ON TOPOLOGICAL LATTICES AND THEIR APPLICATIONS TO MODULE THEORY

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Abstract. Yassemi's "second submodules" are dualized and properties of its spectrum are studied. This is done by moving the ring theoretical setting to a lattice theoretical one and by introducing the notion of a (*strongly*) *topological lattice* $\mathcal{L} = (L, \wedge, \vee)$ with respect to a proper subset X of L. We investigate and characterize (strongly) topological lattices in general in order to apply it to modules over associative unital rings. Given a non-zero left R-module M, we introduce and investigate the spectrum $\mathrm{Spec}^{\mathrm{f}}(M)$ of *first submodules* of M as a dual notion of Yassemi's second submodules. We topologize $\mathrm{Spec}^{\mathrm{f}}(M)$ and investigate the algebraic properties of M by passing to the topological properties of the associated space.

1. Introduction

The Zariski topology plays a prominent role in algebraic geometry and its algebraization has been one of the great motivations in commutative ring theory. The use of modules instead of rings and henceforth the introduction of primeness conditions on them is a classical theme already visible in classical texts of Atiyah-Macdonald [20] and Kaplansky [33]. Since the dawn of non-commutative geometry it has always been of importance to find suitable analogous of the techniques used in the commutative setting. Hence it is natural to look for suitable topologies on non-commutative rings and on modules over them. Several notions of prime (sub)modules and Zariski topologies using these notions have been studied over the last decades; see for example [1, 2, 16, 17, 18, 19, 21, 35, 37, 44]. The most prominent representative of these notions for modules over a commutative ring R is the idea of a "prime submodule" N of a non-zero R-module M as a proper submodule such that any map $M/N \to M/N$ given by multiplication with an element of R is either injective or zero. Dual notions, often stemming from an interest in the category of comodules over coalgebras, e.g. Hopf algebras, have been also investigated lately. See for example [4, 3, 6, 7, 8, 9, 10, 11, 12, 22, 32, 41, 43]. Following ideas of MacDonald who introduced the notion of secondary modules as a dualization of primary modules, Yassemi [43] introduced the concept of a *second (sub)module* N of a non-zero R-module M of a given non-zero module over a commutative ring as a proper submodule such that any map $M/N \to M/N$

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given by multiplication with an element of R is either surjective or zero. This notion was studied for modules over arbitrary associative rings by Annin [9], where a *second module* was called a *coprime module*. Moreover, the notion of *coprime submodules* was investigated by Kazemifard *et al.* [32].

In this paper, we dualize the notion of a coprime submodule to present the spectrum $\mathrm{Spec}^{\mathrm{f}}(M)$ of *first submodules* of a given non-zero left module M over an arbitrary associative, not necessarily commutative, ring R with unity. We topologize this spectrum to obtain a dual Zariski-like topology, study properties of the resulting topological space and investigate the interplay between the properties of that space and the algebraic properties of M as an R-module.

To achieve this goal, we begin in the second section with a more general framework of a *topological complete lattice* $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ with respect to a proper subset $X \subsetneqq L$. We investigate such lattices and characterize them; moreover, we investigate the irreducibility of the closed subsets of X. In Section 3, we apply the results we obtained in Section 2 to the concrete example $\mathcal{L}(M)$, the complete lattice of R-submodules of a given non-zero R-module M, and $X = \mathrm{Spec}^{\mathrm{f}}(M)$, the spectrum of R-submodules of M which are prime as R-modules. In Section 4, we obtain several algebraic properties of ${}_RM$ by passing to the topological properties of $\mathrm{Spec}^{\mathrm{f}}(M)$.

2. Topological Lattices

Throughout, $\mathcal{L} = (L, \wedge, \vee, 0, 1)$ is a complete lattice, $X \subseteq L \setminus \{1\}$ is a non-empty subset and $\mathcal{P} = (\mathcal{P}(X), \cap, \cup, \emptyset, X)$ is the complete lattice on the power set of X. We define an order-reversing map

$$V : L \longrightarrow \mathcal{P}(X), \qquad a \mapsto V(a) = \{p \in X \mid a \leq p\}.$$

It is clear that $V(0) = X$, $V(1) = \emptyset$ and $V\left(\bigvee \mathcal{A}\right) = \bigcap_{a\in A} V(a)$ for every $\mathcal{A} \subseteq L$. This means that the image of V contains X, $\emptyset$ and is closed under arbitrary intersections. If $\mathrm{Im}(V)$ is also closed under finite unions, then the elements of $V(L)$ can be considered the closed sets of a topology on X.

Definition 2.1. We say that $\mathcal{L}$ is a *topological*^{X}*-lattice* (or X*-top*, for short) iff $V(L)$ is closed under finite unions.

The purpose of this section is to characterize X*-top lattices*. Notice that the map V represents the lower adjoint map of a Galois connection between $\mathcal{L}$ and $\mathcal{P}$, where the upper adjoint map is

$$I : \mathcal{P}(X) \longrightarrow L, \ \mathcal{A} \mapsto \bigwedge \mathcal{A}.$$

Since V, I are order reversing and $a \leq I(V(a))$, $\mathcal{A} \subseteq V(I(\mathcal{A}))$ hold for all $a \in L$, $\mathcal{A} \in \mathcal{P}(X)$, we conclude that (V, I) is a Galois connection [29, 3.13] and that

$$V = V \circ I \circ V \qquad \text{and} \qquad I = I \circ V \circ I. \tag{1}$$

The compositions $I \circ V$ and $V \circ I$ are *closure operators* [29, Lemma 32] and the closed elements with respect to this Galois connection are

$$\mathcal{C}(L) = \{a \in L \mid a = I(V(a))\} = \{I(\mathcal{A}) \mid \mathcal{A} \subseteq X\} = \mathrm{Im}(I)$$

and

$$\mathcal{C}(\mathcal{P}(X)) = \{\mathcal{A} \in \mathcal{P}(X) \mid \mathcal{A} = V(I(\mathcal{A}))\} = \{V(a) \mid a \in L\} = \mathrm{Im}(V).$$

Clearly, V is a bijection between $\mathcal{C}(L)$ and $\mathcal{C}(\mathcal{P}(X))$ with inverse I.

A lattice structure on $\mathcal{C}(L)$. Note that $X \subseteq \mathcal{C}(L)$, because for every element $p \in X$ we have $I(V(p)) = \bigwedge([p, 1[\cap X) = p$. Moreover, $(\mathcal{C}(L), \wedge, \bigwedge X)$ is a complete lower semilattice because if $Y \subseteq \mathcal{C}(L)$, then for each $y \in Y$ we have $y = I(\mathcal{A}_y)$ for some subset $\mathcal{A}_y \subseteq X$ and it follows that

$$\bigwedge Y = \bigwedge_{y \in Y} \bigwedge \mathcal{A}_y = \bigwedge \bigcup_{y \in Y} \mathcal{A}_y = I\left(\bigcup_{y \in Y} \mathcal{A}_y\right) \in \mathcal{C}(L).$$

This makes $\mathcal{C}(L)$ a complete lattice by defining a *new* join for each subset $Y \subseteq \mathcal{C}(L)$ as

$$\widetilde{\bigvee} Y := IV\left(\bigvee Y\right) = \bigwedge \{c \in \mathcal{C}(L) \mid y \leq c \ \forall\ y \in Y\}.$$

Notice that this new join $\widetilde{\vee}$ is usually *different* from the original join $\vee$ of L.

Before we characterize X-top lattices, we need to recall the following definition (see for example [5, Definition 1.1.]). An element p in a lower semilattice $(L, \wedge)$ is called *irreducible* iff for all $a, b \in L$ with $p \leq a, b$:

$$a \wedge b \leq p \qquad \Rightarrow \qquad a \leq p \text{ or } b \leq p. \tag{2}$$

The element p is called *strongly irreducible* iff Equation (2) holds for all $a, b \in L$.

Theorem 2.2. *The following statements are equivalent:*

(a) *$\mathcal{L}$ is an X-top lattice;*
(b) *$V : (\mathcal{C}(L), \wedge, \widetilde{\vee}) \to (\mathcal{P}(X), \cap, \cup)$ is an anti-homomorphism of lattices;*
(c) *every element $p \in X$ is strongly irreducible in $(\mathcal{C}(L), \wedge)$;*
(d) *$(\mathcal{C}(L), \wedge, \widetilde{\vee})$ is a distributive lattice and every element $p \in X$ is irreducible in $(\mathcal{C}(L), \wedge)$.*

Proof. (a) $\Rightarrow$ (b) Suppose that $\mathcal{L}$ is X-top, *i.e.* $V(L)$ is closed under finite unions. Let $a, b \in \mathcal{C}(L)$. By assumption, $V(a) \cup V(b) = V(c)$ for some $c \in L$. Hence

$$a \wedge b = I(V(a)) \wedge I(V(b)) = I(V(a) \cup V(b)) = I(V(c))$$

and it follows that $V(a \wedge b) = V(I(V(c))) \overset{(1)}{=} V(c) = V(a) \cup V(b)$. Moreover, it is clear that $V(a \mathbin{\widetilde{\vee}} b) = V(a) \cap V(b)$ for all $a, b \in \mathcal{C}(L)$.

(b) $\Rightarrow$ (c) Let $p \in X$ and $a, b \in \mathcal{C}(L)$. If $a \wedge b \leq p$, then $V(p) \subseteq V(a \wedge b) = V(a) \cup V(b)$ whence $p \in V(a)$ or $p \in V(b)$, *i.e.* $a \leq p$ or $b \leq p$.

(c) $\Rightarrow$ (a) Let $V(a)$ and $V(b)$ be two closed sets. By Equation (1), we can write them as $V(a) = V(a')$ and $V(b) = V(b')$ for some $a', b' \in \mathcal{C}(L)$. Let $p \in V(a' \wedge b')$, whence

$a' \wedge b' \leq p$. Since p is strongly irreducible in $\mathcal{C}(L)$, $a' \leq p$ or $b' \leq p$, *i.e.* $p \in V(a')$ or $p \in V(b')$. Thus $V(a' \wedge b') \subseteq V(a) \cup V(b)$. Since $V(a) \cup V(b) = V(a') \cup V(b') \subseteq V(a' \wedge b')$ always holds, the equality follows.

$(d) \Rightarrow (c)$ holds by [5, Lemma 1.20].

$(b + c) \Rightarrow (d)$ Note that $V : \mathcal{C}(L) \to \mathcal{P}(X)$ is injective and, by (b), the dual lattice $\mathcal{C}(L)^{\circ}$ is isomorphic to a sublattice of the distributive lattice $\mathcal{P}$, whence $(\mathcal{C}(L), \wedge, \widetilde{\vee})$ is distributive as well. On the other hand, every strongly irreducible element is in particular irreducible. ∎

Example 2.3. Let R be an associative, not necessarily commutative, ring with unity, $X = \mathrm{Spec}(R)$ be the spectrum of prime ideals of R and $\mathcal{L}_2(R)$ the lattice of ideals of R. Notice that $\mathrm{Im}(I)$ consists of all ideals that are intersections of prime ideals, *i.e.* the *semiprime ideals* of R [42, 2.5]. It is clear that every prime ideal P is strongly irreducible in $\mathcal{L}_2(R)$; in particular, P is strongly irreducible in $\mathrm{Im}(I)$ whence $\mathcal{L}_2(R)$ is a $\mathrm{Spec}(R)$-top lattice. The topology on $\mathrm{Spec}(R)$ is the ordinary Zariski topology.

Definition 2.4. We say that $\mathcal{L}$ is a *strongly X-top* lattice (or *strongly X-top* for short) iff every element of X is strongly irreducible in $(L, \wedge)$.

The proof of the following result is similar to that of Theorem 2.2: If all elements $p \in X$ are strongly irreducible in $(L, \wedge)$, then it follows by Theorem 2.2 that $\mathcal{L}$ is an X-top lattice. Moreover, for all $a, b \in L$ we have

$$p \in V(a \wedge b) \;\Rightarrow\; [a \wedge b \leq p \Rightarrow a \leq p \text{ or } b \leq p] \;\Rightarrow\; p \in V(a) \cup V(b),$$

i.e. $V(a \wedge b) \subseteq V(a) \cup V(b)$. The reverse inclusion is obvious; this means that $V(a \wedge b) = V(a) \cup V(b)$ for all $a, b \in L$. On the other hand, it is clear that $V(a \vee b) = V(a) \cap V(b)$ for all $a, b \in L$.

Proposition 2.5. *The following statements are equivalent:*

(a) $\mathcal{L}$ *is a strongly X-top lattice;*
(b) $V : \mathcal{L} \to \mathcal{P}$ *is an anti-homomorphism of lattices.*

Example 2.6. Let R be an arbitrary associative ring with unity and $X = \mathrm{Spec}(R)$. As mentioned in Example 2.3, every prime ideal is strongly irreducible in $\mathcal{L}_2(R)$. In particular, if R is commutative (or more generally *left duo*), then the lattice $\mathcal{L}({}_RR)$ of left ideals of R is strongly X-top. However, if $\mathcal{L}_2(R) \neq \mathcal{L}({}_RR)$, then $\mathcal{L}({}_RR)$ might not be strongly X-top. For example, if R is a prime ring which is not uniform as a left R-module, then $\mathcal{L}({}_RR)$ is not strongly X-top because $P = 0$ is a prime ideal and there are non-zero left ideals A, B of R with $A \cap B = 0$. An example of such a ring is given by the full $n \times n$-matrix ring $R = \mathrm{M}_n(K)$ over a field K where $n \geq 2$.

Recall from [24] that for a non-empty topological space $\mathbf{X}$, a non-empty subset $\mathcal{A} \subseteq \mathbf{X}$ is said to be *irreducible* in $\mathbf{X}$ iff for all proper closed subsets A_1, A_2 of $\mathbf{X}$ we have

$$\mathcal{A} \subseteq A_1 \cup A_2 \Rightarrow \mathcal{A} \subseteq A_1 \text{ or } \mathcal{A} \subseteq A_2.$$

A maximal irreducible subset of $\mathbf{X}$ is called an *irreducible component* and is necessarily closed.

Proposition 2.7. *Let $\emptyset \neq \mathcal{A} \subseteq X$.*

(1) *Let $\mathcal{L}$ be X-top. If $I(\mathcal{A})$ is irreducible in $(\mathcal{C}(L), \wedge)$, then $\mathcal{A}$ is an irreducible subset of X.*

(2) *Let $\mathcal{L}$ be strongly X-top. The following are equivalent:*
  - (a) *$I(\mathcal{A})$ is irreducible in $(\mathcal{C}(L), \wedge)$;*
  - (b) *$\mathcal{A}$ is an irreducible subset of X;*
  - (c) *$I(\mathcal{A})$ is (strongly) irreducible in $(L, \wedge)$.*

Proof. (1) By our assumption, X becomes a topological space. Suppose that $\mathcal{A} \subseteq V(a_1) \cup V(a_2)$ for some $a_1, a_2 \in L$. Set $\mathcal{A}_i = V(a_i) \cap \mathcal{A}$ for $i = 1, 2$, so that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Notice that $I(\mathcal{A}) = I(\mathcal{A}_1) \wedge I(\mathcal{A}_2)$, whence $I(\mathcal{A}) = I(\mathcal{A}_i)$ for some $i = 1, 2$ as $I(\mathcal{A})$ is assumed to be irreducible in $\mathcal{C}(L)$, and it follows that

$$\mathcal{A} \subseteq V(I(\mathcal{A})) = V(I(\mathcal{A}_i)) \subseteq V(I(V(a_i))) = V(a_i).$$

(2) Suppose that all elements of X are strongly irreducible in $(L, \wedge)$.

$(a) \Rightarrow (b)$ follows by (1).

$(b) \Rightarrow (c)$ Let $\mathcal{A}$ be an irreducible subset of X and assume that $a_1 \wedge a_2 \leq I(\mathcal{A})$ for some $a_1, a_2 \in L$. It follows that

$$\mathcal{A} \subseteq V(I(\mathcal{A})) \subseteq V(a_1 \wedge a_2) = V(a_1) \cup V(a_2).$$

As $\mathcal{A}$ is irreducible, $\mathcal{A} \subseteq V(a_i)$ for some $i = 1, 2$, whence $I(\mathcal{A}) \geq I(V(a_i)) \geq a_i$ showing that $I(\mathcal{A})$ is strongly irreducible in $(L, \wedge)$.

$(c) \Rightarrow (a)$ is obvious. ∎

Example 2.8. Let R be a simple ring. Then $X = \mathrm{Spec}(R) = \{0\}$. Clearly, $\mathcal{L}({}_RR)$ is an X-top lattice. Notice that X is irreducible since it is a singleton. However, $I(X) = 0$ is irreducible in $(\mathcal{L}({}_RR), \cap)$ if and only if R is uniform as left R-module if and only if $\mathcal{L}({}_RR)$ is strongly X-top. Thus, every simple ring that is not left uniform can be taken as an example to show that the hypothesis on $\mathcal{L}$ to be strongly X-top in Proposition 2.7 (2) cannot be dropped.

Corollary 2.9. *If $\mathcal{L}$ is X-top and $\mathcal{A} \subseteq X$ is such that $I(\mathcal{A}) \in X$, then $\mathcal{A}$ is irreducible.*

The following result will be needed when dealing with *first submodules*.

Corollary 2.10. *Let $\mathcal{L}$ be X-top. If $[x, 1[\subseteq X$ for some $x \in X$, then $[x, 1[$ is a chain. Moreover, if $[x, 1[\subseteq X$ for every $x \in X$, then every non-empty subset $\mathcal{A} \subseteq X$ with $I(\mathcal{A}) \in X$ is a chain.*

Proof. Let $x \in X$ be such that $[x, 1[\subseteq X$ and $a, b \in L$ be such that $x \leq a, b$. By hypothesis, a, b and $c := a \wedge b$ belong to X. Thus, by Theorem 2.2, c is strongly irreducible in $(\mathcal{C}(L), \wedge)$, *i.e.* $a = c$ or $b = c$. Hence, $a \leq b$ or $b \leq a$. Assume now that $[x, 1[\subseteq X$ for every $x \in X$ and let $\mathcal{A} \subseteq X$ be a non-empty subset. If $I(\mathcal{A}) \in X$, then $[I(\mathcal{A}), 1[$ is a chain and hence $\mathcal{A} \subseteq [I(\mathcal{A}), 1[$ is a chain as well. ∎

Example 2.11. Let R be an associative, not necessary commutative, ring with unity and $X = \mathrm{Max}(R)$ the spectrum of maximal ideals of R. The lattice $\mathcal{L}_2(R)$ of all ideals of R is clearly strongly X-top. If R has the property that every proper ideal is contained in a unique maximal ideal (*e.g.* R is local), then every closed set, in particular every *connected component*, is a singleton whence X is totally disconnected.

Example 2.12. Let $X = \mathrm{Max}({}_RR)$ be the spectrum of maximal left ideals of R. In general the lattice $\mathcal{L}({}_RR)$ of left ideals of R is not strongly X-top (*cf.* [5, Example 2.12]).

3. First Submodules

Throughout, R is an associative, not necessarily commutative, ring with unity, M is a non-zero left R-module, $\mathcal{L}(M) = (\mathrm{Sub}(M), \cap, +, 0, M)$ is the complete lattice of R-submodules of M and $\mathcal{S}(M)$ is the (possibly empty) class of simple submodules of M.

Prime modules. Recall from [28] the following definition: ${}_RM$ is *fully faithful* iff every non-zero R-submodule of M is faithful. Moreover, call ${}_RM$ a *prime module* iff M is a non-zero fully faithful $R/\mathrm{ann}_R(M)$-module (see [28, p.48]). It is easy to see that $\mathrm{ann}_R(M)$ is a prime ideal if M is prime module (see [28, Exercise 3I]). For every prime ideal P of R, the cyclic left R-module $M = R/P$ is a left prime module, because if $N = I/P$ is any non-zero left R-submodule of M with I a left ideal of R properly containing P, then $\mathrm{ann}_R(N)I \subseteq P$, *i.e.* $\mathrm{ann}_R(N) \subseteq P = \mathrm{ann}_R(M)$. The class of left prime R-modules is denoted by $\mathbb{P}$ and is clearly closed under non-zero submodules.

Prime submodules. We call a proper submodule N of M a *prime submodule* iff $M/N \in \mathbb{P}$. Taking

$$X = \mathrm{Spec}^{\mathrm{p}}(M) = \{N \in \mathcal{L}(M) \mid N \text{ is a prime submodule of } M\},$$

one defines M to be a *top*$^{\mathrm{p}}$*-module* iff $\mathcal{L}(M)$ is X-top (*cf.* [35]). There are other choices to topologize certain subsets of $\mathcal{L}(M)$. For instance, one could take $X = \mathrm{Spec}^{\mathrm{fp}}(M)$, the class of *fully prime* submodules [2] or $X = \mathrm{Spec}^{\mathrm{f}}(M)$ the class of *fully coprime* submodules [3]. Other choices are $X = \mathrm{Spec}^{\mathrm{c}}(M)$ the class of coprime submodules, or $X = \mathrm{Spec}^{\mathrm{s}}(M)$ the class of *second* submodules [4]. For other possible choices for X, see the (co)primeness notions in the sense of Bican et al. [22].

First submodules. In this work, we are interested in the set X of those submodules of M which belong to $\mathbb{P}$, *i.e.* those which are, *as modules*, prime. We set

$$\mathrm{Spec}^{\mathrm{f}}(M) := \mathbb{P} \cap \mathcal{L}(M)$$

and call its elements *first submodules* of M. We say that ${}_RM$ is *firstless* iff $\mathrm{Spec}^{\mathrm{f}}(M) = \emptyset$.

The following proposition can be easily proved and includes some characterizations of first submodules that will be used in the sequel; more characterizations can be derived from [41, 1.22].

Proposition 3.1. *The following are equivalent for a non-zero R-submodule $0 \neq F \leq_R M$.*

(1) *$F \leq_R M$ is a first submodule;*

(2) $\mathrm{ann}_R(F) = \mathrm{ann}_R(H)$ *for every non-zero (fully invariant) submodule* $0 \neq H \leq_R F$;
(3) *every non-zero (fully invariant) submodule of* F *is a first submodule;*
(4) *For every* $r \in R$ *and* $f \in F$ *we have:* $rRf = 0 \Rightarrow f = 0$ *or* $rF = 0$.

Recall that one calls ${}_RM$ is *colocal* (or *cocyclic* [42]) iff the intersection of all non-zero submodules of M is non-zero.

Remark 3.2. If $0 \neq F \leq_R M$ is simple, then F is indeed a first R-submodule (*i.e.* $\mathcal{S}(M) \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$). So, if ${}_RM$ has an essential socle (called also *atomic* [30]), then $\mathrm{Spec}^{\mathrm{f}}(M) \neq \emptyset$.

Example 3.3. Let $0 \neq F \leq_R M$. If $\mathrm{ann}_R(F) \in \mathrm{Max}(R)$, then ${}_RF$ is first in M : if $H \leq_R F$ is such that $\mathrm{ann}_R(H)F \neq 0$, then $\mathrm{ann}_R(F) + \mathrm{ann}_R(H) = R$ whence $H = (\mathrm{ann}_R(F) + \mathrm{ann}_R(H))H = 0$. It follows that if R is a simple ring, then every non-zero R-submodule of M is first. In particular, every non-zero subspace of a left vector space over a division ring is first.

Examples 3.4. (1) If $0 \neq F \leq_R M$ has no non-trivial fully invariant R-submodules, then F is a first submodule of M. For instance, $\mathbb{Q} \leq_{\mathbb{Z}} \mathbb{R}$ is a first submodule since $\mathbb{Q}$ has no non-trivial fully invariant $\mathbb{Z}$-submodules.
(2) A non-zero semisimple submodule of M need not be first. In case R is commutative, a semisimple R-submodule of M is first if and only if it is non-zero and homogeneous semisimple.
(3) Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{R}$ and $F = \mathbb{Z} \oplus \mathbb{Q}$. Every fully invariant $\mathbb{Z}$-submodule of F is of the form $n\mathbb{Z} \oplus \mathbb{Q}$ for some $n \in \mathbb{N}$ and indeed $\mathrm{ann}_{\mathbb{Z}}(n\mathbb{Z} \oplus \mathbb{Q}) = (0) = \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Q})$. It follows that F is first in M.
(4) Let $M := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. The $\mathbb{Z}$-submodule $F := \bigoplus_{n \in A} \mathbb{Z}/n\mathbb{Z}$, where A is any *infinite* set of prime numbers, is *not* a first submodule since for any $p \in A$ we have $p\mathbb{Z} = \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}) \neq 0 = \mathrm{ann}_{\mathbb{Z}}(F)$.
(5) The *Prüfer* p*-group*

$$\mathbb{Z}_{p^\infty} := \left\{ \frac{n}{p^k} + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} \mid n \in \mathbb{Z} \text{ and } k \in \mathbb{N} \right\}$$

is *not* first in $\mathbb{Q}/\mathbb{Z}$: if $H \lneqq {}_{\mathbb{Z}}\mathbb{Z}_{p^\infty}$, then $H = \mathbb{Z}\left\{\frac{1}{p^k} + \mathbb{Z}\right\}$ for some $k \in \mathbb{N}$ (*e.g.* [42, 17.13]) whence $\mathrm{ann}_{\mathbb{Z}}(H) \neq 0 = \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$.

Following [34, p. 86], we call a (prime) ideal of R an *associated prime* of M iff $\mathfrak{p} = \mathrm{ann}_R(N)$ for some $N \in \mathrm{Spec}^{\mathrm{f}}(M)$; the class of associated primes of M is denoted by $\mathrm{Ass}({}_RM)$. If R is commutative, then $\mathfrak{p} \in \mathrm{Ass}({}_RM)$ if and only if $\mathfrak{p}$ is prime and $\mathfrak{p} = (0 :_R m)$ for some $0 \neq m \in M$ (e.g. [34, Lemma 3.56]).

Example 3.5. Let R be a *commutative* ring. If $\mathfrak{p}$ is an associated prime of M, then $R/\mathfrak{p} \hookrightarrow M$ is a first R-submodule. Notice that we might not have such an embedding if R is non-commutative (*e.g.* [9, Fact 36]).

Remark 3.6. If $F \in \mathrm{Spec}^{\mathrm{f}}(M)$, then $\mathrm{ann}_R(F)$ is a prime ideal: let $I, J \in \mathcal{L}_2(R)$ be such that $IJ \subseteq \mathrm{ann}_R(F)$ and suppose that $J \nsubseteq \mathrm{ann}_R(F)$, i.e. $K := JF \neq 0$. Since ${}_RF$ is first in M and $IK = I(JF) = (IJ)F = 0$, we conclude that $IF = 0$, *i.e.* $I \subseteq \mathrm{ann}_R(F)$. Notice that the converse is not true: for example, $\mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z}) = (0)$ is a prime ideal of $\mathbb{Z}$; however, $\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z}$ is not a first $\mathbb{Z}$-submodule of $\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z}\oplus\mathbb{Z}/3\mathbb{Z}$ since $\mathrm{ann}_{\mathbb{Z}}(0\oplus\mathbb{Z}/8\mathbb{Z}) = 8\mathbb{Z} \neq (0)$.

4. The topological structure of $\mathrm{Spec}^{\mathrm{f}}(M)$

Throughout this section, we fix the general setting of Section 3. In particular, M is a non-zero left R-module over the associative unital ring R and $\mathbb{P}$ is the class of prime R-modules. An R-submodule $N \leq_R M$ is said to be (strongly) hollow iff N is (strongly) irreducible in $\mathcal{L}(M)^\circ = (\mathrm{Sub}({}_RM), +, \cap)$. The class of strongly hollow submodules of M is denoted by $\mathcal{SH}(M)$. In this section, we give some applications of the results in Section 2 to the dual lattice $\mathcal{L}(M)^\circ$.

Top-modules. Since $\mathrm{Sub}({}_RN) \subseteq \mathrm{Sub}({}_RM)$ for every submodule N of M, we have $\mathrm{Spec}^{\mathrm{f}}(N) \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$. Hence, in order to use the map V from the second section, we will use the dual lattice $\mathcal{L}(M)^\circ$ of $\mathcal{L}(M)$ and $X = \mathrm{Spec}^{\mathrm{f}}(M)$. In this case, we have the *order-preserving* map

$$V : \mathrm{Sub}({}_RM) \longrightarrow \mathcal{P}(X), \qquad N \mapsto V(N) = \{P \in \mathbb{P} \mid P \subseteq N\}.$$

The map V forms a Galois connection with the map

$$I : \mathcal{P}(X) \longrightarrow \mathrm{Sub}({}_RM), \qquad \mathcal{A} \mapsto I(\mathcal{A}) = \sum_{P\in\mathcal{A}} P.$$

As before, we have $V = V \circ I \circ V$ and $I = I \circ V \circ I$. Denote the image of V by $\xi^{\mathrm{f}}(M)$. From Section 2, we know that $\xi^{\mathrm{f}}(M)$ contains X, $\emptyset$ and is closed under intersections; note that because of considering the dual lattice of $\mathcal{L}(M)$ one has

$$\bigcap_{\lambda\in\Lambda} V(N_\lambda) = V\left(\bigcap_{\lambda\in\Lambda} N_\lambda\right).$$

The set $\xi^{\mathrm{f}}(M)$ can be described as

$$\xi^{\mathrm{f}}(M) = \{V(I(\mathcal{A})) \mid \mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)\}$$

and depends only on those submodules that are of the form $I(\mathcal{A})$ for some subset $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$. The image of I is

$$\mathcal{I}(M) := \mathcal{C}(\mathcal{L}(M)^\circ) = \{I(\mathcal{A}) \mid \mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)\}$$

which is the set of closed elements relative to the Galois connection (V, I) and forms an lower subsemilattice $(\mathcal{I}(M), +)$ of $\mathcal{L}(M)^\circ$. Note that $\mathrm{Spec}^{\mathrm{f}}(M) = \mathbb{P} \cap \mathrm{Sub}({}_RM) \subseteq \mathcal{I}(M)$.

A lattice structure on $\mathcal{I}(M)$. The lower semilattice of closed elements $(\mathcal{I}(M),+)$ is complete, whence it has a smallest element (which we call the *coradical* of M):

$$\mathrm{Corad}^{\mathrm{f}}(M)=I(\mathrm{Spec}^{\mathrm{f}}(M))=\sum_{P\in\mathrm{Spec}^{\mathrm{f}}(M)}P.$$

This allows defining a *new* join on $\mathcal{I}(M)$ as follows: consider a family $Y=\{C_\lambda\}_{\lambda\in\Lambda}$, where $C_\lambda=I(\mathcal{A}_\lambda)$ and $A_\lambda\subseteq\mathrm{Spec}^{\mathrm{f}}(M)$ for each $\lambda\in\Lambda$, and define

$$\begin{aligned}\widetilde{\bigvee}Y &= IV\left(\bigcap_{\lambda\in\Lambda}C_\lambda\right)=I\left(\bigcap_{\lambda\in\Lambda}V(C_\lambda)\right)\\ &= \sum\{I(\mathcal{A})\mid I(\mathcal{A})\le C_\lambda\ \forall\ \lambda\in\Lambda\}=\sum\left\{F\in\mathrm{Spec}^{\mathrm{f}}(M)\mid F\le\bigcap_{\lambda\in\Lambda}C_\lambda\right\}.\end{aligned}$$

Notice that this new join $\widetilde{\vee}$ is usually *different* from the original join $\cap$.

Definition 4.1. We say that M is a
top$^{\mathrm{f}}$*-module* iff $\mathcal{L}(M)^{\circ}$ is $\mathrm{Spec}^{\mathrm{f}}(M)$-top, *i.e.* iff $\xi^{\mathrm{f}}(M)$ is closed under finite unions;
strongly top$^{\mathrm{f}}$*-module* iff $\mathcal{L}(M)^{\circ}$ is strongly $\mathrm{Spec}^{\mathrm{f}}(M)$-top, *i.e.* iff every first submodule of M is strongly hollow.

From Theorem 2.2 and Corollary 2.10 we get

Theorem 4.2. *The following statements are equivalent:*
(a) *M is a top$^{\mathrm{f}}$-module;*
(b) $V:(\mathcal{I}(M),\widetilde{\vee},+)\to(\xi^{\mathrm{f}}(M),\cap,\cup)$ *is a lattice isomorphism;*
(c) *every first submodule of M is strongly hollow in* $\mathrm{Corad}^{\mathrm{f}}(M)$;
(d) $(\mathcal{I}(M),\widetilde{\vee},+)$ *is a distributive lattice and every first submodule of M is a hollow* (*uniserial*) *module.*

Proof. The equivalence follows from Theorem 2.2.
Every R-submodule of $P\in\mathrm{Spec}^{\mathrm{f}}(M)$ is also a prime module, hence $[P,0[\subseteq\mathrm{Spec}^{\mathrm{f}}(M)$ in $\mathcal{L}(M)^{\circ}$. Thus, Corollary 2.10 applies and proves that every $P\in\mathrm{Spec}^{\mathrm{f}}(M)$ is uniserial. ∎

Lemma 4.3. *If* $\mathrm{Soc}(_RM)\neq 0$, *then the following are equivalent:*
(a) *All isomorphic simple submodules of M are equal.*
(b) $\mathrm{Soc}(M)$ *is a direct sum of non-isomorphic simple modules;*
(c) $\mathrm{Soc}(M)$ *is distributive;*

Proof. $(a)\Rightarrow(b)$ this is clear.
$(b)\iff(c)$ By [39, Proposition 1.3], $\mathrm{Soc}(M)=\bigoplus_{\lambda\in\Lambda}E_\lambda$ is distributive if and only if E_α and E_β are *unrelated* for all $\alpha\neq\beta$ in Λ; the later means for simple modules that $\mathrm{Hom}_R(E_\alpha,E_\beta)=0$.
$(c)\Rightarrow(a)$ By [39, Proposition 1.2], if $\mathrm{Soc}(M)=\bigoplus_{\lambda\in\Lambda}E_\lambda$ (E_λ is simple for each $\lambda\in\Lambda$) and E_α is unrelated to E_β for all $\alpha\neq\beta$ in Λ, then for every submodule $X\subseteq\bigoplus_{\lambda\in\Lambda}E_\lambda$ one has $X=\bigoplus_{\lambda\in\Lambda}(X\cap E_\lambda)$. In particular, if X is simple, then $X=E_\lambda$ for some $\lambda\in\Lambda$. ∎

Corollary 4.4. *If ${}_RM$ is a top$^{\rm f}$-module, then* $\mathrm{Soc}(M)$ *is a (direct) sum of non-isomorphic simple modules.*

Proof. This follows from the fact that $\mathcal{L}(\mathrm{Soc}(M)) = (\mathrm{Sub}(\mathrm{Soc}(M), \cap, +))$ is a sublattice of the distributive lattice $(\mathcal{I}(M), \widetilde{\wedge}, +)$, whence is also distributive. This is equivalent, by Lemma 4.3, to the stated property for $\mathrm{Soc}(M)$. ∎

Remark 4.5. Recall from [3] that ${}_RM$ has the *min-property* iff for every simple R-submodule $H \leq_R M$ we have $H \nsubseteq H_e$, where $H_e := \sum\limits_{K \in S(M)\backslash\{H\}} K$. By Lemma 4.3 and [38, Theorem 2.3], $\mathrm{Soc}(M)$ is distributive if and only if ${}_RM$ has the min-property.

Notation. We set $\mathrm{Sub}_c(M) := \{(0 :_M I) \mid I \in \mathcal{L}_2(R)\}$, $\mathcal{X}(L) := \mathrm{Spec}^{\rm f}(M) \backslash V(L)$ and

$$\begin{array}{lcllcl} \xi^{\rm f}(M) & := & \{V(L) \mid L \in \mathrm{Sub}({}_RM)\}; & \xi^{\rm f}_c(M) & := & \{V(L) \mid L \in \mathrm{Sub}_c(M)\}; \\ \tau^{\rm f}(M) & := & \{\mathcal{X}(L) \mid L \in \mathrm{Sub}({}_RM)\}; & \tau^{\rm f}_c(M) & := & \{\mathcal{X}(L) \mid L \in \mathrm{Sub}_c(M)\}; \end{array}$$

Remark 4.6. Let M be a strongly top$^{\rm f}$-module.

(a) M is a top$^{\rm f}$-module: this follows directly from observation that $\mathrm{Spec}^{\rm f}(M) \subseteq \mathcal{SH}(M)$ if and only if $V(L_1) \cup V(L_2) = V(L_1 + L_2)$ for every pair of submodules $L_1, L_2 \leq_R M$.

(b) $\mathrm{Spec}^{\rm f}(M)$ has a basis of open sets given by

$$\{\mathcal{X}(H) \mid H \leq_R M \text{ is finitely generated}\}$$

Theorem 4.7. $(\mathrm{Spec}^{\rm f}(M), \tau^{\rm f}_c(M))$ *is a topological space.*

Proof. It is obvious that $V(0) = \emptyset$, $V(M) = \mathrm{Spec}^{\rm f}(M)$ and that $\bigcap\limits_{\lambda \in \Lambda} V(L_\lambda) = V(\bigcap\limits_{\lambda \in \Lambda} L_\lambda)$ for every subset $\{L_\lambda\}_\Lambda \subseteq \mathrm{Sub}_c(M)$. We show now that for all ideals $I, \widetilde{I}$ or R we have

$$V((0 :_M I)) \cup V((0 :_M \widetilde{I})) = V((0 :_M I) + (0 :_M \widetilde{I})) = V((0 :_M I \cap \widetilde{I})) = V((0 :_M I\widetilde{I})). \quad (3)$$

Indeed, the following inclusions are obvious

$$V((0 :_M I)) \cup V((0 :_M \widetilde{I})) \subseteq V((0 :_M I) + (0 :_M \widetilde{I})) \subseteq V((0 :_M I \cap \widetilde{I})) \subseteq V((0 :_M I\widetilde{I})) \quad (4)$$

On the other hand, let $F \in V((0 :_M I\widetilde{I}))$ and suppose that $F \nsubseteq (0 :_M \widetilde{I})$. Since $I(\widetilde{I}F) = (I\widetilde{I})F = 0$ and $\widetilde{I}F \neq 0$, we conclude that $IF = 0$ (recall that F is a first submodule of M), *i.e.* $F \subseteq (0 :_M I)$. Consequently, $F \in V((0 :_M I)) \cup \mathcal{V}((0 :_M \widetilde{I}))$. ∎

Example 4.8. For every non-empty set of prime numbers A, the $\mathbb{Z}$-module $M := \oplus_{p \in A} \mathbb{Z}/p\mathbb{Z}$ (with no repetition) is a top$^{\rm f}$-module: it can be easily seen that $\mathrm{Spec}^{\rm f}(M) = \{\mathbb{Z}/p\mathbb{Z} \mid p \in A\}$ and that $\xi^{\rm f}(M)$ is closed under finite unions.

Example 4.9. The $\mathbb{Z}$-modules, i.e. Abelian groups, that are top$^{\rm f}$-module are precisely the submodules of $\mathbb{Q}/\mathbb{Z}$. To see this let M be a non-zero $\mathbb{Z}$-module that is a top$^{\rm f}$-module. If M is not torsion, then there exists a non-zero element $m \in M$ such that $F = \mathbb{Z}m \simeq \mathbb{Z}$. Hence F is a first submodule of M and would be uniserial, by Theorem 4.2. As $\mathbb{Z}$ is not a uniserial $\mathbb{Z}$-module we reach a contradiction. Hence M is a torsion module and can be written as $M = \bigoplus\limits_{p \text{ a prime number}} T_p(M)$, where $T_p(M) = \{m \in M : \exists n > 0, p^n m = 0\}$ is the p-torsion part of M. By Corollary 4.4, the socle of M is a direct some of non-isomorphic simple modules. Hence any non-zero torsion part $T_p(M)$ has a unique simple submodule. Since any non-zero finitely generated submodule of $T_p(M)$ is artinian and has non-zero socle, it must contain the unique simple submodule of $T_p(M)$, i.e. $T_p(M)$ is an essential extension of its simple socle. Thus $T_p(M)$ is isomorphic to a submodule of the injective hull of $\mathbb{Z}_p$, i.e. the Prüfer group $\mathbb{Z}_{p^\infty}$. In particular $M = \bigoplus_p T_p(M)$ embeds into $\bigoplus_p \mathbb{Z}_{p^\infty} \simeq \mathbb{Q}/\mathbb{Z}$.

On the other hand if M is any submodule of $\mathbb{Q}/\mathbb{Z}$, then each non-zero p-torsion part $T_p(M)$ of M is isomorphic to a submodule of $\mathbb{Z}_{p^\infty}$ and hence uniserial. If F is any non-zero first submodule of M and C a cyclic submodule of F, then, as M is torsion, $C \simeq \mathbb{Z}_n$ for some $n > 0$. As C is first we conclude $C \simeq \mathbb{Z}_p$ is simple. Hence $\mathrm{ann}_{\mathbb{Z}}(F) = \mathrm{ann}_{\mathbb{Z}}(C) = p\mathbb{Z}$, which shows that $F \subseteq T_p(M)$. Now it is clear that any finitely generated submodule of F would be isomorphic to a direct sum of copies of $\mathbb{Z}_p$, as F is first and hence F would be semisimple. But since $T_p(M)$ is uniserial and contains a unique simple submodule, F itself must be simple. Hence the first submodules of M are precisely the simple submodules of M and $\mathrm{Corad}^{\mathrm{f}}(M)$ equals the socle of M. We will show that any first submodule of M is strongly hollow in $\mathrm{Corad}^{\mathrm{f}}(M)$. Let $\mathrm{Corad}^{\mathrm{f}}(M) = \bigoplus_{i\in I} F_i$, with $F_i \simeq \mathbb{Z}_{p_i}$ for distinct prime numbers p_i, $i \in I$. Let $i \in I$ and N, L submodules of $\mathrm{Corad}^{\mathrm{f}}(M)$ such that $F_i \subseteq N + L$. Since F_i is cyclic, we have $F_i \subseteq N' + L'$ for finitely generated submodules $N' \subseteq N$ and $L' \subseteq L$. Thus there exists a finite set of indices $J \subseteq I$ such that $N', L' \subseteq \bigoplus_{j\in J} F_j$. Let $D = \bigoplus_{j\in J\setminus\{i\}} F_j$, then $N', L' \subseteq F_i \oplus D$. Furthermore

$$\mathrm{ann}_{\mathbb{Z}}(D) = \bigcap_{j\in J\setminus\{i\}} \mathrm{ann}_{\mathbb{Z}}(F_j) = \left(\prod_{j\in J\setminus\{i\}} p_j\right)\mathbb{Z} \not\subseteq p_i\mathbb{Z} = \mathrm{ann}_{\mathbb{Z}}(F_i).$$

By [5, Lemma 2.17], F_i is strongly hollow in $F_i \oplus D$ and hence $F_i \subseteq N' \subseteq N$ or $F_i \subseteq L' \subseteq L$, i.e. F_i is strongly hollow in $\mathrm{Corad}^{\mathrm{f}}(M)$. By Theorem 4.2, M is a $\mathrm{top}^{\mathrm{f}}$-module.

Example 4.10. Over a simple ring R, every non-zero left R-module is prime. Theorem 4.2 shows that the (strongly) $\mathrm{top}^{\mathrm{f}}$-modules over a simple ring are precisely the non-zero uniserial modules.

Remarks 4.11. Let M be a *top*$^{\mathrm{f}}$*-module,* H a non-zero submodule of M and set $\mathcal{X}(H) = \mathrm{Spec}^{\mathrm{f}}(M)\setminus V(H)$.

(a) $\mathrm{Spec}^{\mathrm{f}}(M)$ is a T_0 (Kolmogorov) space.
(b) The closure of any subset $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ is $\overline{\mathcal{A}} = V(I(\mathcal{A}))$.
(c) $\mathcal{X}(H) = \emptyset$ if and only if $\mathrm{Corad}^{\mathrm{f}}(M) \subseteq H$.
(d) If ${}_RM$ has *essential* socle, then $\mathrm{Spec}^{\mathrm{f}}(H) = \emptyset$ if and only if $H = 0$.
(e) $\mathrm{Spec}^{\mathrm{f}}(H)$ is a subspace of $\mathrm{Spec}^{\mathrm{f}}(M)$.
(f) If $M \simeq N$, then $\mathrm{Spec}^{\mathrm{f}}(M) \approx \mathrm{Spec}^{\mathrm{f}}(N)$ are homeomorphic and $\mathrm{Corad}^{\mathrm{f}}(M) \simeq \mathrm{Corad}^{\mathrm{f}}(N)$.

Recall (*e.g.* [40], [15]) that M is said to be a *multiplication* (resp. *comultiplication*) *module* iff every R-submodule of M is of the form IM (resp. $(0 :_M I)$ for some ideal I of

R, or equivalently iff for every R-submodule $H \leq_R M$ we have $H = (H :_R M)M$ (resp. $H = (0 :_M (0 :_R H))$).

Proposition 4.12. *Let* $0 \neq F \leq_R M$.

(a) *If* ${}_RF$ *is comultiplication, then* F *is first in* M *if and only if* ${}_RF$ *is simple.*
(b) *If* ${}_RF$ *is multiplication, then* F *is first in* M *if and only if* $\mathrm{ann}_R(F)$ *is a prime ideal.*

Proof. (a) If ${}_RF$ is simple, then F is first in M by Remark 3.2. On the other hand, let F be first in M, $0 \neq H \leq_R F$ and consider $I := \mathrm{ann}_R(H)$. Since F is first in M, we have $I = \mathrm{ann}_R(F)$ and so $H = (0 :_F (0 :_R H)) = (0 :_F (0 :_R F)) = F$, i.e. ${}_RF$ is simple.

(b) If F is first in M, then $\mathrm{ann}_R(F)$ is a prime ideal by Remark 3.6. On the other hand, assume that $\mathrm{ann}_R(F) \in \mathrm{Spec}(R)$. Let $0 \neq H \leq_R F$ and consider $I := \mathrm{ann}_R(H)$. Since ${}_RF$ is multiplication, $H = JF$ for some $J \in \mathcal{L}_2(R)$. Notice that $IJ \subseteq \mathrm{ann}_R(F)$, whence $IF = 0$ since $\mathrm{ann}_R(F)$ is a prime ideal and $J \subsetneq \mathrm{ann}_R(F)$. Consequently, ${}_RF$ is first. ∎

Remark 4.13. Let R be zero-dimensional (*i.e.* every prime ideal of R is maximal). It follows by Example 3.3 and Remark 3.6 that

$$\mathrm{Spec}^{\mathrm{f}}(M) = \{F \leq_R M \mid \mathrm{ann}_R(F) \text{ is prime ideal}\}.$$

Examples of zero-dimensional rings include biregular rings [42, 3.18] and left (right) perfect rings.

Definition 4.14. Let $0 \neq H \leq_R M$. A maximal element of $V(H)$, if any, is said to be *maximal under* H. A *maximal element of* $\mathrm{Spec}^{\mathrm{f}}(M)$ is said to be a *maximal first submodule* of M.

Lemma 4.15. *Let* ${}_RM$ *have an essential socle and*

(1) R *is zero-dimensional;* or
(2) *every submodule of* ${}_RM$ *is multiplication.*

For every $0 \neq H \leq_R M$, *there exists* $F \in \mathrm{Spec}^{\mathrm{f}}(M)$ *which is maximal under* H.

Proof. Let $0 \neq H \leq_R M$. Since $\mathrm{Soc}(M) \leq_R M$ is essential, $\emptyset \neq \mathcal{S}(H) \subseteq V(H)$. Let

$$F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots$$

be an ascending chain in $V(H)$ and set $\widetilde{F} := \bigcup_{i=1}^{\infty} F_i$. Then we have a descending chain of prime ideals

$$(0 :_R F_1) \supseteq (0 :_R F_2) \supseteq \cdots \supseteq (0 :_R F_n) \supseteq (0 :_R F_{n+1}) \supseteq \cdots \tag{5}$$

and it follows that $\mathfrak{p} := (0 :_R \widetilde{F}) = \bigcap_{i=1}^{\infty}(0 :_R F_i)$ is a prime ideal. If R is zero-dimensional, then $\widetilde{F} \in \mathrm{Spec}^{\mathrm{f}}(M)$ by Remark 4.13. On the other hand, if ${}_R\widetilde{F}$ is multiplication, then $\widetilde{F} \in V(H)$ by Proposition 4.12 (b). In either case, it follows by Zorn's Lemma that $V(H)$ has a maximal element. ∎

Example 4.16. Recall from [31, p. 128] that ${}_RM$ is *completely cyclic* (or *fully cyclic* [25]) iff every R-submodule of ${}_RM$ is cyclic. If ${}_RM$ is a uniserial module and R is a left (or right) Artinian left duo ring, then ${}_RM$ is completely cyclic by [31, Lemma 13.9], whence every R-submodule of ${}_RM$ is multiplication by [40]; moreover, since ${}_RM$ is cyclic (finitely generated) and R is left Artinian, ${}_RM$ is also Artinian whence $\mathrm{Soc}(M) \leq_R M$ is essential.

4.17. Let M be a *top*$^{\mathrm{f}}$*-module* and consider $\mathrm{Spec}^{\mathrm{f}}(M)$ with the associated topology. Since the lattice $\mathcal{I}(M)$ and the lattice $\xi^{\mathrm{f}}(M)$ of closed subsets are isomorphic, some topological conditions on $\mathrm{Spec}^{\mathrm{f}}(M)$ translate to module theoretical conditions on M. Recall from [24, 23] that a topological space is called *Noetherian* (*Artinian*) iff every descending (ascending) chain of closed sets is stationary. Therefore, $\mathrm{Spec}^{\mathrm{f}}(M)$ is Noetherian (Artinian) if and only if M satisfies the descending (ascending) chain condition on submodules of the form $I(\mathcal{A})$ for subsets $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$. In particular, if M is Noetherian (Artinian), then $\mathrm{Spec}^{\mathrm{f}}(M)$ is Artinian (Noetherian).

Lemma 4.18. *Let M be a top$^{\mathrm{f}}$-module, $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ an irreducible subset and H a non-zero submodule of $I(\mathcal{A})$. If $\mathrm{Spec}^{\mathrm{f}}(H) \neq \emptyset$, then $\mathrm{ann}_R(H) = \mathrm{ann}_R(I(\mathcal{A}))$.*

Proof. Let $P \in \mathrm{Spec}^{\mathrm{f}}(H)$ be a cyclic first submodule of H and hence of $I(\mathcal{A})$. Since P is cyclic, there exist $N_1, \ldots, N_k \in \mathcal{A}$ such that $P \subseteq N_1 + \cdots + N_k$. By Theorem 4.2 P is strongly hollow in $\mathrm{Corad}^f(M)$ and hence $P \subseteq N_i \in \mathcal{A}$ for some i.

Setting
$$\mathcal{A}_0 = \{Q \in \mathcal{A} \mid Q \cap P = 0\},$$
we have
$$\mathcal{A} \subseteq V(I(\mathcal{A}_0)) \cup V(I(\mathcal{A} \setminus \mathcal{A}_0)).$$
By the irreducibility of $\mathcal{A}$ we have that $\mathcal{A}$ is contained in one of the two closed sets. Suppose that $\mathcal{A} \subseteq V(I(\mathcal{A}_0))$, whence $P \subseteq I(\mathcal{A}_0)$ as $P \subseteq N_i \in \mathcal{A}$. Since P is cyclic, there is a finite set $\{Q_1, \cdots, Q_m\} \subseteq \mathcal{A}_0$ with $P \subseteq Q_1 + \cdots + Q_m$. Since P is strongly hollow in $\mathrm{Corad}^f(M)$, we have $P \subseteq Q_i$ for some i. This is a contradiction to P being non-zero. Hence, $\mathcal{A} \subseteq V(I(\mathcal{A} \setminus \mathcal{A}_0))$ and $P \subseteq I(\mathcal{A}) = \sum\{Q \in \mathcal{A} \mid Q \cap P \neq 0\}$. This shows that
$$\mathrm{ann}_R(P) \supseteq \mathrm{ann}_R(I(\mathcal{A})) = \bigcap_{Q \cap P \neq 0} \mathrm{ann}_R(Q) = \bigcap_{Q \cap P \neq 0} \mathrm{ann}_R(Q \cap P) = \mathrm{ann}_R(P).$$
Thus $\mathrm{ann}_R(P) = \mathrm{ann}_R(H) = \mathrm{ann}_R(I(\mathcal{A})$. ∎

Remark 4.19. Note that if $I(\mathcal{A})$ is a distributive module for a non-empty subset $\mathcal{A}$, then $\mathrm{Spec}^{\mathrm{f}}(H) = \emptyset$ if and only if $H = 0$ for all submodules $H \subseteq I(\mathcal{A})$, because if H is non-zero and C is a non-zero cyclic submodule of H, then $C \subseteq I(\mathcal{A})$ implies that there are finitely many first submodules $Q_1, \ldots, Q_n$ such that $C \subseteq Q_1 + \cdots + Q_n$. By distributivity, $C = C \cap Q_1 + \cdots + C \cap Q_n$ and since $C \neq 0$, there must be some $i = 1, \cdots, n$ such that $C \cap Q_i \neq 0$. Thus $C \cap Q_i \in \mathrm{Spec}^{\mathrm{f}}(H)$.

Proposition 4.20. *Let M be a top$^{\mathrm{f}}$-module and let $\emptyset \neq \mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$.*

(1) *If $I(\mathcal{A})$ is a hollow module, then $\mathcal{A}$ is irreducible. The converse holds if M is a strongly top$^{\mathrm{f}}$-module.*

(2) *The following are equivalent:*
   (a) $\mathcal{A}$ *is irreducible and* $\mathrm{Spec}^{\mathrm{f}}(H) \neq \emptyset$ *for any* $0 \neq H \subseteq I(\mathcal{A})$.
   (b) $\mathcal{A}$ *is irreducible and* $I(\mathcal{A})$ *is distributive.*
   (c) $I(\mathcal{A})$ *is a first submodule;*
   (d) $I(\mathcal{A})$ *is uniserial;*
   (e) $\mathcal{A}$ *is a chain.*

Proof. (1) follows from Proposition 2.7 applied to the dual lattice $\mathcal{L}(M)^{\circ}$.

(2) $(a) \Rightarrow (c)$. The hypotheses of Lemma 4.18 are fulfilled for any non-zero submodule of $I(\mathcal{A})$. Hence, all non-zero submodules have the same annihilator, which shows that $I(\mathcal{A})$ is a prime module.

$(c) \Rightarrow (a)$ By Corollary 2.9, $\mathcal{A}$ is irreducible. Clearly any non-zero submodule of a prime module is first; so, if $I(\mathcal{A})$ is a first submodule, then any non-zero submodule of it is first as well.

$(c) \Rightarrow (e)$ follows by Corollary 2.10.

$(e) \Rightarrow (c)$ Assume now that $\mathcal{A}$ is a chain; in particular, $I(\mathcal{A}) = \bigcup_{P \in \mathcal{A}} P$. Since for all $Q, P \in \mathcal{A}$ either $Q \subseteq P$ or $P \subseteq Q$ and since P and Q are prime modules, $\mathrm{ann}_R(P) = \mathrm{ann}_R(Q)$. Every cyclic submodule $U = Rm$ of $I(\mathcal{A})$ lies in one of the members of $\mathcal{A}$ and thus has the same annihilator, *i.e.* $I(\mathcal{A})$ is a prime module or equivalently $I(\mathcal{A}) \in \mathrm{Spec}^{\mathrm{f}}(M)$.

$(d) \iff (e)$ clear.

$(a + d) \Rightarrow (b)$ is clear because a uniserial module is distributive.

$(b) \Rightarrow (a)$ holds by Remark 4.19. ∎

Remark 4.21. Let M be a $\mathrm{top}^{\mathrm{f}}$-module and $\emptyset \neq \mathcal{A} \subseteq \mathcal{S}(M)$. Every non-zero submodule of $I(\mathcal{A}) \subseteq \mathrm{Soc}(M)$ contains a simple (hence first) submodule and so we get as an immediate consequence from Proposition 4.20 that the following statements are equivalent:
   (a) $\mathcal{A}$ is irreducible;
   (b) $I(\mathcal{A})$ is a first submodule of M;
   (c) $\mathcal{A} = \{K\}$ is singleton.

Example 4.22. Let M be a $\mathrm{top}^{\mathrm{f}}$-module. It follows by Remark 4.21 that $\mathcal{S}(M) \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ is irreducible if and only if $\mathrm{Soc}(M)$ is a first submodule of M if and only if M contains a single simple R-submodule.

Remark 4.23. Let M be a $\mathrm{top}^{\mathrm{f}}$-module and $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ be such that $I(\mathcal{A})$ is a first submodule of M. By Theorem 4.2, $I(\mathcal{A})$ is a hollow module (in fact $I(\mathcal{A})$ is moreover a uniserial module). It follows then from Proposition 4.20 (2) that $\mathcal{A}$ is irreducible.

Definition 4.24. We say a $\mathrm{top}^{\mathrm{f}}$-module is *consistent* iff for every $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ we have: $I(\mathcal{A}) \in \mathrm{Spec}^{\mathrm{f}}(M)$ if (and only if) $\mathcal{A}$ is irreducible.

Remark 4.25. From Proposition 4.20 and Remark 4.19 we see that the following statements are equivalent for a $\mathrm{top}^{\mathrm{f}}$-module M:
   (a) M is consistent;
   (b) $\mathrm{Spec}^{\mathrm{f}}(H) \neq \emptyset$ for every non-zero submodule $H \subseteq I(\mathcal{A})$ and every irreducible subset $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$;

(c) $I(\mathcal{A})$ is distributive for every irreducible subset $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$.

For property (c) we use the obvious fact that uniserial modules are distributive.

Example 4.26. Every $\mathrm{top}^{\mathrm{f}}$-module with essential socle is consistent. Moreover, every $\mathrm{top}^{\mathrm{f}}$-module M, for which $\mathrm{Corad}^{\mathrm{f}}(M)$ is distributive, is consistent.

Proposition 4.27. *Let $_RM$ be a consistent $\mathrm{top}^{\mathrm{f}}$-module with $\mathrm{Spec}^{\mathrm{f}}(M) \neq \emptyset$. The following are equivalent for $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$:*

(a) *$\mathcal{A}$ is irreducible;*
(b) *$I(\mathcal{A})$ is a first submodule of M;*
(c) *$I(\mathcal{A})$ is a non-zero hollow (uniserial) module;*
(d) *$\emptyset \neq \mathcal{A}$ is a chain.*

For the special case of $\mathcal{A} = \mathrm{Spec}^{\mathrm{f}}(M)$ we have that $\mathrm{Spec}^{\mathrm{f}}(M)$ is irreducible if and only if $\mathrm{Corad}^{\mathrm{f}}(M)$ is a first submodule of M if and only if the set of first submodules of M is linearly ordered, provided $_RM$ is a consistent $\mathrm{top}^{\mathrm{f}}$-module and $\mathrm{Spec}^{\mathrm{f}}(M) \neq \emptyset$.

Notation. Set

$$\mathrm{Max}(\mathrm{Spec}^{\mathrm{f}}(M)) := \{K \in \mathrm{Spec}^{\mathrm{f}}(M) \mid \ K \text{ is a maximal first submodule of } M\}. \tag{6}$$

Proposition 4.28. *Let $_RM$ be a consistent $\mathrm{top}^{\mathrm{f}}$-module.*

(a) *We have a bijection*

$$\mathrm{Spec}^{\mathrm{f}}(M) \overset{V(-)}{\longleftrightarrow} \{\mathcal{A} \mid \mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M) \textit{ is an irreducible closed subset}\}. \tag{7}$$

(b) *The bijection (7) restricts to a bijection*

$$\mathrm{Max}(\mathrm{Spec}^{\mathrm{f}}(M)) \overset{V(-)}{\longleftrightarrow} \{\mathcal{A} \mid \mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M) \textit{ is an irreducible component}\}.$$

Proof. (a) Let $K \in \mathrm{Spec}^{\mathrm{f}}(M)$. Notice that $K = I(V(K))$ and so the closed set $V(K)$ is irreducible by Proposition 4.27. On the other hand, let $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ be a closed irreducible subset. Notice that $I(\mathcal{A})$ is first in M by Proposition 4.27 and that $\mathcal{A} = \overline{\mathcal{A}} = V(I(\mathcal{A}))$. Clearly, the maps V and I are bijective and the result follows.

(b) This follows from (a), the definitions and the fact that V is order preserving. ∎

A topological space $\mathbf{X}$ is called *sober* if every irreducible closed subset of it is the closure of exactly one point.

Corollary 4.29. *If $_RM$ is a consistent $\mathrm{top}^{\mathrm{f}}$-module, then $\mathrm{Spec}^{\mathrm{f}}(M)$ is a sober space.*

Proof. Let $\mathcal{A} \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ be an irreducible closed subset. By Proposition 4.28 (a), $\mathcal{A} = V(K)$ for some $K \in \mathrm{Spec}^{\mathrm{f}}(M)$. It follows that

$$\mathcal{A} = \overline{\mathcal{A}} = V(I(\mathcal{A})) = V(K) = \overline{\{K\}},$$

i.e. K is a generic point for $\mathcal{A}$. If H is a generic point of $\mathcal{A}$, then $V(K) = V(H)$ whence $K = H$. ∎

Theorem 4.30. *Let $_RM$ be a $\mathrm{top}^{\mathrm{f}}$-module with essential socle.*

(a) *If* $\mathcal{S}(M)$ *is finite, then* $\mathrm{Spec}^{\mathrm{f}}(M)$ *is compact.*
(b) *If* $\mathcal{S}(M)$ *is countable, then* $\mathrm{Spec}^{\mathrm{f}}(M)$ *is countably compact.*

Proof. We prove only (a); the proof of (b) is similar. Assume that $\mathcal{S}(M) = \{N_1, \ldots, N_k\}$. Let $\{V(H_\alpha)\}_{\alpha \in I}$ be an arbitrary collection of closed subsets of $\mathrm{Spec}^{\mathrm{f}}(M)$ with $\bigcap_{\alpha \in I} V(H_\alpha) = \emptyset$. Since $\mathcal{S}(M) \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$, we can pick for each $i = 1, \ldots, k$ some $\alpha_i \in I$ such that $N_i \not\subseteq H_{\alpha_i}$. If $\widetilde{H} := \bigcap_{i=1}^{k} H_{\alpha_i} \neq 0$, then there exists a simple R-submodule $0 \neq N \subseteq \widetilde{H}$ (since $\mathrm{Soc}(\widetilde{H}) = \widetilde{H} \cap \mathrm{Soc}(M) \neq 0$), a contradiction since $N = N_i \not\subseteq H_{\alpha_i}$ for some $i = 1, \ldots, n$. It follows that $\widetilde{H} = 0$, whence $\bigcap_{i=1}^{k} V(H_{\alpha_i}) = V\left(\bigcap_{i=1}^{k} H_{\alpha_i}\right) = V(0) = \emptyset$. ∎

Connectedness Properties. Recall (e.g. [24], [23]) that a non-empty topological space $\mathbf{X}$ is said to be

ultraconnected, iff the intersection of any two non-empty closed subsets is non-empty;

irreducible (or *hyperconnected*), iff $\mathbf{X}$ is not the union of two proper *closed* subsets, or equivalently iff the intersection of any two non-empty open subsets is non-empty;

connected, iff $\mathbf{X}$ is not the *disjoint* union of two proper *closed* subsets; equivalently, iff the only subsets of $\mathbf{X}$ that are *clopen* (*i.e.* closed and open) are $\emptyset$ and $\mathbf{X}$.

Proposition 4.31. *Let* ${}_RM$ *be a top*$^{\mathrm{f}}$*-module and assume that every first submodule of* M *is simple.*

(a) $\mathrm{Spec}^{\mathrm{f}}(M)$ *is discrete.*
(b) M *has a unique simple* R*-submodule if and only if* $\mathrm{Spec}^{\mathrm{f}}(M)$ *is connected.*
(c) ${}_RM$ *is colocal if and only if* $\mathrm{Spec}^{\mathrm{f}}(M)$ *is connected and* $\mathrm{Soc}(M) \leq_R M$ *is essential.*

Proof. (a) Notice that ${}_RM$ has the min-property by Corollary 4.4 and Remark 4.5. It follows that for every $K \in \mathrm{Spec}^{\mathrm{f}}(M) = \mathcal{S}(M)$ we have $\{K\} = \mathcal{X}(\{K_e\})$ an open set.

(b) ($\Rightarrow$) clear.

($\Leftarrow$) By (a), $\mathrm{Spec}^{\mathrm{f}}(M)$ is discrete and so $\mathcal{S}(M) = \mathrm{Spec}^{\mathrm{f}}(M)$ has only one point since a discrete connected space cannot contain more than one-point.

(c) follows directly from the definitions and (b). ∎

Remark 4.32. Let ${}_RM$ be a top$^{\mathrm{f}}$-module with essential socle. Recall that $\mathcal{S}(M) \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$ without any conditions on ${}_RM$. If $\{H\}$ is closed in $\mathrm{Spec}^{\mathrm{f}}(M)$ for some $H \leq_R M$, then $\{H\} = V(K)$ for some $0 \neq K \leq_R M$ and we conclude that ${}_RH$ is simple: if not, then there exists some simple R-submodule $\widetilde{H} \lneqq_R H$ and we would have $\{H, \widetilde{H}\} \subseteq V(K) = \{H\}$, a contradiction. So, $H \leq_R M$ is simple if and only if H is a first submodule of M and $V(H) = \{H\}$ if and only if $\{H\}$ is closed in $\mathrm{Spec}^{\mathrm{f}}(M)$.

Combining Proposition 4.31 and Remark 4.32 we obtain

Theorem 4.33. *For a top*$^{\mathrm{f}}$*-module* M *with essential socle, the following are equivalent:*

(1) $\mathrm{Spec}^{\mathrm{f}}(M) = \mathcal{S}(M)$;
(2) $\mathrm{Spec}^{\mathrm{f}}(M)$ *is discrete;*
(3) $\mathrm{Spec}^{\mathrm{f}}(M)$ *is* T_2 *(Hausdorff space);*
(4) $\mathrm{Spec}^{\mathrm{f}}(M)$ *is* T_1 *(Fréct space).*

Proposition 4.34. *Let* ${}_RM$ *be comultiplication.*

(a) ${}_RM$ *is a strongly top*$^{\mathrm{f}}$*-module; in particular,* ${}_RM$ *is a top*$^{\mathrm{f}}$*-module.*
(b) $\mathcal{S}(M) = \mathrm{Spec}^{\mathrm{f}}(M)$, i.e. *every first submodule of* M *is simple.*
(c) $\mathrm{Spec}^{\mathrm{f}}(M)$ *is discrete.*

Proof. Let ${}_RM$ be comultiplication.

(a) This follows directly from the fact that $\mathrm{Sub}_c(M) = \mathrm{Sub}(M)$, Equation (3) (see Remark 4.6).

(b) This follows from Lemma 4.12 (a) and the fact that all submodules of a comultiplication module are also comultiplication.

(c) This follows from Proposition 4.31 (a). ∎

Example 4.35. If R is a *left dual ring* [36], then ${}_RR$ is a strongly top$^{\mathrm{f}}$-module and $\mathrm{Spec}^{\mathrm{f}}({}_RR) = \mathrm{Min}({}_RR)$ the set of minimal left ideals of R.

Example 4.36. $\mathbb{Z}_{p^\infty}$ is a comultiplication $\mathbb{Z}$-module, whence a strongly top$^{\mathrm{f}}$-module. Any $\mathbb{Z}$-submodule of $\mathbb{Z}_{p^\infty}$ is of the form $\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})$ for some $n \in \mathbb{N}$ and so $\mathbb{Z}_{p^\infty} \notin \mathrm{Spec}^{\mathrm{f}}(\mathbb{Z}_{p^\infty})$ since $\mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty}) = 0 \neq \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}))$ for every $n \in \mathbb{N}$. Moreover, it is evident that $\mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}(\frac{1}{p^{n_1}} + \mathbb{Z})) \supsetneqq \mathrm{ann}_{\mathbb{Z}}(\mathbb{Z}(\frac{1}{p^{n_2}} + \mathbb{Z}))$, whence $\mathbb{Z}(\frac{1}{p^{n_2}} + \mathbb{Z}) \notin \mathrm{Spec}^{\mathrm{f}}(\mathbb{Z}_{p^\infty})$ if $n_1 \lneqq n_2$. Consequently, $\mathrm{Spec}^{\mathrm{f}}(\mathbb{Z}_{p^\infty}) = \{\mathbb{Z}(\frac{1}{p} + \mathbb{Z})\} = \mathcal{S}(\mathbb{Z}_{p^\infty})$. Clearly, $\tau^{\mathrm{f}}(\mathbb{Z}_{p^\infty}) = \{\emptyset, \{\mathbb{Z}(\frac{1}{p} + \mathbb{Z})\}\}$ is the trivial topology and is connected.

Proposition 4.37. *A top*$^{\mathrm{f}}$*-module* M *with essential socle is uniform if and only if* $\mathrm{Spec}^{\mathrm{f}}(M)$ *is ultraconnected.*

Proof. ($\Rightarrow$) Let ${}_RM$ be uniform. For any non-empty closed subsets $V(K_1), V(K_2) \subseteq \mathrm{Spec}^{\mathrm{f}}(M)$, we have indeed $H_1 \neq 0 \neq H_2$ whence $V(H_1) \cap V(H_2) = V(H_1 \cap H_2) \neq \emptyset$, since $H_1 \cap H_2 \neq 0$ by uniformity of ${}_RM$ and so it contains by assumption some simple R-submodule which is indeed first in M.

($\Leftarrow$) Assume that $\mathrm{Spec}^{\mathrm{f}}(M)$ is ultraconnected. Let H_1 and H_2 be non-zero R-submodules of ${}_RM$. It follows that $V(H_1) \neq \emptyset \neq V(H_2)$. By assumption, $V(H_1 \cap H_2) = V(H_1) \cap V(H_2) \neq \emptyset$, hence $H_1 \cap H_2 \neq 0$. ∎

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