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Differential Inclusion Approach for Mixed Constrained Problems Revisited

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Abstract Properties of control systems described by differential inclusions are well established in the literature. Of special relevance to optimal control problems are properties concerning measurability, convexity, compactness of trajectories and Lipschitz continuity of the set valued mapping (or multifunction) defining the differential inclusion of interest. In this work we concentrate on dynamic control systems coupled with mixed state-control constraints. We characterize a class of such systems that can be described by an appropriate differential inclusion defined by a set valued mapping exhibiting "good" properties. We illustrate the importance of our findings with respect to existence of solution of optimal control problems.

Keywords Differential Inclusion · Mixed Constraints · Optimal Control

1 Introduction

Control systems described in terms of differential inclusions have been extensively studied in the literature (see, e.g., [2,3,8,10,17,19,22,23] to name but a few). Differential inclusions appear in control theory when dynamical systems are expressed as

$$\dot{x}(t) \in F(t, x(t)), \tag{1}$$

where $t \in I \subset \mathbb{R}$, $x \in \mathbb{R}^n$ and F is a set valued mapping (or multifunction) with closed values in $\mathbb{R}^n$. Such systems make it possible to study in a uniform way a large number of control problems (in this respect see for example [8]). Differential inclusion control problems have proved to be a useful framework for optimal control. For example, they are convenient to state conditions under

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which existence of solution is ensured and an useful tool to derive optimality conditions (see [8] and [23], for example).

It is commonly accepted that differential inclusions are a natural framework" to study dynamical systems with mixed state-control constraints (see [23], pp. 38). Such approach has been used, for example, in [10], [14] and, recently, in [11], [12] and [13]. In particular, it is essential to establish under which conditions the set of trajectories of control systems described in terms of ordinary differential equations coincide with the set of trajectories satisfying (1). In this respect many questions arise as those of measurability of the set valued mapping defining the differential inclusion (so existence of measurable selections is guaranteed), compactness of trajectories, convexity properties (two subjects relevant for the existence of solution to optimal control problems), etc. Although such aspects are clearly and concisely treated in the literature for control systems of the form

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [a, b], \\ u(t) \in U(t) \quad \text{a.e.} \quad t \in [a, b], \end{cases}$$

(see, for example, Chapter 2 in [23]), the same cannot be said when control systems are coupled with mixed constraints. The system of interest, herein denoted as (Σ), involves the dynamics

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e.} \quad t \in [a, b],$$

mixed constraints

$$(x(t), u(t)) \in S(t) \quad \text{a.e.} \quad t \in [a, b], \tag{2}$$

and boundary conditions

$$(x(a), x(b)) \in E. \tag{3}$$

The data comprises a fixed interval $[a, b]$, a function $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, a set valued mapping $S : [a, b] \to \mathbb{R}^n \times \mathbb{R}^k$ and a set $E \subset \mathbb{R}^n \times \mathbb{R}^n$.

A pair (x, u) comprising an absolutely continuous function x (the state trajectory) and a measurable function u (the control) satisfying all the constraints of (Σ) will be called throughout a *process*.

Our aim is to establish conditions on the data of (Σ) that translate on useful properties for the set valued mappings defining the corresponding differential inclusion. To highlight the required properties, while keeping exposition as simple as possible, we work under assumptions that maybe viewed as too strong. In particular, we restrict our analysis to systems generating bounded set valued mappings. Although unbounded set valued mappings may be of use in some cases (and in this respect we refer the reader to [10], [11], [12] and [13]), systems with bounded set valued mapping remain of interest for applications.

Clearly, conditions under which the state trajectories for (Σ) coincide with the trajectories of a certain differential inclusion

$$\dot{x}(t) \in F_m(t, x(t)) \quad \text{a.e.} \quad t \in [a, b],$$

(where F_m is a set valued mapping to be defined shortly) satisfying the boundaries constraints (3) will be central in our analysis. We will pay particular attention to the case where

$$S(t) := \{(x, u) \in \mathbb{R}^n \times U : \ g(t, x, u) \leq 0\}, \tag{4}$$

where $U \subset \mathbb{R}^k$ and $g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^m$. We, however, do not limit our discussion to this case.

We emphasize that the contribution of this paper does not reside on the novelties of the results but rather on gathering them together as well as on the accompanying discussion and the presentation of its proofs. Our work also highlights the importance of a *bounded slope* condition imposed on the mixed constraints in the vein [10] and its relation with Lipschitz properties of the set valued mappings in our setting. Another aspect of relevance in our discussion resides on convexity assumptions on set valued mappings, a discussion accompanied by simple but illustrative examples.

This paper is organized in the following way. In the next section we state an auxiliary result that will be relevant to our analysis. In section 3, we introduce the main ingredients of our work as well as the main assumptions. More assumptions, this time for the case when $S(t)$ is as in (4), are presented in section 4 followed by results relating these assumptions with those in the previous section. Convexity of $F - m$ and that of various set valued mapping related to our task are discussed in section 6. In section 7 we establish measurability and Lipschitz properties of the relevant set valued mappings. Results on existence of solution to some optimal control problems with mixed constraints are presented in section 8. We finish this paper with conclusions. This is section 9.

Notations: If $g \in \mathbb{R}^m$, the inequality $g \leq 0$ is interpreted component-wise. Define $\mathbb{R}^m_- = \{\xi \in \mathbb{R}^m : \xi \leq 0\}$ and likewise for $\mathbb{R}^m_+$. Also, $|\cdot|$ is the Euclidean norm or the induced matrix norm on $\mathbb{R}^{p\times q}$. The closed unit ball centred at the origin is denoted by $\bar{B}$ whereas B denotes the open unit ball, regardless of the dimension of the underlying space. For any set $A \subset \mathbb{R}^p$, int A and co A denotes the interior and convex hull of C. For any closed set $A \subset \mathbb{R}^p$ the distance of a point $x \in \mathbb{R}^p$ to the set A is defined as

$$d_A(x) = \inf\{|x - a| : \ a \in A\}.$$

If $\Omega \subset \mathbb{R}^p$ and $F : \Omega \to \mathbb{R}^q$ is a set valued mapping, then the graph of F is defined as

$$\text{Gr } F := \{(x, y) \in \Omega \times \mathbb{R}^q : \ y \in F(x)\}.$$

We say that a set $S \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ is $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ -measurable when we refer to measurability relative to the σ-field generated by the products of Lebesgue measurable subsets in $\mathbb{R}$, Borel measurable subsets in $\mathbb{R}^n$ and Borel measurable subsets in $\mathbb{R}^m$.

Consider now a function $h : [a, b] \to \mathbb{R}^p$. We say that $h \in W^{1,1}([a, b]; \mathbb{R}^p)$ if and only if h is absolutely continuous and that $h \in L^1([a, b]; \mathbb{R}^p)$ iff h is integrable.

Take $A \subset \mathbb{R}^p$ to be a closed set with and consider $x^* \in A$. Also let $f : \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. With respect to f, $x^* \in \mathbb{R}^k$ will be such that $f(x^*) < +\infty$. Concerning nonsmooth analysis we use the following notation: $N_A^P(x^*)$ is the *proximal normal cone to* A at x^*, $N_A^L(x^*)$ is the *limiting normal cone* to A at x^* (also known as *Mordukhovich normal cone*), $N_A^C(x^*)$ is the *Clarke normal cone to* A at x^*, $\partial^L f(x^*)$ is *limiting subdifferential* or *Mordukhovich subdifferential* of f at x^* and $\partial^C f(x^*)$ is *(Clarke) subdifferential* of f at x^*. If f is Lipschitz continuous near x^*, the convex hull of the limiting subdifferential, co $\partial^L f(x^*) = \partial^C f(x^*)$.

2 Auxiliary Result

Before proceeding we state an adaptation of Theorem 3.5.2 in [10] that will be important in the forthcoming analysis.

Let $\Gamma : [a, b] \times \mathbb{R}^n \to \mathbb{R}^k$ be a set valued mapping. For each $t \in [a, b]$, set $G(t)$ to be the graph of $x \to \Gamma(t, x)$. Assume $G(t)$ to be closed for each t and let $(x^*, u^*) \in G(t)$.

(BS') There exist $\epsilon > 0$, $R > 0$ and $\mathcal{K} > 0$ such that, for almost every t, the following condition holds

$$x \in B(x^*, \epsilon), \quad u \in B(u^*, R), \quad (\alpha, \beta) \in N^P_{G(t)}(x, u) \implies |\alpha| \leq \mathcal{K}|\beta|.$$

We emphasize that we are assuming the parameters R and $\mathcal{K}$ to be independent of t. As the reader may suspect a more general definition involves such parameters as (measurable) functions of t. Here and throughout this paper, we will remove the dependency of t of many parameters in our assumptions as long as this allows us to skip some technical details while retaining their significance. For a complete discussion on bounded slope condition [BS'] and pseudo- Lipschitz properties of set valued mappings we refer the reader to [10] and [13].

The following theorem asserts that a set valued mapping $x \to \Gamma(t, x)$, satisfying (BS'), is pseudo-Lipschitz.

Theorem 1 (adaptation of Theorem 3.5.2 in [10]) *Assume that the set valued mapping $x \to \Gamma(t, x)$ satisfies (BS') where $(x^*, u^*) \in G(t)$. Then, for any $\xi \in]0, 1[$ and any $x_1, x_2 \in B(x^*, \bar{\epsilon})$, the following holds*

$$\Gamma(t, x_1) \cap \bar{B}(u^*, (1 - \xi)R) \subset \Gamma(t, x_2) + \mathcal{K}|x_1 - x_2|\bar{B},$$

where $\bar{\epsilon} = \min\{\epsilon, \xi R/3\mathcal{K}\}$.

3 Preliminaries and Main Assumptions

Mixed constraints, also known as state dependent control constraints, can be written in the general form (2). Here we associate with S the set valued mapping $S_m : [a,b] \times \mathbb{R}^n \to \mathbb{R}^k$ defined as

$$S_m(t,x) = \left\{ u \in \mathbb{R}^k : \ (x,u) \in S(t) \right\}.$$

For each $t \in [a,b]$ the set $S(t)$ is the graph of $x \to S_m(t,x)$, that is,

$$(x,u) \in S(t) \iff u \in S_m(t,x).$$

Define also the set valued mapping

$$F_m(t,x) := \{ f(t,x,u) : \ u \in S_m(t,x) \} \tag{5}$$

Take any absolutely continuous function $x^* : [a,b] \to \mathbb{R}^n$ and define, for some fixed parameter $\epsilon > 0$

$$X(t) := x^*(t) + \epsilon \bar{B} \quad \text{and} \quad S^*_\epsilon(t) := S(t) \cap \big((x^*(t) + \epsilon B) \times \mathbb{R}^k \big). \tag{6}$$

It is important to notice that $S^*_\epsilon(t)$ is defined as the intersection of $S(t)$ with the open ball $x^*(t) + \epsilon B$, not with $X(t)$.

When appropriate, we will impose that the function x^* satisfies the differential inclusion

$$\dot{x}^*(t) \in F_m(t, x^*(t)). \tag{7}$$

We now state several assumptions that will be use in the forthcoming analysis. These make reference to the parameter ϵ through the definition of (6). Let $\phi : [a,b] \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^p$ be a general function (ϕ may then replaced by f or g).

(B1) The function $t \to \phi(t,x,u)$ is $\mathcal{L}$-measurable for all $(x,u) \in \mathbb{R}^n \times \mathbb{R}^k$.

(B2) The set valued mapping S is $\mathcal{L}$-measurable and, for each $t \in [a,b]$, $S(t)$ is closed.

(B3) The set E is closed.

(B4) For each $t \in [a,b]$ and $x \in X(t)$, there exists $u \in \mathbb{R}^k$ such that $(x,u) \in S(t)$. Furthermore, each $t \in [a,b]$ there exists a constant σ such that

$$(x,u) \in S(t) \implies |u| < \sigma.$$

(BS) There exists a constant $\mathcal{K} > 0$ such that, for almost every $t \in [a,b]$ and all $(x,u) \in S^*_\epsilon(t)$,

$$(\alpha, \beta) \in N^P_{S(t)}(x,u) \implies |\alpha| \le \mathcal{K} |\beta|.$$

(CA) For all $t \in [a,b]$ and $x \in X(t)$, $F_m(t,x)$ is convex.

(LC) There exist constants k_x^ϕ and k_u^ϕ such that, for almost every $t \in [a,b]$ and all $(x_i, u_i) \in S_\epsilon^*(t)$ $(i = 1, 2)$, we have

$$|\phi(t, x_1, u_1) - \phi(t, x_2, u_2)| \leq k_x^\phi |x_1 - x_2| + k_u^\phi |u_1 - u_2|.$$

The forthcoming analysis could be done under assumptions weaker than those above. In particular, the existence of the uniform bound on u's in (B4) could be removed and the parameters $\mathcal{K}$ in (BS) and k_x^ϕ and k_u^ϕ in (LC) could be taken to be merely measurable functions (as in [11]) instead of constants. As mentioned in the Introduction, we opt for the strengthened assumptions above to keep the exposition as simple as possible.

4 On $S(t)$ defined by (4)

We now concentrate on the case where the set $S(t)$ is defined by (4), i.e.,

$$S(t) := \{(x, u) \in \mathbb{R}^n \times U : \ g(t, x, u) \leq 0\}.$$

We state conditions on $U \subset \mathbb{R}^k$ and g that imply the previous assumptions imposed on a general S . They are the following.

(ICU) The set U is compact and, for each $x \in \mathbb{R}^n$, there exists a $u \in U$ such that $g(t, x, u) \leq 0$.

(IMC) There exists a constant M such that, for almost every t, all $(x, u) \in S_\epsilon^*(t)$, $\eta \in N_U^L(u)$, $\gamma \in \mathbb{R}_+^m$ with $\langle \gamma, \ g(t, x, u) \rangle = 0$, we have

$$(\alpha, \beta - \eta) \in \partial_{x,u}^L \langle \gamma, \ g(t, x, u) \rangle \implies |\gamma| \leq M|\beta|.$$

When $(x, u) \to g(t, x, u)$ is continuously differentiable, (IMC) is related to well known linear independence of the gradients of g (we refer the reader to [11] for restatements and discussion of (IMC) under these circumstances).

Our first result, Lemma 1, shows that (B1) and (LC) imposed on g together with (ICU) imply (B2) and (B4).

Lemma 1 *Consider $S(t)$ as defined in (4). Assume g satisfies (B!) and (LC) and that (ICU) holds. Then $t \to S(t)$ is a Lebesgue measurable set valued mapping and, for each t, the set $S(t)$ is nonempty and closed.*

Proof For each $t \in [a, b]$, $S(t)$ is nonempty by (ICU). By (LC) we know that g is a Carathéodory function. Then Proposition 14.33 in [21] asserts that $S(t)$ is a closed set for each t and $t \to S(t)$ is Lebesgue measurable. $\square$

Our next task is to investigate the relation between (IMC) and (BS). The following characterization of $N_{S(t)}^L(x, u)$ will be a cornerstone in this respect.

Lemma 2 *Consider $S(t)$ as defined by (4). Assume (ICU) and (IMC) hold and that g satisfies (B1) and (LC). Then for almost every $t \in [a,b]$, all $(x,u) \in S^*_\epsilon(t)$ and all $(\alpha,\beta) \in N^L_{S^*_\epsilon(t)}(x,u)$, there exists an $\gamma \geq 0$ with $\langle \gamma, g(t,x,u)\rangle = 0$ such that*

$$(\alpha,\beta) \in \partial^L_{(x,u)}\langle \gamma, g(t,x,u)\rangle + \{0\} \times N^L_U(u). \tag{8}$$

We postpone the proof of this Lemma (and it will appear in the end of this section) and we go straight to an important consequence of it, relating (IMC) with (BS).

Lemma 3 *Under the assumptions of Lemma 2, (BS) holds.*

Proof Choose any $t \in [a,b]$ such that the properties in (IMC) and (LC) hold. Let any $(\alpha,\beta) \in N^P_{S(t)}(x,u)$. Since $S^*_\epsilon(t) \subset S(t)$ we have $N^P_{S(t)}(x,u) \subset N^P_{S^*_\epsilon(t)}(x,u)$. On the other hand, we also have $N^P_{S^*_\epsilon(t)}(x,u) \subset N^L_{S^*_\epsilon(t)}(x,u)$ (by Proposition 4.2.6 in [23]). Thus $(\alpha,\beta) \in N^L_{S^*_\epsilon(t)}(x,u)$ and it follows from Lemma 2 and (IMC) that for $\gamma \geq 0$ with $\langle \gamma, g(t,x,u)\rangle = 0$,

$$\eta \in N^L_U(u), \quad (\alpha, \beta - \eta) \in \partial^L_{(x,u)}\langle \gamma, g(t,x,u)\rangle \Longrightarrow |\gamma| \leq M\ |\beta|.$$

By (LC) the function $(x,u) \to \langle \gamma, g(t,x,u)\rangle$ is Lipschitz continuous with constant $|\gamma| \max\{k^g_x, k^g_u\}$. Appealing now to the properties of subdifferentials, we deduce that

$$|\alpha| \leq |(\alpha, \beta-\eta)| \leq \max\{k^g_x, k^g_u\}|\gamma| \leq \max\{k^g_x, k^g_u\}M|\beta|,$$

proving that (BS) holds with $\mathcal{K} = \max\{k^g_x, k^g_u\}M$. $\square$

Proof of Lemma 2: Let $t \in [a,b]$ be such that (IMC) holds. Let $\varphi(x,u) = g(t,x,u)$ and set

$$C_1(t) = \varphi^{-1}(\mathbb{R}^m_-) \quad \text{and} \quad C_2(t) = X(t) \times U.$$

Take any

$$(x,u) \in S^*_\epsilon(t), \quad \text{and} \quad (\alpha,\beta) \in N^L_{S^*_\epsilon(t)}(x,u).$$

The proof is done in two steps. We first characterize $N^L_{C_1(t)}(\varphi(x,u))$ in terms of $\partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle$. This is done appealing to Corollary 10.50 in [21]. In the second step we invoke Theorem 6.42 in [21] to show that

$$N_{S^*_\epsilon(t)}(x,u) \subset N^L_{C_1(t)}(x,u) + N^L_{C_2(t)}(x,u).$$

Step 1: We claim that if $\gamma \in N^L_{\mathbb{R}^m_-}(\varphi(x,u))$ is such that

$$(0,0) \in \partial^L_{(x,u)}\langle \gamma, \varphi(x,u)\rangle,$$

then $\gamma = 0$. To prove our claim, take any such γ. Since $\gamma \in N^L_{\mathbb{R}^m_-}(\varphi(x,u))$ and $\varphi(x,u) \leq 0$, we have

$$\langle \gamma, \varphi(x,u)\rangle = 0, \quad \gamma \geq 0.$$

Appeal to (IMC) and the fact that $0 \in N_U^L(u)$ to deduce that $|\gamma| \leq 0$. Thus $\gamma = 0$ proving our claim. We are then in position to apply Corollary 10.50 in [21] which leads to the conclusion that

$$N^L_{C_1(t)}(x,u) \subset \bigcup \left\{ \partial^L_{(x,u)} \langle \gamma, \varphi(x,u) \rangle : \ \gamma \in N^L_{\mathbb{R}^m_-}(\varphi(x,u)) \right\}. \tag{9}$$

This means that if $(v_1, v_2) \in N^L_{C_1(t)}(x,u)$, then there exists a $\gamma \geq 0$ such that $\langle \gamma, \varphi(x,u) \rangle = 0$ and $(v_1, v_2) \in \partial^L \langle \gamma, \varphi(x,u) \rangle$.

Step 2: We verify the conditions under which Theorem 6.42 in [21] hold. Our first task is then to prove that $N^L_{C_1(t)}(x,u)$ and $N^L_{C_2(t)}(x,u)$ are transversal in (x,u), i.e., that

$$(\xi, \zeta) \in -N^L_{C_1(t)}(x,u) \cap N^L_{C_2(t)}(x,u) \implies (\xi, \zeta) = (0,0). \tag{10}$$

Since $N^L_{C_2(t)}(x,u) = N^L_{X(t)}(x,u) \times N^L_U(x,u)$ and $x \in \text{int} X(t)$ (by definition of $S^*_\epsilon(t)(t)$), we have $\xi = 0$ and $\zeta \in N^L_U(x,u)$. We deduce from (9) that, for some $\gamma \in \gamma \in N^L_{\mathbb{R}^m_-}(\varphi(x,u))$,

$$(0, -\zeta) \in \partial^L \langle \gamma, \varphi(x,u) \rangle.$$

Invoking now (IMC) with $\alpha = 0$, $\beta = 0$ and $\eta = \zeta$, we conclude that $\gamma = 0$. But then $(0, \zeta) = (0,0)$, proving (10). We are now in position to invoke Theorem 6.42 in [21] to conclude that

$$N_{S^*_\epsilon(t)}(x,u) \subset N^L_{C_1(t)}(x,u) + N^L_{C_2(t)}(x,u).$$

It follows from the above that (8) holds, proving the Lemma. □

5 Convexity Assumption on $F_m(t,x)$

We now seek verifiable sufficient conditions for the convexity of the set valued mapping F_m.

We first turn our attention to the case when $S(t)$ is defined by (4). This means that we work with

$$F_m(t,x) := \{ f(t,x,u) : \ u \in S_m(t,x) \} \tag{11}$$

and

$$S_m(t,x) := \{ u \in U : \ g(t,x,u) \leq 0) \}. \tag{12}$$

Given this structure of $S(t)$ one may be tempted to think that easier verifiable conditions for the convexity of F_m would involve S_m and possibly the set valued mappings

$$F(t,y) = \{ (f(t,y,u), g(t,y,u)) : \ u \in U \} \tag{13}$$

and

$$F^f(t,y) = \{ f(t,y,u) : \ u \in U \}, \quad G^g(t,y) = \{ g(t,y,u) : \ u \in U \}. \tag{14}$$

This is however not always the case as the next Lemma shows.

Lemma 4 *Take any $(t,x) \in [a,b] \times \mathbb{R}^n$ and let $F_m(t,x)$, $S_m(t,x)$, F, F^f and G^g be as in (11), (12), (13) and (14). The following relations hold:*

(a) $F(t,x)$ convex $\Longrightarrow F_m(t,x)$ convex, but the opposite implication does not hold.

(b) $F(t,x)$ convex $\Longrightarrow F^f(t,x)$ and $G^g(t,x)$ are convex, but the opposite implication does not hold.

(c) The convexity of $F_m(t,x)$ does not imply the convexity of $F^f(t,x)$ and $G^g(t,x)$ and the opposite implication does not hold.

(d) The convexity of $F_m(t,x)$ does not imply the convexity of $S_m(t,x)$ and the opposite implication does not hold.

Proof

(a) $F(t,x)$ **convex** $\Longrightarrow F_m(t,x)$ **convex.**

Take any $v_1,\ v_2 \in F_m(t,x)$. Then there exist $u_1,\ u_2 \in U$ such that $v_1 = f(t,x,u_1)$, $v_2 = f(t,x,u_2)$, $g(t,x,u_1) \leq 0$ and $g(t,x,u_2) \leq 0$. Set $z_i = g(t,x,u_i)$, $i = 1,2$. We have $(v_i,z_i) \in F(t,x)$, $i = 1,2$. Since $F(t,x)$ is convex, for any $\beta \in [0,1]$, there exists $u \in U$ such that $(v,z) = \beta(v_1,z_1) + (1-\beta)(v_2,z_2) = (f(t,x,u), g(t,x,u))$. But $z = \beta z_1 + (1-\beta) z_2 = g(t,x,u) \leq 0$. Thus $v \in F_m(t,x)$ proving convexity of $F_m(t,x)$.

$F_m(t,x)$ **convex** $\not\Longrightarrow F(t,x)$ **convex.**

Take $U = [-1,1]$, $f(t,x,u) = u$ and $g(t,x,u) = -(u+1)^2$. Then for any x, we have $S_m(t,x) = [-1,1]$ and $F_m(t,x) = [-1,1]$, both convex sets. However,

$$F(t,x) = \left\{(u, -(u+1)^2) : \ u \in U\right\}$$

is not a convex set.

(b) $F(t,y)$ **convex** $\Longrightarrow F^f(t,y)$ **and** $G^g(t,y)$ **are convex.**

Fix y and take any $v_1,\ v_2 \in F(t,x)$. Then there exist $u_1,\ u_2 \in U$ such that $v_1 = f(t,x,u_1)$ and $v_2 = f(t,x,u_2)$. Set $z_1 = g(t,x,u_1)$ and $z_2 = g(t,x,u_2)$. Then, for any $\beta \in [0,1]$ $(v,z) = \beta(v_1,z_1) + (1-\beta)(v_2,z_2)$ is such that $(v,z) \in F(t,x)$, i.e, there exists $u \in U$ such that $(v,z) = (f(t,x,u), g(t,x,u))$. It follows that $v \in F^f(t,x)$ and $z \in G^g(t,x)$ proving convexity of $F^f(t,x)$ and $G^g(t,x)$.

$F^f(t,x)$ **and** $G^g(t,x)$ **convex** $\not\Longrightarrow F(t,x)$ **convex.**

To see this it is enough to define $U = [-1,1]$, $f(t,x,u) = u^2$, and $g(t,x,u) = u$. Then $F^f(t,x) = [0,1]$, $G^g(t,x) = [-1,1]$ and $F(t,x) = \{(u^2,u) : \ u \in [-1,1]\}$, not convex.

(c) $F_m(t,x)$ **convex** $\not\Longrightarrow F^f(t,x)$ **and** $G^g(t,x)$ **convex.**

Take $U = [-1,1]$, $f(t,x,u) = u$ and $g(t,x,u) = (-u, u^3 - u)$. Then both $S_m(t,x) = [0,1]$ and $F_m(t,x) = [0,1]$ are convex. On the other hand, although $F^f(t,x) = [-1,1]$ is convex, $G^g(t,x) = \{(-u, u^3 - u) : \ u \in [-1,1]\}$ is not.

$F^f(t,x)$ **and** $G^g(t,x)$ **are convex** $\not\Longrightarrow F_m(t,x)$ **convex.**

Take $U = [1,1]$, $f(t,x,u) = u$ and $g(t,x,u) = -u^2 + 1/4$. Then $F^f(t,x) = [-1,1]$ and $G^g(t,x) = [-3/4, 1/4]$ are both convex. However, $F_m(t,x) = [-1,-1/2] \cup [1/2,1]$ is not convex.

(d) $F_m(t,x)$ **is convex** $\not\Rightarrow$ $Sm(t,x)$ **convex.**
Let $S_m(t,x) = \{u \in [-2,2] : \ -(u^2-1) \leq 0\} = [-2,-1] \cup [1,2]$ and $F_m(t,x) := \left\{u^2 : u \in S_m(t,x)\right\}$. Then $F_m(t,x) = [1,4]$ is convex although $S_m(t,x)$ is not.
$S_m(t,x)$ **is convex** $\not\Rightarrow$ $F_m(t,x)$ **convex.**
Take $U = [-1,1]$. Then $S_m(t,x) = \{u \in U : u \leq 0\} = [-1,0]$ is convex but $F_m(t,x) = \{(u,u^2) : \ u \in S_m(t,x)\}$ is not; $F_m(t,x)$ coincides with the graph of $m(u) = u^2$ for $u \in [0,1]$.

□

We summarize our findings:

$$\boxed{\begin{array}{lll} F(t,x) \text{ convex} \begin{array}{c}\Longrightarrow\\ \not\Longleftarrow\end{array} & F^f(t,x),\ G^g(t,x) \text{ convex} & \\ \quad\Downarrow\ \not\Uparrow & \quad\not\Uparrow\ \not\Downarrow & \\ F_m(t,x) \text{ convex} \Longleftrightarrow & F_m(t,x) \text{ convex} & \begin{array}{c}\not\Longrightarrow\\ \not\Longleftarrow\end{array} S_m(t,x) \text{ convex} \end{array}}$$

So far the only sufficient condition for the convexity of F_m we have established is the convexity of F. However, convexity of F is quite a strong condition. Indeed, there are simple cases where $F_m(t,x)$ is convex but $F(t,x)$ is not (see, for example, the proof of (d) of Lemma 4).

For $S(t)$ defined as in (4), convexity of the set U and that of the function $u \to g(t,x,u)$ implies convexity of $S_m(t,x)$ (a simple matter to prove)[1]. But this alone is not enough to guarantee that convexity of F_m as proved in Lemma (4). Convexity of F_m is a property of the geometry of the set $F_m(t,x)$ and it does not imply nor is implied by the convexity of the function $u \to f(t,x,u)$, as the following examples illustrate.

Example 1 For the choice of $U = \mathbb{R}$, $g(t,x,u) = |u| - 1$ and $f(t,x,u) = (u,u^2)$, we have $S_m(t,x) = [-1,1]$ convex but $F_m(t,x) = \{(u,u^2) : \ u \in [-1,1]\}$ is not convex. Observe that f is a convex vector valued function since both its components are convex real valued functions. □

Example 2 Recover now the example in the proof of (c) of Lemma 4: $U = [1,1]$, $f(t,x,u) = u$ and $g(t,x,u) = -u^2 + 1/4$. Then $u \to f(t,x,u)$ is a convex function, but $F_m(t,x) = [-1,-1/2] \cup [1/2,1]$ is not. Here the pathological aspect is the fact that $S_m(t,x)$ fails to be convex. □

Example 3 Let $f(t,x,u) = u - u^3$, $g(t,x,u) = |u| - 1$ and $U = \mathbb{R}$. Then we have $S_m(t,x) = [-1,1]$ and $F_m(t,x) = [-\frac{2\sqrt{3}}{9}, \frac{2\sqrt{3}}{9}]$, both convex sets. Here, however, the function f is not convex. □

However, and for any general $S(t)$ (not necessarily defined as in (4)), convexity of $F_m(t,x)$ follows from convexity of $S_m(t,x)$ when f is of the form

$$f(t,x,u) = f_1(t,x) + f_2(t,x)u.$$

[1] However, $S_m(t,x)$ and U can be convex sets while $u \to g(t,x,u)$ is not. Take for example $U = [-1,1]$ and $g(t,x,u) = u - u^3$. Then $S_m(t,x) = [-1,0]$ is convex.

Here $f_1 : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $f_2 : [a,b] \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$. To see this, take $\gamma_1, \gamma_2 \in F_m(t,x)$. Then there exist $u_1, u_2 \in S_m(t,x)$ such that

$$\gamma_1 = f_1(t,x) + f_2(t,x)u_1, \quad \gamma_1 = f_1(t,x) + f_2(t,x)u_2.$$

For any $\alpha \in [0,1]$ there exists a $\tilde{u} = \alpha u_1 + (1-\alpha)u_2 \in S_m(t,x)$ and

$$\alpha\gamma_1 + (1-\alpha)\gamma_2 = f_1(t,x) + f_2(t,x)\tilde{u},$$

proving the convexity of $F_m(t,x)$. We gather our findings in the Lemma below.

Lemma 5 *Consider any set valued mapping F_m as defined in (11) for any general $S_m(t,x)$. If $S_m(t,x)$ is convex and $f(t,x,u) = f_1(t,x) + f_2(t,x)u$, where $f_1 : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ and $f_2 : [a,b] \times \mathbb{R}^n \to \mathbb{R}^{n\times k}$, then $F_m(t,x)$ is convex.*

In particular, if U is convex and $u \to g(t,x,u)$ is convex, then $S_m(t,x)$ is convex.

However, we recall the reader that $F_m(t,x)$ may be convex even when $S_m(t,x)$ fails to be convex (this is (d) of Lemma 4).

In some situations where $F_m(t,x)$ fails to be convex, relaxation can be of help (see, for example, [4] and [23]). In this respect, and as expected, the set valued mapping $\operatorname{co} F_m(t,x)$ plays an important rule.

In the special case where $S(t)$ is as defined in (4), $\operatorname{co} F_m(t,x)$, in view of Carathéodory's Theorem ([23], e.g.), leads to

$$(\Sigma_{rlx}) \begin{cases} \dot{x}(t) = \displaystyle\sum_{i=1}^{n+1} \lambda_i(t) f(t,x(t),u_i(t)), \quad \text{a.e.,} \\ g(t,x(t),u_i(t)) \leq 0, \ \ i = 1,\ldots,n+1, \quad \text{a.e.,} \\ (\lambda_1(t),\ldots,\lambda_{n+1}(t)) \in \Lambda, \quad \text{a.e.} \\ u_i(t) \in U, \quad \text{a.e. for } i = 1,\ldots,n+1 \end{cases} \tag{15}$$

where $\Lambda := \{\lambda' \in \mathbb{R}^{n+1} : \lambda' \geq 0 \text{ and } \sum_{i=1}^{n+1} \lambda'_i = 1\}$.

6 Properties of the Set Valued Mappings S_m and F_m

Under our set of assumptions we now establish important properties of the set valued mappings S and F_m. Throughout this section we focus on a general S. Taking into account the results of section 4, our results also applied when $S(t)$ is defined by (4).

Lemma 6 *Assume that (B2) and (B4) hold and that f satisfies (B1) and (LC). Then*

(a) For almost every $t \in [a,b]$ and each $x \in X(t)$, the sets $S_m(t,x)$ and $F_m(t,x)$ are nonempty and compact.

(b) The set valued mapping F_m is $\mathcal{L} \times \mathcal{B}$-measurable.
(c) The graph of $(t, x) \to S_m(t, x)$ is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$-measurable set.
(d) Assume that x^* satisfies (7). Then there exists an integrable function c such that, for almost every $t \in [a, b]$, $|\gamma(t)| \leq c(t)$ for all $x(t) \in X(t)$ and $\gamma(t) \in F_m(t, x(t))$.

Proof Non emptiness and compactness of $S_m(t, x)$ follows from (B2) and (B4). Also, (B4) guarantees that $F_m(t, x)$ is non empty. Taking into account that $u \to f(t, x, u)$ is continuous by (LC), we get the compactness of the set $F_m(t, x)$.

We now turn to (b) of the Lemma. Take any compact set $A \subset \mathbb{R}^n$. We want to prove that

$$\{(t, x) \in [a, b] \times \mathbb{R}^n : \ F_m(t, x) \cap A \neq \emptyset\} \tag{16}$$

is $\mathcal{L}\times\mathcal{B}$-measurable. Assume that $f^{-1}(A) \neq \emptyset$. By (B1) and (LC), $t \to f(t, x, u)$ is measurable for each (x, u) and $(x, u) \to f(t, x, u)$ is continuous for almost every t. Thus Proposition 2.3.6 in [23] asserts that f is an $\mathcal{L}\times\mathcal{B}\times\mathcal{B}$-measurable function. It follows that

$$f^{-1}(A) = \left\{(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^k : \ f(t, x, u) \in A\right\}$$

is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ measurable set. On the other hand, the set valued mapping $t \to S(t)$ is $\mathcal{L}$-measurable and closed valued by (B2). By Theorem 2.3.7 in [23], the graph of $t \to S(t)$,

$$\Upsilon := \{(t, x, u) \in [a, b] \times \mathbb{R}^n \times \mathbb{R}^k : \ (x, u) \in S(t)\}, \tag{17}$$

is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ measurable set and, consequently, $f^{-1}(A) \cap \Upsilon$ is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$-measurable set. Now, take any $t, x) \in [a, b] \times \mathbb{R}^n$ such that $F_m(t, x) \cap A \neq \emptyset$. Then (B4) guarantees the existence of a $u \in \mathbb{R}^k$ such that $(x, u) \in S(t)$. It follows that $(t, x, u) \in f^{-1}(A) \cap \Upsilon$. Since (t, x) belongs to (16) if and only if there is a $u \in \mathbb{R}^k$ such that $(t, x, u) \in f^{-1}(A) \cap \Upsilon$ we deduce from Proposition 2.34 in [15] the $\mathcal{L} \times \mathcal{B}$ measurability of F_m.

Statement (c) of the Lemma follows from the fact that $(x, u) \in S(t)$ is equivalent to $u \in S_m(t, x)$ and the $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ measurability of the set (17).

Now it remains to prove (d). Take $t \in [a, b]$ such that (LC) holds and $\dot{x}^*(t) \in F_m(t, x^*(t))$. Choose u^* such that $u^* \in S_m(t, x^*(t))$ and $\dot{x}^*(t) = f(t, x^*(t), u^*)$. Take any x such that $x \in X(t)$. We now consider any $\gamma \in F_m(t, x)$. This is possible since $F_m(t, x) \neq \emptyset$ by (B4). By definition of F_m, there exists a $u \in S_m(t, x)$ such that $\gamma = f(t, x, u)$. Appealing to (LC) we get

$$|\gamma| \leq |f(t, x^*(t), u^*)| + 2k_x^f \epsilon + 2k_u^f \sigma = |\dot{x}^*(t)| 2k_x^f \epsilon + 2k_u^f \sigma.$$

Set $c(t) = |\dot{x}^*(t)| + 2k_x^f \epsilon + 2k_u^f \sigma$. Observe that upper bound does not depend on the choice of x or u and it holds for almost every t. Since $\dot{x}^*$ is an integrable function we conclude that $c \in L^1$ proving our claim. $\square$

Remark: In (d) of Lemma 6 the requirement that x^* satisfies (7) is added to guarantee the integrability of c. Alternatively, we can stipulate that f is uniformly bounded by an integrable function. □

We now investigate Lipschitz properties of $x \to S_m(t,x)$ and $x \to F_m(t,x)$ for each t. In this respect, (BS) is essential as we will see. Indeed, conditions (B1), (B2), (B4) and (LC) by themselves, are not enough to guarantee lower semi-continuity of $x \to S_m(t,x)$ or $x \to F_m(t,x)$, let alone Lipschitz continuity, as the following example shows.

Example 4 Let us fix $t \in [a,b]$ (the interval $[a,b]$ here has no relevance) and set $S(t) = \{(x,u) \in \mathbb{R}^n \times \mathbb{R} :\ u \in [-1,1],\ u|x| \le 0\}$. For each t, we have

$$S_m(t,x) = \begin{cases} [-1,1] \text{ if } x = 0, \\ [-1,0] \text{ if } x \neq 0. \end{cases}$$

Set $F_m(t,x) = \{x+u :\ u \in S_m(t,x)\}$. It is a simple matter to see that (B1), (B2), (B4) hold and that $f(x,u) = x+u$ satisfies (LC). However, both F_m and S_m fail to be lower semi-continuous. To see that, consider any sequence $\{x_i\}$ such that $x_i \neq 0$ and $x_i \to 0$. Then $1/2 \in S_m(t,0)$ and $1/2 \in F_m(t,0)$. But there is no convergent sequence $\{u_i\}$ with limit equal to $1/2$, since $u_i \le 0$. Consequently, there is no sequence $\gamma_i \in F_m(t,x_i)$ converging $1/2$.

Assumption (BS) excludes this example from our context. Indeed, for any t, we have $(1,0) \in N^P_{S(t)}(0,1/2)$ and so $1 > 0$, i.e., (BS) does not hold. □

An appeal to Theorem 1 asserts that, in our setting, (BS) guarantees that $x \to S_m(t,x)$ is not merely pseudo-Lipschitz: it is in fact Lipschitz continuous as we show next. For completeness we also state a known result: that under our conditions, Lipschitz continuity of $x \to S_m(t,x)$ implies (BS).

Lemma 7 *Assume that (B2), (B4) and (BS) hold. Then there exists constant $\hat{\varepsilon}$ such that, for almost every t,*

$$x, x' \in x^*(t) + \hat{\varepsilon}B \implies S_m(t,x) \subset S_m(t,x') + \mathcal{K}|x-x'|\bar{B}, \tag{18}$$

where $\mathcal{K}$ is as defined in (BS).

If (B2) and (B4) hold and there exist constant ε and $\mathcal{K}$ such that, for almost every t, (18) is satisfied, then (BS) hold with constant $\mathcal{K}$.

Remark: Before engaging in the proof of this result, it is important to emphasize that the above Lemma is no more than an adaptation of more general results presented and discussed in [10]. We also refer the reader to [13] in this regard.

Proof We only prove the first part of the Lemma since the second part can be found in [13].

Recall that $S(t)$ is the graph of $x \to S_m(t,x)$ and, by (B2), it is a closed set. Now take t such that the property in (BS) holds. Choose $u^* \in S_m(t,x^*(t))$

(u^* exists by (B4)). Set $R = 2\sigma$, $\hat{\varepsilon} = \min\{\epsilon, \frac{\sigma}{3\mathcal{K}}\}$ and $\xi = 1/2$, where σ is as in (B2) and ϵ is the parameter in (6). Then, for any $x \in x^*(t) + \hat{\varepsilon}B$, we have

$$S_m(t,x) \cap \bar{B}(u^*(t), (1-\xi)R) = S_m(t,x) \cap \bar{B}(u^*(t), \sigma) = S_m(t,x)$$

and our first result follows from Theorem 1 and the fact that $S_m(t,x) \cap \bar{B}(u^*(t), \sigma) = S_m(t,x)$. □

As an immediate conclusion of the above Lemma we get the following Corollary.

Corollary 1 *Assume that (B2), (B4) and (BS) hold and that f satisfies (B1) and (LC). Then there exist $\hat{\varepsilon}$ and k_{F_m} such that, for almost every t,*

$$x, x' \in x^*(t) + \hat{\varepsilon}B \implies F_m(t,x) \subset F_m(t,x') + k_{F_m}|x - x'|\bar{B}. \tag{19}$$

Proof Let $t \in [a,b]$ such that the properties in (BS), (LC) and (18) hold. Take any $x, x' \in x^*(t) + \hat{\varepsilon}B$, $\gamma \in F_m(t,x)$ and $\gamma' \in F_m(t,x')$. Here $\hat{\varepsilon}$ is as in Lemma 7. Let u and u' be such that $(x,u) \in S_m(t,x)$, $(x',u') \in S_m(t,x')$, $\gamma = f(t,x,u)$ and $\gamma' = f(t,x',u')$. By Lemma 7 we get

$$|u - u'| \leq \mathcal{K}|x - x'|.$$

It follows from (LC) and the above that

$$\begin{aligned} |f(t,x,u) - f(t,x',u')| &\leq k_x^f|x - x'| + k_u^f|u - u'| \\ &\leq k_x^f|x - x'| + k_u^f\mathcal{K}|x - x'| \\ &= (k_x^f + k_u^f\mathcal{K})|x - x'| \end{aligned}$$

and our result follows with $k_{F_m} = k_x^f + k_u^f\mathcal{K}$. □

We now concentrate on measurable selections. We dwell on S_m, although our results can apply to F_m. Our following and last result of this section asserts the existence of measurable and Lipschitz selection of S_m and it is a direct consequence of Proposition 2.3.10 of [23] and Theorem 9.5.3 in [3] (and in this respect, (a) of Lemma 1 and Lemma 7 are essential).

Lemma 8 *Assume that (B2), (B4) and (BS) hold. Take any measurable function $x : [a,b] \to \mathbb{R}^n$ such that $x(t) \in X(t)$ for each t. Then there exists a measurable function $\mu : [a,b] \to \mathbb{R}^k$ such that*

$$\mu(t) \in S_m(t, x(t)) \quad a.e.\ t \in [a,b]. \tag{20}$$

Furthermore, if for each $t \in [a,b]$ and each $x \in X(t)$, $S_m(t,x)$ is convex, then there exists a constant c and a function $u : [a,b] \times \mathbb{R}^n \to \mathbb{R}^k$ such that $t \to u(t,x)$ is measurable, $x \to u(t,x)$ is Lipschitz with constant $c\mathcal{K}$ and

$$\mu(t) = u(t, x(t)) \quad a.e.\ t \in [a,b].$$

Before presenting the proof it is worth noticing that the convexity of $S_m(t,x)$ is essential in the last statement of the Lemma.

Proof Let Υ denote the graph of $(t,x) \to S_m(t,x)$ and recall that (c) of Lemma 6 guarantees that it is a $\mathcal{L} \times \mathcal{B} \times \mathcal{B}$ set. It follows from Proposition 2.34 in [15] that the graph of $t \to S_m(t,x)$ is a $\mathcal{L} \times \mathcal{B}$-measurable set. For any $x \in \mathbb{R}^n$, the set $S_m(t,x)$ is nonempty and compact by 1 of Lemma 6. Then Theorem 2.3.7 in [23] asserts that $t \to S_m(t,x)$ is a measurable set valued mapping. On the other hand, for each $t \in [a,b]$, Lemma 7 guarantees that $x \to S_m(t,x)$ is Lipschitz. Next we apply Proposition 2.3.10 in [23] and deduce the measurability of the set valued mapping $G(t) = S_m(t,x(t))$. Since $G(t)$ is closed for each t, Theorem 2.3.11 in [23] guarantees the existence of a measurable selection μ of $G(t)$, i.e., μ is a measurable function satisfies (20).

Additionally, suppose that now that $S_m(t,x)$ is convex. Then the last conclusion of follows from Theorem 9.5.3 in [3]. □

7 Differential Inclusions with Mixed Constraints

We are now in position to invoke Chapter 2 in [23] to obtain results concerning compactness of trajectories of the set valued mapping F_m as defined in (5) and the relation between trajectories of F_m and the system (Σ).

Consider the set $X(t)$ to be defined by x^* satisfying (7). Recall that the set valued mapping $X : [a,b] \to \mathbb{R}^n$ is closed and bounded. Under the conditions of Lemma 6 it is a simple matter to deduce the compactness of trajectories of F_m as a direct consequence of Theorem 2.5.3 in [23]. For the sake of completeness we state our findings next.

Theorem 2 *Assume that (CA) and the conditions under which (d) of Lemma 6 hold. Take any sequence $\{x_i\}$, $x_i \in W^{1,1}([a,b];\mathbb{R}^n)$ such that*

$$Gr\ x_i \subset\ Gr\ X, \quad \dot{x}_i(t) \in F_m(t,x_i(t)) \text{ a.e. } t \in [a,b], \quad x_i(0) \in X(0).$$

Then there exists a subsequence (we do not relabel) such that

$$x_i \to x \text{ uniformly} \quad \text{and} \quad \dot{x}_i \to \dot{x} \text{ weakly in } L^1$$

for some $x \in W^{1,1}([a,b];\mathbb{R}^n)$ such that $\dot{x}(t) \in F_m(t,x(t))$ a.e. $t \in [a,b]$.

Our next step is to ensure equivalence between the set of feasible trajectories of system (Σ) and the set of feasible trajectories of F_m.

Definition 1 We say that an absolutely continuous function x is a feasible trajectory of F_m if $x(t) \in X(t)$ for all $t \in [a,b]$ and $\dot{x}(t) \in F_m(t,x(t))$ for almost every $t \in [a,b]$. We denote the set of all F_m-feasible trajectories associated with E to be

$$\mathcal{R}^*_{[a,b]}(E,F_m) := \{x \in C([a,b];\mathbb{R}^n) :\ x \text{ trajectory of } F_m,\ (x(a),x(b)) \in E\}.$$

Definition 2 Define $\mathcal{S}^*_{[a,b]}(E,\Sigma)$ to be the set of all absolutely continuous functions x associated with a control $u : [a,b] \to U$ such that $x(t) \in X(t)$ for all $t \in [a,b]$ and (x,u) solves (Σ).

Theorem 3 *Assume that f satisfies (B1) and (LC) and that (B2)–(B4) and (BS) hold.*

Then $x \in \mathcal{S}^*_{[a,b]}(E,\Sigma)$ if and only if $x \in \mathcal{R}^*_{[a,b]}(E,F_m)$.

Proof The implication $x \in \mathcal{S}^*_{[a,b]}(E,\Sigma) \quad \Longrightarrow \quad x \in \mathcal{R}^*_{[a,b]}(E,F_m)$ is trivial.

To see that the opposite implication holds, take $x \in \mathcal{R}^*_{[a,b]}(E,F_m)$ and set $w(t) = \dot{x}(t)$ and $m(t,u) = f(t,x(t),u)$. Assumption (LC) together with Proposition 2.3.4 in [23] guarantees that $(t,u) \to m(t,u)$ is $\mathcal{L} \times \mathcal{B}$ measurable. Set $G(t) = S_m(t,x(t))$. We have $w(t) \in \{m(t,u) : \ u \in G(t)\}$ for almost every $t \in [a,b]$. Since G is a measurable and closed set valued mapping, Theorem 2.3.13 in [23] asserts the existence of a measurable function $u : [a,b] \to \mathbb{R}^k$ such that

$$u(t) \in G(t) \quad \text{a.e.} \quad \text{and} \quad w(t) = m(t,u(t)) \quad \text{a.e.}$$

It follows that $x \in \mathcal{S}^*_{[a,b]}(E,\Sigma)$, completing our proof. □

It is worth mentioning that the previous result does not require convexity of $F_m(t,x)$ (or $S_m(t,x)$). If however (CA) holds, then it can easily be seen that Theorem 2.6.1 in [23] can be applied (by Lemma 6 and Corollary 1) leading to the following.

Theorem 4 *Assume that x^* defining X satisfies (7), f satisfies (B1) and (LC) and that (B2) – (B4), (BS) and (CA) hold. Then $\mathcal{R}^*_{[a,b]}(E,F_m)$ is compact with respect to the supremum norm topology.*

Remark 1 Observe that (CA) and the existence of the integrable bound for F_m (guaranteed by (d) of Lemma 6) are essential for Theorem 2 to hold. The (LC) and (BS) conditions in Theorem 3 can be replaced by different conditions.

8 On Existence of Solution for Optimal Control Problems

An straightforward consequence of our previous results is the existence of solution to some optimal control problems with mixed constraints.

Let us consider the problem

$$(\mathrm{Q}) \quad \begin{cases} \text{Minimize } l(x(b)) & \\ \text{subject to} & \\ \qquad \dot{x}(t) = f(t,x(t),u(t)) & \text{a.e. } t \in [a,b] \\ (x(t),u(t)) \in S(t) & \text{a.e. } t \in [a,b] \\ \qquad x(b) \in \{x_a\} \times E_b & \end{cases}$$

where $l : \mathbb{R}^n \to \mathbb{R}$ is a lower semi-continuous function. Here $E = \{x_a\} \times E_b$. Suppose that there exists a feasible process (x^*,u^*) for (Q) and consider some $\epsilon > 0$. Define, as before, $X(t) = x^*(t) + \epsilon\bar{B}$.

Additionally, assume that f satisfies (B1) and (LC) and that (B2)-(B4) as well as (BS) and (CA) hold. Then Theorems 3 and 4 allow us to appeal to Proposition 2.6.2 in [23], to deduce existence of a minimizer for (Q).

In this scenario, when the convexity of $F_m(t,x)$ fails (i.e., (CA) does not hold) *relaxation* may be of help. A familiar procedure is to consider the convexified set valued mapping

$$\mathrm{co}\, F_m(t,x).$$

Let us then denote by (Q_{rlx}) the problem we get from (Q) when we replace (Σ) by (Σ_{rlx}) (defined in (15)). Under the above conditions (with the exception of (CA), of course) it follows from Proposition 2.7.3 in [23] and Theorem 3 that (Q_{rlx}) has a solution (x_r,u_r). If, moreover, there exists a $\delta>0$ such that, for all $t\in[a,b]$,

$$x_r(t)+\delta\bar{B}\subset X(t)\quad \text{and } x_r(b)+\delta\bar{B}\subset E_b,$$

then we deduce from Proposition 2.7.3 in [23] and Theorem 3 that

$$\inf(Q_{rlx})=\inf(Q).$$

When $S(t)$ is as defined by (4), another approach can be found in the literature covering a different class of problems. Indeed, when U is compact and convex, the functions f and g satisfy (B1), are uniformly bounded, continuous with respect to x and convex with respect to u, compactness results along the lines of Theorem 2 hold and existence of solution for (Q) is asserted, for example, in [6]. Our approach covers a different class of problems and relies on (CA) or on relaxation procedures as discussed above. Recall that convexity U and of $u\to g(t,x,u)$ implies convexity of $S_m(t,x)$, but convexity of $S_m(t,x)$ does not necessarily implies convexity of $F_m(t,x)$ even when the components of f are convex functions with respect to u (in this respect, see Example 1).

A word of caution when a running cost is added to the cost in (Q), i. e., when we consider the problem

$$(\mathrm{Q}')\qquad \begin{cases} \text{Minimize } l(x(b))+\displaystyle\int_a^b L(t,x(t),u(t))dt \\ \text{subject to} \\ \qquad\qquad \dot{x}(t)=f(t,x(t),u(t)) & \text{a.e. } t\in[a,b] \\ \ (x(t),u(t))\in S(t) & \text{a.e. } t\in[a,b] \\ (x(a),x(b))\in\{x_a\}\times E_b \end{cases}$$

Based on our analysis above, existence of solution of (Q') would follow from the convexity of

$$\tilde{F}_m(t,x)=\{(L(t,x,u),f(t,x,u)):\ u\in S_m(t,x)\}\,.$$

This is indeed a very strong requirement. Existence results are known in the literature under the assumption that the set valued mapping

$$V(t,x)=\{(y,f(t,x,u)):\ y\geq L(t,x,u),\ u\in S_m(t,x)\}$$

is convex (see, e. g., [4,6,8]). Convexity of $F_m(t,x)$ implies convexity of $V(t,x)$ but convexity of $V(t,x)$ does not imply that of $F_m(t,x)$ as the next example shows.

Example 5 We recover here a well known example (see, e.g., [18]) for linear quadratic problems. Consider $L(t,x,u)=u^2$, $f(t,x,u)=u$, $g(t,x,u)=0$ and $U=[0,1]$. Then $V(t,x)$ is convex whereas $F_m(t,x)$ is not. □

In the special case where $S(t)$ is as defined in (4), convexity of $V(t,x)$, in general, does not follow from the convexity of U and that of the functions $u \to L(t,x,u)$, $u \to f(t,x,u)$ and $u \to g(t,x,u)$ as illustrated in the following exaple.

Example 6 Consider $U=\mathbb{R}$, $L(t,x,u)=|u|$, $f(t,x,u)=(u,u^2)$ and $g(t,x,u)=|u|-1$. We have

$$V(t,x)=\left\{(y,u,u^2):\ y \geq |u|,\ u \in [-1,1]\right\}.$$

Setting $\gamma_1=(3,0,0)$ and $\gamma_2=(3,1,1)$, we have $\gamma_i \in V(t,x)$, for $i=1,2$, but $\frac{1}{2}\gamma_1+\frac{1}{2}\gamma_2=(3,\frac{1}{2},\frac{1}{2}) \notin V(t,x)$. This means that $V(t,x)$ is not convex.

9 Conclusions

We established properties of the set valued mapping F_m associated with the system (Σ) that allow us to invoke well known results for differential inclusion. In this respect we followed closely the developments gathered in Chapter 2 of [23]. Special attention was paid to convexity properties of F_m clarifying the relation of convexity of F_m with other set valued mappings linked with (Σ). We also discussed the connection between convexity of F_m and its relaxation with existence of solution to some optimal control problems with mixed constraints.

In the literature, the reformulation of the systems similar to (Σ) into differential inclusions has proved useful to derive necessary conditions of optimality when combined with extended Euler-Lagrange conditions in the vein of [23] and [10] (see also reference within). This is an important tool in [11] and [12], where necessary conditions for mixed constrained optimal control problems are extensively studied.

One last word about the second part of Lemma 8 asserting that under some conditions, any control of our system maybe a feed back control depending Lipschitz continuously on the state. The consequences of such Lemma deserves further investigation. In particular, it may be of use when deriving necessary conditions along the lines of [1].

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