THE OMEGA-INEQUALITY PROBLEM FOR CONCATENATION HIERARCHIES OF STAR-FREE LANGUAGES

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ABSTRACT. The problem considered in this paper is whether an inequality of ω-terms is valid in a given level of a concatenation hierarchy of star-free languages. The main result shows that this problem is decidable for all (integer and half) levels of the Straubing-Thérien hierarchy.

1. INTRODUCTION

With the advent of computers in the 1950's, there was a surge of interest in formal languages, among which regular languages play a key role. After several examples of algebraization of questions on classes (*varieties*) of regular languages had been already discovered in the 1960's and early 1970's, Eilenberg described a general framework for the algebraic formulation of such questions, namely in terms of what is commonly known as Eilenberg's correspondence [14]. The algebraic structures considered by Eilenberg are finite semigroups and monoids and the classes of such structures corresponding to varieties of languages are called *pseudovarieties.* In general, the hope is that the algebraization will convert the membership problem for a given variety of languages in a more manageable membership problem for the corresponding pseudovariety, and indeed, the pioneering examples of instances of the correspondence before its general formulation were found with this purpose and effective application.

Eilenberg's correspondence has been extended in several directions, in particular to capture more general classes of regular languages. One such generalization was developed by Pin [20] with the aim of refining and better understanding a hierarchy of star-free languages introduced by Brzozowski and Cohen [13] and its variant considered by Straubing [29, 30] and Thérien [32]. The algebraic structures emerging in this context are finite semigroups and monoids with a compatible partial order, the relevant classes of such structures being also called pseudovarieties. While pseudovarieties of semigroups and monoids are defined by so-called *pseudoidentities*, which are formal equalities of members of suitable free profinite structures [27, 2], in the ordered counterpart it suffices to replace equality by formal inequality, leading to *pseudoinequalities*, to obtain a similar result [19, 23].

Additional motivation for investigating the Straubing-Thérien hierarchy comes from logic, specifically from finite model theory. Indeed, as has been

shown by Thomas [31], the Straubing-Thérien hierarchy may be viewed as a
skeleton of the hierarchy of languages defined by sentences in the first-order
language with a binary predicate (for ordering positions of letters) and unary
predicates for the letters of the alphabet (to denote the presence of a letter in
a given position), where the complexity of sentences in prenex normal form
is measured in terms of the number of quantifier alternations.

Although significant progress has been recently achieved on the member-
ship problem for the levels of the Straubing-Thérien hierarchy [25, 26, 5], it
remains an open problem whether it is decidable at all levels. A key tool
that has been used in such works is the following *lifting* problem: to deter-
mine when a pair of elements of a finite monoid can be realized as values
in the monoid of the sides of a pseudoinequality which is valid in a given
level of the corresponding hierarchy of pseudovarieties. The decidability of
this condition, which Place and Zeitoun [26] call the *separation property*, en-
tails the decidability of the membership problem for the pseudovariety and
it may be viewed as a form of *hyperdecidability*, namely for the inequality
$x \leqslant y$, in the sense of [3]. Note that in general there are uncountably many
pseudoinequalities to be considered as potential liftings of a given pair of
elements of a finite monoid.

A method introduced by Steinberg and the first author [11] to approach
such decidability questions consists in solving two separate problems involv-
ing an *implicit signature* σ with suitable computational properties:

(1) to solve the σ-inequality problem for the pseudovariety in question;
(2) to show that, if an instance of the lifting problem has a solution,
    then it admits one with a pseudoinequality which is actually a σ-
    inequality.

Although problems such as Problem (2) tend to be very hard, they have the
advantage of being abstract problems which are not of an algorithmic nature.
In the aperiodic case, which is a suitable setting for the problems concern-
ing the Straubing-Thérien hierarchy, the most frequently considered implicit
signature consists of multiplication, 1 and ω-power, and is also denoted ω.

The main result of this paper is a solution of Problem (1) for the sig-
nature ω for every level of the Straubing-Thérien hierarchy. There are two
key ingredients in the proof of this result. The first is the fact that, for a
polynomially closed pseudovariety of ordered monoids V satisfying a pseu-
doinequality $u \leqslant v$, any factorization of u induces a factorization of v of
the same length such that the inequality remains valid in V for factors in
the same position (Section 3). The second is a syntactic analysis of ω-terms
that leads to a description of its factors (Section 5) and to a repetition or
periodicity result (Section 6). These may be viewed as finiteness properties
of ω-words over the pseudovariety of all finite aperiodic monoids. For further
such properties and generalizations, see [8, 15]. The combination of these
results allows us to prove a sort of completeness theorem for ω-inequalities
valid in the Boolean-polynomial closure of any polynomially closed pseudova-
riety of aperiodic ordered monoids: there is a complete deductive calculus to
obtain all such inequalities from a well-determined basis (Section 7). Once
such a calculus has been established, one may effectively enumerate the con-
sequences of a recursively enumerable basis. Since the ω-inequalities which

are not consequences of the basis fail in concrete finite models of the basis, decidability of the ω-inequality problem follows provided both the basis and the pseudovariety of ordered monoids are recursively enumerable. Putting it all together, this yields an inductive argument that shows that all levels of the Straubing-Thérien hierarchy have decidable ω-inequality problem (Section 8).

2. Preliminaries

The reader is referred to the standard literature on finite semigroups for general background [2, 28]. Nevertheless, we recall here some basic notions for the sake of completeness.

2.1. **Pseudovarieties.** A *pseudovariety* of monoids is a class of finite monoids that is closed under taking homomorphic images, submonoids and finite direct products. For finite ordered monoids, that is, monoids equipped with a compatible partial order, pseudovarieties are defined in the same way. A pseudovariety W of monoids may be identified with the pseudovariety of ordered monoids consisting of the members of W equipped with all possible compatible partial orders. Then, for a pseudovariety V of ordered monoids, the class of monoids obtained by forgetting the order of all elements of V and taking homomorphic images is a pseudovariety of monoids, whence it is the pseudovariety of monoids generated by V. This is also the join of V with its *dual*, consisting of the members of V with the orders reversed.

The *trivial* pseudovariety consists of all singleton monoids. Two further examples of pseudovarieties of monoids are M, consisting of all finite monoids, and A, consisting of all finite aperiodic monoids, that is, finite monoids all of whose subgroups are trivial.

By a *pro-V monoid* we mean a monoid with a compact topology for which multiplication is continuous such that distinct points may be separated by continuous homomorphisms into members of V, these being regarded as discrete spaces. For a pseudovariety V of [ordered] monoids and a finite set A, the free pro-V monoid on A is denoted $\overline{\Omega}_A\mathsf{V}$. It may be constructed as the inverse limit of the natural projective family of A-generated monoids from V. For our purposes, the relevant property is the universal property that justifies its name: every function from A to a pro-V monoid M extends uniquely to a continuous homomorphism on $\overline{\Omega}_A\mathsf{V}$, as depicted in the following diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\iota} & \overline{\Omega}_A\mathsf{V} \\
 & \searrow & \big\downarrow \\
 & & M\,.
\end{array}
$$

This property of $\overline{\Omega}_A\mathsf{V}$ entails that each homomorphism $\varphi\colon A^* \to M$ into a pro-V monoid M extends uniquely to a continuous homomorphism $\hat{\varphi}\colon \overline{\Omega}_A\mathsf{V} \to M$, leading to the following commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\iota} & \overline{\Omega}_A\mathsf{V} \\
\big\downarrow & {\scriptstyle\hat{\iota}}\nearrow & \big\downarrow{\scriptstyle\hat{\varphi}} \\
A^* & \xrightarrow{\varphi} & M\,,
\end{array}
$$

where $\hat{\iota} : A^* \to \overline{\Omega}_A \mathsf{V}$ is the unique extension of ι to a homomorphism. In case $\alpha \colon A^* \to B^*$ is a homomorphism, we may also view it as a homomorphism $A^* \to \overline{\Omega}_B \mathsf{V}$, and consider its unique extension to a continuous homomorphism $\hat{\alpha} \colon \overline{\Omega}_A \mathsf{V} \to \overline{\Omega}_B \mathsf{V}$.

Often, the natural monoid homomorphism $\hat{\iota} : A^* \to \overline{\Omega}_A \mathsf{V}$ is an embedding, in which case we identify A^* with its image. This is the case, for instance, for all pseudovarieties of ordered monoids containing the pseudovariety MN consisting of all finite nilpotent semigroups with an identity element adjoined, a property which holds for most nontrivial pseudovarieties of ordered monoids considered in this paper and which will therefore be used freely. Moreover, for such pseudovarieties, the topology on A^* induced from $\overline{\Omega}_A \mathsf{V}$ is discrete.

The elements of $\overline{\Omega}_A \mathsf{V}$ are sometimes called *pseudowords over* V. A *pseudoinequality over* V is a formal inequality of two pseudowords over V; in what follows, they are simply called inequalities. The inequality $u \leqslant v$ is *trivial* if u and v coincide. For $u, v \in \overline{\Omega}_A \mathsf{V}$ and $S \in \mathsf{V}$, the inequality $u \leqslant v$ is said to *hold* in or to be *satisfied* by S if $\varphi(u) \leqslant \varphi(v)$ for every continuous homomorphism $\varphi : \overline{\Omega}_A \mathsf{V} \to S$, and to *hold* in a subclass $\mathcal{C}$ of V if it holds in every member of $\mathcal{C}$; the *pseudoidentity* $u = v$ is *satisfied* by $\mathcal{C}$ if both inequalities $u \leqslant v$ and $v \leqslant u$ are satisfied by the members of $\mathcal{C}$. It is well known that, if an inequality $u \leqslant v$ holds in MN, that is, the pseudoidentity $u = v$ holds in MN, and either u or v are words, then it is trivial.

Note that, for a pseudovariety of ordered monoids V, $\overline{\Omega}_A \mathsf{V}$ is an ordered monoid for the relation $\leqslant$ such that $u \leqslant v$ if and only if the inequality $u \leqslant v$ over V holds in V.

For a pseudovariety V of ordered monoids, a language $L \subseteq A^*$ is said to be V-*recognizable* if there exist a homomorphism $\varphi \colon A^* \to M$ into a monoid M from V and an up-closed (that is, an order filter) subset F of M such that $L = \varphi^{-1}(F)$. Equivalently, the *syntactic ordered monoid* $\mathrm{Synt}(L)$ of L belongs to V, where $\mathrm{Synt}(L)$ is the quotient of A^* by the congruence $\leqslant_L \cap \geqslant_L$, ordered by the partial order induced by the quasi-order $\leqslant_L$, where $u \leqslant_L v$ is defined by the following condition: for every $x, y \in A^*$, $xuy \in L$ implies $xvy \in L$. The natural homomorphism $\varphi \colon A^* \to \mathrm{Synt}(L)$ as well as its extension $\hat{\varphi}$ are both called *syntactic homomorphisms*. It should be noted that, in several papers by Pin and coauthors, such as [24], the syntactic order $\leqslant_L$ is defined to be the dual of the order considered here; see [6] for an explanation as to why our choice should be preferred.

Yet another equivalent formulation of V-recognizability of $L \subseteq A^*$ is that its topological closure $\overline{L}$ in $\overline{\Omega}_A \mathsf{V}$ is open. If $\varphi \colon A^* \to M$ is a homomorphism into $M \in \mathsf{V}$ that recognizes the language L, then $\hat{\varphi} \colon \overline{\Omega}_A \mathsf{V} \to M$ recognizes $\overline{L}$, in the sense that $\overline{L} = \hat{\varphi}^{-1}(\hat{\varphi}(\overline{L}))$. Note that, if $L, K \subseteq A^*$ are V-recognizable languages, then the open set $\overline{L} \cap \overline{K}$ coincides with $\overline{L \cap K}$.

For pseudovariety V of ordered monoids, the V-recognizable languages constitute what is known as a *positive variety of languages*. More precisely, a positive variety of languages is defined as a correspondence $\mathcal{V}$ associating with each finite alphabet A a set $\mathcal{V}(A)$ of regular languages over A that is closed under finite union, finite intersection, and the derivative operations $L \mapsto a^{-1}L = \{w : aw \in L\}$, $L \mapsto La^{-1} = \{w : wa \in L\}$, and such that $L \in \mathcal{V}(B)$ implies $\varphi^{-1}(L) \in \mathcal{V}(A)$ whenever $\varphi : A^* \to B^*$ is a homomorphism.

The above defined correspondence $\mathsf{V} \mapsto \mathcal{V}$ is a bijection [20] and it is the natural analog for ordered monoids of Eilenberg's Correspondence Theorem [14], relating pseudovarieties of monoids and varieties of languages.

2.2. **Polynomial closure.** Given a positive variety of languages $\mathcal{V}$, its *polynomial closure* $\operatorname{Pol} \mathcal{V}$ is defined by letting $\operatorname{Pol} \mathcal{V}(A)$ consist of all unions of finitely many languages of the form

$$L_0 a_1 L_1 \cdots a_n L_n,$$

where the L_i belong to $\mathcal{V}(A)$ and the $a_i \in A$ are letters. Note that the equality $\operatorname{Pol} \operatorname{Pol} \mathcal{V} = \operatorname{Pol} \mathcal{V}$ follows directly from the definition. We say that $\mathcal{V}$ is *polynomially closed* if $\mathcal{V} = \operatorname{Pol} \mathcal{V}$. It is well known that $\operatorname{Pol} \mathcal{V}$ is a positive variety of languages whenever $\mathcal{V}$ is a variety of languages. Therefore, for the pseudovariety V of monoids corresponding to a variety of languages $\mathcal{V}$, we denote by $\operatorname{Pol} \mathsf{V}$ the pseudovariety of ordered monoids corresponding to the positive variety of languages $\operatorname{Pol} \mathcal{V}$. A pseudovariety of ordered monoids corresponding to a polynomially closed positive variety of languages is also called *polynomially closed*. Let us remark that $\operatorname{Pol} \mathsf{V}$ is not defined for a pseudovariety of ordered monoids V in general.

By a result of Pin and Weil [24, Theorem 5.9], if V is a pseudovariety of monoids, then $\operatorname{Pol} \mathsf{V}$ may be described as the Mal'cev product $[\![x^\omega \leqslant x^\omega y x^\omega]\!] \textcircled{m} \mathsf{V}$. Combining with a general basis theorem for Mal'cev products, also due to Pin and Weil [22], we obtain a basis of inequalities for $\operatorname{Pol} \mathsf{V}$ (see [21, Proposition 7.4]).

Proposition 2.1. *Let V be a pseudovariety of monoids. Then, $\operatorname{Pol} \mathsf{V}$ is defined by all inequalities of the form $u^\omega \leqslant u^\omega v u^\omega$, with $u, v \in \overline{\Omega}_A \mathsf{M}$, such that the pseudoidentities $u = v = v^2$ hold in V.*

Also of interest is to consider the *Boolean-polynomial closure* of a pseudovariety V of ordered monoids, denoted $\operatorname{Pol} \mathrm{B} \mathsf{V}$. This means taking first the pseudovariety of monoids generated by V, which recognizes precisely the languages over a finite alphabet A that are Boolean combinations of languages recognized by V, and then taking the polynomial closure of that pseudovariety of monoids. The following basis of inequalities for $\operatorname{Pol} \mathrm{B} \mathsf{V}$ can be found in [5].

Proposition 2.2. *Let W be a pseudovariety of monoids and let $\mathsf{V} = \operatorname{Pol} \mathsf{W}$. Then, $\operatorname{Pol} \mathrm{B} \mathsf{V}$ is defined by all inequalities of the form $u^{\omega+1} \leqslant u^\omega v u^\omega$, with $u, v \in \overline{\Omega}_A \mathsf{M}$, such that the inequality $v \leqslant u$ holds in V.*

The *concatenation hierarchy* based on a pseudovariety of monoids V is the sequence starting at V that alternates polynomial closure with forgetting order plus taking homomorphic images; the pseudovarieties of monoids obtained by forgetting order plus taking homomorphic images are called *levels* of the hierarchy, while the pseudovarieties of ordered monoids obtained by the polynomial closure are called *half levels*. The concatenation hierarchy based on the trivial pseudovariety is known as the *Straubing-Thérien hierarchy*.

2.3. **Validity of inequalities.** The following lemma gives two alternative characterizations of when an inequality holds in a syntactic ordered monoid.

Lemma 2.3. *Let A and B be finite alphabets, $L \subseteq B^*$ be a regular language, and $u, v \in \overline{\Omega}_A \mathsf{M}$ be pseudowords. The following conditions are equivalent:*

(1) *the inequality $u \leqslant v$ holds in the syntactic ordered monoid $\mathrm{Synt}(L)$;*
(2) *for every homomorphism $\alpha \colon A^* \to B^*$ and all words $x, y \in B^*$, $x\hat{\alpha}(u)y \in \overline{L}$ implies $x\hat{\alpha}(v)y \in \overline{L}$;*
(3) *for every homomorphism $\alpha \colon A^* \to B^*$ and all pseudowords $x, y \in \overline{\Omega}_B \mathsf{M}$, $x\hat{\alpha}(u)y \in \overline{L}$ implies $x\hat{\alpha}(v)y \in \overline{L}$.*

Proof. Let $\varphi \colon \overline{\Omega}_B \mathsf{M} \to \mathrm{Synt}(L)$ be the syntactic homomorphism.

(1) $\Rightarrow$ (3) Let $\alpha \colon A^* \to B^*$ be a homomorphism and suppose that $x, y \in \overline{\Omega}_B \mathsf{M}$ are such that $x\hat{\alpha}(u)y \in \overline{L}$. Applying φ and taking into account that $\varphi(\overline{L}) = \varphi(L)$, we obtain the relation $\varphi(x\hat{\alpha}(u)y) \in \varphi(L)$. Recall that $\varphi(L)$ is an order filter in $\mathrm{Synt}(L)$. Since the order in $\mathrm{Synt}(L)$ is stable and $\varphi\hat{\alpha}(u) \leqslant \varphi\hat{\alpha}(v)$ by (1), it follows that $\varphi(x\hat{\alpha}(v)y) \in \varphi(L)$. Since φ recognizes $\overline{L}$, we deduce that $x\hat{\alpha}(v)y \in \overline{L}$, as required.

The implication (3) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let $\psi \colon \overline{\Omega}_A \mathsf{M} \to \mathrm{Synt}(L)$ be an arbitrary continuous homomorphism. We need to show that $\psi(u) \leqslant \psi(v)$. Choose a homomorphism $\alpha \colon A^* \to B^*$ such that $\psi = \varphi \circ \hat{\alpha}$. By definition of the syntactic order and since $\varphi|_{B^*}$ recognizes L, the preceding inequality is equivalent to the property that, for all $p, q \in \mathrm{Synt}(L)$, $p\psi(u)q \in \varphi(L)$ implies $p\psi(v)q \in \varphi(L)$. To establish this property, note first that, since $\varphi|_{B^*}$ is onto, given $p, q \in \mathrm{Synt}(L)$ such that $p\psi(u)q \in \varphi(L)$, there exist $x, y \in B^*$ such that $\varphi(x) = p$ and $\varphi(y) = q$. Since φ recognizes $\overline{L}$, we deduce that $x\hat{\alpha}(u)y \in \overline{L}$. By (2), it follows that $x\hat{\alpha}(v)y \in \overline{L}$, and so indeed $p\psi(v)q \in \varphi(L)$, as claimed. $\square$

Lemma 2.3 serves to establish a simple profinite characterization of when an inequality holds in a pseudovariety of ordered monoids.

Proposition 2.4. *Let V be a pseudovariety of ordered monoids and let $u, v \in \overline{\Omega}_A \mathsf{M}$. The following conditions are equivalent:*

(1) *the inequality $u \leqslant v$ holds in V;*
(2) *whenever $L \subseteq A^*$ is a V-recognizable language, $u \in \overline{L}$ implies $v \in \overline{L}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that (1) holds and let $L \subseteq A^*$ be a V-recognizable language such that $u \in \overline{L}$. By assumption, $\mathrm{Synt}(L)$ satisfies the inequality $u \leqslant v$. By Lemma 2.3(2), taking $x = y = 1$ and α to be the identity mapping, we deduce from $u \in \overline{L}$ that $v \in \overline{L}$, as required.

(2) $\Rightarrow$ (1) Since V is generated by the syntactic ordered monoids $\mathrm{Synt}(K)$ in V of regular languages K, it suffices to show that, for every V-recognizable language $K \subseteq B^*$, the ordered monoid $\mathrm{Synt}(K)$ satisfies the inequality $u \leqslant v$. For this purpose, we establish the condition (2) of Lemma 2.3. Thus, we should show that, for every homomorphism $\alpha \colon A^* \to B^*$ and all words $x, y \in B^*$, $x\hat{\alpha}(u)y \in \overline{K}$ implies $x\hat{\alpha}(v)y \in \overline{K}$.

We claim that, for $w \in \overline{\Omega}_A \mathsf{M}$, $x\hat{\alpha}(w)y \in \overline{K}$ is equivalent to $w \in \overline{L}$, where $L = \alpha^{-1}(x^{-1}Ky^{-1})$. Suppose first that $x\hat{\alpha}(w)y \in \overline{K}$ and let w_n be a sequence of words of A^* converging to w. Since $\overline{K}$ is an open set

such that $\overline{K} \cap B^* = K$ (cf. [2, Theorem 3.6.1]) and the limit $x\hat{\alpha}(w)y$ of the sequence $x\alpha(w_n)y$ belongs to $\overline{K}$, passing to a subsequence if necessary, we may assume that all terms in the sequence belong to K. Then every w_n belongs to L, so $w \in \overline{L}$. The converse follows from the continuity of both $\hat{\alpha}$ and multiplication.

It remains to apply the hypothesis (2) to the language $L = \alpha^{-1}(x^{-1}Ky^{-1})$, which is V-recognizable because the class of all V-recognizable languages constitutes a positive variety of languages. $\square$

3. Lifting factorizations

The next result shows that factorizations may be lifted along inequalities. This property plays a key role in the sequel. The proof uses nets in compact spaces, which is a classical tool in Topology (see, for instance [33]).

Theorem 3.1. *Let* V *be a polynomially closed pseudovariety of ordered monoids and let* $u, v \in \overline{\Omega}_A$M. *If the inequality* $u \leqslant v$ *holds in* V *then, for every factorization* $u = u_0au_1$ *with* $a \in A$, *there is a factorization* $v = v_0av_1$ *such that each inequality* $u_i \leqslant v_i$ *holds in* V *$(i = 0, 1)$.*

Proof. Let $u = u_0au_1$ be an arbitrary factorization with $a \in A$. Let I be the set of all pairs (L_0, L_1) of V-recognizable languages $L_i \subseteq A^*$ such that $u_i \in \overline{L_i}$ $(i = 0, 1)$. We consider on I the partial order defined by $(L_0, L_1) \leqslant (K_0, K_1)$ if $K_i \subseteq L_i$ $(i = 0, 1)$. Since the positive variety of languages corresponding to V is closed under intersection, the above partial order on the set I is upper directed. For each $(L_0, L_1) \in I$, since $u = u_0au_1 \in \overline{L_0}a\overline{L_1} = \overline{L_0aL_1}$ and the language L_0aL_1 is V-recognizable because V is polynomially closed, it follows from Proposition 2.4 that $v \in \overline{L_0a\overline{L_1}}$, and so we may choose a factorization $v = v_0^{(L_0,L_1)}av_1^{(L_0,L_1)}$ such that each $v_i^{(L_0,L_1)}$ belongs to $\overline{L_i}$ $(i = 0, 1)$.

The mapping $\eta\colon I \to (\overline{\Omega}_AM)^2$ defined by $\eta(L_0, L_1) = (v_0^{(L_0,L_1)}, v_1^{(L_0,L_1)})$ may be viewed as a net in the compact product space $(\overline{\Omega}_A$M$)^2$. Hence, there is a convergent subnet $\eta \circ \lambda$ determined by a mapping $\lambda\colon J \to I$ from another upper directed set J into I such that, for every $(L_0, L_1) \in I$ there is some $j \in J$ such that $(L_0, L_1) \leqslant \lambda(j)$. Let (v_0, v_1) be the limit of the subnet $\eta \circ \lambda$. By continuity of multiplication, since $v = v_0^{\lambda(j)}av_1^{\lambda(j)}$ for every $j \in J$, it follows that $v = v_0av_1$.

We claim that each inequality $u_i \leqslant v_i$ holds in V $(i = 0, 1)$. To establish these inequalities, we apply again Proposition 2.4: it suffices to show that, for all pairs of V-recognizable languages $L_i \subseteq A^*$ such that $u_i \in \overline{L_i}$ $(i = 0, 1)$, we have $v_i \in \overline{L_i}$ $(i = 0, 1)$. Let $j_0 \in J$ be such that $(L_0, L_1) \leqslant \lambda(j_0)$. Now, for every $i \in \{0, 1\}$ and $j \in J$ such that $j \geqslant j_0$, the pseudoword $v_i^{\lambda(j)}$ belongs to the closure of the ith component of $\lambda(j)$, which is contained in $\overline{L_i}$. Hence $v_i^{\lambda(j)} \in \overline{L_i}$ for every $j \geqslant j_0$, which implies that the limit v_i also belongs to the closed set $\overline{L_i}$, as claimed. $\square$

Most of the time, it will be inconvenient to keep referring to the letter a in the factorizations to be lifted along inequalities considered in Theorem 3.1. The following result avoids it and further extends the lifting to an arbitrary number of factors.

Corollary 3.2. *Let* V *be a polynomially closed pseudovariety of ordered monoids and let* $u, v \in \overline{\Omega}_A \mathsf{M}$. *If the inequality* $u \leqslant v$ *holds in* V *then, for every factorization* $u = u_1 \cdots u_n$, *there is a factorization* $v = v_1 \cdots v_n$ *such that each inequality* $u_i \leqslant v_i$ *holds in* V *(i = 1, \ldots, n)*.

Proof. Proceeding by induction, it suffices to consider the case $n = 2$. So, suppose that the inequality $u \leqslant v$ holds in V and consider a factorization $u = u_1 u_2$. If $u_2 = 1$, then the factorization $v = v \cdot 1$ has the required properties. Otherwise, taking into account that u_2 is the limit of a sequence of nonempty words, a standard compactness argument shows that there is a factorization $u_2 = au'$, where a is some letter from A. We may then apply Theorem 3.1 to lift the factorization $u = u_1 a u'$ along the inequality $u \leqslant v$ to a factorization $v = v_1 a v'$ such that the inequalities $u_1 \leqslant v_1$ and $u' \leqslant v'$ hold in V. It remains to take $v_2 = av'$ and observe that the inequality $u_2 \leqslant v_2$ also holds in V. $\square$

4. Omega-inequalities

It is well known that elements of the free profinite monoid $\overline{\Omega}_A \mathsf{M}$ may be viewed as operations with a natural interpretation on each profinite monoid in such a way that the interpretation is preserved under continuous homomorphisms (see, for instance, [4]). More precisely, each $w \in \overline{\Omega}_A \mathsf{M}$ defines an A-ary operation symbol which is *naturally interpreted* in a profinite monoid M as the operation $w_M \colon M^A \to M$ that maps each function $\varphi \colon A \to M$ to $\hat{\varphi}(w)$. By an *implicit signature*, we mean a set of such operation symbols including the binary multiplication and the (nullary) identity element, that is, the standard signature for working with monoids. For an implicit signature σ, each profinite monoid M has thus a natural structure of a σ-algebra by interpreting each operation symbol naturally. It is a simple exercise to show that, for a pseudovariety V of ordered monoids, the σ-subalgebra of $\overline{\Omega}_A \mathsf{V}$ generated by A, denoted by $\Omega_A^{\sigma} \mathsf{V}$, is a V-free σ-algebra on A. Elements of $\Omega_A^{\sigma} \mathsf{V}$ will be called *σ-words (over V)*.

The absolutely free σ-algebra on a generating set A is the term σ-algebra $T_{\sigma}(A)$. The members of $T_{\sigma}(A)$ are obtained recursively from the elements of A by formally applying successively an operation from σ. As is standard, they may be visualized as finite rooted trees in which the leaves are labeled by members of A or the constant 1 and the non-leaf nodes are labeled by elements of σ; for each non-leaf node with operation w the sons are written in the order they are taken as arguments of the operation w. A node is called a *right descendant* whenever it is the second son of a node labeled by a binary operation.

Such construction and representation are unique for each σ-term t. For a pseudovariety V of ordered monoids, there is a *natural* homomorphism of σ-algebras $\theta \colon T_{\sigma}(A) \to \Omega_A^{\sigma} \mathsf{V}$ mapping each free generator to itself. For a σ-word u over V, a σ-term in $\theta^{-1}(u)$ is said to *represent u*.

We will be concerned with the signature ω consisting of binary multiplication and the usual ω-power, whose natural interpretation on a profinite monoid M maps each element s to the unique idempotent in the closed subsemigroup of M generated by s. As an example, consider the ω-word $(a^2 b^{\omega})^{\omega} a b^{\omega}$. One of its ω-term representations is described by the tree in

Figure 1. Note that the tree has four right descendants, which are graphically indicated by the lower end of an edge going down from a node towards the right.
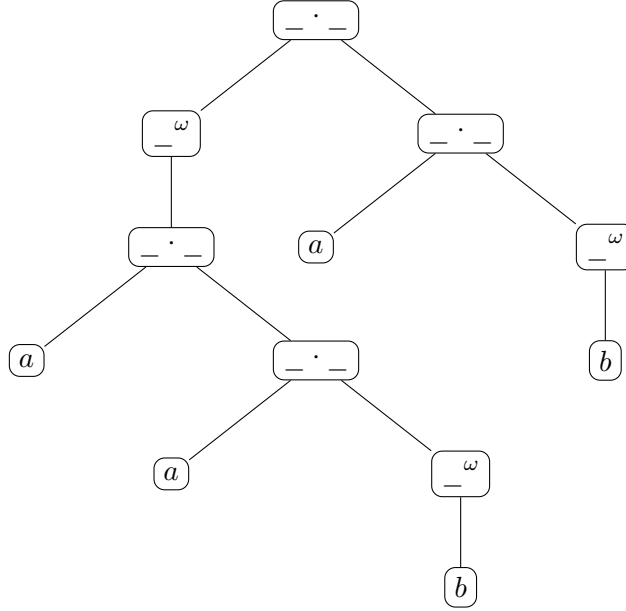

FIGURE 1. A tree representing the ω-word $(a^2 b^\omega)^\omega a b^\omega$.

By the ω-*inequality problem* for a pseudovariety V of ordered monoids we mean the problem that takes as input a pair (u, v) of ω-terms and asks whether the inequality $u \leqslant v$ is valid in V. Decidability of this problem amounts to being able to algorithmically calculate in the ordered monoid $\Omega_A^\omega \mathsf{V}$. Replacing inequalities by equalities, one may analogously define the corresponding notions such as *decidability* of the ω-*equality problem*.

Our aim is to show that the ω-inequality problem is decidable for all levels of the Straubing-Thérien hierarchy of aperiodic monoids. Rather than trying to construct efficient algorithms for such an infinite class of problems, we concentrate on "theoretical decidability", proving simply that both the instances of the problem with positive solution and those with negative solution may be recursively enumerated. For the purpose of enumerating inequalities with certain properties, it is convenient to consider only finite alphabets contained in a fixed countable set of variables, which we do from hereon without further mention.

We say that a pseudovariety of ordered monoids V is *recursively enumerable* if there is a Turing machine that successively produces as outputs precisely the elements of V, up to isomorphism, and nothing else. Equivalently, the corresponding positive variety of languages is recursively enumerable.

Proposition 4.1. *Let V be a recursively enumerable pseudovariety of ordered monoids. Then the ω-inequality problem for V is co-recursively enumerable.*

Proof. There is a Turing machine that successively enumerates all pairs consisting of a member M of V, up to isomorphism, and an ω-inequality $u \leqslant v$,

and nothing else. Note that the ω-power of an element s is computable in M: for instance, one may compute it as $s^\omega = s^{n!}$, where $n = |M|$. Hence, one may effectively test for each such pair $(M, u \leqslant v)$ whether M satisfies $u \leqslant v$ and output the pair $u \leqslant v$ in the negative case. This way, we recursively enumerate precisely the inputs for the ω-inequality problem with negative output. $\qquad\square$

We are thus left with enumerating the positive cases of the ω-inequality problem for suitable pseudovarieties of ordered monoids V, a task that is accomplished in Theorem 8.2. We are actually going to deal with inequalities of ω-words from $\Omega_A^\omega \mathsf{A}$, instead of inequalities of ω-terms, in order to be able to employ known properties of the monoids $\Omega_A^\omega \mathsf{A}$. With this aim, we generalize the ω-inequality problem from ω-terms to ω-words in a given pseudovariety. Let W be a pseudovariety of monoids with decidable ω-equality problem, that is, such that calculations in the ordered monoids $\Omega_A^\omega \mathsf{W}$ can be performed algorithmically. By the ω-*inequality problem over* W for a pseudovariety V of ordered monoids contained in W we mean the problem that takes as input a pair (u, v) of ω-words from $\Omega_A^\omega \mathsf{W}$ and asks whether the inequality $u \leqslant v$ is valid in V. The following result allows us to deal only with such restricted ω-inequality problems, when convenient.

Proposition 4.2. *Let* W *be a pseudovariety of monoids with decidable ω-equality problem and let* V *be a pseudovariety of ordered monoids contained in* W *such that the ω-inequality problem for* V *over* W *is recursively enumerable. Then, the ω-inequality problem for* V *is recursively enumerable.*

Proof. Let $\pi \colon \overline{\Omega}_A \mathsf{M} \to \overline{\Omega}_A \mathsf{W}$ be the unique continuous homomorphism mapping each generator to itself. By assumption, there is a Turing machine enumerating the quadruples (u, v, w, z), where $u, v \in \Omega_A^\omega \mathsf{W}$ and $w, z \in \Omega_A^\omega \mathsf{M}$ are such that $u \leqslant v$ holds in V. Using the decidability of the ω-equality problem for W, for each such quadruple, algorithmically calculate $\pi(u)$ and $\pi(v)$, and check whether they are equal to w and z, respectively. In the affirmative case, output the ω-inequality $w \leqslant z$. The resulting Turing machine recursively enumerates the positive cases of the ω-inequality problem for V. $\qquad\square$

5. EQUIDIVISIBILITY AND FACTORIALITY

We recall here the notion of equidivisibility, which was introduced in [18]. A semigroup S is said to be *equidivisible* if, for all $s, t, u, v \in S$ such that $st = uv$, there is $w \in S^1$ such that either $s = uw$ and $wt = v$, or $sw = u$ and $t = wv$. A pseudovariety of monoids V is *equidivisible* if $\overline{\Omega}_A \mathsf{V}$ is equidivisible for every finite set A. A pseudovariety of monoids V is said to be *closed under concatenation* if, for all V-recognizable languages $K, L \subseteq A^*$, the language KL is also V-recognizable. Similar notions may be considered for pseudovarieties of semigroups.

As shown in [7, Lemma 2.3] for pseudovarieties of semigroups, such a pseudovariety V containing all finite nilpotent semigroups is closed under concatenation if and only if the multiplication in $\overline{\Omega}_A \mathsf{V}$ is an open mapping for each finite set A. As mentioned in Section 2, the requirement that V contain all finite nilpotent semigroups is made to ensure that the subsemigroup

of $\overline{\Omega}_A \mathsf{V}$ generated by the free generators is free and discrete. In the case of pseudovarieties of monoids, the corresponding sufficient condition is that V contain MN. Under such an assumption for a pseudovariety of monoids V, the argument in the proof of [7, Lemma 2.3] also yields that V is closed under concatenation if and only if the multiplication in $\overline{\Omega}_A \mathsf{V}$ is an open mapping for each finite set A.

Another relevant result from the same paper is [7, Lemma 4.8], whose proof shows that, if the multiplication in $\overline{\Omega}_A \mathsf{V}$ is an open mapping, then $\overline{\Omega}_A \mathsf{V}$ is equidivisible. In particular, since the pseudovariety A is closed under concatenation, the multiplication in each profinite aperiodic monoid $\overline{\Omega}_A \mathsf{A}$ is an open mapping and A is equidivisible.

For an ω-term s and a positive integer k, the ω-term s^k is defined recursively by $s^1 = s$ and $s^{k+1} = s^k \cdot s$. The need for expliciting this definition is due to the fact that the multiplication of ω-terms is not associative, as we prefer to keep track of the order in which the operations are performed. Additionally, we assume that, for all ω-terms s and t, both $s^0 \cdot t$ and $t \cdot s^0$ denote t.

Decompositions of an ω-term t are pairs of ω-terms, defined inductively with respect to the structure of t. If t is either 1 or a letter, then its decompositions are $(1, t)$ and $(t, 1)$. If $t = t_1 \cdot t_2$, then t has decompositions of two symmetric forms:

- for every decomposition (s_1, s_2) of t_1, the pair $(s_1, s_2 \cdot t_2)$ is a decomposition of t;
- for every decomposition (s_1, s_2) of t_2, the pair $(t_1 \cdot s_1, s_2)$ is a decomposition of t.

If $t = s^\omega$, then the decompositions of t are defined to be the pairs of the form $(s^k \cdot s_1, s_2 \cdot s^\ell)$, where (s_1, s_2) is a decomposition of s and k, ℓ are non-negative integers or ω, with at least one of them equal to ω.

In order to illustrate the definition with a concrete example, let a, b be a pair of letters. Decompositions of the term $a \cdot b$ are exactly $(1, a \cdot b)$, $(a, 1 \cdot b)$, $(a \cdot 1, b)$ and $(a \cdot b, 1)$. Consequently, decompositions of the term $(a \cdot b)^\omega$ are

$$((a \cdot b)^k \cdot 1, (a \cdot b) \cdot (a \cdot b)^\ell), \qquad ((a \cdot b)^k \cdot a, (1 \cdot b) \cdot (a \cdot b)^\ell),$$
$$((a \cdot b)^k \cdot (a \cdot 1), b \cdot (a \cdot b)^\ell), \qquad ((a \cdot b)^k \cdot (a \cdot b), 1 \cdot (a \cdot b)^\ell),$$

where k and ℓ are non-negative integers or ω, with at least one of them equal to ω. In particular, the decompositions obtained for $\ell = 0$ and $k = \omega$ are

$$((a \cdot b)^\omega \cdot 1, a \cdot b), \qquad ((a \cdot b)^\omega \cdot a, 1 \cdot b),$$
$$((a \cdot b)^\omega \cdot (a \cdot 1), b), \qquad ((a \cdot b)^\omega \cdot (a \cdot b), 1).$$

Note that $((a \cdot b)^\omega, 1)$, $((a \cdot b)^\omega \cdot a, b)$ and $((a \cdot b)^\omega, (a \cdot b)^\omega)$ fail to be decompositions of $(a \cdot b)^\omega$. However, as the next result shows, all factorizations of ω-words may be obtained from decompositions of ω-terms representing them. It appears to be intimately related with results of [10], but not quite to follow from them in any direct way. We thus include here a direct proof. The key ingredients are the facts that $\overline{\Omega}_A \mathsf{A}$ is equidivisible and its multiplication is an open mapping.

Theorem 5.1. *Let t be an ω-term and $u, v \in \overline{\Omega}_A\mathsf{A}$. Then, t represents the product uv if and only if there exists a decomposition (t_1, t_2) of t such that t_1 represents u and t_2 represents v.*

Proof. That a decomposition of t yields a factorization of the ω-word in $\Omega_A^\omega\mathsf{A}$ represented by t amounts to a straightforward induction on the construction of t as an ω-term. For the converse, we also proceed by induction on the construction of t as an ω-term. At the basis of the induction are the identity term 1 and the letters, for which the result is obvious: indeed, both 1 and the letters admit no nontrivial factorizations in $\overline{\Omega}_A\mathsf{A}$ and this corresponds exactly to the decompositions allowed in these cases.

Suppose that $t = s_1 \cdot s_2$ is obtained by multiplying two (shorter) terms and let w_i be the ω-word in $\Omega_A^\omega\mathsf{A}$ represented by s_i $(i = 1, 2)$. From the equality $w_1 w_2 = uv$, by equidivisibility of $\overline{\Omega}_A\mathsf{A}$ we deduce that there is some pseudoword $z \in \overline{\Omega}_A\mathsf{A}$ such that either $w_1 = uz$ and $zw_2 = v$, or $w_1 z = u$ and $w_2 = zv$. In the first case, applying the induction hypothesis to s_1, we obtain a decomposition $(s_{1,1}, s_{1,2})$ of s_1 such that $s_{1,1}$ represents u and $s_{1,2}$ represents z. Hence, the ω-terms $t_1 = s_{1,1}$ and $t_2 = s_{1,2} \cdot s_2$ represent u and v, respectively. The other case is similar.

Suppose next that $t = s^\omega$ where s is a (shorter) ω-term. In view of the preceding case, we may assume inductively that the result holds for each ω-term s^n with n a positive integer. Let w be the ω-word in $\Omega_A^\omega\mathsf{A}$ represented by s. Since the sequence $(w^n)_n$ converges to the product uv and the multiplication in $\overline{\Omega}_A\mathsf{A}$ is open, there is a strictly increasing sequence of indices $(n_k)_k$ such that there is a factorization $w^{n_k} = u_k v_k$ with $\lim u_k = u$ and $\lim v_k = v$ (cf. [7, Lemma 2.5]). By the induction hypothesis applied to s^{n_k}, for each k there is a decomposition $(t_{k,1}, t_{k,2})$ of s^{n_k} such that $t_{k,1}$ represents u_k and $t_{k,2}$ represents v_k. Moreover, by the definition of decomposition and a simple induction, for each k there exists ℓ_k, with $0 \leqslant \ell_k \leqslant n_k - 1$, such that $t_{k,1} = s^{\ell_k} \cdot t'_{k,1}$ and $t_{k,2} = (\dots((t'_{k,2} \cdot s) \cdot s) \dots) \cdot s$, where $(t'_{k,1}, t'_{k,2})$ is a decomposition of s and in the latter term the number of multiplications by s is $n_k - \ell_k - 1$. Let $w_{k,i}$ be the ω-word in $\Omega_A^\omega\mathsf{A}$ represented by $t'_{k,i}$ $(i = 1, 2)$. Then the equalities $u_k = w^{\ell_k} \cdot w_{k,1}$ and $v_k = w_{k,2} \cdot w^{n_k - \ell_k - 1}$ hold in $\Omega_A^\omega\mathsf{A}$, and the decomposition of s yields the factorization $w = w_{k,1} w_{k,2}$. Up to taking a subsequence of $(n_k)_k$, we may assume that each of the sequences $(w_{k,i})_k$ converges to some w_i $(i = 1, 2)$ and that each of the sequences $(\ell_k)_k$ and $(n_k - \ell_k - 1)_k$ is either constant or strictly increasing. Let ℓ denote the common value of ℓ_k if the sequence $(\ell_k)_k$ is constant, and ω if the sequence is strictly increasing. Similarly, let m denote either the common value of $n_k - \ell_k - 1$ or ω. Continuity of multiplication in $\overline{\Omega}_A\mathsf{A}$ yields the equalities $u = w^\ell w_1$, $v = w_2 w^m$, and $w = w_1 w_2$. The latter equality and the induction hypothesis applied to the ω-term s provide a decomposition (s_1, s_2) of s such that s_i represents w_i $(i = 1, 2)$. It remains to take $t_1 = s^\ell \cdot s_1$ and $t_2 = s_2 \cdot s^m$ to obtain the required decomposition. $\qquad\square$

A subset X of a semigroup S is said to be *factorial in S* if $st \in X$ with $s, t \in S$ implies $s, t \in X$. For an implicit signature σ, a pseudovariety V is said to be *σ-factorial* if $\Omega_A^\sigma\mathsf{V}$ is factorial in $\overline{\Omega}_A\mathsf{V}$ for every finite set A.

The following result is an immediate application of Theorem 5.1. It was first proved in [9] as a by-product of a language-theoretical proof of the correctness of McCammond's algorithm for solving the ω-identity problem for A [17]. An alternative proof and a generalization to all Burnside pseudovarieties $B_n = [\![x^{\omega+n} = x^\omega]\!]$ can be found in [10].

Corollary 5.2. *The pseudovariety* A *is* ω-*factorial.*   □

Combining Corollary 5.2 with Corollary 3.2, we obtain the following result.

Corollary 5.3. *Let* V *be a polynomially closed pseudovariety of aperiodic ordered monoids and let* $u, v \in \Omega_A^\omega \mathsf{A}$. *If the inequality* $u \leqslant v$ *holds in* V *then, for every factorization* $u = u_1 \cdots u_n$, *there is a factorization* $v = v_1 \cdots v_n$ *into* ω-*words such that each inequality* $u_i \leqslant v_i$ *holds in* V $(i = 1, \ldots, n)$.   □

6. Repetitions in ω-words

This section is devoted to establishing another reflection of the intuitively expected phenomenon that there are not many ways to factorize an ω-word. A precise formulation of how such factorizations are obtained is already given by Theorem 5.1. For application in the next section, we also need the following repetition result for iterated factorizations.

Proposition 6.1. *Suppose, for each* $i \geqslant 0$, v_i, x_i *are* ω-*words in* $\Omega_A^\omega \mathsf{A}$ *such that* $v_i = x_i v_{i+1}$. *Then there exist indices* i *and* j *with* $i \leqslant j$ *such that* $v_i = (x_i \cdots x_j)^\omega v_{j+1}$.

As an example, take $v_0 = (a^2 b^\omega)^\omega a b^\omega$ and consider the factorizations $v_i = x_i v_{i+1}$ given by the ω-words $v_i = b^i v_0$ and $x_i = b^i (a^2 b^\omega)^\omega$ $(i \geqslant 0)$. Note that there are infinitely many ways to factorize v_0. However, Proposition 6.1 says that some repetition is always possible. In this concrete example, this is quite easy, because one can take $i = j$ an arbitrary index. In the general case we need to be more careful to choose appropriate indices and we need to overcome certain technical obstacles.

In order to prove this key proposition, we are going to study, instead of the factorizations $v_i = x_i v_{i+1}$, the corresponding syntactic decompositions of an ω-term representing v_0, with the aim of finding certain repetitions, which would allow to repeat several consecutive ω-terms in the decomposition without changing the resulting ω-word $v_i = x_i \cdots x_j v_{j+1}$.

Lemma 6.2. *Let* $v \in \Omega_A^\omega \mathsf{A}$ *be an* ω-*word, and for each* $i \geqslant 1$ *let* $v = v_i \tilde{v}_i$ *be a factorization in* $\Omega_A^\omega \mathsf{A}$. *Then there exist indices* i *and* j *with* $i < j$ *such that* $v = v_j \tilde{v}_i$.

Proof. Let t be an ω-term representing v. We proceed by induction on the construction of t. If t is a letter from A, then the only factorizations of v are $1 \cdot v$ and $v \cdot 1$ by Theorem 5.1, and the claim is obvious.

In case $t = t_1 \cdot t_2$, denote by u_1 and u_2 the ω-words represented by t_1 and t_2, respectively. By equidivisibility, for each i there exists $y_i \in \Omega_A^\omega \mathsf{A}$ such that either $u_1 = v_i y_i$ and $y_i u_2 = \tilde{v}_i$, or $u_1 y_i = v_i$ and $u_2 = y_i \tilde{v}_i$. By symmetry, we may assume that the former case occurs infinitely often. Applying the induction hypothesis to the factorizations of u_1, we obtain indices i and j

with $i < j$ such that $u_1 = v_j y_i$. It follows that $v = u_1 u_2 = v_j y_i u_2 = v_j \tilde{v}_i$, as required.

Finally, consider the case $t = s^\omega$. According to Theorem 5.1, for each i there exists a decomposition $(s_{i,1}, s_{i,2})$ of the ω-term s and exponents k_i, ℓ_i, that are either non-negative integers or ω, such that v_i is represented by the ω-term $s^{k_i} \cdot s_{i,1}$ and $\tilde{v}_i$ is represented by the ω-term $s_{i,2} \cdot s^{\ell_i}$. Since for each i at least one of k_i and ℓ_i is equal to ω, without loss of generality we may assume that $\ell_i = \omega$ for all i. Consider the ω-words w_i and $\tilde{w}_i$ represented by the ω-terms $s_{i,1}$ and $s_{i,2}$, respectively. Then, all products $w_i \tilde{w}_i$ are equal to the ω-word u represented by s. Using the induction hypothesis, we obtain indices i and j with $i < j$ such that $u = w_j \tilde{w}_i$. Then $v = u^\omega = u^{k_j} w_j \tilde{w}_i u^\omega = v_j \tilde{v}_i$, which concludes the proof.   □

For the proof of Proposition 6.1, it is convenient to introduce a measure of depth of ω-terms. With this aim, let t be an arbitrary ω-term. In terms of the tree of t, we consider two parameters:

- the maximum number $\mu_\omega(t)$ of occurrences of the ω-power in a branch of the tree of t;
- for each branch of the tree of t with $\mu_\omega(t)$ occurrences of the ω-power, let n be the number of right descendants above the top occurrence of the ω-power in the branch; the number $\mu_\ell(t)$ is the maximum of all such n (if $\mu_\omega(t) = 0$, then $\mu_\ell(t)$ is equal to the maximum number of right descendants in a branch).

Let $\mu(t)$ stand for the pair $(\mu_\omega(t), \mu_\ell(t))$. For example, for the term t depicted in Figure 1, we have $\mu(t) = (2, 0)$, while if the term s is obtained by replacing in t the rightmost leaf b by the term b^ω, then $\mu(s) = (2, 1)$.

Using the lexicographic ordering $\leqslant$ of pairs of non-negative integers, the definition of μ can be rephrased inductively with respect to the structure of terms as follows: $\mu(1) = \mu(a) = (0, 0)$ for all $a \in A$; $\mu(s^\omega) = (\mu_\omega(s) + 1, 0)$; $\mu(t_1 \cdot s^\omega) = \max\{\mu(t_1), \mu(s^\omega)\}$; $\mu(t_1 \cdot t_2) = \max\{\mu(t_1), \mu(t_2) + (0, 1)\}$ if t_2 is not of the form s^ω, where the addition of pairs is taken component-wise.

Lemma 6.3. *If t is an ω-term and $(\bar{t}, t')$ is a decomposition of t, then $\mu(t) \geqslant \mu(t')$ and the equality can only occur in the following four cases:*

(1) t is 1 or a letter;
(2) $t = s_1 \cdot s_2$ and $(\bar{t}, t') = (\bar{s}, s' \cdot s_2)$ for some decomposition $(\bar{s}, s')$ of s_1;
(3) $t = s^\omega$ and $(\bar{t}, t') = (s^k \cdot \bar{s}, s' \cdot s^\omega)$ for some decomposition $(\bar{s}, s')$ of s and $k \in \{0, 1, 2, \ldots\} \cup \{\omega\}$;
(4) $t = s_1 \cdot s^\omega$ and $(\bar{t}, t') = (s_1 \cdot (s^k \cdot \bar{s}), s' \cdot s^\omega)$ for some decomposition $(\bar{s}, s')$ of s and $k \in \{0, 1, 2, \ldots\} \cup \{\omega\}$.

Proof. We proceed by induction on the construction of the term t. The claim is obvious for t equal to 1 or a letter, which means that we fall in Case (1).

Suppose that $t = s_1 \cdot s_2$ and $(\bar{t}, t') = (\bar{s}, s' \cdot s_2)$, with $(\bar{s}, s')$ a decomposition of s_1. Since these assumptions fall in Case (2), it is enough to prove the inequality $\mu(t) \geqslant \mu(t')$. We distinguish two cases. For s_2 of the form s^ω we get

$$\mu(t) = \max\{\mu(s_1), \mu(s^\omega)\} \geqslant \max\{\mu(s'), \mu(s^\omega)\} = \mu(s' \cdot s_2),$$

using the induction hypothesis on the ω-term s_1. Similarly, if s_2 is not of the form s^ω, we obtain

$$\mu(t) = \max\{\mu(s_1), \mu(s_2) + (0,1)\} \geqslant \max\{\mu(s'), \mu(s_2) + (0,1)\} = \mu(s' \cdot s_2).$$

If $t = s_1 \cdot s_2$ and $(\bar{t}, t') = (s_1 \cdot \bar{s}, s')$, with $(\bar{s}, s')$ a decomposition of s_2, then we also distinguish two cases. For s_2 of the form s^ω we get

$$\mu(t) = \max\{\mu(s_1), \mu(s^\omega)\} \geqslant \mu(s^\omega) = \mu(s_2) \geqslant \mu(s'),$$

using the induction hypothesis on the ω-term s_2; in fact, the induction hypothesis yields that the last inequality is strict unless $(\bar{s}, s')$ is of the form $(s^k \cdot \tilde{s}, s'' \cdot s^\omega)$ for some decomposition $(\tilde{s}, s'')$ of s and $k \in \{0, 1, 2, \ldots\} \cup \{\omega\}$, which shows that the equality $\mu(t) = \mu(s')$ only holds in Case (4). In the second case, assuming that s_2 is not of the form s^ω, we obtain strict inequality:

$$\mu(t) = \max\{\mu(s_1), \mu(s_2) + (0,1)\} > \mu(s_2) \geqslant \mu(s').$$

If $t = s^\omega$ and the decomposition $(\bar{t}, t')$ of t has $t' = s' \cdot s^\omega$, with $(\bar{s}, s')$ a decomposition of s, meaning that Case (3) holds, then

$$\mu(s' \cdot s^\omega) = \max\{\mu(s'), \mu(s^\omega)\} = \mu(s^\omega),$$

because $\mu(s') \leqslant \mu(s) < \mu(s^\omega)$ by the induction hypothesis. Finally, if $t = s^\omega$ and the decomposition $(\bar{t}, t')$ of t has $t' = s' \cdot s^k$, with $(\bar{s}, s')$ a decomposition of s and k a non-negative integer, then

$$\mu(s' \cdot s^k) \leqslant \max\{\mu(s'), (\mu_\omega(s), \mu_\ell(s) + 2)\} < \mu(s^\omega),$$

because $\mu(s') \leqslant \mu(s) < \mu(s^\omega)$ by the induction hypothesis, while a simple induction taking into account that $s^{k+1} = s^k \cdot s$ shows that the equality $\mu(s^k) = \mu(s) + (0,1)$ holds whenever $k > 1$ in case s is not of the form $s = r^\omega$, and $\mu(s^k) = \mu(s)$ otherwise. $\qquad\square$

We may now proceed with the proof of the main result of this section.

Proof of Proposition 6.1. Let t_0 be an ω-term representing v_0. Using Theorem 5.1, choose successively for each i some decomposition (s_i, t_{i+1}) of the ω-term t_i such that s_i represents x_i and t_{i+1} represents v_{i+1}.

The proof of the proposition proceeds by induction with respect to $\mu(t_0)$, and for ω-terms t_0 with the same value of $\mu(t_0)$ by induction with respect to the size of the term. By Lemma 6.3, we know that $\mu(t_{i+1}) \leqslant \mu(t_i)$ for every $i \geqslant 0$. If $\mu(t_i) < \mu(t_0)$ for some $i \geqslant 1$, then the statement follows directly from the induction assumption applied to t_i. Thus, assume that $\mu(t_i) = \mu(t_0)$ for all i, which implies that all decompositions (s_i, t_{i+1}) are of one of the four forms described in Lemma 6.3.

If some term t_i is 1 or a letter, then the same is true for all terms t_i, and all terms s_i, except for at most one, are equal to 1; then the statement is obviously true. It remains to deal with the situation when every decomposition (s_i, t_{i+1}) is of one of the forms (2) to (4), which is assumed for the remainder of the proof. Let us first formulate a useful observation.

Claim. Assume that i is such that the decomposition (s_i, t_{i+1}) of t_i is of the form (3) with $t_i = s^\omega$ or (4) with $t_i = \bar{t}_i \cdot s^\omega$ for some ω-terms s and $\bar{t}_i$. Then, for every $j > i$, the decomposition (s_j, t_{j+1}) is of the form (2) or (4),

and there exist ω-terms $\bar{t}_j$ for $j > i$ such that $t_j = \bar{t}_j \cdot s^\omega$. Moreover, the inequality $\mu(\bar{t}_{i+1}) < \mu(t_{i+1})$ holds.

Proof of Claim. We prove the statement that $t_j = \bar{t}_j \cdot s^\omega$ for some ω-terms $\bar{t}_j$ by induction with respect to $j > i$. Note that since $\mu(t_{j+1}) = \mu(t_j)$, this form of t_j ensures that, for $j > i$, the decomposition (s_j, t_{j+1}) of t_j is of the form (2) or (4).

By the assumption that the decomposition (s_i, t_{i+1}) of t_i is of the form (3) or (4), we know that the ω-term t_{i+1} has the form $\bar{t}_{i+1} \cdot s^\omega$ where $\bar{t}_{i+1} = s'$ in the notation of Lemma 6.3. Thus, we have proved the base of the induction $j = i + 1$. Moreover, in both cases the term s' is the second component of a decomposition $(\bar{s}, s')$ of s and, therefore, $\mu(\bar{t}_{i+1}) = \mu(s') \leqslant \mu(s)$ by Lemma 6.3. Hence we get the last statement of the claim, because $\mu(s) < \mu(s^\omega) \leqslant \mu(t_{i+1})$.

To prove the induction step, we first assume that the decomposition (s_j, t_{j+1}) of $t_j = \bar{t}_j \cdot s^\omega$, with $j > i$, is of the form (2). Then, we have $t_{j+1} = t' = s' \cdot s^\omega$ in the notation of case (2) of Lemma 6.3, where $s^\omega = s_2$. Hence, we may take $\bar{t}_{j+1} = s'$ and we are done in this case. We may proceed in the same way when we assume that the decomposition (s_j, t_{j+1}) of $t_j = \bar{t}_j \cdot s^\omega$, with $j > i$, is of the form (4). □

We distinguish two cases for which the proof proceeds in different ways. First, assume that there exists an index k such that for all $j \geqslant k$ the decomposition (s_j, t_{j+1}) of t_j is of the form (2), that is, there exist ω-terms t' and $\bar{t}_j$ for $j \geqslant k$ such that $t_j = \bar{t}_j \cdot t'$ and $(s_j, \bar{t}_{j+1})$ is a decomposition of $\bar{t}_j$. Choose k to be the least of such indices. Denote by $\bar{v}_j$ the ω-word represented by $\bar{t}_j$. Our goal is to apply the induction hypothesis to the sequence of factorizations $\bar{v}_j = x_j \bar{v}_{j+1}$ for $j \geqslant k$. This would be possible, once we verify either that $\mu(\bar{t}_k) < \mu(t_0)$ or that $\mu(\bar{t}_k) = \mu(t_0)$ and the size of $\bar{t}_k$ is smaller than that of t_0. At least one of these conditions is certainly true if $k = 0$, since then $\bar{t}_k$ is a proper subterm of t_0. If $k \geqslant 1$, then the minimality of choice of k ensures that the decomposition (s_{k-1}, t_k) of t_{k-1} is of the form (3) or (4). Hence, we may apply the Claim with $i = k - 1$ to deduce that $\mu(\bar{t}_k) < \mu(t_k)$. Since we have also assumed that $\mu(t_k) = \mu(t_0)$, the induction hypothesis now provides us with indices i and j, with $i < j$, such that $\bar{v}_i = (x_i \cdots x_j)^\omega \bar{v}_{j+1}$. Multiplication by the ω-word w represented by the ω-term t' turns this equality into the required repetition $v_i = \bar{v}_i w = (x_i \cdots x_j)^\omega \bar{v}_{j+1} w = (x_i \cdots x_j)^\omega v_{j+1}$.

Second, assume that such an index k does not exist, so that, according to the claim, for infinitely many indices i the decomposition (s_i, t_{i+1}) is of the form (4). Let $i_1 < i_2 < \cdots$ be all such indices. Then for every $m \geqslant 1$, all decompositions with indices between i_m and i_{m+1} are of the form (2). For all $i \geqslant i_1$, the term t_i can be written as $t_i = \bar{t}_i \cdot s^\omega$, for a fixed ω-term s and certain ω-terms $\bar{t}_i$. If i is not one of the indices i_m, then $(s_i, \bar{t}_{i+1})$ is a decomposition of $\bar{t}_i$. On the other hand, for all indices i_m we have $s_{i_m} = \bar{t}_{i_m} \cdot (s^{k_m} \cdot \bar{s}_m)$, with $(\bar{s}_m, \bar{t}_{i_m+1})$ a decomposition of s and $k_m \in \{0, 1, 2, \ldots\} \cup \{\omega\}$. Denoting ω-words in $\Omega_A^\omega \mathsf{A}$ represented by ω-terms $\bar{t}_i$, s and $\bar{s}_m$ by $\bar{v}_i$, w and $\bar{w}_m$, respectively, the above relations between these ω-terms translate into the following equalities of ω-words: $v_i = \bar{v}_i w^\omega$ for all $i \geqslant i_1$, $\bar{v}_i = x_i \bar{v}_{i+1}$ for indices i other than i_m, $x_{i_m} = \bar{v}_{i_m} w^{k_m} \bar{w}_m$, and

$w = \bar{w}_m \bar{v}_{i_m+1}$. By Lemma 6.2, there exist indices $\ell, n \geqslant 1$, with $\ell < n$, such that $w = \bar{w}_n \bar{v}_{i_\ell+1}$. We are going to verify that $i_\ell + 1$ and i_n are the required indices i and j. First we calculate the product of ω-terms x_i between two consecutive indices i_m as

$$x_{i_m+1} \cdots x_{i_{m+1}} = x_{i_m+1} \cdots x_{i_{m+1}-1} \bar{v}_{i_{m+1}} w^{k_{m+1}} \bar{w}_{m+1} = \bar{v}_{i_m+1} w^{k_{m+1}} \bar{w}_{m+1},$$

using successively the equalities $\bar{v}_i = x_i \bar{v}_{i+1}$ for $i = i_{m+1} - 1, \ldots, i_m + 1$. Then, we obtain

$$x_{i_\ell+1} \cdots x_{i_n} = \bar{v}_{i_\ell+1} w^{k_{\ell+1}+\cdots+k_n+n-\ell-1} \bar{w}_n,$$

using all equalities $w = \bar{w}_m \bar{v}_{i_m+1}$ for $m = \ell + 1, \ldots, n - 1$. Denoting the number $k_{\ell+1} + \cdots + k_n + n - \ell - 1$ by q, we finally calculate

$$\begin{aligned}
(x_{i_\ell+1} \cdots x_{i_n})^\omega v_{i_n+1} &= (\bar{v}_{i_\ell+1} w^q \bar{w}_n)^\omega \bar{v}_{i_n+1} w^\omega \\
&= \bar{v}_{i_\ell+1} (w^q \bar{w}_n \bar{v}_{i_\ell+1})^\omega w^q \bar{w}_n \bar{v}_{i_n+1} w^\omega \\
&= \bar{v}_{i_\ell+1} w^\omega = v_{i_\ell+1},
\end{aligned}$$

using the equalities $\bar{w}_n \bar{v}_{i_\ell+1} = \bar{w}_n \bar{v}_{i_n+1} = w$. $\qquad\square$

7. Syntactic proofs of ω-inequalities

The aim of this section is to show that the positive cases of the ω-inequality problem for an aperiodic pseudovariety of ordered monoids of the form $\mathsf{W} = \mathrm{Pol}\,\mathsf{B}\,\mathsf{V}$ may be derived from the positive cases of the ω-inequality problem for V. Since we deal only with aperiodic monoids, it is convenient to consider only ω-inequality problems over A. By Proposition 4.2, the general ω-inequality problem reduces to this one since the ω-equality problem for A is decidable as was shown by McCammond [17]. For this reason, all ω-inequalities considered in this section are taken over A.

In order to enumerate the positive cases of the ω-inequality problem for W, we show that, under suitable assumptions on V, if an inequality $u \leqslant v$ of ω-words is valid in W, then there is a finite syntactic proof of this fact from a convenient set of hypotheses, or axioms. This is thus a sort of completeness result for the choice of the set of axioms. Provided the axioms can be enumerated, since formal proofs can then be enumerated, so can be the provable facts.

To fulfill the above program, we need to make precise what kind of formal proofs we will be considering. By a *formal proof* of an ω-inequality $u \leqslant v$ from a given set Γ of hypotheses, we mean a finite sequence $u_i \leqslant v_i$ ($i = 1, \ldots, n$) of ω-inequalities such that each member of the sequence satisfies one of the following conditions:

- $u_i \leqslant v_i$ is a member of Γ;
- $u_i \leqslant v_i$ is of the form $u_j u_k \leqslant v_j v_k$ with $j, k < i$;
- $u_i \leqslant v_i$ is of the form $u_j^\omega \leqslant v_j^\omega$ with $j < i$;
- there are $j, k < i$ such that $u_i = u_j$, $v_j = u_k$, and $v_k = v_i$.

If there is such a formal proof, then we say that $u \leqslant v$ is provable from Γ and we write $\Gamma \vdash u \leqslant v$. Note that we do not include in our proof rules the possibility of making substitutions of variables by ω-words. We will not need them because we will consider sets of hypotheses that are closed under such substitutions. Similarly, we do not need to take into account the possibility

of multiplying both sides of an inequality on the left and on the right by the same ω-words because we will consider sets of hypotheses containing all trivial ω-inequalities.

Proposition 7.1. *Let V be a polynomially closed pseudovariety of aperiodic ordered monoids and let Γ be the set of all trivial ω-inequalities together with all inequalities of the form $u^\omega \leqslant u^\omega v u^\omega$, with $u, v \in \Omega_A^\omega \mathsf{A}$, such that the ω-inequality $v \leqslant u$ is valid in V. If an ω-inequality $u \leqslant v$ is valid in $\mathsf{W} = \operatorname{Pol} \mathsf{B} \mathsf{V}$, then $\Gamma \vdash u \leqslant v$.*

Proof. Let t be an ω-term representing u. We proceed by induction on the construction of t. If t is a letter or 1, then $u = v$, and so $u = v$ belongs to Γ.

In case $t = t_1 \cdot t_2$, we obtain the corresponding factorization $u = u_1 u_2$, where t_i represents u_i $(i = 1, 2)$. By Corollary 5.3, there is a factorization $v = v_1 v_2$ in $\Omega_A^\omega \mathsf{A}$ such that the inequality $u_i \leqslant v_i$ is valid in W. The induction hypothesis yields $\Gamma \vdash u_i \leqslant v_i$ $(i = 1, 2)$ and so also $\Gamma \vdash u \leqslant v$.

Suppose next that $t = s^\omega$. Let $v_0 = v$ and let $w \in \Omega_A^\omega \mathsf{A}$ be the ω-word represented by s. Inductively, we apply Corollary 5.3 to the inequality $u \leqslant v_i$ $(i \geqslant 0)$ and the factorization $u = wu$, to obtain a factorization $v_i = x_i v_{i+1}$ in $\Omega_A^\omega \mathsf{A}$ such that the inequalities $w \leqslant x_i$ and $u \leqslant v_{i+1}$ hold in W. By Proposition 6.1, there exist i, j such that $0 \leqslant i \leqslant j$ and

$$(1) \qquad v = x_0 \cdots x_{i-1} (x_i \cdots x_j)^\omega v_{j+1}.$$

Similarly, by the left-right dual of Proposition 6.1, there exists a factorization $v_{j+1} = v'(y_n \cdots y_m)^\omega y_{m-1} \cdots y_0$ in $\Omega_A^\omega \mathsf{A}$ such that each of the inequalities $u \leqslant v'$ and $w \leqslant y_p$ $(p = 0, \ldots, n)$ holds in W. Combining with (1), we obtain the factorization

$$(2) \qquad v = x_0 \cdots x_{i-1} (x_i \cdots x_j)^\omega v'(y_n \cdots y_m)^\omega y_{m-1} \cdots y_0.$$

Since the pseudovariety of monoids generated by V is contained in W, it must satisfy the equality $u = v'$, whence V satisfies the inequality $v' \leqslant u$. By the choice of Γ, it follows that the inequality $u^\omega \leqslant u^\omega v' u^\omega$ belongs to Γ. The preceding inequality may be written as

$$(3) \qquad u \leqslant (w^{j-i+1})^\omega v'(w^{m-n+1})^\omega.$$

Since the inequality (3) belongs to Γ and $u = wu = uw$ by aperiodicity, we deduce that

$$(4) \qquad \Gamma \vdash u \leqslant w^i (w^{j-i+1})^\omega v'(w^{m-n+1})^\omega w^m.$$

On the other hand, the term induction hypothesis yields the relations

$$(5) \qquad \Gamma \vdash w \leqslant x_p \text{ and } \Gamma \vdash w \leqslant y_q \quad (p = 0, \ldots, j; q = 0, \ldots, n).$$

Combining (2), (4), and (5), we conclude that $\Gamma \vdash u \leqslant v$, which completes the induction step and the proof. $\square$

8. MAIN RESULTS

By an *ordered ω-monoid* we mean an ordered monoid with a unary operation of ω-power such that $s \leqslant t$ implies $s^\omega \leqslant t^\omega$. A *variety* of ordered ω-monoids is a class of such structures that is closed under taking homomorphic images, ordered ω-submonoids, and arbitrary direct products. It is

well known that varieties of ordered ω-monoids are defined by ω-inequalities [12, 16].

For a pseudovariety V of ordered monoids, we denote by V^ω the variety of ordered ω-monoids generated by V. Taking into account the basis of ω-identities of A^ω obtained by McCammond [17], Proposition 7.1 yields the following basis result.

Theorem 8.1. *Let W be a pseudovariety of aperiodic monoids and let $\mathsf{V} = \mathrm{Pol}\,\mathsf{W}$. Then the variety $(\mathrm{Pol}\,\mathrm{B}\,\mathsf{V})^\omega$ of ordered ω-monoids is defined by the following inequalities:*

(1) $x(yz) = (xy)z$, $x(yx)^\omega = (xy)^\omega x$;
(2) $(x^\omega)^\omega = (x^r)^\omega = xx^\omega = x^\omega x = x^\omega$ for every $r \geqslant 2$;
(3) $u^\omega \leqslant u^\omega v u^\omega$ whenever the inequality $v \leqslant u$ is valid in V. $\square$

Proof. The equalities in (1) and (2) hold in the variety A^ω, whence also in the subvariety $(\mathrm{Pol}\,\mathrm{B}\,\mathsf{V})^\omega$. The inequalities in (3) hold in $(\mathrm{Pol}\,\mathrm{B}\,\mathsf{V})^\omega$ by Proposition 2.2. That every inequality valid in $(\mathrm{Pol}\,\mathrm{B}\,\mathsf{V})^\omega$ is a consequence of the inequalities (1)–(3) follows from Proposition 7.1. $\square$

The following is the announced recursive enumerability of the ω-identity problem for suitable pseudovarieties of ordered monoids. It is formulated as a transfer result of that property along the Boolean-polynomial closure.

Theorem 8.2. *Let W be a pseudovariety of aperiodic monoids and let $\mathsf{V} = \mathrm{Pol}\,\mathsf{W}$. If the ω-inequality problem for V is recursively enumerable, then so is it for $\mathrm{Pol}\,\mathrm{B}\,\mathsf{V}$.*

Proof. Consider the set Γ defined in the statement of Proposition 7.1. By Proposition 7.1, each ω-inequality over A valid in $\mathsf{W} = \mathrm{Pol}\,\mathrm{B}\,\mathsf{V}$ is provable from Γ. Since the converse follows from Proposition 2.2, we conclude that the ω-inequalities over A that are valid in W are precisely those that are provable from Γ. As Γ is recursively enumerable by hypothesis, the inequalities that are provable from Γ constitute a recursively enumerable set. Hence the ω-inequality problem for W over A is recursively enumerable, and therefore the ω-inequality problem for W is also recursively enumerable by Proposition 4.2. $\square$

Note that, from the definitions, it follows immediately that, if V is a recursively enumerable pseudovariety of ordered monoids, then so is $\mathrm{Pol}\,\mathrm{B}\,\mathsf{V}$. Combining Theorem 8.2 with Proposition 4.1, we obtain the following main theorem of this paper.

Theorem 8.3. *Let W be a recursively enumerable pseudovariety of aperiodic monoids and let $\mathsf{V} = \mathrm{Pol}\,\mathsf{W}$. If the ω-inequality problem for V is decidable, then so is it for $\mathrm{Pol}\,\mathrm{B}\,\mathsf{V}$.* $\square$

In particular, this leads to the following application.

Corollary 8.4. *Let V be a recursively enumerable pseudovariety of aperiodic monoids such that the ω-inequality problem is decidable for $\mathrm{Pol}\,\mathsf{V}$. Then, the ω-inequality problem is decidable for every level and half level of the concatenation hierarchy starting from V.* $\square$

To obtain our main application, we need to be able to start the induction process with a pseudovariety of the form $\mathsf{Pol}\,\mathsf{W}$ for which the ω-inequality problem is decidable. For the Straubing-Thérien hierarchy, we should start with W as the trivial pseudovariety, for which $\mathsf{Pol}\,\mathsf{W}$ is the pseudovariety of ordered monoids defined by the inequality $1 \leqslant x$ [24, Proposition 8.4], and sometimes and henceforth denoted J^+. The join with its dual, defined by the inequality $1 \geqslant x$, is the pseudovariety J of all finite $\mathcal{J}$-trivial monoids. The ω-equality problem for J has been solved by the first author [1]. The solution consists in writing ω-words in a canonical form, namely products of letters and ω-powers of words that are products of distinct letters in increasing order (assuming a total order on the alphabet), in such a way that any factor adjacent to an ω-power u^ω has at least one letter that does not appear in u.

Given a word $w = a_1 \cdots a_n \in A^*$, with $a_i \in A$, we denote by $w{\uparrow}$ the language $A^* a_1 A^* \cdots a_n A^*$, consisting of all words that admit w as a subword. We also say that w is a *subword* of a pseudoword $u \in \overline{\Omega}_A\mathsf{M}$ if $u \in \overline{w{\uparrow}}$. If $u \in \Omega_A^\omega\mathsf{M}$ is an ω-word in canonical form over J then w is a subword of u if and only if it is a subword of $u^{(k)}$ for some positive integer k, where $u^{(k)}$ is obtained from u by replacing each exponent ω by k [2, Lemma 8.2.3], or equivalently, if w is a subword of some word from the language $u^{(*)}$ defined by the regular expression obtained from u by replacing each exponent ω by $*$.

Proposition 8.5. *The ω-inequality problem for J^+ is decidable.*

Proof. Let $u \leqslant v$ be an ω-inequality. Since the transformation of an ω-word to its canonical form over J is effective, without loss of generality we may assume that u and v are in canonical form.

Since every J^+-recognizable language is a finite union of languages of the form $w{\uparrow}$, we conclude from Proposition 2.4 that J^+ satisfies the inequality $u \leqslant v$ if and only if every subword of u is also a subword of v. In view of the observation preceding the statement of this proposition, the latter condition can be decided by checking whether the regular language consisting of all subwords of $u^{(*)}$ is a subset of the regular language of subwords of $v^{(*)}$. $\square$

Finally, here is the main application of our results.

Corollary 8.6. *The ω-inequality problem is decidable for all members of the Straubing-Thérien hierarchy.* $\square$

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