Lyapounov Stability for Measure Driven Differential Inclusions

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In this paper, we will derive stability conditions of the Lyapunov ty[pe] for impulsive dynamic control systems whose dynamics are given b[y] differential inclusions of the form:

$$(D)\qquad \begin{aligned} dx(t) &\in F(t,x(t))dt + G(t,x(t))\mu(dt) \quad t\in[0,\infty)\\ x(0) &= x_0. \end{aligned}$$

Here, $F:[0,\infty)\times\mathbb{R}^n \hookrightarrow \mathcal{P}(\mathbb{R}^n)$, and $G:[0,\infty)\times\mathbb{R}^n\hookrightarrow\mathcal{P}(\mathbb{R}^{n\times q})$ a[re] given set-valued functions and $\mu\in C^*([0,\infty);K)$, the set in the du[al] space of continuous functions from $[0,\infty)$ to $\mathbb{R}^q$ with values in K. Th[e] set K is the positive pointed convex cone in $\mathbb{R}^q$.

We consider the solution concept adopted to (D) in [1]. That is, evolves due to the contribution of an absolutely continuous compone[nt] x_{ac} and a singular component x_s, i.e., $x(t) = x_{ac}(t) + x_s(t)\ \forall t\in[0,$ with

$$\left\{\begin{aligned} x(0) &= x_0\\ \dot{x}_{ac}(t) &\in F(t,x(t)) + G(t,x(t))\cdot w_{ac}(t)\ \ \mathcal{L}\text{-a.e.}\\ x_s(t) &= \int_{[0,t]} g(\tau)d\bar{\mu}_s(\tau) \qquad \forall t\in[0,\infty). \end{aligned}\right. \quad (1$$

Here, $\bar{\mu}$ is the total variation measure associated with the vector-value[d] measure μ, μ_s and μ_{ac} are, respectively, the singular and the absolutel[y] continuous components of μ, w_{ac} is the time derivative of μ_{ac}, and $g($ is a $\bar{\mu}_s$-a.e. measurable selection of a certain multifunction

$$\tilde{G}(t,x(t^-);\mu(\{t\})) : [0,\infty)\times\mathbb{R}^n\times K \hookrightarrow \mathcal{P}(\mathbb{R}^n),$$

defined by

$$\left\{\begin{array}{l} \{G(t,z)w(t)\} \quad \text{if}\,|\alpha|=0\\[1ex] \{\frac{[\xi(\eta(t))-\xi(\eta(t^-))]}{\bar{\alpha}} : \dot{\xi}(s)\in G(t,\xi(s))\dot{\gamma}(s)\ \ \bar{\eta}(t)\text{-a.e.},\ \xi(0)=z \text{ and}\\ \qquad\qquad \gamma(\eta(t)) - \gamma(\eta(t^-))) = \alpha\} \quad \text{if}\,\bar{\alpha}>0, \end{array}\right.$$

where $|\alpha| = \sum_{i=1}^q \alpha^i$, $w(\cdot)$ is the Radon-Nicodym derivative of μ w.r.t. $\bar{\mu}$, (ξ,γ) are in $AC([0,1];\mathbb{R}^n\times\mathbb{R}^q)$, and the pair (θ,γ) is a *graph completion* with $\dot{\theta}(s)\equiv 0$ on $\bar{\eta}(t)$. In order to define this multifunction, the concept of graph completion of a time reparametrized function is required (see [1]).

Definition 1. A family of *graph completions* associated with a measure $\mu\in C^*(0,1;K)$ is a pair $(\theta,\Gamma):[0,\infty)\hookrightarrow\mathbb{R}^+\times\mathcal{P}(K)$ where $\theta:[0,\infty)\to\mathbb{R}$ is the "inverse" of $\bar{\eta}$ (i.e., $\theta(s)=t, \forall s\in\bar{\eta}(t)$) and the set-valued function Γ takes values $\gamma(s)$, being $\gamma:\bar{\eta}(t)\to\mathbb{R}^q$ given by

$$\gamma(s) := \left\{\begin{array}{ll} M(\theta(s)) & \text{if } \bar{\mu}(\{t\})=0\\ M(t^-) + \int_{\eta(t^-)}^{s} v(\sigma)d\sigma & \text{if } \bar{\mu}(\{t\})>0, \end{array}\right.$$

where $v(\cdot)\in V^t := \{v:\bar{\eta}(t)\to\mathbb{R}^q | v(s)\in K, \forall s\in\bar{\eta}(t), \int_{\bar{\eta}(t)} v(s)ds = \mu(\{t\})\}$ and $\bar{\mu}(dt) = \sum_{i=1}^q \mu_i(dt)$. Here, $M(\cdot) = \mathrm{col}(M_1(\cdot),\ldots,M_q(\cdot))$, with $M_i(t) = \int_{[0,t]}\mu_i(ds)$, $\forall t>0$, and $M_i(0)=0$, and $\bar{\eta}(t) = [\eta(t^-),\eta(t)]$ if $\bar{\mu}(\{t\})>0$, and $\{\eta(t)\}$, otherwise, with $\eta(t) := t + \sum_{i=1}^q M_i(t)$.

In analogous way to Lyapunov stability theory for conventional control systems (see, for example [2]), we will consider x^* to be an equilibrium point for the impulsive dynamic system (D).

Definition 2. Functions (V,W) form a Lyapunov pair for (D) at x^* if they satisfy the following properties:

1) Positive definiteness: $V\geq 0$, $W\geq 0$ with $W(x)=0$ iff $x=x^*$.
2) The set $\{x\in\mathbb{R}^n | V(x)\leq r\}$ is compact for all $r\geq 0$.
3) For all $z\in\mathbb{R}^n$ and for all $\xi\in\partial_p V(z)$, we have

$$\mathrm{Min}\{<\xi,\nu>: \nu\in F(t,z)v_0 + G(t,z)v : (v_0,v)\in\bar{V}\} \leq -W(z).$$

Here, $\bar{V} := \{(v_0, v) : v_0 \geq 0, v \in K, \sum_{i=0}^{q} v_i = 1\}$ and $\partial_P V(z)$ denotes the set of proximal gradients of V at z.
The last item requires some explanation. Two distinct situations may arise in the minimization process: The control measure is either absolutely continuous or singular with respect to the Lebesgue measure. In the first case, we have $v_0 > 0$ and $z = x(t)$, and condition 3) coincides with the conventional one. However, in the second situation, $v_0 = 0$ and this condition has to considered along the graph completion of the trajectory between the jump endpoints, that is, $z = \tilde{x}(s)$ for $s \in \bar{\eta}(t)$.

Since $V(x(t^+)) - V(x(t^-)) = \int_{\bar{\eta}(t)} < \xi(s), \zeta(s)w(s) > ds$ for some selections

- $\xi(s)$ of $\partial_P V(\tilde{x}(s))$,
- $\zeta(s)$ of $G(t, \tilde{x}(s))$, with $\tilde{x}$ satisfying $\dot{\tilde{x}}(s) = \zeta(s)w(s)$ and $\tilde{x}(\eta(t^-)) = x(t^-)$, and
- $w(s)$ of K, with $\int_{\bar{\eta}(t)} w(s)ds = \mu(t)$,

a somewhat weaker expression of the decreasing condition in terms of the original parametrization is given by

$$-\bar{w} \geq \text{Min}\{\text{Min}\{V(x(t^+) - V(x(t^-)), < \xi, \nu >\} | \nu \in F(t, z)v_0 + +G(t, z)v(v_0, v) \in \bar{V}, \ x(t^+) \in \mathcal{R}_G(\eta(t); \eta(t^-), x(t^-))\}.$$

where $\bar{w} = \text{Max}\{\int_{\bar{\eta}(t)} W(\tilde{x}(s))ds, W(x(t))\}$.
Here, $\bar{V}$ is as above and $\mathcal{R}_G(s_f; s_i, z)$ denotes the reachable set at the parameter[1] value s_f of the singular dynamics when the trajectory path was passing through z at the parameter value s_i.

Main result. Informally, our main result states that, under some natural mild assumptions, for any given point $\bar{x}$ for which $0 \in \bar{F}(t, \bar{x}) := \{F(t, \bar{x})v_0 + G(t, \bar{x})v : (v_0, v) \in \bar{V}\}$, the existence of a pair of Lyapunov functions implies that, for x_0 in the domain of V, there exists a control process (x, μ) such that $x(0) = x_0$ and $x(t) \to \bar{x}$ as t goes to infinity.

[1] According to the above parametrization, this parameter is the total variation of the control measure plus the current value of time.

References

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