Strong controlled invariance of behavioral nD systems

ABSTRACT

In this paper we give a stronger version of the notion of behavioral controlled invariance introduced in (Pereira & Rocha, 2017) in the context of regular partial interconnections. In such interconnections, the variables are divided into two sets: the variables to-be-controlled and the variables on which it is allowed to enforce restrictions (control variables); moreover, regularity means that the restrictions of the controller do not overlap with the ones already implied by the laws of the original behavior. A complete characterization of strong controlled invariance for nD behaviors is derived making use of a special controller behavior known as the canonical controller.

KEYWORDS
Behaviors; Invariance; Controlled invariance; Canonical controllers

1. Introduction

Controlled invariance in the behavioral context was introduced in (Pereira & Rocha, 2017), extending the notion of invariance to the control setting. Roughly speaking, a sub-behavior $\mathcal{V}$ of a behavior $\mathcal{B}$ is said to be $\mathcal{B}$-invariant if the freedom of the trajectories of $\mathcal{B}$ is "captured" by $\mathcal{V}$, i.e., if $\mathcal{B}$ has no free variables modulo $\mathcal{V}$. This means that the system trajectories whose restriction to a sufficiently large portion of the domain (the past, in the 1D case) lies in $\mathcal{V}$ are in fact contained in the sub-behavior $\mathcal{V}$ (herefrom the term $\mathcal{B}$-invariant). If $\mathcal{V}$ is not $\mathcal{B}$-invariant, one may wish to control the system in order to obtain a restricted dynamics with respect to which $\mathcal{V}$ is invariant. When this is possible, $\mathcal{V}$ is said to be controlled invariant.

In this context it is important to recall that the behavioral approach to control consists in interconnecting a given behavior with a suitable controller behavior in order to obtain a desired controlled behavior. There are two main situations to be considered: *full interconnection* (where all the system variables are available for control) and *partial interconnection* (where the variables are divided into *to-be-controlled variables* and *control variables*). Of particular importance are *regular controllers* which are characterized by imposing restrictions on the control variables that do not overlap with the ones already implied by the laws of the original behavior.

The full interconnection control problem was firstly addressed for 1D systems in (Willems, 1997). In (Rocha & Wood, 2001), further results have been obtained not only for the 1D case, but also for multidimensional (nD) systems.

As concerns partial interconnections, in (Belur & Trentelman, 2002) the solvability of a 1D partial interconnection problem has been related to the solvability of a suitable associated full control problem. Results for the corresponding nD case have been obtained in (Rocha, 2002) and (Rocha, 2005), considering a special behavior, the *canonical controller*, introduced in (Willems, Belur, Julius & Trentelman, 2003) for the 1D case.

Here we introduce a strong version of the notion of controlled invariance in the context of regular partial interconnections and study this property from the (easier) standpoint of full interconnection by resorting to the associated canonical controller.

2. Preliminaries

In this paper we consider *nD behaviors* $\mathcal{B}$ defined over the continuous nD domain $\mathbb{R}^n$ that can be described by a set of linear constant coefficient partial differential equations, i.e.,

$$\mathcal{B} = \ker H(\underline{\partial}) := \{z \in \mathcal{U} : H(\underline{\partial})z = 0\},$$

where $\mathcal{U} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{q}})$, for some $\mathtt{q} \in \mathbb{N}$, $\underline{\partial} = (\partial_1, \ldots, \partial_n)$, the ∂_i's are the elementary partial differential operators and $H(\underline{s})$, with $\underline{s} = (s_1, \ldots, s_n)$, is an nD polynomial matrix, (i.e, it belongs to the set $\mathbb{R}^{\bullet \times \mathtt{q}}[\underline{s}]$ of $\bullet \times \mathtt{q}$ matrices with entries in the ring $\mathbb{R}[\underline{s}]$ of nD polynomials), known as a *(kernel) representation* of $\mathcal{B}$. For short, whenever the context is clear we omit the indeterminate $\underline{s}$ and the operator $\underline{\partial}$. We shall refer to $\mathcal{B}$ as a *kernel behavior* or simply as a *behavior*.

Note that different representations may give rise to the same behavior. In particular $\ker H = \ker UH$ for any unimodular nD polynomial matrix U. Moreover, $\mathcal{B}_1 = \ker H_1 \subseteq \mathcal{B}_2 = \ker H_2$ if and only if there exists an nD polynomial matrix $\bar{H}$ such that $H_2 = \bar{H}H_1$.

Instead of characterizing $\mathcal{B}$ by means of a representation matrix H, it is also possible to characterize it by means of its *orthogonal module* $\mathrm{Mod}(\mathcal{B})$, which consists of all the nD polynomial rows r such that $\mathcal{B} \subset \ker r$, and can be shown to coincide with the polynomial module generated by the rows of H, i.e., $\mathrm{Mod}(\mathcal{B}) = \mathcal{RM}(H)$, where $\mathcal{RM}$ stands for row module, see (Oberst, 1990; Wood, 2000) for details.

The notion of autonomy plays an important role in the context of controlled invariance. Although there are several (equivalent) ways of defining this property (Rocha & Wood, 2001; Willems, 1997; Zerz, 2000), here we simply define autonomy as the absence of free variables, in the following sense: given a behavior $\mathcal{B}$ in the universe $\mathcal{U} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^{\mathtt{q}})$ and trajectories w with components w_i, $i \in \{1, \ldots, q\}$, w_i is said to be a *free variable* of $\mathcal{B}$ if

$$\forall w_i^* \in C^\infty(\mathbb{R}^n, \mathbb{R}), \ \exists w \in \mathcal{B} \text{ s.t. } w_i = w_i^*.$$

Definition 2.1. An nD behavior $\mathcal{B}$ is called *autonomous* if $\mathcal{B}$ has no free variables.

The next proposition provides a characterization of autonomy in terms of kernel representations. This was proven in (Rocha & Wood, 1997) and (Wood, Rogers & Owens, 1999) for the discrete domain case, but the proofs are also valid in the case of continuous domains (Zerz, 2000).

Proposition 2.2. *Given an nD behavior $\mathcal{B} = \ker H$, then $\mathcal{B}$ is autonomous if and only if the nD polynomial matrix H has full column rank.*

Minimal left annihilators will be relevant in the sequel. They are defined as follows (Zerz, 2000).

Definition 2.3. Let $H \in \mathbb{R}^{g\times \mathtt{q}}[\underline{s}]$. Then $X \in \mathbb{R}^{m\times g}[\underline{s}]$ is called a *minimal left annihilator (MLA)* of H if the following conditions hold:

(1) X is a left annihilator of H, i.e., $XH = 0$.
(2) If $X_1 H = 0$, with $X_1 \in \mathbb{R}^{p\times g}[\underline{s}]$, then $X_1 = MX$, for some nD polynomial matrix M.

In (Oberst, 1990), it was shown that the *quotient of two behaviors* admits the structure of a behavior (see also (Wood, 2000)). Indeed, if $\mathcal{B}$ and $\mathcal{B}'$ are behaviors such that $\mathcal{B}' \subseteq \mathcal{B}$, choosing a kernel representation H' of $\mathcal{B}'$ the following isomorphism holds:

$$\mathcal{B}/\mathcal{B}' \cong H'(\mathcal{B}).$$

The kernel representation of the quotient of two behaviors can be related with the kernel representations of the latter as stated in the following result, (Rocha & Wood, 2001; Wood, Oberst, Rogers & Owens, 2000).

Proposition 2.4. *Let $\mathcal{B}' \subseteq \mathcal{B}$ be two nD behaviors, where $\mathcal{B}' = \ker H'$ and $\mathcal{B} = \ker EH'$, for some nD polynomial matrices H' and E. Let C be a MLA of H', and set*

$$L = \begin{bmatrix} E \\ C \end{bmatrix}.$$

Then $\mathcal{B}/\mathcal{B}' \cong \ker L$. In the case where H' has full row rank, clearly $\mathcal{B}/\mathcal{B}' \cong \ker E$.

3. Behavioral control

In the behavioral approach, in order to control a behavior one imposes suitable restrictions to its variables so as to obtain a new desired behavior. This is achieved by interconnecting (intersecting) the given behavior with another behavior called controller. As mentioned in the Introduction, two situations can be considered, namely *full interconnection*, where all the system variables are available for control, (Rocha & Wood, 2001; Willems, 1997) and *partial interconnection*, where the variables are divided into *to-be-controlled variables* and *control variables*, (Belur & Trentelman, 2002).

To make the notations more precise, if a behavior $\mathcal{B}$ has variables z we denote it by $\mathcal{B}_z$. In set theoretic terms, control by full interconnection can be formulated as follows. If $\mathcal{B}_z$ is the behavior of the system to be controlled (the plant) and $\mathcal{C}_z$ is the *full controller*, i.e, the set of all signals compatible with the additional restrictions to be imposed, then the resulting controlled behavior is the interconnection given by

$$\mathcal{B}_z \cap \mathcal{C}_z. \tag{1}$$

A desired controlled behavior $\mathcal{D}_z$ is said to be *implementable* (from $\mathcal{B}_z$) *by full interconnection* if there exists a full controller $\mathcal{C}_z$ that *implements* it, i.e., such that

$$\mathcal{D}_z = \mathcal{B}_z \cap \mathcal{C}_z.$$

In order to define partial interconnections, we denote the to-be-controlled variables by w and the control variables by c. We assume that the joint behavior of these variables, i.e., the (w, c)-behavior, is given as:

$$\mathcal{B}_{(w,c)} := \{(w, c) \in \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}} \mid R(\underline{\partial})w = M(\underline{\partial})c\}\,, \tag{2}$$

where, for $\mathtt{q} \in \mathbb{N}$, $\mathcal{U}^{\mathtt{q}} := \mathcal{C}^{\infty}(\mathbb{R}^n, \mathbb{R}^{\mathtt{q}})$ and $R(\underline{s}) \in \mathbb{R}^{g\times\mathtt{w}}[\underline{s}], M(\underline{s}) \in \mathbb{R}^{g\times\mathtt{c}}[\underline{s}]$ are nD polynomial matrices.

The w-behavior induced by $\mathcal{B}_{(w,c)}$ is defined as $\mathcal{B}_w = \pi_w\left(\mathcal{B}_{(w,c)}\right)$, where π_w denotes the projection into $\mathcal{U}^{\mathtt{w}}$, and is obtained by eliminating c from the equation $R(\underline{\partial})w = M(\underline{\partial})c$, which is achieved by applying to both sides of the equation a minimal left annihilator $L(\underline{\partial})$ of $M(\underline{\partial})$ (Oberst, 1990, Corollary 2.38). This yields $\mathcal{B}_w = \ker(LR)$. Analogously, $\mathcal{B}_c = \ker(NM)$ where N is a MLA of R.

The control action then consists in restricting the behavior of the control variables c in order to obtain a desired effect on w, this is, given a behavior to be controlled $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$ and a desired behavior $\mathcal{D}_w \subset \mathcal{U}^{\mathtt{w}}$, a controller behavior $\mathcal{C}_c \subset \mathcal{U}^{\mathtt{c}}$ (given by $\mathcal{C}_c = \{c \in \mathcal{U}^{\mathtt{c}} \;:\; C(\underline{\partial})c = 0\} = \ker C$, for some adequate nD polynomial matrix $C(\underline{s})$) has to be determined such that

$$\mathcal{D}_w = \pi_w\left(\mathcal{B}_{(w,c)} \cap \mathcal{C}^*_{(w,c)}\right), \tag{3}$$

where $\mathcal{C}^*_{(w,c)}$ stands for the lifted behavior

$$\mathcal{C}^*_{(w,c)} := \{(w, c) \in \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}} \mid w \text{ is free and } c \in \mathcal{C}_c\}.$$

If (3) holds, we say that $\mathcal{D}_w$ is *implementable* by partial interconnection from $\mathcal{B}_{(w,c)}$, or, equivalently, that $\mathcal{C}_c$ *implements* $\mathcal{D}_w$.

Regular controllers play an important role in this context. They are characterized by imposing restrictions on the control variables that do not overlap with the ones already implied by the laws of the original behavior.

Given two behaviors $\mathcal{B}^1_z = \ker H_1(\underline{\partial})$ and $\mathcal{B}^2_z = \ker H_2(\underline{\partial})$ their interconnection $\mathcal{B}^1_z \cap \mathcal{B}^2_z = \ker\left(\begin{bmatrix} H_1(\underline{\partial}) \\ H_2(\underline{\partial}) \end{bmatrix}\right)$ is said to be a *regular full interconnection* if

$$\operatorname{rank}\begin{bmatrix} H_1(\underline{s}) \\ H_2(\underline{s}) \end{bmatrix} = \operatorname{rank} H_1(\underline{s}) + \operatorname{rank} H_2(\underline{s}).$$

In terms of modules, the previous equation is equivalent to

$$\mathrm{Mod}(\mathcal{B}_z^1) \cap \mathrm{Mod}(\mathcal{B}_z^2) = \{0\}.$$

A full controller $\mathcal{C}_z$ is called a *regular full controller*, if its interconnection (1) with the plant $\mathcal{B}_z$ is regular. A behavior $\mathcal{D}_z$ is *regularly implementable by full interconnection* if it is implemented by a regular full controller.

In partial interconnections, given the nD polynomial matrices $R(\underline{s}), M(\underline{s})$ and $C(\underline{s})$ that respectively describe the to-be-controlled behavior $\mathcal{B}_{(w,c)}$ and the controller $\mathcal{C}_c$, the regularity of the corresponding partial interconnection is equivalent to the following condition:

$$\mathrm{rank} \begin{bmatrix} R(\underline{s}) & M(\underline{s}) \\ 0 & C(\underline{s}) \end{bmatrix} = \mathrm{rank} \begin{bmatrix} R(\underline{s}) & M(\underline{s}) \end{bmatrix} + \mathrm{rank} \begin{bmatrix} 0 & C(\underline{s}) \end{bmatrix}.$$

In terms of modules, the previous equation is equivalent to

$$\mathrm{Mod}(\mathcal{B}_{(w,c)}) \cap \mathrm{Mod}(\mathcal{C}^*_{(w,c)}) = \{0\}.$$

Thus, in particular, every controller $\mathcal{C}_c = \ker C$ is regular if the nD polynomial matrix $R(\underline{s})$ has full row rank. In turn, this condition means that all the control variables are free in the to-be-controlled behavior $\mathcal{B}_{(w,c)}$.

A controller $\mathcal{C}_c$ is called a *regular partial controller*, if the interconnection (3) is regular. In the same way, a behavior $\mathcal{D}_w$ is *regularly implementable by partial interconnection* if it is implemented by a regular partial controller.

It is not difficult to see that only sub-behaviors $\mathcal{D}_w$ of $\mathcal{B}_w$ are implementable from $\mathcal{B}_{(w,c)}$ by partial interconnection. Moreover, the smallest sub-behavior of $\mathcal{B}_w$ implementable by partial interconnection is clearly obtained by setting all the control variables to be zero. This gives rise to the behavior

$$\mathcal{N}_w := \left\{ w \in \mathcal{U}^{\mathtt{w}} \mid (w, 0) \in \mathcal{B}_{(w,c)} \right\},$$

whose kernel representation is $\mathcal{N}_w = \ker R$, known as *hidden behavior* (Belur & Trentelman, 2002). As the following result shows, $\mathcal{N}_w$ plays an important role in the characterization of (the possibility of) implementation by partial interconnection (Rocha, 2002).

Proposition 3.1. *An nD behavior $\mathcal{D}_w$ is implementable from $\mathcal{B}_{(w,c)}$ by partial interconnection if and only if $\mathcal{N}_w \subset \mathcal{D}_w \subset \mathcal{B}_w$.*

As concerns regular implementation, the partial interconnection case is more difficult to investigate than the full interconnection case. For the 1D case, this difficulty has been overcome in (Belur & Trentelman, 2002) by proving that a behavior $\mathcal{D}_w$ is regularly implementable by partial interconnection from $\mathcal{B}_{(w,c)}$ if and only if is regularly implemented by full interconnection from $\mathcal{B}_w$. However, as shown in (Rocha, 2002), this no longer holds in the nD case, ($n \geq 2$).

In order to analyze the problem of nD regular implementation by partial interconnection a new kind of controller, called *canonical controller*, was used in (Rocha, 2005) based on (Willems et al., 2003).

Definition 3.2. Let $\mathcal{B}_{(w,c)}$ be a given plant behavior and $\mathcal{D}_w$ a desired behavior (control objective). The *canonical controller* associated with $\mathcal{B}_{(w,c)}$ and $\mathcal{D}_w$ is defined as

$$\mathcal{C}_c^{can} := \{c \mid \exists w : (w, c) \in \mathcal{B}_{(w,c)} \text{ and } w \in \mathcal{D}_w\}.$$

Thus, the canonical controller consists of all the control variable trajectories compatible with the desired behavior for the variables to be controlled.

Based on the canonical controller, a characterization of regular implementation by partial interconnection in terms of full interconnection in the nD case is given next (Rocha, 2005).

Theorem 3.3. *Let $\mathcal{B}_{(w,c)}$ be a given plant behavior and $\mathcal{D}_w$ a control objective. Let further $\mathcal{C}_c^{can}$ be the associated canonical controller. Assume that $\mathcal{D}_w$ is implementable by partial interconnection from $\mathcal{B}_{(w,c)}$. Then $\mathcal{D}_w$ is regularly implementable by partial interconnection from $\mathcal{B}_{(w,c)}$ if and only if $\mathcal{C}_c^{can}$ is regularly implementable by full interconnection from $\mathcal{B}_c$ (the c-behavior induced from $\mathcal{B}_{(w,c)}$).*

This result is crucial for our study in the sequel.

4. Behavioral controlled-invariance

Before introducing the notion of behavioral controlled-invariance, following (Pereira & Rocha, 2017; Rocha & Wood, 1997) we adopt the next definition for behavioral invariance.

Definition 4.1. Given an nD behavior $\mathcal{B}_w$, a sub-behavior $\mathcal{V}_w$ of $\mathcal{B}_w$ is said to be $\mathcal{B}_w$-*invariant* if the quotient behavior $\mathcal{B}_w/\mathcal{V}_w$ is autonomous.

Since autonomy is the absence of free variables, this intuitively means that all the freedom of the trajectories of $\mathcal{B}_w$ is captured by $\mathcal{V}_w$. By Propositions 2.2 and 2.4 the following corollary is immediate.

Corollary 4.2. *Let $\mathcal{V}_w \subseteq \mathcal{B}_w$ be two nD behaviors, where $\mathcal{V}_w = \ker V$ and $\mathcal{B}_w = \ker EV$, for some nD polynomial matrices V and E, with V full row rank. Then $\mathcal{V}_w$ is $\mathcal{B}_w$-invariant if and only if E is full column rank.*

In the previous setting, controlled-invariance was defined in (Pereira & Rocha, 2017) as follows.

Definition 4.3. Let $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$ be an nD behavior. A sub-behavior $\mathcal{V}_w$ of the induced w-behavior $\mathcal{B}_w \subset \mathcal{U}^{\mathtt{w}}$ is said to be $\mathcal{B}_{(w,c)}$-*controlled-invariant* if there exists a behavior $\mathcal{D}_w$ implementable by partial interconnection from $\mathcal{B}_{(w,c)}$, such that $\mathcal{V}_w \subset \mathcal{D}_w$ and $\mathcal{V}_w$ is $\mathcal{D}_w$-invariant.

As mentioned before, when the matrix $R(\underline{s})$ of the (w, c)-behavior description (2) is a full row rank polynomial matrix, every partial controller is regular. For this case, controlled invariance for nD behaviors was characterized in (Pereira & Rocha, 2017) as follows.

Proposition 4.4. *Consider the nD behavior $\mathcal{B}_{(w,c)}$ described by $Rw = Mc$ with R full row rank. Let $\mathcal{B}_w = \pi_w\left(\mathcal{B}_{(w,c)}\right)$, $\mathcal{N}_w = \ker R$ and $\mathcal{V}_w = \ker V \subset \mathcal{B}_w$. Then*

(1) Defining $\overline{\mathcal{B}}_w := \mathcal{N}_w + \mathcal{V}_w$,

$$\mathcal{V}_w \text{ is } \mathcal{B}_{(w,c)}\text{-controlled-invariant} \Leftrightarrow \overline{\mathcal{B}}_w/\mathcal{V}_w \text{ is autonomous.}$$

(2) If, in addition, V has full row rank,

$$\mathcal{V}_w \subset \mathcal{B}_w \text{ is } \mathcal{B}_{(w,c)}\text{-controlled-invariant} \Leftrightarrow \operatorname{rank}\begin{bmatrix} R \\ V \end{bmatrix} = \operatorname{rank} R.$$

Remark 1. Note that if R has full row rank, $\overline{\mathcal{B}}_w := \mathcal{N}_w + \mathcal{V}_w$ is the smallest implementable behavior by regular partial interconnection from $\mathcal{B}_{(w,c)}$ containing $\mathcal{V}_w$. Thus, in this case, $\mathcal{V}_w$ is $\mathcal{B}_{(w,c)}$-controlled-invariant if and only if $\mathcal{V}_w$ is invariant with respect to the smallest regularly implementable behavior that contains it.

Example 4.5. Consider the 2D behavior $\mathcal{B}_{(w,c)}$ described by $Rw = Mc$ with

$$R = \begin{bmatrix} s_1+1 & 0 \\ 0 & s_2+1 \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} s_2+1 \\ -(s_1+1) \end{bmatrix}.$$

Since $L = \begin{bmatrix} s_1+1 & s_2+1 \end{bmatrix}$ is a MLA of M then

$$\mathcal{B}_w = \ker LR = \ker\begin{bmatrix} (\partial_1+1)^2 & (\partial_2+1)^2 \end{bmatrix}.$$

Define $\mathcal{V}_w = \ker V \subset \mathcal{B}_w$ with $V = \begin{bmatrix} (s_1+1)^2 & 0 \\ 0 & 1 \end{bmatrix}$.
Since $\overline{\mathcal{B}}_w = \mathcal{N}_w + \mathcal{V}_w = \ker R + \ker V$, it follows from (Rocha & Wood, 2001, Lemma 2.14) that $\overline{\mathcal{B}}_w = \ker F$ with $F = AR = BV$ and $\begin{bmatrix} -A & B \end{bmatrix}$ a MLA of $\begin{bmatrix} R \\ V \end{bmatrix}$. It is easy to check that

$$A = \begin{bmatrix} s_1+1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & s_2+1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} (s_1+1)^2 & 0 \\ 0 & s_2+1 \end{bmatrix}.$$

By Proposition 2.4, $\overline{\mathcal{B}}_w/\mathcal{V}_w \cong \ker B$ and by Proposition 2.2 this quotient behavior is autonomous since B has full column rank. Hence, by Proposition 4.4, $\mathcal{V}_w$ is $\mathcal{B}_{(w,c)}$-controlled-invariant. Moreover, by (Pereira & Rocha, 2017), a controller behavior that regularly implements $\overline{\mathcal{B}}_w$ is $\mathcal{C}_c = \ker C$ with

$$C = AM = \begin{bmatrix} s_1+1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s_2+1 \\ -(s_1+1) \end{bmatrix} = \begin{bmatrix} (s_1+1)(s_2+1) \\ -(s_1+1) \end{bmatrix}. \quad \square$$

When the matrix $R(\underline{s})$ has not full row rank, Proposition 4.4 does not hold since it may be impossible to implement $\overline{\mathcal{B}}_w$ by regular partial interconnection, as shown in the following example.

Example 4.6. Consider the 2D behavior $\mathcal{B}_{(w,c)}$ described by $Rw = Mc$ with

$$R = \begin{bmatrix} s_1 - 1 & -(s_1 - 1) \\ s_2 - 1 & -(s_2 - 1) \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} s_1 + 1 \\ s_2 + 1 \end{bmatrix}.$$

Since $N = \begin{bmatrix} s_2 - 1 & -(s_1 - 1) \end{bmatrix}$ is a MLA of R and $L = \begin{bmatrix} -(s_2 + 1) & s_1 + 1 \end{bmatrix}$ is a MLA of M then

$$\mathcal{B}_c = \ker NM = \ker \big(-2(\partial_1 - \partial_2)\big)$$

and

$$\mathcal{B}_w = \ker LR = \ker \begin{bmatrix} 2(\partial_2 - \partial_1) & -2(\partial_2 - \partial_1) \end{bmatrix}.$$

Define $\mathcal{V}_w = \ker V \subset \mathcal{B}_w$ with $V = \begin{bmatrix} \partial_2 - \partial_1 & 0 \\ 0 & \partial_2 - \partial_1 \end{bmatrix}$.
Analogously to the previous example we have that $\overline{\mathcal{B}}_w = \ker F$ with $F = AR = BV$, where

$$A = \begin{bmatrix} 1 & -1 \\ \partial_2 - 1 & \partial_1 - 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} \partial_1 - \partial_2 & \partial_2 - \partial_1 \\ 0 & 0 \end{bmatrix}.$$

We prove next that $\overline{\mathcal{B}}_w$ is not implemented by regular partial interconnection from $\mathcal{B}_{(w,c)}$. By Proposition 3.3, $\overline{\mathcal{B}}_w$ is not regularly implementable by partial interconnection from $\mathcal{B}_{(w,c)}$ if and only if the canonical controller associated to $\overline{\mathcal{B}}_w$, $\mathcal{C}_c^{can}$, is not regularly implementable by full interconnection from $\mathcal{B}_c$. By Definition 3.2, $\mathcal{C}_c^{can}$ is defined by the equations

$$\begin{cases} Rw = Mc \\ ARw = 0 \end{cases}$$

By eliminating the variable w, one obtains as describing equations for the associated c-behavior:

$$\begin{bmatrix} N & 0 \\ A & -I \end{bmatrix} \begin{bmatrix} M \\ 0 \end{bmatrix} c = 0 \Leftrightarrow \begin{bmatrix} N \\ A \end{bmatrix} Mc = 0,$$

and, therefore,

$$\mathcal{C}_c^{can} = \ker \begin{bmatrix} NM \\ AM \end{bmatrix} = \ker \begin{bmatrix} -2(\partial_1 - \partial_2) \\ \partial_1 - \partial_2 \\ 2(\partial_1 \partial_2 - 1) \end{bmatrix}.$$

Moreover, by (Rocha & Wood, 2001, Theorems 4.1 and 4.5) and (Zerz, 2000, Definition 4), if $\mathcal{C}_c^{can}$ is regularly implementable by full interconnection from $\mathcal{B}_c$ then $\mathcal{B}_c/\mathcal{C}_c^{can}$ can be represented by a generalized factor left prime (GFLP) polynomial matrix[1]. Since

$$\mathcal{B}_c = \ker NM,\ \mathcal{C}_c^{can} = \ker \begin{bmatrix} NM \\ AM \end{bmatrix},\ NM = \begin{bmatrix} 1 \mid 0 & 0 \end{bmatrix} \left[\frac{NM}{AM}\right]$$

and $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2(\partial_1\partial_2 - 1) & \partial_1 - \partial_2 \end{bmatrix}$ is a MLA of $\begin{bmatrix} NM \\ AM \end{bmatrix}$, by Proposition 2.4

$$\mathcal{B}_c/\mathcal{C}_c^{can} \cong \ker \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -2(\partial_1\partial_2 - 1) & \partial_1 - \partial_2 \end{bmatrix} = \ker \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \partial_1 - \partial_2 \end{bmatrix}.$$

Since this latter matrix is not GFLP, then $\mathcal{C}_c^{can}$ is not regularly implementable by full interconnection from $\mathcal{B}_c$ and therefore $\overline{\mathcal{B}}_w$ is not implemented by regular partial interconnection from $\mathcal{B}_{(w,c)}$. □

So, in the case $R(\underline{s})$ has not full row rank one should find, if possible, a behavior $\mathcal{D}_w$ containing $\overline{\mathcal{B}}_w$ which is "large" enough to be regularly implementable, but sufficiently "small" so that $\overline{\mathcal{B}}_w/\mathcal{V}_w$ is autonomous. This is a difficult problem, which is currently under investigation.

Here we focus on the possibility of taking $\mathcal{D}_w = \overline{\mathcal{B}}_w$ in Definition 4.3, and therefore define the following stronger notion of controlled invariance.

Definition 4.7. Let $\mathcal{B}_{(w,c)} \subset \mathcal{U}^{\mathtt{w}} \times \mathcal{U}^{\mathtt{c}}$ be an nD behavior. A sub-behavior $\mathcal{V}_w$ of the induced w-behavior $\mathcal{B}_w \subset \mathcal{U}^{\mathtt{w}}$ is said to be $\mathcal{B}_{(w,c)}$-*strongly controlled-invariant* if $\overline{\mathcal{B}}_w$ is implementable from $\mathcal{B}_{(w,c)}$ by regular partial interconnection and $\overline{\mathcal{B}}_w/\mathcal{V}_w$ is autonomous.

Remark 2. It easily follows from this definition, together with Proposition 4.4 and Remark 1, that strong controlled-invariance and controlled-invariance are equivalent when the matrix R has full row rank.

Although strong controlled-invariance is a more restrictive property than controlled-invariance, it is easier to characterize the former than the latter. The following theorem, which is the main result of this paper, gives such a characterization in terms of the canonical controller (see Definition 3.2).

Theorem 4.8. *Consider the behavior $\mathcal{B}_{(w,c)}$ described by $Rw = Mc$ and let $\mathcal{V}_w = \ker V$ be a sub-behavior of the induced w-behavior $\mathcal{B}_w$. Let A and B be polynomial matrices such that $\begin{bmatrix} -A & B \end{bmatrix}$ is a MLA of $\begin{bmatrix} R \\ V \end{bmatrix}$. If N is a MLA of R and $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ is a MLA of $\begin{bmatrix} NM \\ AM \end{bmatrix}$, then $\mathcal{V}_w$ is $\mathcal{B}_{(w,c)}$-strongly controlled-invariant if and only if the following two conditions hold:*

- *(i) the matrix $\begin{bmatrix} B \\ E \end{bmatrix}$ has full column rank, where E is a MLA of V;*

[1]Recall (Zerz, 2000, Definition 3) that an nD polynomial matrix $H(\underline{s})$ is GFLP, if the existence of a factorization $H = DH_1$ (D not necessarily square) with $rank(H) = rank(H_1)$ implies the existence of an nD polynomial matrix E such that $H_1 = EH$.

(*ii*) *there exists a polynomial matrix* Y *such that*

$$\begin{bmatrix} Q_1 & Q_2 \\ I & 0 \end{bmatrix} \left(Y \begin{bmatrix} I & 0 \end{bmatrix} - I \right) \begin{bmatrix} NM \\ AM \end{bmatrix} = 0.$$

Moreover, if an nD polynomial matrix Y *as in* (*ii*) *exists, then the regular partial controller can be taken as* $\mathcal{C}^c = \ker C$ *with*

$$C = \left(Y \begin{bmatrix} I & 0 \end{bmatrix} - I \right) \begin{bmatrix} NM \\ AM \end{bmatrix}.$$

Remark 3. If the matrix V has full row rank, then the condition (i) of the previous theorem should be replaced by "The matrix B has full column rank".

Proof. Considering the behavior $\mathcal{B}_{(w,c)}$ described by $Rw = Mc$, one has that $\mathcal{B}_w = \ker LR$ where L is a MLA of M and $\mathcal{B}_c = \ker NM$, where N is a MLA of R. Moreover, $\mathcal{N}_w = \ker R$ and since $\begin{bmatrix} -A & B \end{bmatrix}$ is a MLA of $\begin{bmatrix} R \\ V \end{bmatrix}$, by (Rocha & Wood, 2001, Lemma 2.14), $\overline{\mathcal{B}}_w = \mathcal{N}_w + \mathcal{V}_w = \ker F$ with $F = AR = BV$.

By definition, $\mathcal{V}_w$ is said to be $\mathcal{B}_{(w,c)}$-*strongly controlled-invariant* if $\overline{\mathcal{B}}_w$ is implementable from $\mathcal{B}_{(w,c)}$ by regular partial interconnection and $\overline{\mathcal{B}}_w/\mathcal{V}_w$ is autonomous.

If E is a MLA of V, then by Proposition 2.2 and Proposition 2.4, $\overline{\mathcal{B}}_w/\mathcal{V}_w$ is autonomous if and only if the matrix $\begin{bmatrix} B \\ E \end{bmatrix}$ has full column rank.

On the other hand, by Theorem 3.3, $\overline{\mathcal{B}}_w$ is regularly implementable by partial interconnection from $\mathcal{B}_{(w,c)}$ if and only if $\mathcal{C}_c^{can}$ is regularly implementable by full interconnection from $\mathcal{B}_c$. By Definition 3.2, the canonical controller $\mathcal{C}_c^{can}$ associated with $\mathcal{B}_{(w,c)}$ and $\overline{\mathcal{B}}_w$ is defined by the equations

$$\begin{cases} Rw = Mc \\ ARw = 0 \end{cases}$$

By eliminating the variable w, one obtains:

$$\begin{bmatrix} N & 0 \\ A & -I \end{bmatrix} \begin{bmatrix} M \\ 0 \end{bmatrix} c = 0 \Leftrightarrow \begin{bmatrix} N \\ A \end{bmatrix} Mc = 0,$$

and so $\mathcal{C}_c^{can} = \ker \begin{bmatrix} NM \\ AM \end{bmatrix}$.
Hence, $\mathcal{C}_c^{can}$ is regularly implementable by full interconnection from $\mathcal{B}_c$ if there exists a full controller $\mathcal{C}_c = \ker C$ such that

$$\mathcal{RM}\left(\begin{bmatrix} NM \\ AM \end{bmatrix} \right) = \mathcal{RM}(NM) \oplus \mathcal{RM}(C)$$

$$\Leftrightarrow \quad \mathcal{RM} \begin{bmatrix} NM \\ AM \end{bmatrix} = \mathcal{RM}\left(\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} NM \\ AM \end{bmatrix} \right) \oplus \mathcal{RM}(C).$$

By (Zerz & Lomadze , 2001) this is equivalent to the existence of a polynomial matrix Y such that

$$\begin{bmatrix} Q_1 & Q_2 \\ I & 0 \end{bmatrix} \left(Y \begin{bmatrix} I & 0 \end{bmatrix} - I \right) \begin{bmatrix} NM \\ AM \end{bmatrix} = 0,$$

where $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ is a MLA of $\begin{bmatrix} NM \\ AM \end{bmatrix}$.

Moreover, if such matrix exists, then one may take $C = \left(Y \begin{bmatrix} I & 0 \end{bmatrix} - I \right) \begin{bmatrix} NM \\ AM \end{bmatrix}$. □

Remark 4. For details on the existence and computation of the polynomial matrix Y we refer to Zerz & Lomadze (2001).

Example 4.9. Consider the 2D behavior $\mathcal{B}_{(w,c)}$ described by $Rw = Mc$ with

$$R = \begin{bmatrix} s_2+3 & s_1+s_2+6 & s_1+s_2+6 \\ s_2 & s_1+s_2 & s_1+s_2 \\ 3 & 6 & 6 \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} s_2+1 & s_1+1 \\ s_1+1 & s_2+1 \\ s_1+1 & s_2+1 \end{bmatrix}.$$

Note that $\operatorname{rank} R = 2$, and hence R has not full row rank. Since $N = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}$ is a MLA of R and $L = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$ is a MLA of M then

$$\mathcal{B}_c = \ker NM = \ker \left(\begin{bmatrix} 1 & -1 & -1 \end{bmatrix} M \right) = \ker \begin{bmatrix} \partial_2 - 2\partial_1 - 1 & \partial_1 - 2\partial_2 - 1 \end{bmatrix}$$

and

$$\mathcal{B}_w = \ker LR = \ker \begin{bmatrix} \partial_2 - 3 & \partial_1 + \partial_2 - 6 & \partial_1 + \partial_2 - 6 \end{bmatrix}.$$

Moreover, let $\mathcal{V}_w = \ker V$ with $V = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Since $LR = \begin{bmatrix} s_2 - 3 & s_1 - 3 \end{bmatrix} V$, $\mathcal{V}_w$ is a sub-behavior of $\mathcal{B}_w$. Further, $\mathcal{V}_w$ is also contained in the hidden behavior $\mathcal{N}_w$, since

$$\mathcal{N}_w = \ker R \quad \text{and} \quad R = BV \quad \text{with} \quad B = \begin{bmatrix} s_2+3 & s_1+3 \\ s_2 & s_1 \\ 3 & 3 \end{bmatrix}.$$

Therefore $\overline{\mathcal{B}}_w = \mathcal{V}_w + \mathcal{N}_w = \mathcal{N}_w$ and, by Proposition 2.4, $\overline{\mathcal{B}}_w / \mathcal{V}_w \cong \ker B$. Since B has full column rank it follows from Proposition 2.2 that $\overline{\mathcal{B}}_w / \mathcal{V}_w$ is autonomous.

To show that $\mathcal{V}_w$ is $\mathcal{B}_{(w,c)}$-strongly controlled-invariant, by Definition 4.7 it remains to prove that $\overline{\mathcal{B}}_w$ is implementable from $\mathcal{B}_{(w,c)}$ by regular partial interconnection which, by Proposition 3.3, is equivalent to show that the canonical controller associated to $\overline{\mathcal{B}}_w$, $\mathcal{C}_c^{can}$, is regularly implementable by full interconnection from $\mathcal{B}_c$. By Definition 3.2, $\mathcal{C}_c^{can}$ is defined by the equations

$$\begin{cases} Rw = Mc \\ Rw = 0 \end{cases}$$

and hence $\mathcal{C}_c^{can} = \ker M$. Considering a controller behavior $\mathcal{C}_c = \ker C$ with

$$C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M = \begin{bmatrix} s_1+1 & s_2+1 \\ s_1+1 & s_2+1 \end{bmatrix}$$

it follows that

$$\mathcal{B}_c \cap \mathcal{C}_c = \ker \begin{bmatrix} NM \\ C \end{bmatrix} = \ker \left(\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} M \right) = \ker M = \mathcal{C}_c^{can}.$$

Moreover, since $\operatorname{rank} NM = 1$, $\operatorname{rank} C = 1$ and $\operatorname{rank} M = 2$, we have that $\mathcal{B}_c \cap \mathcal{C}_c = \mathcal{C}_c^{can}$ is a regular full interconnection and thus $\mathcal{V}_w$ is $\mathcal{B}_{(w,c)}$-strongly controlled-invariant.

It is easy to check that condition (ii) of Theorem 4.8 hold with the matrices

$$A = I_3, \quad \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} = \begin{bmatrix} 0 & L \\ 1 & N \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 & 3 & 1 & 1 \end{bmatrix}^T. \quad \square$$

5. Conclusions

In this paper, the property of strong controlled invariance of nD behavioral systems was introduced in the context of partial interconnections, and completely characterized from the point of view of full interconnections by resorting to the associated canonical controller. The obtained conditions can be easily checked my means of computer algebra tools. The property of controlled invariance, which is less restrictive but more difficult to characterize than strong controlled invariance, is currently under investigation.

References

M. Belur, H. Trentelman (2002): Stabilization, pole placement, and regular implementability, *IEEE T. Automat. Contr.*, 47(5), 735–744.

U. Oberst (1990): Multidimensional constant linear systems, *Acta Appl. Math.*, 20, 1–175.

R. Pereira, P. Rocha (2017): A note on controlled invariance for behavioral nD systems *Algebraic and Symbolic Computation Methods in Dynamical Systems*, in press.

P. Rocha (2002): Regular implementability of nD behaviors, *Proceedings of the 15th International Symposium on the Mathematical Theory of Networks and Systems, MTNS'02*, University of Notre Dame, South Bend, USA.

P. Rocha (2005): Canonical controllers and regular implementation of nD behaviors, *Proceedings of the 16th IFAC World Congress*, Prague, Czech Republic.

P. Rocha, J. Wood (1997): A new perspective on controllability properties for dynamical systems, *Int. J. Appl. Math. Comp. Sci.*, 7(4), 869–879.

P. Rocha, J. Wood (2001): Trajectory control and interconnection of 1D and nD systems, *SIAM J. Control Optim.*, 40(1), 107–134.

J.C. Willems (1997): On interconnections, control and feedback, *IEEE T. Automat. Contr.*, 42(3), 326–339.

J.C. Willems, M. Belur, A.A. Juliu, H. Trentelman (2003): The canonical controller and its regularity, *Proceedings of the 42nd IEEE Conference on Decision and Control*, Maui, Hawaii, USA.

J. Wood (2000): Modules and behaviors in nD systems theory, *Multidim. Syst. Sign. P.*, 11, 11–48.

J. Wood, E. Rogers, D. H. Owens (1999): Controllable and Autonomous nD Linear Systems, *Multidim. Syst. Sign. P.*, 10, 33–69.

J. Wood, U. Oberst, E. Rogers, D. H. Owens (2000): A behavioral approach to the pole structure of one-dimensional and multidimensional linear systems, *SIAM J. Control Optim.*, 38, 627–661.

E. Zerz (2000): Topics in multidimensional linear system theory, *Springer Lecture Notes in Control and Information Theory*, 256, Springer, London.

E. Zerz, V. Lomadze (2001): A Constructive Solution to Interconnection and Decomposition Problems, *SIAM J. Control Optim.*, 40(4), 1072–1086.