REDUCIBILITY OF POINTLIKE PROBLEMS

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ABSTRACT. We show that the pointlike and the idempotent pointlike problems are reducible with respect to natural signatures in the following cases: the pseudovariety of all finite semigroups in which the order of every subgroup is a product of elements of a fixed set π of primes; the pseudovariety of all finite semigroups in which every regular $\mathcal{J}$-class is the product of a rectangular band by a group from a fixed pseudovariety of groups that is reducible for the pointlike problem, respectively graph reducible. Allowing only trivial groups, we obtain ω-reducibility of the pointlike and idempotent pointlike problems, respectively for the pseudovarieties of all finite aperiodic semigroups (A) and of all finite semigroups in which all regular elements are idempotents (DA).

1. INTRODUCTION

For a pseudovariety V of semigroups, the effective computation of V-pointlike subsets of finite semigroups intervenes in the solution of various decision problems. One famous example is the case of G-pointlike sets, where G is the pseudovariety of all finite groups, for which a concrete algorithm was proposed by Henckell and Rhodes [15] and later proved by Ash [11]. It was conceived as a successful tool to effectively decide whether a finite semigroup divides the power semigroup of some finite group (see [20, 14] for a history of the problem). The computation of V-idempotent pointlikes sets in turn yields easily a solution of the membership problem for pseudovarieties given by Mal'cev products of the form $\mathsf{W} \circledm \mathsf{V}$, provided membership in W is decidable [13, Proposition 4.3].

The computation of pointlike and idempotent pointlike sets has been carried out for several pseudovarieties. A general approach for obtaining theoretical computability was devised by Steinberg and the first author through tameness [9, 10]. The idea is that there is an obvious semi-algorithm to generate subsets of a given finite semigroup that are not V-pointlike (respectively not V-idempotent pointlike), provided V is decidable. To generate the favorable cases, one needs witnesses in the profinite semigroup, namely pseudowords that evaluate to the given elements in the semigroup

and are equal over V. The essential property for tameness is *reducibility*, which means that such witnesses may be found among pseudowords of a restricted, effectively enumerable form, such as ω-words; more generally, terms in an implicit signature should be taken. This general approach may and has been considered for arbitrary finite systems of equations, the cases of pointlike and idempotent pointlike sets corresponding respectively to systems of the forms $x_1 = \cdots = x_n$ and $x_1 = \cdots = x_n = x_n^2$.

In this paper, we show that the reducibility property holds for pointlike and idempotent pointlike problems for certain pseudovarieties under simple assumptions. The two cases that we consider here are the pseudovarieties of the form $\overline{\mathsf{G}_\pi}$, of all finite semigroups whose subgroups have orders whose prime factors belong to the set π of primes; and the pseudovarieties of the form $\mathsf{DO} \cap \overline{\mathsf{H}}$ of all finite semigroups in which all regular $\mathcal{J}$-classes are products of rectangular bands by groups from the pseudovariety H. In both instances, the case where all subgroups are trivial is of interest and represents, respectively, the pseudovarieties A, of all finite aperiodic semigroups, and DA, of all finite semigroups in which all regular elements are idempotents.

2. Preliminaries

The reader is referred to the standard bibliography [1, 23] on finite semigroups for undefined terminology and background.

By an *implicit signature* we mean a set σ of pseudowords (also called implicit operations) over the pseudovariety S of all finite semigroups, that is, elements of some finitely generated free profinite semigroup $\overline{\Omega}_A\mathsf{S}$, the only requirement being that binary multiplication is one of them. Since all implicit signatures are assumed to contain binary multiplication, we omit reference to it when describing implicit signatures. By definition, pseudowords have a natural interpretation in every finite (whence also in every profinite) semigroup, so that every profinite semigroup has a natural structure of σ-semigroup. The σ-subalgebra of $\overline{\Omega}_A\mathsf{S}$ generated by A is denoted $\Omega_A^\sigma\mathsf{S}$. Examples of implicit signatures are given by ω, consisting of the unary operation ω-power $\_^\omega$, and κ, consisting of the unary operation $(\omega-1)$-power $\_^{\omega-1}$.

A subset P of a finite semigroup S is said to be *pointlike* with respect to a pseudovariety V if, for every onto continuous homomorphism $\varphi : \overline{\Omega}_A\mathsf{S} \to S$, there is a subset P' of $\overline{\Omega}_A\mathsf{S}$ such that $\varphi(P') = P$ and $p_\mathsf{V}(P')$ is a singleton [3], where $p_\mathsf{V} : \overline{\Omega}_A\mathsf{S} \to \overline{\Omega}_A\mathsf{V}$ denotes the canonical projection from the free profinite semigroup to the relatively free pro-V semigroup. An equivalent formulation in terms of relational morphisms and further discussion on the interest of this notion can be found in [23]. A simple and fruitful interpretation in terms of formal languages has been formulated and established in [3]. The set of all nonempty V-pointlike subsets of S constitutes a semigroup under set multiplication and is denoted $\mathcal{P}_\mathsf{V}(S)$. The semigroup of all nonempty subsets of S will be denoted by $\mathcal{P}(S)$. The problem of computing

V-pointlikes is said to be σ-*reducible* for an implicit signature σ if the above set P' can always be chosen to be a subset of $\Omega_A^\sigma \mathsf{S}$.

As shown in [9], under suitable additional assumptions, the κ-reducibility of the pointlike problem implies its algorithmic solution. However, the resulting algorithm is merely theoretical and completely impractical. In the case of the pseudovariety A, a structural algorithm for computing pointlike sets of finite semigroups has been obtained by Henckell [12]. Generalizations and more transparent and shorter proofs can be found in [16] and in [22] (the latter paper being based on the interpretation of pointlike sets from [3]). Algorithms with the same flavor have been obtained in [7] for the pseudovarieties J and R. Similar techniques to the ones developed in [22] have been applied to show that the pointlike sets of size 2 of the pseudovariety DA are also effectively computable [21].

Henckell's algorithm actually intervenes in the proof of κ-reducibility for the A-pointlike problem presented in Section 3 in the form of the extension obtained in [16]. Also essential in our treatment of idempotent pointlike sets in the aperiodic case is Henckell's result that every A-idempotent pointlike subset of a finite semigroup is contained in some A-pointlike set that is idempotent [13]. The extension of this result to pseudovarieties of the form $\mathsf{A} \textcircled{m} \mathsf{V}$ is also attributed to Henckell in [17, Theorem 4.5].

3. REDUCIBILITY OF A-POINTLIKE SETS

Given a set π of primes, its complement in the set of all primes is denoted π'. A π-*group* is a finite group whose order factors into primes from π. The pseudovariety of all finite π-groups is denoted G_π. For a pseudovariety H of groups, $\overline{\mathsf{H}}$ stands for the pseudovariety consisting of all finite semigroups all of whose subgroups lie in H. Note that $\overline{\mathsf{G}_\emptyset} = \mathsf{A}$.

Since the additive semigroup of positive integers $\mathbb{Z}_+$ is free, its profinite completion $\widehat{\mathbb{Z}_+}$ is a free profinite semigroup. For $\nu \in \widehat{\mathbb{Z}_+}$ and an element s of a profinite semigroup S, denote by s^ν the image of ν under the unique continuous homomorphism $\widehat{\mathbb{Z}_+} \to S$ that maps 1 to s.

In case $\pi = \emptyset$, we let $\nu_\pi = \omega + 1$. Otherwise, let ν_π be any accumulation point of the sequence $((p_1 \cdots p_n)^{n!})_n$ in $\widehat{\mathbb{Z}_+}$, where $p_1, p_2, \dots$ is an enumeration of the set π, possibly with repetitions. In case π consists of all primes, it is easy to see that $\nu_\pi = \omega$. Note that $\overline{\mathsf{G}_\pi}$ is defined by the pseudoidentity $x^{\nu_\pi} = x^\omega$ while $\mathsf{G}_{\pi'}$ is defined by $x^{\nu_\pi - 1} = 1$. Denote by κ_π the implicit signature obtained by enriching the signature κ with the operations x^μ, if it is not already expressible by an ω-term, whenever there exists $k \in \mathbb{Z}_+$ such that $k(\mu + 1) = \nu_\pi - 1$. In particular, one can easily check that κ coincides with both $\kappa_\emptyset$ and κ_π if π consists of all primes.

A subset of $\mathcal{P}(S)$ is said to be *downward closed* if, whenever it contains P and $Q \subseteq P$, it also contains Q. For a subset P of a finite semigroup S, we denote by $P^{\omega + *}$ the set $P^\omega \bigcup_{n \geq 1} P^n$. Given a set of primes π and a semigroup S, we define $\mathcal{CP}_\pi(S)$ to be the smallest downward closed subsemigroup

of $\mathcal{P}(S)$ containing all singleton subsets of S and which contains $P^{\omega+*}$ whenever it contains an element P which generates a cyclic π'-subgroup of $\mathcal{P}(S)$. It follows from [16, Theorem 2.3] that the equality $\mathcal{CP}_\pi(S) = \mathcal{P}_{\overline{\mathsf{G}_\pi}}(S)$ holds for every finite semigroup S, which implies that $\mathcal{P}_{\overline{\mathsf{G}_\pi}}(S)$ is computable in case π is a recursive set of primes. The following theorem strengthens the easy direction of this result, showing not only that elements of $\mathcal{CP}_\pi(S)$ are $\overline{\mathsf{G}_\pi}$-pointlike subsets, but also that this can be witnessed by κ_π-terms.

Theorem 3.1. *Let π be a set of primes and let $\varphi : \overline{\Omega}_A \mathsf{S} \to S$ be a continuous homomorphism onto a finite semigroup. Then every $P \in \mathcal{CP}_\pi(S)$ has the following property:*

(3.1) *there exists a function $\alpha_P : P \to \Omega_A^{\kappa_\pi}\mathsf{S}$ such that the equality $\varphi\,\alpha_P = \mathrm{id}_P$ holds and $p_{\overline{\mathsf{G}_\pi}}\alpha_P$ is constant.*

Proof. To prove the theorem, we proceed by induction on the construction of the semigroup $\mathcal{CP}_\pi(S)$ that is immediately derived from its definition. At the base of the induction, we take the subset of $\mathcal{P}(S)$ consisting of the singleton subsets of S, for which property (3.1) obviously holds, as the restriction of φ to $\Omega_A^{\kappa_\pi}\mathsf{S}$ is onto. For the induction steps, we need to distinguish three types of transformations on subsets of S.

For taking subsets, it suffices to observe that, if $Q \subseteq P$, α_P verifies (3.1), and α_Q is taken to be the restriction of α_P to Q, then α_Q verifies (3.1) for Q in the place of P.

For taking products, suppose that $P, Q \in \mathcal{P}(S)$ are such that α_P and α_Q verify the corresponding properties (3.1) for P and Q. Let $R = PQ$ and define $\alpha_R : R \to \Omega_A^{\kappa_\pi}\mathsf{S}$ by letting, for $r \in R$, $\alpha_R(r) = \alpha_P(p)\alpha_Q(q)$ where $r = pq$ is any chosen factorization of r with $p \in P$ and $q \in Q$. Given $r \in R$, consider its chosen factorization $r = pq$ with $p \in P$ and $q \in Q$. Then we have

$$\varphi(\alpha_R(r)) = \varphi(\alpha_P(p)\alpha_Q(q)) = \varphi(\alpha_P(p))\varphi(\alpha_Q(q)) = pq = r.$$

Similarly, one shows that $p_{\overline{\mathsf{G}_\pi}}\alpha_R$ is a constant mapping. This shows that R also has property (3.1).

Finally, suppose that P is a subset of S for which there is a function α_P satisfying property (3.1) and such that P generates a cyclic π'-subgroup of $\mathcal{P}(S)$. Let $Q = P^{\omega+*}$ and note that $Q = \bigcup_{n=1}^m P^n$ for some m.

By the assumption that the cyclic subgroup of $\mathcal{P}(S)$ generated by P is a π'-group, the equality $P^{\nu_\pi} = P$ holds. Hence, there is some $\mu \in \widehat{\mathbb{Z}_+}$ such that $\nu_\pi - 1 = k(\mu+1)$, where k denotes the order of the π'-group generated by P. This implies that the unary operation x^μ belongs to κ_π.

Note that P^k is a subsemigroup of S. Given $p \in P$, since $p \in P^{k\ell+1}$ for every $\ell \geq 0$, there exists a factorization $p = q_1 \cdots q_\ell p'$, where each $q_i \in P^k$ and $p' \in P$. Choosing $\ell = |P^k| + 1$, there are integers i and j such that $1 \leq i < j \leq \ell$ such that $q_1 \cdots q_i = q_1 \cdots q_j$. Hence, there are factorizations

$$p = q_1 \cdots q_i \cdot (q_{i+1} \cdots q_j)^\omega \cdot q_{j+1} \cdots q_\ell p' = s \cdot e \cdot t,$$

where $s = s_1 \cdots s_k$, $e = e^2 = e_1 \cdots e_k$, and t together with all s_m and all e_m belong to P. Having chosen such a factorization, we now define

$$\beta(p) = \alpha_P(s_1) \cdots \alpha_P(s_k) \big(\alpha_P(e_1) \cdots \alpha_P(e_k)\big)^\mu \alpha_P(t).$$

Note that $\beta(p)$ is given by a κ_π-term. Moreover, we obtain the equalities

$$\varphi(\beta(p)) = s_1 \cdots s_k (e_1 \cdots e_k)^\mu t = se^\mu t = set = p,$$

$$p_{\overline{\mathsf{G}_\pi}}(\beta(p)) = p_{\overline{\mathsf{G}_\pi}}(\alpha_P(p))^{k+\mu k+1} = p_{\overline{\mathsf{G}_\pi}}(\alpha_P(p))^{\nu_\pi} = p_{\overline{\mathsf{G}_\pi}}(\alpha_P(p))^\omega.$$

It follows that we may assume that the function α_P is such that the constant value of $p_{\overline{\mathsf{G}_\pi}}\alpha_P$ is an idempotent f. This entails that, for the function α_{P^n} defined on P^n according to the product case of the induction, the composite $p_{\overline{\mathsf{G}_\pi}}\alpha_{P^n}$ still has the same constant value f. Hence, we may extract from the relation $\bigcup_{n=1}^{m} \alpha_{P^n}$ a function $\alpha_{P^{\omega+*}}$ which has the required property (3.1) for $P^{\omega+*}$. This concludes the inductive step and the proof of the theorem. $\qquad\square$

Corollary 3.2. *If π is an arbitrary set of primes, then the pseudovariety $\overline{\mathsf{G}_\pi}$ is κ_π-reducible for pointlike sets and also κ_π-reducible for idempotent pointlike sets. In particular, A is κ-reducible for both pointlike sets and idempotent pointlike sets.*

Proof. Recall that by [16, Theorem 2.3], the equality $\mathcal{CP}_\pi(S) = \mathcal{P}_{\overline{\mathsf{G}_\pi}}(S)$ holds for every finite semigroup S. The result for pointlike sets then follows directly from Theorem 3.1. For idempotent pointlike sets, it suffices to invoke, additionally, [17, Theorem 4.5] since $\mathsf{A} \textcircled{m} \overline{\mathsf{G}_\pi} = \overline{\mathsf{G}_\pi}$. $\qquad\square$

4. Reducibility of DA-pointlike sets

For the pseudovariety DA, the proof below of κ-reducibility of the pointlike problem is inspired by [5, Lemma 5.10]. It is of a more syntactical nature than the proof of Theorem 3.1, using central basic factorizations as introduced in [2]. A further parameter that we proceed to introduce plays a key role. Given $u \in \overline{\Omega}_A\mathsf{S}$, there may or may not be a central basic factorization of the form $u = u_0 a_0 \cdot u_0' \cdot b_0 u_0''$ with $c(u) = c(u_0) \uplus \{a_0\} = c(u_0'') \uplus \{b_0\}$. We are interested in iterating such a factorization on the middle factor u_0' while it is possible to do so without reducing the content, that is, while $c(u_0') = c(u)$. The supremum of the number of times we can keep iterating such a factorization on the middle factor is denoted $\|u\|$.

Theorem 4.1. *Let H be a pseudovariety of groups and suppose that σ_0 is an implicit signature such that the H-pointlike problem is σ_0-reducible. Let $\mathsf{V} = \mathsf{DO} \cap \overline{\mathsf{H}}$ and let $\sigma = \sigma_0 \cup \{\_^\omega\}$. Suppose S is a finite semigroup, $\varphi : \overline{\Omega}_A\mathsf{S} \to S$ is an onto homomorphism, and $\{s_1, \ldots, s_n\}$ is a V-pointlike subset of S. Given $u_1, \ldots, u_n \in \overline{\Omega}_A\mathsf{S}$ such that $\varphi(u_i) = s_i$ $(i = 1, \ldots, n)$ and*

(4.1) $$p_\mathsf{V}(u_1) = \cdots = p_\mathsf{V}(u_n),$$

there exist $w_1, \ldots, w_n \in \Omega_A^\sigma\mathsf{S}$ such that

(4.2) $$\varphi(w_i) = s_i \ (i = 1, \ldots, n) \ and \ p_\mathsf{V}(w_1) = \cdots = p_\mathsf{V}(w_n).$$

For shortness, we say that the n-tuple $(w_1, \ldots, w_n)$ of σ-terms is a (V, σ)-*reduction* of $(u_1, \ldots, u_n)$ if it satisfies property (4.2).

Proof. Without loss of generality, we may assume that S has a content function, that is that the content function $c : \overline{\Omega}_A \mathsf{S} \to \mathcal{P}(A)$ factors through φ. Note that, by (4.1), we must have $c(u_1) = \cdots = c(u_n)$.

We show by induction on $|c(u_1)|$ that the u_i may be replaced by κ-terms w_i satisfying properties (4.2). The case $c(u_1) = \{a\}$ is rather easy. Indeed, in case u_1 is a finite word, by (4.1) so are all u_i and, therefore, they are σ-terms. Otherwise, for each i, we have $u_i = a^{\omega} u_i$. As (4.1) entails $p_\mathsf{H}(u_1) = \cdots = p_\mathsf{H}(u_n)$, there exists an (H, σ_0)-reduction $(w_1', \ldots, w_n')$ of $(u_1, \ldots, u_n)$. Then, $w_i = a^{\omega} w_i'$ $(i = 1, \ldots, n)$ defines a (V, σ)-reduction of $(u_1, \ldots, u_n)$.

Suppose that the claim holds whenever $|c(u_1)| < K$ and suppose an instance of the problem is given in which $|c(u_i)| = K$. Factorize u_i as

$$(4.3) \qquad u_i = u_{i,0} a_0 u_{i,1} a_1 \cdots u_{i,l} a_l \cdot u_{i,l}' \cdot b_l u_{i,l}'' \cdots b_1 u_{i,1}'' b_0 u_{i,0}''$$

where $c(u_i) = c(u_{i,p}) \uplus \{a_p\} = c(u_{i,q}'') \uplus \{b_q\}$ $(p, q = 0, \ldots, l)$. We take here l to be arbitrary if $\|u_i\| = \infty$ or $l = \|u_i\|$ otherwise. By (4.1) and using [2], we deduce that the $\|u_i\|$ are the same for all i and, moreover, the sequences of markers $a_0, a_1, \ldots, a_l$ and $b_0, b_1, \ldots, b_l$ are also the same for all i and V satisfies each of the following pseudoidentities for all $i, j \in \{1, \ldots, n\}$ and $p \in \{0, \ldots, l\}$:

$$u_{i,p} = u_{j,p}, \ u_{i,l}' = u_{j,l}', \ u_{i,p}'' = u_{j,p}''$$

By construction, $|c(u_{i,p})| = |c(u_{i,p}'')| = |c(u_1)| - 1$ and so, by the induction hypothesis, for each p there exist (V, σ)-reductions $(w_{1,p}, \ldots, w_{n,p})$ of $(u_{1,p}, \ldots, u_{n,p})$ and $(w_{1,p}'', \ldots, w_{n,p}'')$ of $(u_{1,p}'', \ldots, u_{n,p}'')$.

We next distinguish two cases. In the first case, we assume $\|u_1\| < \infty$. By the choice of $l = \|u_i\| = \|u_1\|$, either $|c(u_{i,l}')| < |c(u_1)|$ or there is a factorization of $u_{i,l}'$ of one of the forms $\alpha x \beta$ or $\alpha y \beta x \gamma$ with $c(u_{i,l}') = c(\alpha) \uplus \{x\} = c(\beta) \uplus \{x\}$ or $c(u_{i,l}') = c(\alpha y \beta) \uplus \{x\} = c(\beta x \gamma) \uplus \{y\}$, respectively. Moreover, the same case occurs for all $i \in \{1, \ldots, n\}$ and the factors in the same positions must have the same value under the projection p_V. Applying the induction hypothesis again to each of the factors of the $u_{i,l}'$ thus determined, we deduce that there exists a (V, σ)-reduction $(w_{1,l}', \ldots, w_{n,l}')$ of $(u_{1,l}', \ldots, u_{n,l}')$. One may then verify that, taking

$$w_i = w_{i,0} a_0 w_{i,1} a_1 \cdots w_{i,l} a_l \cdot w_{i,l}' \cdot b_l w_{i,l}'' \cdots b_1 w_{i,1}'' b_0 w_{i,0}''$$

defines a (V, σ)-reduction $(w_1, \ldots, w_n)$ of the original n-tuple $(u_1, \ldots, u_n)$.

It remains to handle the case where $\|u_1\| = \infty$. Consider, for each l, the n-tuple of pairs of $S \times S$

$$\left(\varphi(w_{i,0} a_0 w_{i,1} a_1 \cdots w_{i,l} a_l), \varphi(b_l w_{i,l}'' \cdots b_1 w_{i,1}'' b_0 w_{i,0}'') \right) \quad (i = 1, \ldots, n).$$

Since S is finite, there are indices k and l such that $k < l$ and the n-tuples corresponding to these two indices coincide. Thus, for every $i = 1, \ldots, n$, we

have

$$\varphi(w_{i,0}a_0w_{i,1}a_1\cdots w_{i,k}a_k)$$
$$= \varphi(w_{i,0}a_0w_{i,1}a_1\cdots w_{i,k}a_k(w_{i,k+1}a_{k+1}\cdots w_{i,l}a_l))$$
$$= \varphi(w_{i,0}a_0w_{i,1}a_1\cdots w_{i,k}a_k(w_{i,k+1}a_{k+1}\cdots w_{i,l}a_l)^\omega),$$

and, similarly,

$$\varphi(b_k w''_{i,k}\cdots b_1 w''_{i,1} b_0 w''_{i,0})$$
$$= \varphi((b_l w''_{i,l}\cdots b_{k+1} w''_{i,k+1})^\omega b_k w''_{i,k}\cdots b_1 w''_{i,1} b_0 w''_{i,0}).$$

Let $(w'_{1,k},\ldots,w'_{n,k})$ be an (H,σ)-reduction of $(u'_{1,k},\ldots,u'_{n,k})$. Note that since S is assumed to have a content function, the content of $w'_{i,k}$ is the same as that of $u'_{i,k}$. Take

$$w_i = w_{i,0}a_0w_{i,1}a_1\cdots w_{i,k}a_k(w_{i,k+1}a_{k+1}\cdots w_{i,l}a_l)^\omega$$
$$\cdot w'_{i,k}\cdot(b_l w''_{i,l}\cdots b_{k+1} w''_{i,k+1})^\omega b_k w''_{i,k}\cdots b_1 w''_{i,1} b_0 w''_{i,0}.$$

Then again one verifies that $(w_1,\ldots,w_n)$ is a (V,σ)-reduction of the original n-tuple $(u_1,\ldots,u_n)$. □

Corollary 4.2. *If the pseudovariety of groups H is σ-reducible for the pointlike problem, then $\mathsf{V} = \mathsf{DO}\cap\overline{\mathsf{H}}$ is $\sigma\cup\{\_^\omega\}$-reducible for the pointlike problem.*

Proof. Let S be a finite semigroup and let $\{s_1,\ldots,s_n\}$ be a V-pointlike subset. Fix an onto homomorphism $\varphi:\overline{\Omega}_A\mathsf{S}\to S$. By a general compactness result [3] there are $u_1,\ldots,u_n\in\overline{\Omega}_A\mathsf{S}$ such that $\varphi(u_i) = s_i\ (i = 1,\ldots,n)$ and $p_\mathsf{V}(u_1) = \cdots = p_\mathsf{V}(u_n)$. The result now follows immediately from Theorem 4.1. □

The same approach allows us to deal with idempotent pointlike sets. However, a stronger assumption is needed on the pseudovariety of groups H. Recall that a system of equations may be associated with a directed graph Γ by viewing both vertices and edges as variables and assigning to each edge $x\xrightarrow{y}z$ the equation $xy = z$. For such a system, we may consider *constraints* in a finite semigroup, given by a function $\psi:\Gamma\to S$. Given a continuous onto homomorphism $\varphi:\overline{\Omega}_A\mathsf{S}\to S$, a V-*solution* of a thus constrained system is a function $\gamma:\Gamma\to\overline{\Omega}_A\mathsf{S}$ such that $\varphi(\gamma(x)) = \psi(x)$ for every $x\in\Gamma$ and the pseudoidentity $\gamma(x)\gamma(y) = \gamma(z)$ holds in V for every edge $x\xrightarrow{y}z$. A pseudovariety V is said to be σ-*reducible for systems of graph equations*, or *graph σ-reducible* for shortness, if every constrained system of equations associated with a finite directed graph Γ that admits a V-solution also admits a V-solution $\gamma:\Gamma\to\Omega_A^\sigma\mathsf{S}$.

Theorem 4.3. *Let H be a pseudovariety of groups that is graph σ_0-reducible. Let $\mathsf{V} = \mathsf{DO}\cap\overline{\mathsf{H}}$ and let $\sigma = \sigma_0\cup\{\_^\omega\}$. Suppose S is a finite semigroup, $\varphi:\overline{\Omega}_A\mathsf{S}\to S$ is an onto homomorphism, and $\{s_1,\ldots,s_n\}$ is a V-idempotent*

pointlike subset of S. Given $u_1, \ldots, u_n \in \overline{\Omega}_A \mathsf{S}$ such that $\varphi(u_i) = s_i$ $(i = 1, \ldots, n)$ and

$$p_\mathsf{V}(u_1) = \cdots = p_\mathsf{V}(u_n) = p_\mathsf{V}(u_n^2),$$

there exist $w_1, \ldots, w_n \in \Omega_A^\sigma \mathsf{S}$ such that

$$\varphi(w_i) = s_i \ (i = 1, \ldots, n) \ \text{and} \ p_\mathsf{V}(w_1) = \cdots = p_\mathsf{V}(w_n) = p_\mathsf{V}(w_n^2).$$

Proof. The proof follows the same lines as that of Theorem 4.1. We only mention in detail the necessary adaptations. First, the hypothesis that the $p_\mathsf{V}(u_i)$ are (the same) idempotent implies that $\|u_i\| = \infty$, which restricts the type of cases that need to be considered in the main induction step. However, the induction argument does not reduce the idempotent pointlike problem to the same problem on smaller content, but rather to the pointlike problem, which has already been treated in Theorem 4.1. This is why we need to assume again that the H-pointlike problem is σ_0-reducible.

The other point where a modification is needed is when handling the construction of the σ-terms $w_{i,k}'$. Since the singleton subsets $\{a\}$ of A^* are recognizable, by replacing S by a suitable finite semigroup, we may assume that $\varphi^{-1}(\varphi(a)) = \{a\}$ for every $a \in A$. By assumption, we know that, for each $i \in \{1, \ldots, n\}$, the following pseudoidentity holds in H:

$$u_{i,0} a_0 \cdots u_{i,k} a_k \cdot u_{i,k}' \cdot b_k u_{i,k}'' \cdots b_0 u_{i,0}'' = 1.$$

Consider the directed graph Γ with n cycles of length $4k + 5$ based at the same vertex, where the ith cycle has successive edges

$$x_{i,0}, y_{i,0}, \ldots, x_{i,k}, y_{i,k}, z_i, y_{i,k}'', x_{i,k}'', \ldots, y_{i,0}'', x_{i,0}''.$$

The corresponding system of equations is constrained as follows. The edge constraints are given by:

$$\psi(x_{i,j}) = \varphi(u_{i,j}), \ \psi(y_{i,j}) = \varphi(a_j),$$
$$\psi(x_{i,j}'') = \varphi(u_{i,j}''), \ \psi(y_{i,j}'') = \varphi(b_j),$$
$$\psi(z_i) = \varphi(u_{i,k}').$$

For the vertex constraints, we may take an arbitrary constraint s at the common vertex v_0 of the n cycles and then take for the constraint of any other vertex v the product of s by the constraints of successive edges of the unique path leading from v_0 to v.

An H-solution of the above system is obtained by assigning to the edge variables the values given by

$$x_{i,j} \mapsto u_{i,j}, \ y_{i,j} \mapsto a_j, \ x_{i,j}'' \mapsto u_{i,j}'', \ y_{i,j}'' \mapsto b_j, \ z_i \mapsto u_{i,k}',$$

as well as adequate values to the vertex variables. Since H is graph σ_0-reducible by hypothesis, the above constrained system admits a solution $\gamma : \Gamma \to \Omega_A^\sigma \mathsf{S}$. We use γ to define

$$\bar{w}_{i,j} = \gamma(x_{i,j}), \ \bar{w}_{i,j}'' = \gamma(x_{i,j}''), \ w_{i,k}' = \gamma(z_i).$$

Finally, we let

$$w_i = \bar{w}_{i,0} a_0 \bar{w}_{i,1} a_1 \cdots \bar{w}_{i,k} a_k (w_{i,k+1} a_{k+1} \cdots w_{i,l} a_l)^\omega$$
$$\cdot \bar{w}'_{i,k} \cdot (b_l w''_{i,l} \cdots b_{k+1} w''_{i,k+1})^\omega b_k \bar{w}''_{i,k} \cdots b_1 \bar{w}''_{i,1} b_0 \bar{w}''_{i,0}.$$

Then, each $p_V(w_i)$ is a group element and it must in fact be an idempotent because γ is an H-solution of the system determined by the graph Γ. Moreover, $\varphi(w_i) = \varphi(u_i)$ because of the choice of the pair k, l and of the constraints on the edges of Γ. Hence V is σ-reducible for idempotent pointlike sets. $\qquad\square$

Corollary 4.4. *If the pseudovariety of groups* H *is graph σ-reducible, then* $V = DO \cap \overline{H}$ *is $\sigma \cup \{\_^\omega\}$-reducible for the idempotent pointlike problem.* $\qquad\square$

The case of DA has deserved the most interest among the pseudovarieties of the form $DO \cap \overline{H}$.

Corollary 4.5. *The pseudovariety* DA *is ω-reducible both for the pointlike and idempotent pointlike problems.* $\qquad\square$

It is natural to expect that the essential ingredients in the proof of complete κ-tameness of the pseudovariety R should apply to DA. Yet the highly technical proof in [6] remains to be adapted as only part of such a program has been carried out [18, 19].

Further examples of pseudovarieties for which one may apply Corollaries 4.2 and 4.4 are recorded in the following corollaries.

Corollary 4.6. *The pseudovariety* DO *is κ-reducible for both pointlike and idempotent pointlike sets.*

Proof. To apply Corollaries 4.2 and 4.4, one just needs to observe that $DO \subseteq S = \overline{G}$ and G is graph κ-reducible by [9, Theorem 4.9], which depends on Ash's seminal results [11]. $\qquad\square$

Denote by Ab the pseudovariety of all finite Abelian groups.

Corollary 4.7. *The pseudovariety* $DO \cap \overline{Ab}$ *is κ-reducible for both pointlike and idempotent pointlike sets.*

Proof. The proof is similar to that of Corollary 4.6, taking into account that in fact the pseudovariety Ab is κ-reducible for arbitrary finite systems of equations [8] and, thus, in particular, it is graph κ-reducible. $\qquad\square$

For a prime p, denote by G_p the pseudovariety of all finite p-groups. While G_p is not graph κ-reducible, as was observed in [9], the first author has constructed a signature σ containing κ and such that G_p is graph σ-reducible [4]. The proof of this result depends on an earlier weaker result of Steinberg [24].

Corollary 4.8. *The pseudovariety* $DO \cap \overline{G_p}$ *is σ-reducible for both pointlike and idempotent pointlike sets, where σ is the implicit signature constructed in* [4]. $\qquad\square$

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