real axis on the complex plane.

We also introduce a seminorm

$$\|f\|^2 = \sum_n (|n| + a|n|^3) |\hat{f}_n|^2$$

equivalent to $\|\cdot\|_{3/2}$.

Theorem 2. Suppose $\varphi_0(x)$ and $e(x, t)$ satisfy the conditions of Theorem 1 and are analytic in x : for some constants $\sigma > 0$ and $\beta > 0$, $\|\varphi_0(x)\|_{\sigma, 3/2} < \infty$ and $\|e(x, t)\|_{\beta, 0}$ is uniformly bounded in time. Then the solution to the RLWE is analytic in x at any $t \geq 0$, and the width of its analyticity strip around the real axis decreases in time at most exponentially. For $d \geq 0$, the width decreases in time at most algebraically.

Proof. We will show that at any time t the solutions to the Fourier–Galerkin system of equations (2), that were considered in Theorem 1, for some $\kappa(t) > 0$ have Gevrey–Sobolev norms $\|\varphi^{(N)}(x, t)\|_{\kappa(t), 3/2}$, that are bounded uniformly in N . This will imply that the solution to the RLWE, $\varphi(x, t)$, also have finite norms $\|\varphi(x, t)\|_{\kappa(t), 3/2}$, this proving Theorem 2.

For a given N , we consider a transformation

$$\widehat{\varphi}_n^{(N)}(t) = \widehat{w}_n^{(N)}(t) \exp \left(-\frac{\beta|n|}{1 + \|w^{(N)}(x, t)\|^{1+\varepsilon}} \right), \quad (14)$$

$$w^{(N)}(x, t) = \sum_n \widehat{w}_n^{(N)}(t) e^{inx},$$

where $\varepsilon \leq 1$ is a positive constant. For brevity, we henceforth omit the superscript (N) in $\widehat{w}_n^{(N)}$. We seek a solution to the system of nonlinear equations (14) in the form

$$\widehat{w}_n(t) = \widehat{\varphi}_n(t) \exp(\psi(t)|n|),$$

where $\psi(t) > 0$ satisfies the equation

$$\psi(t) \left(1 + \left(\sum_{|n| \leq N} (|n| + a|n|^3) e^{2\psi(t)|n|} |\widehat{\varphi}_n^{(N)}(t)|^2 \right)^{(1+\varepsilon)/2} \right) = \beta.$$

It has a unique solution for any $t \geq 0$, because the l.h.s. is a continuous monotonically increasing unbounded function of ψ , that vanishes for $\psi = 0$. We assume without any loss of generality that

$$\beta \leq \sigma \left(1 + (\|\varphi_0\|_{\sigma, 1/2}^2 + a\|\varphi_0\|_{\sigma, 3/2}^2)^{(1+\varepsilon)/2} \right),$$

whereby $\|w^{(N)}(x, t)\|_{3/2}$ are bounded uniformly in N at $t = 0$.

Substitution (14) transforms the Fourier–Galerkin equations (2) into the system of equations

$$\begin{aligned} (1 + an^2)\dot{\widehat{w}}_n + \beta(1 + \varepsilon)(|n| + a|n|^3) \widehat{w}_n \frac{\|w\|^\varepsilon}{(1 + \|w\|^{1+\varepsilon})^2} \frac{d}{dt} \|w\| + (d + ibn)\widehat{w}_n \\ + ic \sum_m m \widehat{w}_m \widehat{w}_{n-m} e^{\gamma(|n|-|m|-|n-m|)} + \widehat{e}_n(t) e^{\gamma|n|} = 0, \end{aligned}$$

where it is denoted $\gamma = \beta / (1 + \|w\|^{1+\varepsilon})$.

Multiplying the equation by $w_{-n} = \overline{\widehat{w}_n}$ and summing up the results over n , we find

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left(\|w\|_0^2 + a\|w\|_1^2 + 2\beta(1 + \varepsilon)I(\|w\|) \right) + d\|w\|_0^2 \\ + ic \sum_{m,n} m \widehat{w}_m \widehat{w}_{n-m} \widehat{w}_{-n} e^{\gamma(|n|-|m|-|n-m|)} + \sum_n \widehat{e}_n(t) e^{\gamma|n|} \widehat{w}_{-n} = 0, \end{aligned} \quad (15)$$

where it is denoted

$$I(q) = \int_0^q \frac{u^{2+\varepsilon}}{(1 + u^{1+\varepsilon})^2} du.$$

For $0 < \varepsilon < 1$ and large q ,

$$I(q) = (1 - \varepsilon)^{-1} q^{1-\varepsilon} + O(q^{-2\varepsilon}). \quad (16)$$

We transform now the sum

$$\sum \equiv \text{ic} \sum_{m,n} m \hat{w}_m \hat{w}_{n-m} \hat{w}_{-n} e^{\gamma(|n|-|m|-|n-m|)}.$$

It remains unaltered when we change the index $m \rightarrow n - m$, as well as when we change the indices $m \rightarrow -n$, $n \rightarrow m - n$. Summing the two sums obtained by these changes of indices with the original sum, we find

$$\sum = \frac{\text{ic}}{3} \sum_{m,n} n \hat{w}_m \hat{w}_{n-m} \hat{w}_{-n} (e^{\gamma(|n|-|m|-|n-m|)} - e^{\gamma(|n-m|-|n|-|m|)}).$$

By virtue of the inequalities $|e^{\mu'} - e^{\mu''}| \leq |\mu' - \mu''|$ that holds true for any $\mu' \leq 0$ and $\mu'' \leq 0$, and $|n|^\mu \leq |m|^\mu + |n - m|^\mu$ for any $0 \leq \mu \leq 1$, the above relation implies

$$|\sum| \leq \frac{2|c|}{3} \gamma \sum_{m,n} |n|^{1-\varepsilon/2} (|m|^{\varepsilon/2} + |n - m|^{\varepsilon/2}) |m| |\hat{w}_m| |\hat{w}_{n-m}| |\hat{w}_{-n}|. \quad (17)$$

By the Cauchy-Bunyakovsky-Schwarz inequality, for $\varepsilon > 0$

$$\left| \sum_n |n|^{1-\varepsilon/2} |\hat{w}_{-n}| \right| = \left| \sum_n |n|^{-(1+\varepsilon)/2} |n|^{3/2} |\hat{w}_{-n}| \right| \leq \left(\sum_n |n|^{-1-\varepsilon} \right)^{1/2} \|w\|_{3/2}. \quad (18)$$

By the Cauchy-Bunyakovsky-Schwarz and Hölder's inequalities, for $0 \leq \varepsilon \leq 1$

$$\begin{aligned} \left| \sum_m |m|^{1+\varepsilon/2} |\hat{w}_m| |\hat{w}_{n-m}| \right| &\leq \|w\|_0 \|w\|_{1+\varepsilon/2} \\ &= \|w\|_0 \left(\sum_m (|m|^3 |\hat{w}_m|^2)^\varepsilon |m|^2 |\hat{w}_m|^2 \right)^{1/2} \leq \|w\|_0 \|w\|_{3/2}^\varepsilon \|w\|_1^{1-\varepsilon} \end{aligned} \quad (19)$$

and

$$\left| \sum_m |m| |n - m|^{\varepsilon/2} |\hat{w}_m| |\hat{w}_{n-m}| \right| \leq \|w\|_{\varepsilon/2} \|w\|_1. \quad (20)$$

Inequalities (17)–(20) imply

$$\begin{aligned} |\sum| &\leq \frac{2|c|}{3} \gamma (\|w\|_0 \|w\|_{3/2}^\varepsilon \|w\|_1^{1-\varepsilon} + \|w\|_{\varepsilon/2} \|w\|_1) \left(\sum_n |n|^{-1-\varepsilon} \right)^{1/2} \|w\|_{3/2} \\ &\leq Q_5 (\|w\|_0 + \|w\|_{\varepsilon/2}) \|w\|_1^{1-\varepsilon}, \end{aligned}$$

where $Q_5 = \frac{2\beta|c|}{3} \left(\sum_n |n|^{-1-\varepsilon} \right)^{1/2}.$

Thus, we find from (15):

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|w\|_0^2 + a\|w\|_1^2 + 2\beta(1+\varepsilon)I(\|w\|) \right) \\ & \leq -d\|w\|_0^2 + Q_5(\|w\|_0 + \|w\|_{\varepsilon/2})\|w\|_1^{1-\varepsilon} + \|e\|_{\beta,0}\|w\|_0, \end{aligned}$$

whereby

$$\frac{d\zeta}{dt} \leq -d\zeta + \frac{Q_5}{a}\zeta^{1-\varepsilon} + \|e\|_{\beta,0}, \quad (21)$$

where it is denoted

$$\zeta^2 = \|w^{(N)}\|_0^2 + a\|w^{(N)}\|_1^2 + 2\beta(1+\varepsilon)I(\|w^{(N)}\|).$$

For $d < 0$, by Gronwall's lemma $\zeta \leq \zeta_0 e^{\mu t}$ for any $\mu > -d$; since also $2\beta(1+\varepsilon)I(\|w^{(N)}\|) \leq \zeta^2$ and by virtue of (16), we have $\|w^{(N)}\| \leq \zeta_1 e^{2\mu t/(1-\varepsilon)}$; here ζ_0 and ζ_1 are suitable positive constants. Consequently,

$$\|\varphi^{(N)}\|_{\beta/(1+\zeta_1^{1+\varepsilon} e^{(2\mu(1+\varepsilon)/(1-\varepsilon))t}),1} \leq \|\varphi^{(N)}\|_{\beta/(1+\|w^{(N)}\|^{1+\varepsilon}),1} = \|w^{(N)}\|_1 \leq \frac{\zeta_0}{\sqrt{a}} e^{\mu t}.$$

This bound is uniform in N , and therefore in the limit $N_k \rightarrow \infty$ we obtain

$$\|\varphi\|_{\beta/(1+\zeta_1^{1+\varepsilon} e^{(2\mu(1+\varepsilon)/(1-\varepsilon))t}),1} \leq \frac{\zeta_0}{\sqrt{a}} e^{\mu t}.$$

Hence the width of the analyticity strip of φ around the real axis is bounded from below by an exponentially decaying quantity, $\beta/(1+\zeta_1^{1+\varepsilon} e^{(2\mu(1+\varepsilon)/(1-\varepsilon))t})$.

For $d \geq 0$, (21) reduces to

$$\frac{d\zeta}{dt} \leq Q_5 \zeta^{1-\varepsilon} + \|e\|_{\beta,0}.$$

Integrating this inequality yields $\zeta^\varepsilon \leq \zeta_2 t + \zeta_3$. Since $2\beta(1+\varepsilon)I(\|w^{(N)}\|) \leq \zeta^2$, (16) implies $\|w^{(N)}\|^{1+\varepsilon} \leq \zeta_4 t^{2(1+\varepsilon)/(\varepsilon(1-\varepsilon))} + \zeta_5$. Consequently,

$$\begin{aligned} \|\varphi^{(N)}\|_{\beta/(1+\zeta_5+\zeta_4 t^{2(1+\varepsilon)/(\varepsilon(1-\varepsilon))}),1} & \leq \|\varphi^{(N)}\|_{\beta/(1+\|w^{(N)}\|^{1+\varepsilon}),1} \\ & = \|w^{(N)}\|_1 \leq (\zeta_2 t + \zeta_3)^{1/\varepsilon} / \sqrt{a}, \end{aligned}$$

where all ζ_i are suitable positive constants. Since this bound is uniform in N , we obtain in the limit $N_k \rightarrow \infty$

$$\|\varphi\|_{\beta/(1+\zeta_5+\zeta_4 t^{2(1+\varepsilon)/(\varepsilon(1-\varepsilon))}),1} \leq (\zeta_2 t + \zeta_3)^{1/\varepsilon} / \sqrt{a}.$$

Therefore, for $d \geq 0$ the width of the analyticity strip of φ around the real axis is bounded from below by the quantity $\beta/(1+\zeta_5+\zeta_4 t^{2(1+\varepsilon)/(\varepsilon(1-\varepsilon))})$, which

decays algebraically. Within the allowed interval $0 < \varepsilon < 1$, the exponent $2(1 + \varepsilon)/(\varepsilon(1 - \varepsilon))$ takes the minimal value for $\varepsilon = \sqrt{2} - 1$. The optimal exponent that we have thus found is $2(\sqrt{2} + 1)^2$. Q.E.D.

4 Existence and uniqueness of travelling-wave solutions

When the forcing has the form $e(x, t) = e(\xi)$ for $\xi = x - \Omega t$, the RLWE may have travelling-wave solutions such that $\varphi(x, t) = \varphi(\xi)$. We establish now their existence.

Substituting $\varphi(x, t) = \varphi(\xi)$ into the RLWE we obtain an equation for the wave profile φ :

$$-\Omega(\varphi' - a\varphi''') + b\varphi' + c\varphi\varphi' + d\varphi + e(\xi) = 0, \quad (22)$$

where $'$ denotes henceforth differentiation in $\xi \in \mathbb{R}^1$. 2π -periodicity in x translates to 2π -periodicity in ξ .

Theorem 3. Suppose $a\Omega d \neq 0$. If $e(\xi) \in C^\infty(\mathbb{R}^1)$ is 2π -periodic, then there exists a 2π -periodic solution to (22), $\varphi(\xi) \in C^\infty(\mathbb{R}^1)$, for any constants $a > 0$, b , c and $d \neq 0$. If the forcing is weak:

$$\|e\|_0 < \frac{|d|}{2\sqrt{\sum_{n \neq 0} |p_n|^2}}, \quad (23)$$

where quantities p_n are defined in (26) below, the travelling-wave solution to the RLWE is unique.

Proof.

i. We seek a solution to (22) in the form of a Fourier series

$$\varphi(\xi) = \sum_n \hat{\varphi}_n e^{in\xi}.$$

The travelling-wave RLWE then reduces to the system of equations

$$-in(1 + an^2)\Omega\hat{\varphi}_n + (d + ibn)\hat{\varphi}_n + \frac{icn}{2} \sum_m \hat{\varphi}_m \hat{\varphi}_{n-m} + \hat{e}_n = 0. \quad (24)$$

Equation (24) for $n = 0$ (i.e., the average of (22) over ξ) yields $\hat{\varphi}_0 = -\hat{e}_0/d$.

Dividing (24) by $-i(a\Omega n^3 + (\Omega + c\hat{e}_0/d - b)n + id)$, we obtain for $n \neq 0$

$$\hat{\varphi}_n = p_n \sum_{0 \neq m \neq n} \hat{\varphi}_m \hat{\varphi}_{n-m} + q_n, \quad (25)$$

where it is denoted

$$p_n = \frac{cn}{2(a\Omega n^3 + (\Omega + c\widehat{e}_0/d - b)n + 1d)}, \quad (26)$$

$$q_n = - \frac{1\widehat{e}_n}{a\Omega n^3 + (\Omega + c\widehat{e}_0/d - b)n + 1d}. \quad (27)$$

The system of equations (25) does not involve an equation for $n = 0$. To simplify notation, we henceforth formally assume that $\widehat{\varphi}_0 = 0$ in (25).

ii. We have thus rendered the travelling-wave RLWE as a fixed-point problem $\varphi = \mathcal{A}\varphi$, where the operator \mathcal{A} is defined by the r.h.s. of (25):

$$\mathcal{A} : \sum_{n \neq 0} \varphi_n(t) e^{in\xi} \mapsto \sum_{n \neq 0} \left(p_n \sum_{0 \neq m \neq n} \widehat{\varphi}_m \widehat{\varphi}_{n-m} + q_n \right) e^{in\xi}.$$

We will seek a solution in the subspace of zero-mean functions of the Sobolev space $H^1(\mathbb{T}^1)$ (the norm $\|\cdot\|_2$ in $H^s(\mathbb{T}^1)$ was defined in the previous section). Existence of solutions to the fixed-point problem (25) is guaranteed by the Leray–Schauder principle ([26], see also [25]) under two conditions:

1) Any solution to the equation

$$\varphi = \mu \mathcal{A}\varphi \quad (28)$$

belongs to a ball in $H^1(\mathbb{T}^1)$ of a radius independent of μ for $0 \leq \mu \leq 1$.

2) The operator $\mathcal{A} : H^1(\mathbb{T}^1) \rightarrow H^1(\mathbb{T}^1)$ is compact, i.e., $\mathcal{A}(\varphi^n)$ strongly converges in $H^1(\mathbb{T}^1)$ for any sequence φ^n , weakly converging in $H^1(\mathbb{T}^1)$.

To establish 1), note that (28) is equivalent to the system of equations

$$(-in(1 + an^2)\Omega + d + 1bn)\widehat{\varphi}_n + \frac{1\mu cn}{2} \sum_m \widehat{\varphi}_m \widehat{\varphi}_{n-m} + \mu \widehat{e}_n = 0.$$

We multiply this equation for $n \neq 0$ by $\widehat{\varphi}_{-n}$ and sum the results over n to find

$$d\|\varphi\|_0^2 = -\mu \sum_n \widehat{\varphi}_n \widehat{e}_n,$$

which implies

$$\|\varphi\|_0 \leq \mu \|e\|_0 / |d|.$$

We multiply now (25) by $|n|^{2s+2}\widehat{\varphi}_{-n}$; summation over n then yields

$$\|\varphi\|_{s+1}^2 = \mu \sum_{n \neq 0} p_n |n|^{2s+2} \sum_{0 \neq m \neq n} \widehat{\varphi}_m \widehat{\varphi}_{n-m} \widehat{\varphi}_{-n} + \mu \sum_{n \neq 0} q_n |n|^{2s+2} \widehat{\varphi}_{-n}.$$

By virtue of the inequality $|n|^{2s} \leq R_s(|m|^{2s} + |n-m|^{2s})$, valid for all $s > 0$ and $R_s = \max(2^{2s-1}, 1)$,

$$\begin{aligned}
\|\varphi\|_{s+1}^2 &\leq \mu R_{s+1} \sum_{0 \neq m \neq n \neq 0} |p_n| |n|^{s+1} (|m|^{s+1} + |n-m|^{s+1}) |\widehat{\varphi}_m| |\widehat{\varphi}_{n-m}| |\widehat{\varphi}_{-n}| \\
&\quad + \mu \left(\sum_{n \neq 0} |q_n|^2 |n|^{2s+2} \right)^{1/2} \|\varphi\|_{s+1} \\
&\leq \mu R_{s+1} \sum_{n \neq 0} |p_n| |n|^{s+1} |\widehat{\varphi}_{-n}| 2 \|\varphi\|_0 \|\varphi\|_{s+1} + \mu \left(\sup_{n \neq 0} |q_n| |n|^3 \right) \|e\|_{s-2} \|\varphi\|_{s+1} \\
&\leq 2\mu R_{s+1} \|\varphi\|_0 \|\varphi\|_{s+1} \|\varphi\|_s \left(\sum_{n \neq 0} |p_n|^2 |n|^2 \right)^{1/2} + \mu \left(\sup_{n \neq 0} |q_n| |n|^3 \right) \|e\|_{s-2} \|\varphi\|_{s+1},
\end{aligned}$$

whereby

$$\|\varphi\|_{s+1} \leq 2\mu R_{s+1} \left(\sum_{n \neq 0} |p_n|^2 |n|^2 \right)^{1/2} \|\varphi\|_0 \|\varphi\|_s + \mu \left(\sup_{n \neq 0} |q_n| |n|^3 \right) \|e\|_{s-2}. \quad (29)$$

Assuming here $s = 0$ we find that any solution φ to the problem (28) for $0 \leq \mu \leq 1$ belongs to the ball

$$\|\varphi\|_1 \leq \left(\frac{2\|e\|_0}{|d|} \right)^2 \left(\sum_{n \neq 0} |p_n|^2 |n|^2 \right)^{1/2} + \left(\sup_{n \neq 0} |q_n| |n|^3 \right) \|e\|_{-2},$$

as required.

To establish 2), consider a weakly converging sequence $\varphi^k(\xi)$ in $H^1(\mathbb{T}^1)$. By properties of weak convergence, functions $\varphi^k(\xi)$ are uniformly bounded in $H^1(\mathbb{T}^1)$: $\|\varphi^k\|_1 \leq A$. By the Sobolev embedding theorem, weak convergence in $H^1(\mathbb{T}^1)$ implies strong convergence in $H^1(\mathbb{T}^1)$: for any $\epsilon > 0$ and $s < 1$ there exists $K(s)$ such that $\|\varphi^{k'} - \varphi^{k''}\|_s \leq \epsilon$ provided $k' > K(s)$ and $k'' > K(s)$. We need to show that

$$\|\mathcal{A}(\varphi^{k'}) - \mathcal{A}(\varphi^{k''})\|_1 = \left\| \sum_{n \neq 0} \left(p_n \sum_{0 \neq m \neq n} (\widehat{\varphi}_m^{k'} \widehat{\varphi}_{n-m}^{k'} - \widehat{\varphi}_m^{k''} \widehat{\varphi}_{n-m}^{k''}) \right) e^{in\xi} \right\|_1 \rightarrow 0$$

for $k', k'' \rightarrow \infty$. In terms of $\theta^{k', k''} = \varphi^{k'} - \varphi^{k''}$ and $\widehat{\theta}^{k', k''} = \widehat{\varphi}^{k'} - \widehat{\varphi}^{k''}$,

$$\begin{aligned}
&\|\mathcal{A}(\varphi^{k'}) - \mathcal{A}(\varphi^{k''})\|_1^2 \\
&= \sum_{n \neq 0} \left(|p_n|^2 |n|^2 \sum_{0 \neq m \neq n} (\widehat{\varphi}_m^{k'} \widehat{\theta}_{n-m}^{k', k''} + \widehat{\varphi}_{n-m}^{k''} \widehat{\theta}_m^{k', k''}) \sum_{0 \neq l \neq -n} (\widehat{\varphi}_l^{k'} \widehat{\theta}_{-n-l}^{k', k''} + \widehat{\varphi}_{-n-l}^{k''} \widehat{\theta}_l^{k', k''}) \right) \\
&\leq \sum_{n \neq 0} |p_n|^2 |n|^2 (\|\varphi^{k''}\|_0 + \|\varphi^{k'}\|_0)^2 \|\theta^{k', k''}\|_0^2 \leq 4A^2 \left(\sum_{n \neq 0} |p_n|^2 |n|^2 \right) \|\theta^{k', k''}\|_0^2.
\end{aligned}$$

This proves the required strong convergence of $\mathcal{A}(\varphi^k)$ in $H^1(\mathbb{T}^1)$ for $k \rightarrow \infty$.

iii. Solutions φ to the travelling-wave RLWE have finite norms in any Sobolev space $H^s(\mathbb{T}^1)$ and hence are infinitely differentiable. This follows directly from inequality (29) for $\mu = 1$ in combination with induction in integer $s > 0$.

iv. The number of solutions to the travelling-wave RLWE for given parameter values is unknown, unless the coefficients p_n and/or the energy $\|\varphi\|_0$ are small, in which case the solution is unique.

Suppose there exist two solutions φ' and φ'' . We denote $\theta = \varphi' - \varphi''$ and $\hat{\theta} = \hat{\varphi}' - \hat{\varphi}''$ and find

$$\begin{aligned} \|\theta\|_0^2 &= \|\mathcal{A}(\varphi') - \mathcal{A}(\varphi'')\|_0^2 \\ &= \sum_{n \neq 0} \left(|p_n|^2 \sum_{0 \neq m \neq n} (\hat{\varphi}'_m \hat{\theta}_{n-m} + \hat{\varphi}''_{n-m} \hat{\theta}_m) \sum_{0 \neq l \neq -n} (\hat{\varphi}'_l \hat{\theta}_{-n-l} + \hat{\varphi}''_{-n-l} \hat{\theta}_l) \right) \\ &\leq \sum_{n \neq 0} |p_n|^2 (\|\varphi'\|_0 + \|\varphi''\|_0)^2 \|\theta\|_0^2. \end{aligned}$$

Thus coexistence of distinct solutions satisfying

$$\|\varphi\|_0 < \left(4 \sum_{n \neq 0} |p_n|^2 \right)^{-1/2}$$

is ruled out. Since any solution to the travelling-wave RLWE has a bounded norm $\|\varphi\|_0 \leq \|e\|_0/|d|$, the problem has a unique solution provided inequality (23) holds true. Q.E.D.

5 Non-well-posedness of the non-damped travelling-wave RLWE

A problem of physical relevance is said, following Hadamard, to be well-posed, if it possesses a solution that is unique and depends continuously on the data. The evolutionary problem for the non-damped ($d = 0$) RLWE (the BBM equation) is well-posed [3]. Instead of developing the existence theory for travelling waves for $d = 0$, we show here that, by contrast, the travelling-wave problem for the RLWE (22) for $d = 0$ is not well-posed, since arbitrarily large solutions can exist for a forcing of whichever small amplitude.

We consider fast oscillating (both in space and time) solutions to (22) of the form

$$\varphi(\xi) = \Omega^\beta \Phi(\eta), \quad \eta \equiv \Omega^\alpha \xi \quad (30)$$

in the limit $\Omega \rightarrow \infty$. Substituting the ansatz (30) into (22) yields

$$-\Omega^{1+\alpha+\beta}\Phi' + a\Omega^{1+3\alpha+\beta}\Phi''' + b\Omega^{\alpha+\beta}\Phi' + c\Omega^{\alpha+2\beta}\Phi\Phi' + e(\eta) = 0, \quad (31)$$

the prime $'$ denoting in this section differentiation in the fast variable η . In this section we assume $a > 0$ and $c < 0$ (the important condition here is $c \neq 0$; the convention about the sign of c is technical since (31) has the symmetry $\Phi \rightarrow -\Phi$, $c \rightarrow -c$). Note that for such coefficients we might reduce equation (31) to the one for $a = b = c = 1$ (provided $b \neq 0$) by appropriately rescaling Ω , η and Φ .

The two terms defining the nature of the problem, i.e., the ones involving the third-order derivative and the nonlinearity, balance each other if $\beta = 1 + 2\alpha$. If α and β are rational, it is natural to seek Φ in the form of power series in Ω^{-1} in an appropriate fractional power. The simplest case is $\alpha = 1$, $\beta = 3$. For these parameter values we consider the series

$$\Phi(\eta) = \sum_{i \geq 0} \Omega^{-i} \Phi_i(\eta). \quad (32)$$

Substituting (32) into (31), expanding and collecting all the terms involving Ω^{7-i} for some $i \geq 0$, we obtain a hierarchy of equations for $\Phi_i(\eta)$:

$$-\Phi'_{i-2} + a\Phi'''_i + b\Phi'_{i-3} + c \sum_{j=0}^i \Phi_j \Phi'_{i-j} = 0 \quad (33)$$

(assuming that the amplitude of the forcing $e(\eta)$ is so small that it does not contribute to (33) at this level).

i. For $i = 0$, (33) reduces to

$$\Phi'''_0 - Q\Phi_0\Phi'_0 = 0, \quad (34)$$

where it is denoted $Q = -c/a$ (by our convention $Q > 0$). Integrating (34) in η once, we find

$$\Phi''_0 = \frac{1}{2}(Q\Phi_0^2 + C_1). \quad (35)$$

Using the standard techniques, we reduce the order of (35) by regarding Φ_0 as a new independent variable and Φ'_0 as an unknown function of this variable; integrating (35) in Φ_0 then yields

$$(\Phi'_0)^2 = \mathcal{P}(\Phi_0) \equiv \frac{Q}{3}\Phi_0^3 + C_1\Phi_0 + C_0, \quad (36)$$

where C_1 and C_0 are some constants. We assume henceforth $4C_1^3 < -9QC_0^2$, whereby the polynomial $\mathcal{P}(\Phi_0)$ in the r.h.s. of (36) has three distinct real roots $\lambda_1 < \lambda_2 < \lambda_3$ (see a sketch of the plot of $\mathcal{P}(\Phi_0)$ in Fig. 1).

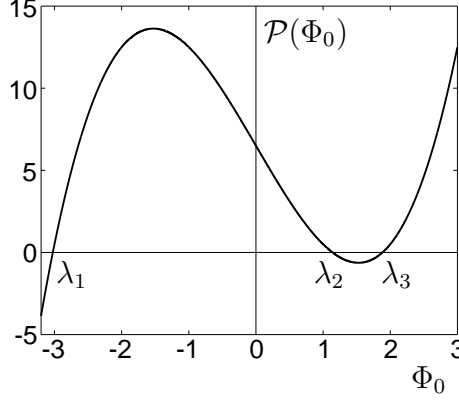


Fig. 1. A plot of the cubic polynomial $\mathcal{P}(\Phi_0)$ in the r.h.s. of (36) for $Q = 3$, $C_0 = 6.5$, $C_1 = -7$.

Solutions to (36) can be expressed in terms of the Weierstrass elliptic function (see [2,1])

$$\wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{|n_1|+|n_2| \neq 0} \left(\frac{1}{(z + 2n_1\omega_1 + 2n_2\omega_2)^2} - \frac{1}{(2n_1\omega_1 + 2n_2\omega_2)^2} \right)$$

that is holomorphic and double-periodic, the periods being $2\omega_1$ and $2\omega_2$, and solves equations

$$\left(\frac{d}{dz} \wp \right)^2 = 4\wp^3 - g_2\wp - g_3 \quad (37)$$

and

$$z = \int_{\wp(z; \omega_1, \omega_2)}^{\infty} (4\Phi^3 - g_2\Phi - g_3)^{-1/2} d\Phi \quad (38)$$

on the complex plane $z \in \mathbb{C}^1$. The Weierstrass elliptic function was employed to solve a generalised BBM equation in [30]. The rescaled function $\tilde{\Phi} = (Q/12)\Phi_0$ satisfies ODE (37) for $g_2 = -QC_1/12$ and $g_3 = -Q^2C_0/144$. The half-periods ω_i can be found from the conditions

$$g_2 = 60 \sum_{|n_1|+|n_2| \neq 0} (2n_1\omega_1 + 2n_2\omega_2)^{-4}, \quad g_3 = 140 \sum_{|n_1|+|n_2| \neq 0} (2n_1\omega_1 + 2n_2\omega_2)^{-6}.$$

Since the roots of the r.h.s. of (37), $e_i = (Q/12)\lambda_i$, are real, one of the half-periods (say, ω_1) is real, and the other one (respectively, ω_2) is imaginary. Separating variables in (37) and taking into account (38), we find

$$\tilde{\Phi}(\eta) = \wp \left(\int_{\tilde{\Phi}(\eta_0)}^{\infty} (4\Phi^3 - g_2\Phi - g_3)^{-1/2} d\Phi + \eta_0 - \eta; \omega_1, \omega_2 \right). \quad (39)$$

The three quantities $\wp(\omega_1; \omega_1, \omega_2)$, $\wp(\omega_2; \omega_1, \omega_2)$ and $\wp(\omega_1 + \omega_2; \omega_1, \omega_2)$ coincide with the roots e_i , and hence, by (38),

$$\omega_1 = \int_{e_3}^{\infty} (4\Phi^3 - g_2\Phi - g_3)^{-1/2} d\Phi, \quad \omega_2 = \int_{e_2}^{e_3} (4\Phi^3 - g_2\Phi - g_3)^{-1/2} d\Phi.$$

This removes the ambiguity in the choice of branches of the square root in the path of integration in the r.h.s. of (39). Using the addition formula for the Weierstrass elliptic function and the relations $\wp(\omega_1 + \omega_2; \omega_1, \omega_2) = e_2$, $\wp'(\omega_1 + \omega_2; \omega_1, \omega_2) = 0$, we obtain from (39) the solution in the form that does not involve complex numbers:

$$\tilde{\Phi}(\eta) = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{\wp \left(\int_{\tilde{\Phi}(\eta_0)}^{e_2} (4\Phi^3 - g_2\Phi - g_3)^{-1/2} d\Phi + \eta_0 - \eta; \omega_1, \omega_2 \right) - e_2}.$$

Furthermore, we can represent the solution in terms of the Jacobi elliptic functions of modulus $k = \sqrt{(e_2 - e_1)/(e_3 - e_1)}$ using the identities (see [2,1])

$$\wp \left(\frac{q}{\sqrt{e_3 - e_1}}; \omega_1, \omega_2 \right) = e_1 + \frac{e_3 - e_1}{\text{sn}^2(q)} = e_2 + (e_3 - e_1) \frac{\text{dn}^2(q)}{\text{sn}^2(q)} = e_3 + (e_3 - e_1) \frac{\text{cn}^2(q)}{\text{sn}^2(q)}.$$

However, rather than applying the above results of the theory of elliptic functions, it appears more instructive to establish the properties of the solution that are important for our purposes by directly inspecting (34)–(36). Consider a solution to the ODE (36) such that $\lambda_1 < \Phi_0(0) < \lambda_2$. To be specific, let $\Phi'_0(0)$ satisfying (36) be positive. Thus, on increasing η , Φ_0 is growing till it approaches the value λ_2 . The ODE (36) can be expressed as

$$\Phi'_0 = C(\Phi_0) \sqrt{\lambda_2 - \Phi_0}.$$

For $\Phi_0 \approx \lambda_2$, $C(\Phi_0) \approx \sqrt{Q(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_2)/3}$ is a smooth function bounded from below by a positive constant. Consequently, Φ_0 takes the limit value λ_2 at a finite $\eta = \eta_0$. For $\Phi_0 = \lambda_2$, the r.h.s. of (35) is non-zero, and hence at $\eta = \eta_0$ the sign of Φ'_0 changes and Φ_0 begins to decrease. Separation of variables in (36) yields

$$\pm \int_{\Phi_0(\eta)}^{\lambda_2} \left(\frac{Q}{3} \Phi^3 + C_1 \Phi + C_0 \right)^{-1/2} d\Phi = \eta - \eta_0,$$

where the sign in the l.h.s. is ‘–’ for $\eta < \eta_0$ and ‘+’ for $\eta > \eta_0$. By virtue of this relation, $\Phi_0(\eta)$ is a symmetric function of η about η_0 : $\Phi_0(\eta_0 + \eta) = \Phi_0(\eta_0 - \eta)$.

By a similar argument, Φ_0 continues to decrease till $\Phi_0(\eta_1) = \lambda_1$ for some $\eta = \eta_1$, and subsequently the process repeats itself: there exists an infinite sequence η_k such that $\Phi_0(\eta_{2k}) = \lambda_2$ and $\Phi_0(\eta_{2k+1}) = \lambda_1$. Moreover, $\Phi_0(\eta)$ is symmetric in η about each η_k . Thus, $\Phi_0(\eta)$ is periodic in η , with the half-period $E/2 = \eta_{k+1} - \eta_k$ (this value being independent of k). In what follows we fix the origin of the variable η by letting $\eta_1 = 0$. Plots of a sample solution $\Phi_0(\eta)$ to (36)–(34) computed for $Q = 3$, $C_0 = 6.5$, $C_1 = -7$ and its derivative $\Phi'_0(\eta)$ are shown in Fig. 2.

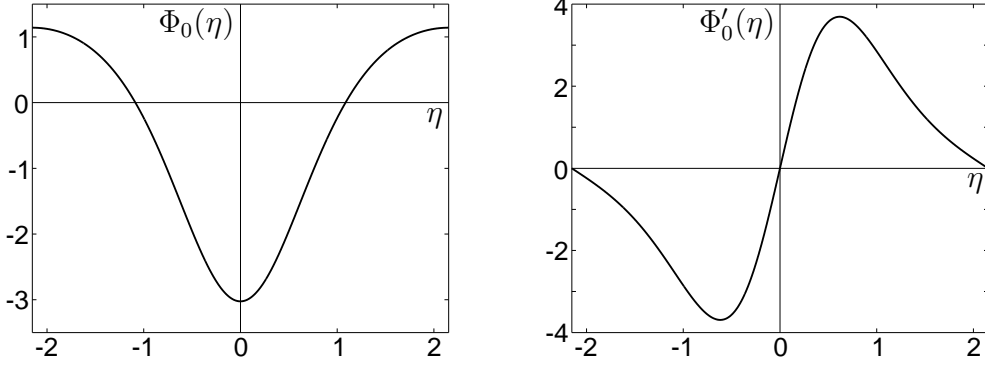


Fig. 2. Plots of a sample solution $\Phi_0(\eta)$ to (36) for $Q = 3$, $C_0 = 6.5$, $C_1 = -7$ (left panel) and its derivative $\Phi'_0(\eta)$ (right panel).

ii. For $i > 0$, (33) becomes

$$a\mathcal{L}\Phi_i - \Phi'_{i-2} + b\Phi'_{i-3} + c \sum_{j=1}^{i-1} \Phi_j \Phi'_{i-j} = 0, \quad (40)$$

for $i = 1$ reducing to

$$\mathcal{L}\Phi_1 = 0. \quad (41)$$

Here \mathcal{L} is the operator of linearisation of (34) in the vicinity of $\Phi_0(\eta)$:

$$\mathcal{L}f = f_i''' - Q(\Phi_0 f)'.$$

It is assumed to act in the Lebesgue space of zero-mean functions that have the same period E in η as Φ_0 . The adjoint operator is

$$\mathcal{L}^* f = -f_i''' + Q\{\Phi_0 f'\},$$

where

$$\langle f \rangle = \frac{1}{E} \int_{-E/2}^{E/2} f(\eta) d\eta \quad \text{and} \quad \{f\} = f - \langle f \rangle$$

denote the average of function f over the period E of Φ_0 and its oscillatory part, respectively. Evidently, operators \mathcal{L} and \mathcal{L}^* map the subspace of even functions (i.e., such that $f(\eta) = f(-\eta)$), into the subspace of odd functions (i.e., $f(\eta) = -f(-\eta)$), and vice versa.

In order to determine the solvability conditions for equations (40), we need to examine the kernel of \mathcal{L}^* . By (34), $\mathcal{L}^*\{\Phi_0\} = 0$. Differentiating (34) in η yields $\mathcal{L}\Phi'_0 = 0$ (this is a manifestation of translation invariance of equations (36)–(34)). Thus, the kernels of \mathcal{L} and \mathcal{L}^* are at least one-dimensional. Actually, generically $\dim \ker \mathcal{L} = \dim \ker \mathcal{L}^* = 2$, the kernels involving generalised eigenfunctions and the operators having 2×2 Jordan cells associated with the eigenvalue 0. To see this, consider solutions to the problems

$$S''_\nu - Q(\Phi_0 S_\nu - \nu) = 0, \quad S_\nu(0) = 1, \quad S'_\nu(0) = 0$$

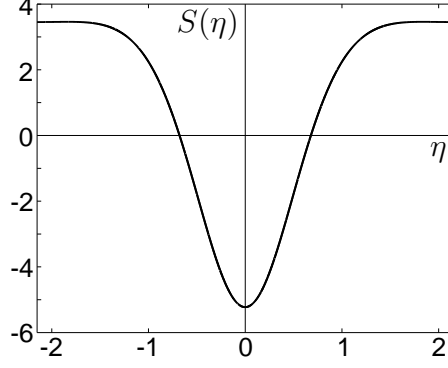


Fig. 3. A plot of the function S computed for a sample solution Φ_0 to (36) shown in Fig. 2.

and the linear combination $S(\eta) = \mu S_0(\eta) + (1-\mu)S_1(\eta)$, where μ is found from the condition $S'(E/2) = 0$. As we have established, Φ_0 is symmetric about the points $kE/2$, where k is integer. Using this, it is easy to show that $S(\eta)$ is also symmetric about these points, and thus is E -periodic. By construction, $S(\eta)$ satisfies the equations

$$S_i''' - Q(\Phi_0 S)' = 0 \quad \Leftrightarrow \quad \mathcal{L}\{S\} = Q\langle S \rangle \Phi_0'. \quad (42)$$

Thus, $\mathcal{L}\{S\} \neq 0$ unless $\langle S \rangle = 0$, but $\mathcal{L}^2\{S\} = 0$, i.e., $\{S\}$ is a generalised eigenfunction associated with the eigenvalue 0 (clearly, Φ_0' and $\{S\}$ are linearly independent: the former eigenfunction is odd while the latter is even). The respective odd generalised eigenfunction from the kernel of \mathcal{L}^* is $\{\int_0^\eta \{S\} d\eta\}$.

We present in Fig. 3 a plot of the function S that was computed for a sample solution to (34)–(36) Φ_0 shown in Fig. 2. $\langle S \rangle = 0.93314$ is non-zero beyond numerical accuracy (the Lebesgue norm of S is 6.54875). We have checked numerically that the kernel of \mathcal{L} is two-dimensional.

The theorem on the Fredholm alternative implies that an equation of the form

$$\mathcal{L}f = u \quad (43)$$

is solvable in the space of zero-mean E -periodic functions whenever

$$\int_{-E/2}^{E/2} u(\eta) \{\Phi_0\} d\eta = 0, \quad (44)$$

and then the solution can be found up to an arbitrary additive term $\kappa \Phi_0'$. (More precisely, the Fredholm alternative theorem is stated for linear problems where the operator is a sum of the identity operator and a compact one [21,27]; however, by considering (44) in the Fourier space, it is simple to show that after integrating the equation three times, we obtain a problem equivalent to (44), for which the theorem on Fredholm alternative is readily applicable.) In particular, the problem (41) has a general solution $\Phi_1 = \kappa_1 \Phi_0'$, where κ_1 is an arbitrary constant.

iii. Thus, (40) specifies Φ_i up to an arbitrary additive term $\kappa_i\Phi'_0$. In principle, one starts solving (40) for a given $i > 1$ by satisfying the solvability condition (44) and determining from it the coefficient κ_j for an appropriate $j < i$. However, we can just set all $\kappa_i = 0$. Then all functions Φ_i are even, equations (40) have odd non-homogeneous parts, and, $\{\Phi_0\}$ being even, the solvability conditions (44) are trivially satisfied. In particular, $\Phi_1 = 0$ and $\Phi_2 = \{S\}/(Q\langle S\rangle)$.

We have therefore shown that one can recursively solve equations (40) in all orders and determine all terms in the power series (32). By construction, a truncated series (32)

$$\Phi_I(\eta) = \sum_{i=0}^I \Omega^{-i} \Phi_i(\eta)$$

is a solution to (31) for the forcing

$$e(\eta) = \sum_{i=\min(7-2I, 4-I)}^{6-I} \Omega^i e_i(\eta)$$

(where all $e_i(\eta)$ are of the order of unity). Thus, we have found an oscillatory solution to the original RLWE (22) for waves for $d = 0$, whose amplitude grows as $O(\Omega^3)$, despite it is sustained by the forcing $O(\Omega^{6-I})$ which, for large I and Ω can be made arbitrarily small with any fixed number of derivatives. This shows that the undamped RLWE for waves gives rise to a problem that is not well-posed.

Several remarks are in order. Our construction is not applicable for $d \neq 0$ technically because the damping term breaks the symmetry of the solution, and we cannot argue any more that the solvability conditions are automatically satisfied. One might try to overcome this by employing the general procedure, whereby one reintroduces the terms $\kappa_i\Phi'_0$ from the kernel of the operator of linearisation \mathcal{L} into Φ_i for $i > 0$ and satisfies the solvability conditions by solving the respective equations in κ_i . However, the system of equations obtained from the solvability conditions does not have a solution. The reason for this failure lies in the fact that while we are constructing a family of solutions to the travelling-wave RLWE that are supposed to grow with Ω unboundedly as Ω^β , any travelling-wave solution to the RLWE for $d \neq 0$ has a bounded norm $\|\varphi\|_0 \leq \|e\|_0/|d|$.

The family of travelling waves that we have constructed for $d = 0$ is non-unique: asymptotic solutions can be obtained for any $\alpha > 0$, $\beta = 1 + 2\alpha$ with the leading-order term Φ_0 satisfying equations (34)–(36). A similar analysis can also be attempted for $\alpha \leq 0$, but in this case the equation for the leading term in the expansion of Φ differs from (34).

6 Asymptotic expansion for a weak forcing

We consider now the travelling-wave RLWE (22) for the forcing proportional to a small parameter ϵ , i.e., we assume in this section that the term $e(\xi)$ in (22) is changed to $\epsilon e(\xi)$. In this case a solution to (22) can be sought as an asymptotic power series

$$\varphi(\xi) = \sum_{k \geq 0} \varphi^{(k)}(\xi) \epsilon^k. \quad (45)$$

Substituting the series into (22), we obtain a transport system of equations

$$\mathcal{M}\varphi^{(1)} = -e(\xi); \quad (46)$$

$$\mathcal{M}\varphi^{(k)} = -\frac{c}{2} \frac{d}{d\xi} \sum_{l=1}^{k-1} \varphi^{(l)} \varphi^{(k-l)}, \quad k > 1. \quad (47)$$

Here \mathcal{M} denotes the operator $\mathcal{M} : \varphi \mapsto -\Omega(\varphi' - a\varphi''') + b\varphi' + d\varphi$, where $'$ denotes the derivative in ξ . Existence of solutions to these problems follows from Theorem 3 applied for $c = 0$.

In terms of the Fourier coefficients of $\varphi^{(k)}$ these equations take the form, respectively,

$$\hat{\varphi}_n^{(1)} = q_n; \quad (48)$$

$$\hat{\varphi}_n^{(k)} = p_n \sum_{m=1}^{k-1} \sum_{l=1}^{k-1} \hat{\varphi}_m^{(l)} \hat{\varphi}_{n-m}^{(k-l)}, \quad k > 1, \quad (49)$$

where

$$p_n = \frac{cn}{2(a\Omega n^3 + (\Omega - b)n + id)}, \quad q_n = -\frac{i\hat{e}_n}{a\Omega n^3 + (\Omega - b)n + id}.$$

Unlike in the previous section, now we do not single out the equation for $n = 0$, since that would imply an undesirable dependence of p_n and q_n on ϵ , as in (26)–(27). Note that $\hat{\varphi}_0^{(k)} = 0$ for $k > 1$.

These relations imply

$$\hat{\varphi}_n^{(k)} = \sum_{\substack{m_1, \dots, m_k \\ m_1 + \dots + m_k = n}} \zeta_{m_1, \dots, m_k} q_{m_1} \dots q_{m_k}.$$

By (48), for $k = 1$ just a single term for $m_1 = n$ is present in this sum, which is $\zeta_n = 1$ for any n . By (49), the recurrence relation

$$\begin{aligned} \zeta_{m_1, \dots, m_{k+1}} = p_{m_1 + \dots + m_{k+1}} & \left(\zeta_{m_1} \zeta_{m_2, \dots, m_{k+1}} + \dots + \zeta_{m_1, \dots, m_l} \zeta_{m_{l+1}, \dots, m_{k+1}} \right. \\ & \left. + \dots + \zeta_{m_1, \dots, m_k} \zeta_{m_{k+1}} \right) \end{aligned}$$

holds (there are k terms in the sum in parenthesis here).

Theorem 4. Power series (45) is an asymptotic expansion in ϵ of the solution $\varphi(\xi)$ to the travelling-wave RLWE.

Proof. For $K > 1$, the residual

$$\theta(\xi) = \varphi(\xi) - \sum_{k=1}^{K-1} \varphi^{(k)}(\xi) \epsilon^k$$

satisfies the equation

$$\mathcal{M}\theta = -\frac{c}{2} \frac{d}{d\xi} \sum_{k=K}^{2K-2} \left(\epsilon^k \sum_{l=1}^{k-1} \varphi^{(l)} \varphi^{(k-l)} \right). \quad (50)$$

Multiplying (46) by $\varphi^{(1)}$, we find $\|\varphi^{(1)}\|_0 \leq \|e\|_0/|d|$. Multiplying (49) by $\widehat{\varphi}_{-n}^{(k)}$ and summing over $n \neq 0$, we obtain

$$\begin{aligned} \|\varphi^{(k)}\|_0^2 &= \sum_{l=1}^{k-1} \sum_{n \neq 0} p_n \widehat{\varphi}_{-n}^{(k)} \left(\sum_m \widehat{\varphi}_m^{(l)} \widehat{\varphi}_{n-m}^{(k-l)} \right) \\ &\leq \sum_{l=1}^{k-1} \left(\sum_{n \neq 0} |p_n|^2 \right)^{1/2} \|\varphi^{(k)}\|_0 \|\varphi^{(l)}\|_0 \|\varphi^{(k-l)}\|_0, \end{aligned}$$

whereby

$$\|\varphi^{(k)}\|_0 \leq \sum_{l=1}^{k-1} \left(\sum_{n \neq 0} |p_n|^2 \right)^{1/2} \|\varphi^{(l)}\|_0 \|\varphi^{(k-l)}\|_0.$$

This establishes (using induction in k) that all $\varphi^{(k)}$ have finite norms $\|\varphi^{(k)}\|_0$.

In the Fourier space, equation (50) in

$$\theta(\xi) = \sum_n \widehat{\theta}_n e^{in\xi}$$

takes the form

$$\widehat{\theta}_n = p_n \sum_{k=K}^{2K-2} \epsilon^k \left(\sum_{l=1}^{k-1} \sum_m \widehat{\varphi}_m^{(l)} \widehat{\varphi}_{n-m}^{(k-l)} \right).$$

Multiplying it by $\widehat{\theta}_{-n}$ and summing over $n \neq 0$, we find

$$\begin{aligned} \|\theta\|_0^2 &= \sum_{k=K}^{2K-2} \left(\epsilon^k \sum_{l=1}^{k-1} \sum_{n \neq 0} p_n \widehat{\theta}_{-n} \left(\sum_m \widehat{\varphi}_m^{(l)} \widehat{\varphi}_{n-m}^{(k-l)} \right) \right) \\ &\leq \sum_{k=K}^{2K-2} \left(\epsilon^k \sum_{l=1}^{k-1} \left(\sum_{n \neq 0} |p_n|^2 \right)^{1/2} \|\theta\|_0 \|\varphi^{(l)}\|_0 \|\varphi^{(k-l)}\|_0 \right), \end{aligned}$$

and hence

$$\|\theta\|_0 \leq \sum_{k=K}^{2K-2} \left(\epsilon^k \sum_{l=1}^{k-1} \left(\sum_{n \neq 0} |p_n|^2 \right)^{1/2} \|\varphi^{(l)}\|_0 \|\varphi^{(k-l)}\|_0 \right) = O(\epsilon^K).$$

Q.E.D.

7 Concluding remarks

We have presented mathematical results concerning existence, uniqueness, spatial analyticity and well-posedness of space-periodic evolutionary and travelling-wave solutions to the RLWE with forcing and damping. This work has been necessitated by the ongoing intensive numerical study of various regimes exhibited by solutions to this equation [13,14,15,16,17,33,34,39].

The techniques used here to analyse the RLWE can also be applied to the closely related Korteweg-de Vries equation. Well-posedness of problems that can be stated for this equation is still a topic of active investigation. Under the condition of spatial periodicity, the Cauchy problem for the KdV equation was recently proved to be locally well-posed in a class of analytic functions that can be extended holomorphically in a symmetric strip of the complex plane around the x -axis [18]. While we have proved (section 3) that the width of the analyticity strip decays at most polynomially, it was shown in [18] that the uniform radius of spatial analyticity of solutions to the KdV equation does not shrink as time goes by.

In the limit of high wave speed, power series expansions of travelling-wave solutions to the RLWE and the KdV equation differ only in minor details. Thus, upon introduction of the necessary but non-essential modifications (in particular, $\beta = 2\alpha$ is now required in the ansatz (30), the simplest case being $\alpha = 1$, $\beta = 2$), our construction (see section 5) establishes the lack of continuity of space-periodic travelling-wave solutions to the KdV equation with respect to small-amplitude forcing. (Other exact travelling-wave solutions to the KdV equation with external forcing were recently derived, that involve Jacobi elliptic functions [35,24]; see also [12].)

The following problems remain open: Does no shrinking of the width of the analyticity strip occur for solutions to the RLWE as this happens for space-periodic solutions to the KdV equation? We have not proved convergence of the asymptotic power series that we have constructed for travelling-wave solutions in sections 3 and 6; do they converge? Are evolutionary solutions to the RLWE analytic in time? For the sake of completeness, one would like to extend our results on existence of travelling waves to cover the case of

the absence of damping ($d = 0$). Finally, we have not studied stability of our travelling-wave solutions to short- or large-scale perturbations; while the former problem can be addressed numerically, the latter one can be tackled by using the homogenisation methods similar to those employed in the study of the large-scale magnetic field generation [44].

Another extension of our work would be an investigation of the shallow-water wave equation proposed by Camassa and Holm [6], which could be modified by adding linear damping and external forcing.

Acknowledgments

ACLC and ELR thank the support of CNPq (Brazil) and FAPESP (Brazil). ACLC is grateful for the award of a Marie Curie International Incoming Fellowship and the hospitality of Paris Observatory (France). RC, OP and VZ were financed in part by the grant 11-05-00167-a from the Russian foundation for basic research. The two-month visits of OP and VZ to the Institute of Aeronautical Technology (Brazil) were supported by FAPESP (Brazil).

References

- [1] M. Abramowitz, I.A. Stegun, eds. Handbook of mathematical functions with formulas, graphs, and mathematical tables. Dover, New York (1965).
- [2] N.I. Akhiezer. Elements of the theory of elliptic functions. AMS Translations of Mathematical Monographs vol. 79. AMS, Rhode Island (1990).
- [3] T.B. Benjamin, J.L. Bona, J.J. Mahony. Model equations for long waves in nonlinear dispersive systems. Phil. Trans. R. Soc. London, Ser. A 272 (1972) 47–78.
- [4] J. Bona. On the stability theory of solitary waves. Proc. R. Soc. Lond. A 344 (1975) 363–374.
- [5] J.L. Bona, N. Tzvetkov. Sharp well-posedness results for the BBM equation. Discrete and Continuous Dynamical Systems 23 (2009) 1241–1252.
- [6] R. Camassa, D.D. Holm. An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. 71 (1993) 1661–1664.
- [7] A.C.-L. Chian, R.A. Miranda, E.L. Rempel, Y. Saiki, M. Yamada. Amplitude-phase synchronization at the onset of permanent spatiotemporal chaos. Phys. Rev. Lett. 104 (2010) 254102.

- [8] A.O. Çelebi, V.K. Kalantarov, M. Polat. Attractors for the generalized Benjamin–Bona–Mahony equation. *J. Diff. Equations* 157 (1999) 439–451.
- [9] I. Chuesov, M. Polat, S. Siegmund. Gevrey regularity of global attractor for generalized Benjamin–Bona–Mahony equation. *Matematicheskaya fizika, analiz, geometriya* 11 (2004) 226–242.
- [10] R.K. Dodd, J.C. Eilbeck, J.D. Gibbon, H.C. Morris. *Solitons and nonlinear wave equations*. Academic, London (1982).
- [11] P.G. Estévez, Ş. Kuru, J. Negro, L.M. Nieto. Travelling wave solutions of the generalized Benjamin–Bona–Mahony equation. *Chaos, Solitons and Fractals* 40 (2009) 2031–2040.
- [12] M.L. Gandarias, M.S. Bruzón. Some conservation laws for a forced KdV equation. *Nonlinear Analysis* 13 (2012) 2692–2700.
- [13] K. He. Crisis-induced transition to spatiotemporally chaotic motions. *Phys. Rev. E* 80 (1998) 696–699.
- [14] K. He, A.C.-L. Chian. On-off collective imperfect phase synchronization and bursts in wave energy in a turbulent state. *Phys. Rev. Lett.* 91 (2003) 034102.
- [15] K. He, A.C.-L. Chian. Critical dynamic events at the crisis of transition to spatiotemporal chaos. *Phys. Rev. E* 69 (2004) 026207.
- [16] K. He, A.C.-L. Chian. Nonlinear dynamics of turbulent waves in fluids and plasmas. *Nonlinear Processes in Geophysics* 12 (2005) 13–24.
- [17] K. He, A. Salat. Hysteresis and onset of chaos in periodically driven nonlinear drift waves. *Plasma Phys. Controlled Fusion* 31 (1989) 123–141.
- [18] A.A. Himonas, G. Petronilho. Analytic well-posedness of periodic gKdV. *J. Diff. Equations* 253 (2012) 3101–3112.
- [19] W. Horton. Nonlinear drift waves and transport in magnetized plasma. *Phys. Rep.* 192 (1990) 1–177.
- [20] H. Jafari, N. Kadkhoda, C.M. Khalique. Travelling wave solutions of nonlinear evolution equations using the simplest equation method. *Computers and Mathematics with Applications* 64 (2012) 2084–2088.
- [21] A.N. Kolmogorov, S.V. Fomin. *Elements of the theory of functions and functional analysis*. Dover, New York (1999).
- [22] N.A. Kudryashov. Simplest equation method to look for exact solutions of nonlinear differential equations. *Chaos, Solitons and Fractals* 24 (2005) 1217–1231.
- [23] N.A. Kudryashov. Exact solitary waves of the Fisher equation. *Phys. Lett. A* 342 (2005) 99–106.
- [24] N. Kudryashov. On new travelling wave solutions of the KdV and the KdV-Burgers equations. *Communications in Nonlinear Science and Numerical Simulation* 14 (2009) 1891–1900.

- [25] O.A. Ladyzhenskaya. The mathematical theory of viscous incompressible flow. Gordon and Breach, New York, London (1969); 2nd edn. Nauka, Moscow (1970) (in Russian).
- [26] J. Leray, J. Schauder. Topologie et equations fonctionelles. *Ann. Sci. École Norm. Sup.* 13 (1934) 45–78.
- [27] L.A. Liusternik, V.I. Sobolev. Elements of functional analysis. Frederick Ungar Publ., New York (1961).
- [28] L.A. Medeiros, G.P. Menzala. Existence and uniqueness for periodic solutions of the Benjamin–Bona–Mahony equation. *SIAM J. Math. Anal.* 8 (1977) 792–799.
- [29] L.A. Medeiros, M.M. Miranda. Weak solutions for a nonlinear dispersive equation. *J. Math. Analysis and Applications* 59 (1977) 432–441.
- [30] J. Nickel. Elliptic solutions to a generalised BBM equation. *Phys. Lett. A* 364 (2007) 221–226.
- [31] M. Panthee. On the ill-posedness result for the BBM equation. *Discrete and Continuous Dynamical Systems* 30 (2011) 253–259.
- [32] D.H. Peregrine. Calculations of the development of an undular bore. *J. Fluid Mech.* 25 (1966) 321–330.
- [33] E.L. Rempel, A.C.-L. Chian. Origin of transient and intermittent dynamics in spatiotemporal chaotic systems. *Phys. Rev. Lett.* 98 (2007) 014101.
- [34] E.L. Rempel, R.A. Miranda, A.C.-L. Chian. Spatiotemporal intermittency and chaotic saddles in the regularized long-wave equation. *Phys. Fluids* 21 (2009) 074105.
- [35] A.H. Salas. Computing solutions to a forced KdV equation. *Nonlinear Analysis. Real World Applications* 12 (2011) 1314–1320.
- [36] R.E. Showalter. Sobolev equations for nonlinear dispersive systems. *Appl. Anal.* 7 (1978) 297–308.
- [37] M. Stanislavova. On the global attractor for the damped Benjamin–Bona–Mahony equation. *Discrete and continuous dynamical systems. Suppl.* vol. (2005) 824–832.
- [38] M. Stanislavova, A. Stefanov, B. Wang. Asymptotic smoothing and attractors for the generalized Benjamin–Bona–Mahony equation on R^3 . *J. Diff. Equations* 219 (2005) 451–483.
- [39] B.A. Toledo, A.C.-L. Chian, E.L. Rempel, R.A. Miranda, P.R. Muñoz, J.A. Valdivia. Wavelet-based multifractal analysis of nonlinear time-series: the earthquake-driven tsunami of 27 February 2010 in Chile. *Phys. Rev. E* 87 (2013) 022821.
- [40] B. Wang. Strong attractors for the Benjamin–Bona–Mahony equation. *Appl. Math. Lett.* 10 (1997) 23–28.

- [41] B. Wang, W. Yang. Finite-dimensional behaviour for the Benjamin–Bona–Mahony equation. *J. Phys. A: Math. Gen.* 30 (1997) 4877–4885.
- [42] L. Zeng. Existence and stability of solitary-wave solutions of equations of Benjamin–Bona–Mahony type. *J. Diff. Equations* 188 (2003) 1–32.
- [43] V. Zheligovsky. A priori bounds for Gevrey–Sobolev norms of space-periodic three-dimensional solutions to equations of hydrodynamic type. *Advances in differential equations* 16 (2011) 955–976 [<http://arxiv.org/abs/1001.4237>].
- [44] V. Zheligovsky. Large-scale perturbations of magnetohydrodynamic regimes: linear and weakly nonlinear stability theory. *Lecture Notes in Physics*, vol. 829. Springer-Verlag, Heidelberg (2011).