ON FREE TIME OPTIMAL CONTROL PROBLEMS

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Abstract: Here we consider free time optimal control problems involving mixed state-control constraints. For such problems we derive necessary conditions of optimality in the form of a Maximum Principle appealing to recent results on mixed constraints where necessary conditions in the form of a nonsmooth maximum principles are derived under minimal regularity assumptions. Applications to optimal control problems involving differential algebraic equations are also considered.

Keywords: Optimal Control, free time, mixed constraints.

1. INTRODUCTION

We derive necessary conditions for free end time problems involving mixed state-control constraints. In contrast to previous results the regularity assumptions on the mixed constraints imposed are in some sense minimal.

The free time optimal problem we focus on problems of the form:

$$(P)\quad \begin{cases} \text{Minimize } l(a,x(a),b,x(b)) \\ \text{subject to} \\ \dot{x}(t)=f(t,x(t),u(t)) \quad \text{a.e. } t\in[a,b] \\ 0=b(t,x(t),u(t)) \quad \text{a.e. } t\in[a,b] \\ 0\geqslant g(t,x(t),u(t)) \quad \text{a.e. } t\in[a,b] \\ u(t)\in U \quad \text{a.e. } t\in[a,b] \\ (a,x(a),b,x(b))\in E. \end{cases}$$

Here the function $f\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ describe the system dynamics, $b\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^{m_b}$ and $g\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^{m_g}$ describe the equality and inequality mixed constraints, the set $U\subset\mathbb{R}^k$ is the pointwise control set while the given closed set $E\subset\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$ and function $l\colon\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ specify the endpoint constraints and costs.

A special feature of the problem above is the fact that the endpoints of the time interval $[a,b]$ are now choice variables: an *admissible process* for (P) is now a triple $([a,b],x,u)$ in which $[a,b]$ is an interval, x, the state function, is an absolutely continuous function ($x\in W^{1,1}([a,b];\mathbb{R}^n)$) and u, the control function, is a measurable function $u:[a,b]\to\mathbb{R}^k$, satisfying the constraints of the problem.

Necessary conditions of optimality for free time problems have been the focus of attention since the work of Pontryagin and his associates [25] and their applicability has been extended by a number of authors (see, for example, [1, 3, 18, 24, 30]). On earlier work on such problems some continuity of the data with respect to the time variable is required. Necessary conditions for free time problems with data merely measurable in the time variable were first derived in [5].

A simple and common technique used to derive necessary conditions when smoothness of the data with respect to the time variable is assumed is a well known transformation of the independent variable t. This transformation permits the association of the problem under study with a fixed time problem (P'). If necessary conditions are known for (P'), the required conditions for (P) are then obtained.

The aim of this paper is the derivation of necessary conditions for (P) with mixed constraints via the time transformation and under hypotheses that can be considered in some sense minimal. For reasons that will discussed later on we assume that the control set U does not depend on t. Under suitable hypotheses of the data we will show that derivation of necessary conditions is possible via application of necessary conditions of optimality recently developed in [12].

A note of caution is called for. Here we will assume that the data of (P) with respect to t is smooth instead of merely Lipschitz continuous as customary in some literature (see, for example, [29]). This may come as a surprise for some familiar with the main and most general result in [12], where only local Lipschtiz continuity with respect to the state variable is assumed. We choose to assume smooth time dependence in this paper since, for mixed constraints of the form

$$(x(t),u(t))\in S(t),$$

necessary conditions in [12] are given in terms of *normal cone* to the set $S(t)$. Even when S is defined via inequalities or/and equalities (when, as in our case, $S(t):=\{(x,u):\ b(t,x,u)=0,\ g(t,x,u)\leqslant 0,\ u\in U\}$) necessary conditions are expressed in terms of multipliers related to the functions defining S only when some differentiability of the data is assumed (see [12][Theorem 7.1]; also see section 3 below).

We point out that necessary conditions for (P) are obtained in [22] under full rank conditions on the mixed constraints. We improve on [22] since we assume merely Mangasarian- Fromowitz type conditions.

This paper is organized in the following way. In the next section we introduce some preliminaries. We then give a brief review of the main techniques associated with the above mentioned t transformation.

Before reporting in section 5 on some of the main results of [12] , amply used in our analysis in section 6, we present a short discussion on regularity assumption on the mixed constraints under which previous results have been proved.

Necessary conditions for (P) are stated in section 6 as Theorem 6.1 together with our main assumptions on the data. Also in this section we point out that Theorem 6.1 directly provides a maximum principle for standard optimal problem with pointwise control constraints of the form $u \in V(t)$ where the set V is defined via equalities, inequalities and autonomous set control constraints. A discussion on future research to extend our results to problems with nonsmooth data comes in the end of this section.

In section 7 we show that application of Theorem 6.1 permits the derivation for free time optimal control problems involving semi-explicit differential and algebraic equations (DAE). Notably, when the interval $[a, b]$ is fixed, we get a general and simple smooth maximum principle for optimal control problems with DAE's.

The proof of the Theorem 6.1 is presented in the following section. Conclusions and future directions of research are in the last section.

2. PRELIMINARIES

For g in $\mathbb{R}^m$, inequalities like $g \leqslant 0$ are interpreted componentwise. We set

$$R^m_+ := \{x \in \mathbb{R}^m : \ x_i \geqslant 0, \ i = 1, \dots, m\}.$$

Here and throughout, $|\cdot|$ represents the Euclidean norm or the induced matrix norm on $\mathbb{R}^{p\times q}$. The *Euclidean distance function* with respect to a given set $A \subset \mathbb{R}^m$ is

$$d_A \colon \mathbb{R}^k \to \mathbb{R}, \qquad y \mapsto d_A(y) = \inf \{|y - x| \ : \ x \in A\}.$$

A function $h\colon [a, b] \to \mathbb{R}^p$ lies in $W^{1,1}([a, b]; \mathbb{R}^p)$ if and only if it is absolutely continuous; in $L^1([a, b]; \mathbb{R}^p)$ iff it is integrable; and in $L^\infty([a, b]; \mathbb{R}^p)$ iff it is essentially bounded. The norm of $L^\infty([a, b]; \mathbb{R}^p)$ is $\|\cdot\|_\infty$.

We make use of standard concepts from nonsmooth analysis. Let $A \subset \mathbb{R}^k$ be a closed set with $\bar{x} \in A$. The *proximal normal cone to* A *at* $\bar{x}$ is denoted by $N^P_A(\bar{x})$, while $N^L_A(\bar{x})$ denotes the *limiting normal cone* and $N^C_A(\bar{x})$ is the *Clarke normal cone*.

Given a lower semicontinuous function $f\colon \mathbb{R}^k \to \mathbb{R} \cup \{+\infty\}$ and a point $\bar{x} \in \mathbb{R}^k$ where $f(\bar{x}) < +\infty$, $\partial^L f(\bar{x})$ denotes the *limiting subdifferential* of f at $\bar{x}$. When the function f is Lipschitz continuous near x, the convex hull of the limiting subdifferential, co $\partial^L f(x)$, coincides with the *(Clarke) subdifferential* $\partial^C f(\bar{x})$. For details on such nonsmooth analysis concepts, see [3, 6, 23, 26, 29].

3. THE TIME TRANSFORMATION

Here we briefly review the well known time transformation. To keep the exposition simple we focus attention to a special case of problem (P), denoted here as (S), where the mixed constraints $b(t, x, u) = 0$ and $g(t, x, u) \leqslant 0$ are absent:

$$(S) \quad \begin{cases} \text{Minimize } l(a), x(a)), b, x(b)) \\ \text{subject to} \\ \ \dot{x}(t) = \alpha(s) f(t, x(t), u(t)) \ \text{ a.e.} \\ \ u(t) \in U \ \text{ a.e.} \\ \ (\tau(a), x(a)), b, x(b)) \in E. \end{cases}$$

Suppose that $([\bar{a}, \bar{b}], \bar{x}, \bar{u})$ is a local minimum for (S) (defined in some sense). Assume for the time being that $U(t) = U$ and consider the *fixed time problem*:

$$(S') \quad \begin{cases} \text{Minimize } l(\tau(\bar{a}), y(\bar{a})), \bar{b}, y(\bar{b})) \\ \text{subject to} \\ \ \dot{\tau}(s) = \alpha(s) \text{ a.e.} \\ \ \dot{y}(s) = \alpha(s) f(\tau(s), y(s), v(s)) \ \text{ a.e.} \\ \ (\alpha(s), v(s)) \in [1/2, 3/2] \times U \ \text{ a.e.} \\ \ (\tau(\bar{a}), y(\bar{a})), \bar{b}, y(\bar{b})) \in E. \end{cases}$$

Problem (S') is a fixed time problem where the state variable is now (τ, y) while the control is (α, v). It is not difficult to relate the admissible processes of (S') with those of (P). Take (τ, y, α, v) to be any admissible process for (S'). Set $a = \tau(\bar{a})$ and $b = \tau(\bar{b})$. Since $\alpha > 0$, we have $a < b$. Consider the transformation $\psi : [\bar{a}, \bar{b}] \to [a, b]$ defined as

$$\psi(s) := \tau(\bar{a}) + \int_{\bar{a}}^{s} \alpha(\sigma) d\sigma.$$

One can easily see that $([a, b], x, u)$, with $x : [a, b] \to \mathbb{R}^n$ and $u : [a, b] \to \mathbb{R}^k$ defined as

$$x(t) := y \circ \psi^{-1}(t), \quad u(t) := v \circ \psi^{-1}(t),$$

is an admissible process for (S).

The association between (S) and (S') plays an important role in optimal control. Under certain hypotheses on (S) the data of $(S')'$ may satisfy conditions allowing the application of known necessary conditions, which in turn can be rewritten in terms of data of (S). This association can only work when the data of (S) satisfy some generalized differentiability properties and, moreover, necessary conditions are available for (S').

The last issue is of foremost importance when the control set U is time dependent, i.e., is $U(t)$. The change of the independent variable t into a state variable leads the point wise control constraint

$$u(t) \in U(t)$$

into the mixed state-control constraint of the form

$$v(s) \in V(\tau(s)).$$

It is then clear that necessary conditions for problems involving mixed constraints play a crucial role in the derivation of necessary conditions for free time optimal control problems in the form of (S). Our interest in free time problem was aroused by the role that mixed constrained problems play in this respect.

4. ON REGULAR MIXED CONSTRAINED PROBLEMS

Before engaging on the latest necessary conditions for mixed constrained problem we give an brief overview of some regularity assumptions under which necessary conditions have previously been derived for fixed time problems. For the sake of simplicity we choose to dwell on two types of regularity assumptions. Obviously we leave out some assumptions of great interest in the literature; in this respect we refer the reader to the bibliography.

In what follows in this section, and when referring to problem (P), we assume $U = \mathbb{R}^k$, i.e., problems with mixed constraints and without pointwise set control constraints.

In most of the literature derivation of necessary conditions for fixed time optimal control problems with mixed constraints has been done under *regular* conditions, usually invoking the full rank of matrix

$$\begin{bmatrix} D_u b(t,x,u) \\ D_u g(t,x,u) \end{bmatrix}.$$

Loosely speaking those are conditions that allow the the association of (P) (seen as a fixed time problem) with smooth data, with an auxiliary problem, (P_{aux_1}) where the inequality constraint $g(t,x(t),u(t)) \leqslant 0$ is replaced by equality constraints by considering

$$g\big(t,x(t),u(t)\big) + \varpi^2(t) = 0, \tag{1}$$

ϖ being a new control (this transformation is known as Valentine's transform). Implicit function theorems could then be used so that problem (P_{aux_1}) would be further associated with the following standard optimal control problem

$$\textbf{(S)} \qquad \begin{cases} \text{Minimize } l(x(0),x(1)) \\ \text{st} \\ \quad \dot{x}(t) = \tilde{f}(t,x(t),w(t)) \quad \text{a.e.} \\ \quad (x(0),x(1)) \in E \end{cases}$$

where w comprises now the control u and the new control variable. This approach can be applied, for example, when, for some $K > 0$ and for all admissible processes (x,u) , we have

$\det \Gamma(t)\Gamma^T(t) > K$ where

$$\Gamma(t) = \begin{bmatrix} D_u b(t,x,u) & 0 \\ \nabla_u g(t,x,u) & \text{diag}\{\sqrt{-g_i(t,x,u)}\}_{i=1}^m \end{bmatrix}.$$

or, alternatively, under stronger smoothness assumptions, on

$$\det \begin{bmatrix} D_u b(t,x,u) \\ D_u g(t,x,u) \end{bmatrix} \begin{bmatrix} D_u b(t,x,u) \\ D_u g(t,x,u) \end{bmatrix}^T > K$$

Usually such conditions apply when *weak local minimums* are considered and produce "weak forms" of the maximum principle.

Observe that the problem we consider above has no pointwise set control constraints. In fact, and as mentioned before, we assume that $U(t) = \mathbb{R}^k$. In the literature, the approaches we describe can also be adapted to deal with situations when the original control u has two components, i.e., $u = (u_1, u_2)$, where $u_1 \in \mathbb{R}^m$ and $u_2 \in U_2 \subset \mathbb{R}^{k-m}$. Besides, one can find necessary conditions for mixed constrained problems under assumptions weaker than the full rank conditions where the use of implicit function theorem may not be so clearly stated (see for example [17]).

Another alternative condition involves the linear positive independence of the vectors $\nabla_u g_i(t,x,u)$, $i = 1,\ldots,m$ (see [17]). A generalization of the linear positive independence assumption on g in [17] is extended to nonsmooth problems with only inequalities in [16] where the following assumption is assumed:

(R2) There exist a constant $K_1 > 0$, a function $h \in L^\infty([0,1];\mathbb{R}^k)$, with $|h(t)| = 1$ a.e., such that for almost every $t \in [0,1]$, all $(x,u) \in T_{\delta'}(t)$ (for some $\delta' > 0$) and all $(\gamma^i,\psi^i) \in$ co $\partial_{x,u} g_i(t,x,u)$, $i = 1,\ldots,m$ we have

$$\psi^i \cdot h(t) > K_1, \quad i = 1,\ldots,m.$$

Under this hypothesis a weak form of the nonsmooth maximum principle (which applies to weak local minimums) for (P) is obtained under some convexity assumptions.

This small digression into the world of regular mixed constrained optimal control problems would not be complete without a reference to the latest developments in the area obtained in [12], which although not yet published, have been announced. We do that in the following section.

5. AUXILIARY RESULTS

In this section we present simplified versions of two of the main results in [12]. We concentrate on those that will be of importance in the forthcoming developments. Those are necessary conditions of optimality for mixed constrained optimal control problems under minimal regularity assumptions.

One of the most innovative aspect of the work in [12] is the fact that *local minimum of radius R* is considered. Here, and for the sake of simplicity we focus instead on the well known notion of on *strong local minimum*. We also do some simplifications pertained to the main hypotheses; while here some parameters, including the Lipschitz parameters, are constants, in [12] they are assumed to be measurable functions.

Take a fixed interval $[a,b]$ and a set S of $[a,b]\times\mathbb{R}^n\times\mathbb{R}^k$. Define

$$S(t) := \{(x,u) : (t,x,u) \in S\}.$$

Assume for the time being that $E \subset \mathbb{R}^n \times \mathbb{R}^n$ and $l : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. Consider the following problem:

$$(C) \begin{cases} \text{Minimize } l(x(a), x(b)) \\ \text{subject to} \\ \quad \dot{x}(t) = f(t, x(t), u(t)) \text{ a.e. } t \in [a, b] \\ \quad (x(t), u(t)) \in S(t) \text{ a.e. } t \in [a, b] \\ \quad (x(a), x(b)) \in E. \end{cases}$$

This problem involves measurable control functions u and absolutely continuous function x. The pair (x, u) is called an admissible process for (C) if it satisfies the constraints of the problem with finite cost.

We say that the process $(\bar{x}, \bar{u})$ is a *strong local minimum* if it minimizes the cost over admissible processes (x, u) such that

$$|x(t) - \bar{x}(t)| \leqslant \epsilon, \quad \text{for all } t \in [a, b].$$

Define

$$S_*^\epsilon(t) = \{(x, u) \in S(t) : \ |x - \bar{x}(t)| \leqslant \epsilon\}.$$

We consider the *basic hypotheses* on the problem data throughout. They are the following: f is $\mathcal{L} \times \mathcal{B}^{n+k}$, S is $\mathcal{L} \times \mathcal{B}$, E is closed and l is locally Lipschitz.

In generic terms we assume that a function $\phi(t, x, u)$ satisfies $[L_*^\epsilon]$ if:

$[\mathbf{L}_*^\epsilon]$ There exist constants k_x^ϕ and $k_u^f\phi$ such that for almost every $t \in [a, b]$ and every $(x_i, u_i) \in S_*^\epsilon(t)$ $(i = 1, 2)$ we have

$$|\phi(t, x_1, u_1) - \phi(t, x_2, u_2)| \leqslant k_x^\phi |x_1 - x_2| + k_u^\phi |u_1 - u_2|.$$

If this assumption is imposed on f, then the Lipschitz constants are denoted by k_x^f and k_u^f.

As for $S(t)$ we consider the following **bounded slope condition**:

$[\mathbf{BS}_*^\epsilon]$ There exists a constant k_S such that for almost every $t \in [a, b]$ the following condition holds

$$(x, u) \in S_*^\epsilon(t), \ (\alpha, \beta) \in N_{S(t)}^P(x, u) \Longrightarrow |\alpha| \leqslant K_s |\beta|.$$

Necessary conditions of optimality for (C) are given by the following theorem:

Theorem 5.1 (see Theorem 7.1 in [12]) *Let $(\bar{x}, \bar{u})$ be a strong local minimum for problem (C). Assume that the set $S_*^\epsilon(t)$ is closed, the basic hypotheses and that f satisfies $[L_*^\epsilon]$ and that $[BS_*^\epsilon]$ holds.*

Then there exist an absolutely continuous function $p \colon [a, b] \to \mathbb{R}^n$, and a scalar $\lambda_0 \geqslant 0$ such that

$$(p(t), \lambda_0) \neq 0 \ \forall t \in [a, b], \tag{2}$$

$$(-\dot{p}(t), 0) \in \partial_{x,u}^C \{\langle p, f \rangle\} (t, \bar{x}(t), \bar{u}(t)) - N_{S(t)}^C(\bar{x}(t), \bar{u}(t)) \ \text{ a.e.} \tag{3}$$

$$(\bar{x}(t), u) \in S(t) \Longrightarrow \langle p(t), f(t, \bar{x}(t), u) \rangle \leqslant \langle p(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle \ \text{ a.e.} \tag{4}$$

$$(p(a), -p(b)) \in N_E^L(\bar{x}(a), \bar{x}(b)) + \lambda_0 \partial l(\bar{x}(a), \bar{x}(b)). \tag{5}$$

The main features of the above theorem are (3), the *Euler adjoint inclusion*, and (4), the *Weierstrass condition*. A particular and of important feature of (3) is that it involves the *joint* Clarke subdifferential with respect to (x, u). This inclusion is in the vein of that of the earlier work [13] where a "weak" form of a nonsmooth maximum principle (without the Weierstrass condition) is obtained for standard optimal control problems (see problem (S) in section 3). Notably, and as proved in [13], nonsmooth maximum principle involving the Euler adjoint inclusion provide sufficient condition for convex linear problems. We observe that (3) can only hold when the data with respect to the control variable u is assumed to be locally lipschitz. For discussions on the Euler adjoint inclusion we refer the reader to [11, 13, 15].

Let us now specify the above theorem for the special case in which

$$\begin{aligned} S := \{(t, x, u) : \ t \in [a, b], u \in U, \\ g(t, x, u) \leqslant 0, \ b(t, x, u) = 0\}. \end{aligned} \tag{6}$$

Here g and b take values in $\mathbb{R}^{m_g}$ and $\mathbb{R}^{m_b}$ and $U \subset \mathbb{R}^k$. Mixed constraints in this form have dominated the literature on mixed constraints. [1]

In this context we take U to be compact and we assume the basic hypotheses and that the functions f, g and h satisfy $[L_*^\epsilon]$.

We also consider the nonsmooth extension of the classical Mangasarian-Fromowitz condition on the mixed constraints:

$[\mathbf{M}_*^\epsilon]$ There exists a constant M such that for almost every $t \in [a, b]$ the following condition holds

$$\begin{gathered} (x, u) \in S_*^\epsilon(t), \ \lambda \in \mathbb{R}^{m_b}, \ \mu \in N_U^L(u), \\ \gamma \in \mathbb{R}_+^{m_g} : \ \langle \gamma, g(t, x, u) \rangle = 0 \\ (\alpha, \beta - \mu) \in \partial^L \langle (\lambda, \gamma), (b(t, x, u), g(t, x, u)) \rangle \\ \Longrightarrow |(\lambda, \gamma)| \leqslant M |\beta|. \end{gathered}$$

As shown in [12], if $[M_*^\epsilon]$ holds, then $[BS_*^\epsilon]$ holds for S as defined in (6). Under these assumptions Theorem 5.1 holds.

We now turn to the case when smooth assumptions on the data of our problem are imposed. As we will see in this case the Euler adjoint inclusion (3) can be expressed in terms of multipliers associated with the equality and inequality constraints. This case will play a crucial role in the forthcoming analysis.

Corollary 5.2 *Let $(\bar{x}, \bar{u})$ be a strong local minimum for problem (C). Assume that the set $S_*^\epsilon(t)$ is closed, all the functions are C^1 and $\bar{u}$ is bounded. Suppose also that for every t, at each $(x, u) \in S_*^\epsilon(t)$ the following condition, denoted by $[\mathbf{MF}_*^\epsilon]$, holds*

$$\begin{gathered} \lambda \in \mathbb{R}^{m_b}, \gamma \in \mathbb{R}_+^{m_g}, \langle \gamma, g(t, x, u) \rangle = 0, \\ D_u \{\langle (\lambda, \gamma), (b, g) \rangle\} (t, x, u) \in N_U^L(u) \Longrightarrow \\ \gamma = 0, \lambda = 0, \end{gathered}$$

[1] In [12] the case where the control set U depends on t is considered.

Then there exist an absolutely continuous function $p\colon [a,b] \to \mathbb{R}^n$, a scalar $\lambda_0 \geqslant 0$ and measurable functions

$$\lambda : [a,b] \to \mathbb{R}^{m_b},\ \gamma : [a,b] \to \mathbb{R}^{m_g}_+,\ \mu : [a,b] \to \mathbb{R}^k$$

with

$$\langle \gamma(t), g(t, \bar{x}(t), \bar{u}(t)) \rangle = 0,\ \mu \in N^C_U(\bar{u}(t)) \quad \textit{a.e.}$$

and

$$|(\lambda(t), \gamma(t))| \leqslant M|p(t)|k^f_u \quad \textit{a.e.}$$

such that (2), (4), (5) hold and

$$(-\dot{p}(t), \mu(t)) = D_{x,u}\{\langle (p, \lambda, \gamma), (f, -b, -g) \rangle\}(t, \bar{x}(t), \bar{u}(t)) \textit{ a.e.} \tag{7}$$

A novel feature of this corollary is the weak regularity hypothesis $[MF^\epsilon_*]$ imposed on the mixed constraints. Indeed $[MF^\epsilon_*]$ is weaker than well known *positive linear independent* conditions on the gradients of the mixed constraints, a condition, in turn, weaker than customary full rank conditions (see [19]).

6. PROBLEMS WITH MIXED CONSTRAINTS

Recall that the problem under consideration is

$$(P) \quad \begin{cases} \text{Minimize } l(a, x(a), b, x(b)) \\ \text{subject to} \\ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a,b] \\ b(t, x(t), u(t)) = 0 \quad \text{a.e. } t \in [a,b] \\ g(t, x(t), u(t)) \leqslant 0 \quad \text{a.e. } t \in [a,b] \\ u(t) \in U \quad \text{a.e. } t \in [a,b] \\ (a, x(a), b, x(b)) \in E. \end{cases}$$

If the interval $[a,b]$ were fixed, this problem could be viewed as the special case of (C) of the previous section in which

$$S := \{(t,x,u) : t \in [a,b],\ u \in U,\ b(t,x,u) = 0,\ g(t,x,u) \leqslant 0\}.$$

However, since the interval $[a,b]$ is a variable of the problem we define the set S instead as

$$S := \{(t,x,u) : t \in \mathbb{R},\ u \in U,\ b(t,x,u) = 0,\ g(t,x,u) \leqslant 0\}$$

Now the variable t is free to take any value in the real line.

Next we clarify first what we mean by a *strong local minimum* for such problem when the underlying time interval $[a,b]$ is a choice variable.

We identify a function $x : [a,b] \to \mathbb{R}^n$ with its extension x_e to all $]-\infty, +\infty[$ by constant extrapolation of end values to the left and right: if, for example, $\widetilde{x} \in \mathbb{R}^n$ and $t < a$, then $|\widetilde{x} - x_e(t)| := |\widetilde{x} - x(a)|$.

Accordingly, given two absolutely continuous functions $x : [a,b] \to \mathbb{R}^n$ and $x' : [a',b'] \to \mathbb{R}^n$ we define

$$\| x - x' \|_{L^\infty} := \| x_e - x'_e \|_{L^\infty},$$

where x_e and x'_e are the extensions.

An admissible *process* for (P) is taken to be a triple $([a,b], x, u)$ in which $[a,b]$ is an interval, $x \in W^{1,1}([a,b]; \mathbb{R}^n)$ and $u : [a,b] \to \mathbb{R}^k$ is a measurable function satisfying the constraints of the problem. An admissible process $([\bar{a},\bar{b}], \bar{x}, \bar{u})$ is a **strong local minimum** of (P) if there exists a $\varepsilon > 0$ such that

$$l(a, x(a), b, x(b)) \leqslant l(\bar{a}, \bar{x}(\bar{a}), \bar{b}, \bar{x}(\bar{b}))$$

over all admissible processes $([a,b], x, u)$ of (P) satisfying

$$|a - \bar{a}| \leqslant \varepsilon,\ |b - \bar{b}| \leqslant \varepsilon,\ \| x - x' \|_{L^\infty} \leqslant \varepsilon.$$

Assume the following assumptions that make reference to a process $([\bar{a},\bar{b}], \bar{x}, \bar{u})$ and a parameter $\varepsilon > 0$:

(H1) The function l is locally Lipschitz.

(H2) The set E is closed and $U \subset \mathbb{R}^k$ is a compact set.

(H3) The functions f, b and g are C^1 functions.

(H4) For all $(\tau, y) \in \mathbb{R} \times \mathbb{R}^n$ such that $|\tau - t| \leqslant \varepsilon$ and $|y - \bar{x}(t)| \leqslant \varepsilon$ for all $t \in [\bar{a},\bar{b}]$ and all $v \in U$ the following condition hold

$$\begin{gathered} \lambda \in \mathbb{R}^{m_b}, \gamma \in \mathbb{R}^{m_g}_+, \langle \gamma, g(\tau, y, v) \rangle = 0 \\ D_v\{\langle (\lambda, \gamma), (b, g) \rangle\}(\tau, y, v) \in N^L_U(v) \\ \Longrightarrow \gamma = 0, \lambda = 0 \end{gathered}$$

We are now in position to state our main results.

Theorem 6.1 *Let $([\bar{a},\bar{b}], \bar{x}, \bar{u})$ be a strong local minimum for problem (P). Assume that hypotheses (H1)–(H4) hold together with our basic assumptions. Then there exist absolutely continuous functions $p\colon [\bar{a},\bar{b}] \to \mathbb{R}^n$, $r\colon [\bar{a},\bar{b}] \to \mathbb{R}$, a scalar $\lambda_0 \geqslant 0$ and measurable functions*

$$\lambda : [\bar{a},\bar{b}] \to \mathbb{R}^{m_b},\ \gamma : [\bar{a},\bar{b}] \to \mathbb{R}^{m_g}_+,\ \mu : [\bar{a},\bar{b}] \to \mathbb{R}^k$$

with

$$\langle \gamma(t), g(t, \bar{u}(t)) \rangle = 0,\ \mu \in N^C_U(\bar{u}(t)) \ \textit{ a.e. } t \in [\bar{a},\bar{b}],$$

and

$$|(\lambda(t), \gamma(t))| \leqslant M|(p(t), r(t))|k^f_u \ \textit{ a.e. } t \in [\bar{a},\bar{b}]$$

such that

(i) $(p(t), \lambda_0) \neq 0\ \forall t \in [\bar{a},\bar{b}]$,

(ii) $(\dot{r}(t), -\dot{p}(t), \mu(t)) = D_{t,x,u}\{\langle (p, \lambda, \gamma), (f, -b, -g) \rangle\}(t, \bar{x}(t), \bar{u}(t)),$

(iii) $\langle p(t), f(t, \bar{x}(t), u)\rangle \leqslant \langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle$
for all $u \in U,\ g(t, \bar{x}(t), u) \leqslant 0,\ b(t, \bar{x}(t), u) = 0,$

(iv) $r(t) = \langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle,$

(v) $(-r(\bar{a}), p(\bar{a}), r(\bar{b}), -p(\bar{b})) \in N_E^L(\bar{a}, \bar{x}(\bar{a}), \bar{b}, \bar{x}(\bar{b})) + \lambda_0 \partial^L l(\bar{a}, \bar{x}(\bar{a}), \bar{b}, \bar{x}(\bar{b})).$

where the inclusions and equalities in (ii)–(iv) hold for almost every $t \in [\bar{a}, \bar{b}]$.

The proof of this theorem is presented in the next section. It relies on Corollary 5.2 and on the time transformation mentioned in the Introduction.

As mentioned before necessary conditions for (P) in the same vein are obtained in [22]. The novel aspect of our result is that Theorem 6.1 holds under (H4), a weaker condition than full rank conditions imposed in [22].

6.1. Problems with Pointwise Control Constraints

When both functions b and g are independent of x, Theorem 6.1 provides necessary conditions for problems with *pointwise control constraints*. This is a particular situation of of nonautonomous control constraints $u(t) \in V(t)$ where the set V is

$$V(t) = \left\{u \in \mathbb{R}^k : u \in U,\ b(t, u) = 0,\ g(t, u) \leqslant 0\right\}.$$

We summarize our findings in the for of the following corollary:

Corollary 6.2 *Let* $([\bar{a}, \bar{b}], \bar{x}, \bar{u})$ *be a strong local minimum for problem*

$$(\widetilde{P}) \quad \begin{cases} \text{Minimize } l(a, x(a), b, x(b)) \\ \text{subject to} \\ \ \dot{x}(t) = f(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \ b(t, u(t)) = 0 \quad \text{a.e. } t \in [a, b] \\ \ g(t, u(t)) \leqslant 0 \quad \text{a.e. } t \in [a, b] \\ \ u(t) \in U \quad \text{a.e. } t \in [a, b] \\ \ (a, x(a), b, x(b)) \in E. \end{cases}$$

Assume that hypotheses (H1)–(H4) hold. Then there exist absolutely continuous functions $p\colon [\bar{a}, \bar{b}] \to \mathbb{R}^n$, $r\colon [\bar{a}, \bar{b}] \to \mathbb{R}$, *a scalar* $\lambda_0 \geqslant 0$ *and measurable functions*

$$\lambda : [\bar{a}, \bar{b}] \to \mathbb{R}^{m_b},\ \gamma : [\bar{a}, \bar{b}] \to \mathbb{R}_+^{m_g},\ \mu : [\bar{a}, \bar{b}] \to \mathbb{R}^k$$

with

$$\langle \gamma(t), g(t, \bar{u}(t))\rangle = 0,\ \mu \in N_U^C(\bar{u}(t)) \ \text{ a.e. } t \in [\bar{a}, \bar{b}],$$

and

$$|(\lambda(t), \gamma(t))| \leqslant M|(p(t), r(t))| k_u^f \ \text{ a.e. } t \in [\bar{a}, \bar{b}]$$

such that (i), (iv), (v) of Theorem 6.1 hold together with

(ii) $(\dot{r}(t), -\dot{p}(t), \mu(t)) = D_{t,x,u}\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - D_{t,x,u}\langle \lambda, b(t, \bar{u}(t))\rangle - D_{t,x,u}\langle \gamma), g(t, \bar{u}(t))\rangle,$

(iii) $\langle p(t), f(t, \bar{x}(t), u)\rangle \leqslant \langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle$
for all $u \in U,\ g(t, u) \leqslant 0,\ b(t, u) = 0,$

where the inclusions and equalities in (ii)–(iv) hold for almost every $t \in [\bar{a}, \bar{b}]$.

6.2. On nonsmooth Problems

It is a simple matter to see that a maximum principle in the same vain of Theorem 6.1 can be obtained for nonsmooth problem (P) when H3 above is replaced by $[L_*^\epsilon]$ and H4 is replaced by its nonsmooth counter part $[BS_*^\epsilon]$. However, and taking into account Theorem 5.1, the Euler adjoint inclusion will be stated in terms of the normal cone to the set S and not in terms of multipliers directly associated with the functions describing the equality and inequality mixed constraints.

7. PROBLEMS WITH DIFFERENTIAL ALGEBRAIC EQUATIONS

Dynamic models which take the form of a coupled set of differential and algebraic equations "DAEs" are widespread in many areas of engineering, in particular in process systems engineering. Although in theoretic terms one can expect to reduce such models to standard differential equation models by elimination of variables, in which case separate treatment of DAE systems would be superfluous, this is often not an option for a number of reasons; the lack of differentiability properties to do , it may be numerically inefficient to do so, etc. It is then important to look for costate equations which themselves come in the form of a DAE system, with a view to employing DAE solvers.

Free time optimal control problems involving differential algebraic equations are now the focus of our attention. We concentrate our analysis on problems in the form:

$$(D) \quad \begin{cases} \text{Minimize } l(a, x(a), b, x(b)) \\ \text{subject to} \\ \ \dot{x}(t) = f(t, x(t), z(t), u(t)) \quad \text{a.e. } t \in [a, b] \\ \ b(t, x(t), z(t), u(t)) = 0 \quad \text{a.e. } t \in [a, b] \\ \ u(t) \in U \quad \text{a.e. } t \in [a, b] \\ \ (a, x(a), b, x(b)) \in E. \end{cases}$$

Here we have $f\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^n$ and $b\colon \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$. We call attention to the fact that b takes values in $\mathbb{R}^m$ while $z \in \mathbb{R}^m$. This is an important characteristic of the type of problems we will consider here.

For applications it is customary and quite useful to divide the large class of differential algebraic systems into subclasses. In this respect the notion of *differential index* is of interest (see, for example, [2]). Roughly speaking one says that systems involving differential algebraic of the form

$$\begin{cases} \dot{x}(t) &= f(t, x(t), z(t), u(t)) \\ 0 &= b(t, x(t), z(t), u(t)) \end{cases}$$

are of *index one* if the following condition holds

$$\det D_z g(t, x, z, u) \neq 0. \tag{8}$$

There is a substantial literature on optimality conditions for optimal control problems with differential algebraic equations in the linear quadratic case and, in more general cases, on numerical methods. For fixed time problems of necessary conditions are derived in [14]. There problems with nonsmooth dynamics but some smooth assumptions on the function b describing the algebraic equation are considered. The main result in [14] applies when the *index one* condition is assumed along the optimal solution. In the smooth case, necessary conditions for index one, index two and index three optimal control problems are derived in [28]. An alternative approach can be found, for autonomous problems, in [20]. In a more general setting see also [12].

Problem (D) can be rewritten in various ways. When, for example, endpoint constraints to z are present we can view the state as comprising (x, z). In such situations it may be of interest to reformulate the dynamics and the algebraic equation as

$$h(t, w, u, \dot{w}) = 0$$

where $w = (x, z)$ and

$$h(t, w, u, v) = \begin{bmatrix} v - f(t, w, u) \\ b(t, w, u) \end{bmatrix}.$$

But there is cases where z can be treated as a control component. Then (D) can be viewed as a special case of mixed constraints. This last approach is the one we consider in the remaining of this section. Further research on (D) is currently under research.

Let us consider (D). A process for this problem comprises $([a, b], x, z, u)$, where, as before, $[a, b]$ is an interval, x is absolutely continuous and u is a measurable function while the "fast" variable z is a bounded function. A strong local minimum for (D) is now $([\bar{a}, \bar{b}], \bar{x}, (\bar{z}, \bar{u}))$.

We shall impose the basic hypotheses as well as H1-H4 to (D) keeping in mind that our control variable is now (z, u). To clarify the exposition notice that H4 now reads:

(H4') For all $(\tau, y) \in \mathbb{R} \times \mathbb{R}^n$ such that $|\tau - t| \leqslant \varepsilon$ and $|y - \bar{x}(t)| \leqslant \varepsilon$ for all $t \in [\bar{a}, \bar{b}]$ and all $(z, u) \in \mathbb{R}^m \times U$ the following condition hold

$$\text{For all } \lambda \in \mathbb{R}^{m_b}: \quad \left\{ \begin{array}{l} D_z\langle \lambda, b(\tau, y, z, v)\rangle = 0 \\ D_v\langle \lambda, b(\tau, y, z, v)\rangle \in N_U^L(v) \end{array} \right. \Longrightarrow \lambda = 0$$

A strong local minimum for (D), now denoted as $([\bar{a}, \bar{b}], \bar{x}, (\bar{z}, \bar{u}))$, is defined as before. Since u takes values in a compact set, we can assert that $(\bar{z}, \bar{u})$ is bounded.

Corollary 7.1 *Let $([\bar{a}, \bar{b}], \bar{x}, \bar{z}, \bar{u})$ be a strong local minimum for problem (D). Assume that hypotheses (H1)–(H3) and (H4') hold. Then there exist absolutely continuous functions $p\colon [\bar{a}, \bar{b}] \to \mathbb{R}^n$, $r\colon [\bar{a}, \bar{b}] \to \mathbb{R}$, a scalar $\lambda_0 \geqslant 0$ and measurable functions*

$$\lambda : [\bar{a}, \bar{b}] \to \mathbb{R}^m, \ \mu : [\bar{a}, \bar{b}] \to \mathbb{R}^k$$

with

$$\mu \in N_U^C(\bar{u}(t)) \ \text{a.e. } t \in [\bar{a}, \bar{b}],$$

and

$$|\lambda(t)| \leqslant M|(p(t), r(t))|k_u^f \ \text{a.e. } t \in [\bar{a}, \bar{b}]$$

such that (i), (iv), (v) of Theorem 6.1 hold together with

(b) $(\dot{r}(t), -\dot{p}(t), \mu(t)) =$
$$D_{t,x,u}\langle p(t), f(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle - D_{t,x,u}\langle \lambda, b(t, \bar{z}(t), \bar{u}(t))\rangle,$$
$$0 = D_z\langle p(t), f(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle - D_z\langle \lambda, b(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle,$$

(c) $\langle p(t), f(t, \bar{x}(t), \bar{z}(t), u)\rangle \leqslant$
$$\langle p(t), f(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle$$
for all $u \in U, b(t, \bar{x}(t), \bar{z}(t), u) = 0,$

where the equalities and inequalities above hold for almost every $t \in [\bar{a}, \bar{b}]$.

An interesting feature of this corollary is that, when applied to smooth *fixed end time* problem it gives us a smooth maximum principle for problems with differential algebraic equation that subsumes Theorems 3.1 and Theorem 3.1 in [14].

Sketch of the Proof: Since $\bar{z}$ is bounded there exists a $K_z > 0$ such that $|z(t)| \leqslant K_z$ for all $t \in [\bar{a}, \bar{b}]$. Define a compact set $Z \subset \mathbb{R}^m$ as the closed ball centered in the origin with radius $2K_z$. It is a simple matter to show that $([\bar{a}, \bar{b}], \bar{x}, (\bar{z}, \bar{u}))$ is a local minimum for

$$(D') \quad \left\{ \begin{array}{l} \text{Minimize } l(a, x(a), b, x(b)) \\ \text{subject to} \\ \quad \dot{x}(t) = f(t, x(t), (z(t), u(t))) \text{ a.e. } t \in [a, b] \\ \quad b(t, x(t), (z(t), u(t))) = 0 \text{ a.e. } t \in [a, b] \\ \quad (z(t), u(t)) \in Z \times U \text{ a.e. } t \in [a, b] \\ \quad (a, x(a), b, x(b)) \in E. \end{array} \right.$$

Then (D') is a problem in the form of (P) with control variable (z, u). The conditions under which Theorem 6.1 are satisfied. Then conclusions (i)–(v) of Theorem 6.1 hold, where, taking into account that $N_Z^L(\bar{z}(t)) = \{0\}$, (ii) and (iii) are now

$$(\dot{r}(t), -\dot{p}(t), 0, \mu(t)) = \tag{9}$$
$$D_{t,x,z,u}\langle p(t),\ f(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle - D_{t,x,z,u}\langle \lambda(t),\ b(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle,$$

and

$$\langle p(t), f(t, \bar{x}(t), z, u)\rangle \leqslant \langle p(t), f(t, \bar{x}(t), \bar{z}(t), \bar{u}(t))\rangle, \tag{10}$$
$$\text{for all } (z, u) \in Z \times U,\ b(t, \bar{x}(t), z, u) = 0.$$

From (9) we get

$$(\dot r(t), -\dot p(t), \mu(t)) = \\ D_{t,x,u}\langle p(t),\ f(t,\bar x(t),\bar z(t),\bar u(t))\rangle - \\ \langle \lambda(t),\ b(t,\bar x(t),\bar z(t),\bar u(t))\rangle, \\ 0 = D_z\langle p(t),\ f(t,\bar x(t),\bar z(t),\bar u(t))\rangle - \\ \langle \lambda(t),\ b(t,\bar x(t),\bar z(t),\bar u(t))\rangle$$

This is (b) of the Corollary.

Now take $z = \bar z$ in (10). We get (c) of the Corollary. Turn again to the original (10). The proof is complete.

8. PROOF OF THEOREM 6.1

Choose $\varepsilon > 0$ such that $([\bar a, \bar a], \bar x, \bar u)$ is a strong local minimum for (P) and that (H1)–(H4) are satisfied

Consider the fixed time optimal control problem with mixed constraints:

$$(Q) \quad \begin{cases} \text{Minimize } l(\tau(\bar a), y(\bar a)), \bar b, y(\bar b)) \\ \text{subject to} \\ \ \dot\tau(s) = \alpha(s) \text{ a.e.} \\ \ \dot y(s) = \alpha f(\tau(s), y(s), v(s)) \text{ a.e.} \\ \ 0 = b(\tau(s), y(s), v(s)) \text{ a.e.} \\ \ 0 \geqslant g(\tau(s), y(s), v(s)) \text{ a.e.} \\ \ (\alpha(s), v(s)) \in [1/2, 3/2] \times U \text{ a.e.} \\ \ (\tau(\bar a), y(\bar a)), \bar b, y(\bar b)) \in E. \end{cases}$$

Here (τ, y) and $\alpha, v)$ are the state and control variables respectively.

To facilitate the analysis define, for (Q), the following sets

$$S := \\ \{(s,\tau,y,\alpha,v):\ s \in [\bar a, \bar b],\ (\alpha, v) \in [1/2, 3/2] \times U, \\ b(\tau,y,v) = 0,\ g(\tau,y,v) \leqslant 0\}, \\ S(s) := \{(\tau,y,\alpha,v):\ (s,\tau,y,\alpha,v) \in S\}, \\ S_*^\varepsilon(s) := \{(\tau,y,\alpha,v) \in S(s):\ |\tau(s) - s| \leqslant \varepsilon, \\ |y(s) - \bar x(s)| \leqslant \varepsilon\}.$$

We claim that

$$((\bar\tau(s) \equiv s, \bar x), (\bar u, \bar\alpha \equiv 1))$$

is a strong local minimum for (Q). To prove our claim consider any admissible process $((\tau, y), (\alpha, v))$ for (Q) such that

$$(\tau(s), y(s), \alpha(s), v(s)) \in S_*^\varepsilon(s) \quad \text{a.e. } s \in [\bar a, \bar b].$$

Take the transformation $\psi : [\bar a, \bar b] \to [a, b]$:

$$\psi(s) := \tau(\bar a) + \int_{\bar a}^{s} \alpha(\sigma) d\sigma.$$

This transformation is strictly increasing, Lipschitz continuous with Lipschitz inverse. We prove that $([a, b], x, u)$, where $a = \tau(\bar a)$, $b = \tau(\bar b)$ and $x : [a, b] \to \mathbb{R}^n$, $u : [a, b] \to \mathbb{R}^k$, defined as

$$x(t) := y \circ \psi^{-1}(t), \quad u(t) := v \circ \psi^{-1}(t),$$

satisfy the constraints of (P). We have

$$(a, x(a), b, x(b)) = (\tau(\bar a), y(\bar a), \tau(\bar b), y(\bar b)) \in E, \\ u(t) \in U \text{ a.e. } t \in [a, b], \\ b(t, x(t), u(t)) = 0, \quad g(t, x(t), u(t)) \leqslant 0 \text{ a.e. } t \in [a, b] \\ \dot x(t) = f(t, x(t), u(t)) \text{ a.e. } t \in [a, b] \\ |a - \bar a| \leqslant \varepsilon,\ |b - \bar b| \leqslant \varepsilon \\ |x(t) - \bar x_e(t)| \leqslant \varepsilon\ \forall\, t \in [a, b]$$

It follows that $([a, b], x, u)$ is an admissible process for (P). Taking into account the optimality of $([\bar a, \bar b], \bar x, \bar u)$ we deduce that

$$l(a, x(a), b, x(b)) \leqslant l(\bar a, \bar x(\bar a), \bar b, \bar x(\bar b)),$$

proving our claim.

It is an easy task to see that the hypotheses are satisfied for application of Corollary 5.2 to (Q) with reference to the process $(\bar\tau, \bar x, \bar u, \bar\alpha)$. We deduce the existence of absolutely continuous functions $p : [\bar a, \bar b] \to \mathbb{R}^n$, $\widetilde r : [\bar a, \bar b] \to \mathbb{R}$, measurable functions $\lambda : [\bar a, \bar b] \to \mathbb{R}^{m_b}$, $\gamma : [\bar a, \bar b] \to \mathbb{R}_+^{m_g}$ and $\mu : [\bar a, \bar b] \to \mathbb{R}^k$ such that $\langle \gamma(s), g(\tau(s), y(s), v(s))\rangle = 0$, $\mu(s) \in N_U^C(\bar u(s))$ for almost every $t \in [\bar a, \bar b]$, and a scalar $\lambda_0 \geqslant 0$ such that the following conditions hold

(a) $\quad (p(s), \widetilde r(s), \lambda_0) \neq 0\ \forall s \in [\bar a, \bar b]$

(b) $\quad (-\dot{\widetilde r}(s), -\dot p(s), 0, \mu(s)) \in$
$D_{(\tau,y,\alpha,v)}\{\widetilde r(s)\bar\alpha(s) +$
$\langle p(s), \bar\alpha(s) f(\bar\tau(s), \bar x(s), \bar u(s))\rangle$
$-\langle \lambda, b(\bar\tau(s), \bar x(s), \bar u(s))\rangle$
$-\langle \gamma, g(\bar\tau(s), \bar x(s), \bar u(s))\rangle\}$ a.e.

(c) $\quad \widetilde r(s) + \langle p(s), f(\bar\tau(s), \bar x(s), \bar u(s))\rangle \geqslant$
$\widetilde r(s)\alpha + \langle p(s), \alpha(s) f(\bar\tau(s), \bar x(s), u)\rangle$
for all $u \in U, g(\bar\tau(s), \bar x(s), u) \leqslant 0$,
$b((\bar\tau(s), \bar x(s), u) = 0$ and $\alpha \in [1/2, 3/2]$ a.e.

(d) $\quad (\widetilde r(\bar a), p(\bar a), -\widetilde r(\bar b), p(\bar b)) \in$
$N_E^L(\bar a, \bar x(\bar a), \bar b, \bar x(\bar b)) + \lambda_0 \partial^L l(\bar a, \bar x(\bar a), \bar b, \bar x(\bar b))$.

From (c) above we deduce that

$$\langle p(t),\ f(t, \bar x(t), \bar u(t))\rangle \geqslant \langle p(t), f(t, \bar x(t), u)\rangle$$

for all $u \in U$, $g(t, \bar x(t), u) \leqslant 0$ and $b(t, \bar x(t), u) = 0$ almost everywhere. This is (iii) of Theorem 6.1. Also from (c) we deduce that

$$\widetilde r(t) = -\langle p(t),\ f(t, \bar x(t), \bar u(t))\rangle. \tag{11}$$

Take $r := -\widetilde r$. Then (11) is (iv) of the Theorem. Introducing r in (a), (b) and (d) above we deduce that all the conclusions of the Theorem hold.

9. CONCLUSIONS AND FUTURE WORK

We have derived necessary conditions for smooth free time optimal control constraints with mixed constraints given in terms of equalities, inequalities and point wise control constraints. A well known time transformation as well as recently developed results in [12] were essential in our analysis. From such conditions it is a simple matter to obtain necessary conditions for standard optimal control with pointwise control constraints of the form

$$u(t) \in V(t)$$

in situations when

$$V(t) = \left\{u \in \mathbb{R}^k : u \in U,\ b(t,u) = 0,\ g(t,u) \leqslant 0\right\}.$$

Those are captured in Corollary 6.2.

We also applied Theorem 6.1 to derive necessary conditions for free time optimal control problems involving semi-explicit differential algebraic equations stated as Theorem 7.1.

To be best of our knowledge Theorem 6.1 and Corollary 6.2 are valid under minimal regularity assumptions on the mixed constraints. This work also seems to be the first with the complete statement of first order necessary conditions for free time problems with differential algebraic equations.

As explained before and for the sake of simplicity we opt to consider here only problems with data smooth with respect to the time variable. These problems are of interest in their own since this is exactly what happens in many engineering application, namely in robotics and process systems engineering.

A thorough analysis of the proof and the results reported here will convince the reader that similar results could be derived under weaker assumptions and for more general problems. Among those are the following:

(i) Necessary conditions for free time optimal control problems with mixed constraints with data Lipschitz continuous.

(ii) Necessary conditions for free time optimal control problems with mixed constraints for *local minimum of radius R* with smooth or Lipschitz time dependence.

(iii) Necessary conditions for problems with mixed set constraints of the form

$$(x(t), u(t)) \in S(t)$$

and without special structure.

(iv) Necessary conditions for problems with control constraints and/or mixed constraints with measurable time dependence.

Case (i) and (iii) are of foremost importance and under research.

Derivation of necessary conditions for such cases will certainly follow from [12]. A setback to the analysis is that the necessary conditions will not be easy to deal with in applications since they will be stated in terms of the normal cones to the set of constraints in most of the cases. Although this is to be expected in case (iii) it is nevertheless a drawback when dealing with case (i). Further research on mixed constraints in the vein of [12] is thus called for.

Case (ii), which is expected to be an easy case, is currently under research.

That leaves us case (iv). When some smoothness or Lipschitz time dependence is assumed it is expected that boundary conditions on the pseudo Hamiltonian

$$H(t, x, p, u) = \langle p, f(t, x, u)\rangle$$

are expressed in terms of the multiplier r (see Theorem 6.1; see also Chapter 8 in [29]). The pseudo Hamiltonian along the optimal solution should also satisfy

$$\widetilde{r}(t) = -\langle p(t),\ f(t, \bar{x}(t), \bar{u}(t))\rangle.$$

This does not hold true when measurably time dependent data is considered. In view of [29] a more demanding approach involving essential values will be called for. This will also be the focus of future research.

On last word about an important application of the analysis done in this paper concerns Optimal multiprocesses as described in [4]. Optimal multiprocesses are of relevance in their own and since many hybrid optimal control problems can be reformulated as such. Thus the need to extend necessary conditions derived in [4] to cover problems involving nonstandard constraints. In [9] such is done for problems with state constraints with smooth data with respect to t via the time transformation. Our analysis also demonstrates that we are in position to do the same concerning optimal multiprocesses with mixed constraints. The time measurable case as well as problems with both mixed and pure state constraints are under research.

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