Statistical properties and rare events for chaotic dynamical systems

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Todas as correções determinadas pelo júri, e só essas, foram efetuadas.

O Orientador,

Porto, ______/______/_________

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Sworn Statement

I, Raquel Brás Sá Couto, enrolled in the Doctor's Degree UC|UP Joint PhD Program in Mathematics at the Faculty of Sciences of the University of Porto in a cotutelle agreement with the University of St Andrews hereby declare, in accordance with the provisions of paragraph a) of Article 14 of the Code of Ethical Conduct of the University of Porto, that the content of this thesis reflects perspectives, research work and my own interpretations at the time of its submission.

By submitting this thesis, I also declare that it contains the results of my own research work and contributions that have not been previously submitted to this or any other institution.

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Raquel Brás Sá Couto

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Abstract

The application of Extreme Value Theory to Dynamical Systems has been a topic of interest for a few years now (see the influential work in [FFT10] which built on [Col01] and [FF08]) opening up a whole new framework for the statistical study of chaotic systems.

In the early stages, the relationship between orbit visits to small sets of the ambient space and extreme values (of suitable random variables) provided statistical laws for recurrence (*eg.* [FFT10],[FFT11],[FFT12],[CFF$^+$15]). In particular, strong recurrence properties such as periodicity directly impact on the limiting law due to them being responsible for the clustering of extreme observations. This study extends the scope of the classical Poincaré Recurrence Theorem in ergodic theory.

Recently, with the focus on functional limit theorems, very strong results for the convergence of (appropriately scaled) sums of heavy-tailed dynamically defined random variables have been deduced ([FFT20]). The most well-known distributional result for an appropriately scaled sum is the Central Limit Theorem which can't be used with heavy tails.

The main purpose of our work is the application of the enriched functional limit theorem for heavy-tailed dynamical sums proved in [FFT20] (Theorem 2.2.6) to two different contexts which have previously been investigated from the point of view of extreme value laws: correlated maximal sets ([AFFR16] and [AFFR17]) and a Cantor maximal set ([FFRS20]). That essentially demands the convergence of some point processes, the key being the understanding of the clustering patterns of the tail observations of such processes. These patterns are well described by means of a structure introduced in [FFT20] and tailored to the dynamical context, which we prove to be, in the correlated maxima setting, as in Theorem 3.1.2, Theorem 3.1.14 or Theorem 3.2.1, and in the Cantor setting as in Theorem 4.5.1. Prior to our work, only a maximal set consisting of a single repelling periodic point had been considered. As we will see, the clustering patterns that we capture in our study are significantly richer (than for a maximal set reduced to a single point) and more accurately described (compared to the framework available for [AFFR16], [AFFR17] and [FFRS20]) via the new tool kit at hand.

We structure this thesis as follows. In Chapter 2, we summarise the background theory from [FFT20] which is required to the statement of the main Theorem 2.2.6 as well as to providing some insight into it. Then, we justify that the theorem can be deduced in the setting of correlated maximal sets, in Chapter 3, and in the setting of a Cantor maximal set, in Chapter 4.

Keywords: extreme values, dynamical extremes, clustering, heavy tails, point processes, functional limit theorems

Resumo

A aplicação da Teoria de Valores Extremos aos Sistemas Dinâmicos tem sido alvo de interesse nos últimos tempos (ver o importante trabalho [FFT10] baseado em [Col01] e [FF08]) revelando toda uma nova estrutura para o estudo estatístico dos sistemas caóticos.

Numa fase inicial, a relação entre visitas das órbitas de pontos a conjuntos pequenos no espaço ambiente e valores extremos (de variáveis aleatórias adequadas) resultou em distribuições de probabilidade para a recorrência (*eg.* [FFT10],[FFT11],[FFT12],[CFF$^+$15]). Em particular, fortes propriedades de recorrência tais como a periodicidade têm um impacto direto na distribuição limite por serem responsáveis pelo clustering de observações extremas. Este estudo amplia o âmbito do clássico Teorema da Recorrência de Poincaré na teoria ergódica.

Recentemente, com o foco nos teoremas funcionais do limite, foram deduzidos resultados muito fortes para a convergência de somas (devidamente normalizadas) de variáveis aleatórias de cauda pesada dinamicamente definidas ([FFT20]). O resultado mais conhecido para a convergência em distribuição de uma soma devidamente normalizada é o Teorema do Limite Central que não pode ser usado com caudas pesadas.

O principal objetivo deste trabalho é a aplicação do teorema funcional do limite enriquecido provado em [FFT20] (Teorema 2.2.6) a somas the caudas pesadas dinamicamente definidas em dois contextos que foram previamente investigados do ponto de vista das leis de valores extremos: conjuntos maximais correlacionados ([AFFR16] and [AFFR17]) e um conjunto maximal de Cantor ([FFRS20]). Essencialmente, é exigida a convergência de certos processos pontuais, a chave estando na perceção dos padrões de clustering das observações de cauda desses mesmos processos. Estes padrões são bem descritos através de uma estrutura introduzida em [FFT20] e feita à medida do contexto dinâmico, que provamos ser, no contexto de máximos correlacionados, como no Teorema 3.1.2, Teorema 3.1.14 ou Teorema 3.2.1, e no contexto de Cantor como no Teorema 4.5.1. Anteriormente ao nosso trabalho, apenas um conjunto maximal reduzido a um único ponto periódico repulsor tinha sido considerado. Como veremos, os padrões de clustering capturados pelo nosso estudo são significativamente mais ricos (do que para um conjunto maximal reduzido a um único ponto) e descritos com mais precisão (comparativamente com a estrutura disponível para [AFFR16], [AFFR17] e [FFRS20]) através do novo conjunto de ferramentas à disposição.

Estruturamos esta tese do seguinte modo. No Capítulo 2, resumimos a teoria de [FFT20] necessária para a escrita do Teorema 2.2.6 bem como para alguma intuição sobre o mesmo. De seguida, justificamos que o teorema pode ser deduzido no contexto de conjuntos maximais correlacionados, no Capítulo 3, e no contexto de um conjunto maximal de Cantor, no Capítulo 4.

Palavras-chave: valores extremos, extremos dinâmicos, clustering, caudas pesadas, processos pontuais, teoremas funcionais do limite

Contents

Chapter 1

Introduction

The present work lies on the interplay of two vast areas of pure mathematics which contribute with countless applications to the real world: dynamical systems and probability.

Our focus is on the theory of dynamical extremes.

The theory of extreme values, from a probabilistic point of view, looks into the tails of the probability distributions. An abnormal event, seen as one which falls far away from the mean, has the potential to cause a big impact on a system (*eg.* climate, financial market) and can't be gauged by most of the traditional models which disregard such occurrences for them being marked as outliers.

The dynamical perspective, especially when dealing with chaotic dynamical systems, is that one may attempt to understand a system by observing the behaviour of random variables defined on it.

Suppose we have a dynamical system defined on some ambient space where we take a random variable whose high values correspond to point entries in small sets of the space. Clearly, if we are interested in such orbit visits then the extreme values assumed by the random variable must be taken into account.

Observe that a pure dynamical problem, tied to the recurrence properties of a dynamical system, just got formulated in terms of a statistical query bound to the study of extremal behaviour.

The theory of dynamical extremes may have emerged with such ideas but has developed well beyond them. For an extensive account of the early work on the field we refer to [LFdF+16].

1.1 Extreme Value Theory

Extreme Value Theory (EVT), in the classical sense, emerges with the following question: given a sequence $(X_n)_{n\in\mathbb{N}}$ of independent and identically distributed (iid) random variables, is there a limiting distribution to the sequence $M_n = \max\{X_1, \dots, X_n\}$ of partial maxima?

It is known that if for some normalising sequences $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ and $\{b_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$, $\mathbb{P}(a_n(M_n - b_n) \leq y) \to H(y)$ for some (non-degenerate) distribution H, then H must be of one of the three following types:

(i) Type I (Gumbel): $H(y) = e^{-e^{-y}}$, where $y \in \mathbb{R}$;

(ii) Type II (Fréchet): $H(y) = e^{-y^{-\alpha}}$ if $y > 0$ and $H(y) = 0$ otherwise, where $\alpha > 0$ is a parameter;

(iii) Type III (Weibull): $H(y) = e^{-(-y)^{\alpha}}$ if $y \leq 0$ and $H(y) = 1$ otherwise, where $\alpha > 0$ is a parameter;

often referred to as the classical Extreme Value Laws (EVL).

A necessary and sufficient condition for a limiting distribution to M_n is the existence of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ with property $n\mathbb{P}(X_1 > u_n) \to \tau$. However, that does not guarantee that the sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ exist and, as a result, that the limiting distribution is of Type I, Type II or Type III. Now, if $u_n(y) = a_n^{-1}y + b_n$ and $n\mathbb{P}(X_1 > u_n(y)) \to \tau(y)$, then $\tau(y) = e^{-y}$ or $\tau(y) = y^{-\alpha}$ or $\tau(y) = (-y)^{\alpha}$ and so $H(y) = H(\tau(y))$ of Type I or Type II or Type III, respectively.

For a detailed account of the classical EVT see, for example, Chapter 1 of [LLR83].

The statements just presented still hold true for non-independent sequences verifying certain requirements on their dependence structure. Such requirements are expressed by Conditions $D(u_n)$ and $D'(u_n)$ in Chapter 3 of [LLR83]. $D(u_n)$ is a strong mixing condition whereas $D'(u_n)$ is a non-clustering condition. Essentially, $D(u_n)$ and $D'(u_n)$ play the role of some long-term and short-term independence, respectively. Under $D(u_n)$ and $D'(u_n)$ the Type I, Type II and Type III EVL can be derived.

1.2 Functional Limit Theorems

Donsker's Theorem is probably the first example of a Functional Limit Theorem (FLT) one comes across.

Theorem (Donsker)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be an iid sequence of random variables with finite second moments. Define*

$$S_n(t) = \sum_{j=1}^{\lfloor nt \rfloor} X_j, t \in [0,1].$$

Then

$$n^{-\frac{1}{2}} S_n(t) \to W(t), \ t \in [0,1]$$

where $W(t)$ is a Brownian motion, in the space of continuous functions on $[0,1]$, $C([0,1])$, equipped with the uniform topology.

In the absence of a finite second moment which is the case, for example, when $(X_n)_{n\in\mathbb{N}}$ is made up of heavy-tailed random variables, Donsker's Theorem is no longer of use. However, it is know that in some circumstances $n^{-\frac{1}{\alpha}} S_n(t)$ converges to an α-stable Lévy process, in the space of càdlàg functions on $[0,1]$, $D([0,1])$, with Skorohod's J_1 topology.

Observe that Donsker's Theorem implies a Central Limit Theorem and, analogously, a limiting α-stable Lévy process implies an α-stable Law.

Taking sums of heavy-tailed random variables will result in a very large value showing up to dominate a partial sum. This not only justifies the discontinuities in the limit but suggests that the study of heavy-tailed partial sums can benefit from the framework developed for maxima.

1.3 Dynamical Systems

Let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, f)$ be a probability preserving (discrete) dynamical system which stands for $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu)$ a probability space and $f : \mathcal{X} \to \mathcal{X}$ a transformation that preserves μ.

In the most general setting, we take $\mathcal{X}$ a d-dimensional compact manifold with a norm $\|\cdot\|$, $\mathcal{B}_{\mathcal{X}}$ is the corresponding Borel σ-algebra and μ is a probability measure.

1.3.1 Extreme Value Laws

In [FFT10], the authors established a link between the existence of Hitting Time Statistics (HTS) and Extreme Value Laws (EVL) for (discrete time) dynamical systems.

Let the *first hitting time function* to a set A be defined as $r_A(x) = \inf\{j \in \mathbb{N} : f^j(x) \in A\}$.

Looking into orbit visits to shrinking families of sets in the phase space is tied to the study of recurrence properties. To be precise, define the observable $\varphi(x) = g(\mathrm{dist}(x, \zeta))$, where $\zeta \in \mathcal{X}$, and $g : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ is such that:

(i) 0 is a global maximum ($g(0) = +\infty$ is allowed);

(ii) g is a strictly decreasing bijection in a neighbourhood of 0;

(iii) g has one of three types of behaviour (see [FFT10]) for which examples are $g_1(x) = -\log(x)$, $g_2(x) = x^{-1/\alpha}$ for some $\alpha > 0$ and $g_3(x) = D - x^{1/\alpha}$ for some $D \in \mathbb{R}$ and $\alpha > 0$.

The stochastic process $X_0, X_1, \ldots$ given by $X_n = \varphi \circ f^n$, for each $n \in \mathbb{N}_0$, is stationary.

Now, in a neighbourhood of $\zeta \in \mathcal{X}$,

$$\begin{aligned}\{x : f^n(x) \in B_{g^{-1}(u)}(\zeta)\} &= \{x : \mathrm{dist}(f^n(x), \zeta) < g^{-1}(u)\} \\ &= \{x : g(\mathrm{dist}(f^n(x), \zeta)) > u\} \\ &= \{x : \varphi(f^n(x)) > u\} \\ &= \{x : X_n(x) > u\}\end{aligned}$$

that is, visits to balls correspond to *exceedances* of the dynamically defined process $(X_n)_{n\in\mathbb{N}_0}$. Replacing u by $\{u_n\}_{n\in\mathbb{N}}$ such that $g^{-1}(u_n) =: \delta_n \to 0$, one can then talk about visits to the sequence $\{B_{\delta_n}(\zeta)\}_{n\in\mathbb{N}}$ of shrinking balls.

Observe that

$$f^{-1}(\{x : X_0(x) \leq u_n, \ldots, X_{n-1}(x) \leq u_n\}) = \{x : r_{B_{\delta_n}(\zeta)}(x) > n\}$$

which is equivalent to

$$f^{-1}(\{x : M_n(x) \leq u_n\}) = \{x : r_{B_{\delta_n}(\zeta)}(x) > n\}.$$

The first two results in [FFT10] assure the equivalence between the existence of HTS (to balls) and EVL. Furthermore, to exponential HTS correspond EVL of Types I, II and III (as in Section 1.1) for g_1, g_2 and g_3, respectively, in the definition of φ, and the converse also holds. Important classes of dynamical systems with good mixing properties, such as Axiom A diffeomorphisms, transitive Markov chains and uniformly expanding maps of the interval, are known to have exponential HTS. Therefore, in light of [FFT10], all those systems admit EVL.

We remark that the dynamically defined sequence $(X_n)_{n\in\mathbb{N}_0}$ is not iid. In addition, the mixing condition of [LLR83] (referred to as $D(u_n)$ in Section 1.1) is not easily checked for processes $(X_n)_{n\in\mathbb{N}_0}$ arising from dynamical systems. An alternative condition, that together with Leadbetter's $D'(u_n)$ guarantees an EVL, is condition $D_2(u_n)$ proposed in [FF08]. $D_2(u_n)$ follows from a sufficiently fast decay of correlations so, because rates of decay of correlations are known for many dynamical systems, checking $D_2(u_n)$ is straightforward in many cases.

So far, the HTS-EVL discussion was motivated by the original setting in Extreme Value Theory where, in particular, clustering of exceedances is prevented from happening (in the probabilistic sense) by the requirement that D' is satisfied. D' forces the exceedances of the increasingly high levels u_n to be scattered in the time line and hence, in the dynamical setting, the same applies to the visits to the shrinking target sets $B_{\delta_n}(\zeta)$. In the classical probabilistic setting, in the presence of D but absence of D' an EVL is still obtained although affected by a parameter $\theta \in [0, 1]$ called the *extremal index*. θ equal to 1 means absence of clustering of extreme observations which becomes more intense as θ tends to 0.

For stochastic processes arising from dynamical systems, the existence of $\theta < 1$ has been related to self-recurrence properties of the maximal set, namely when it consists of a single periodic point ([FFT12]) or a finite or countable number of points in the same orbit ([AFFR16] and [AFFR17]).

Intuitively, if, for instance, the maximal set consists of a single periodic point, ζ, of prime period p, then an arbitrarily small neighbourhood of ζ that is visited at time 0 self-recurs at times which are integer multiples of p. The rate of expansion of the system at ζ determines the amount of time required for such self-recurrence to cease as well as the amount of overlap between the original neighbourhood of ζ and its p-th iterate which, in turn, dictates θ. Specifically, when ζ is hyperbolic with Df^p_ζ expanding and μ is an absolutely continuous invariant probability (*i.e.* acip), we have $\theta = 1 - \frac{1}{|Df^p_\zeta|}$.

In [FFT12], the authors proved that, for the full shift (on a finite alphabet) with Bernoulli measure, θ can only differ from 1 at periodic points. That was done for cylinders, hence, in terms of cylinder EVL, an important class of systems ended up completely characterised. The same dichotomy was extended for balls in [AFV15] (see also [CFF+15]).

When the maximal set consists of more than one point in the same dynamical orbit, as was considered in [AFFR16] and [AFFR17], one observes what is sometimes called a fake periodic effect.

In cases like this (*i.e.* with (fake) periodicity), θ^{-1} corresponds to the mean number of exceedances in a cluster. However, there are scenarios where such agreement is not verified (see [AFF20]).

Having exponential HTS implies that the waiting time between consecutive hits to a shrinking target is given by an exponential law, increasing as the target shrinks. In the absence of D', the hits come in bulk/clusters and the time frame must be rescaled by θ. More precisely, with the notation fixed in Section 1.1, $H(\tau) = e^{-\tau}$ is replaced by $H(\tau) = e^{-\theta\tau}$.

1.3.2 Rare Events Point Processes

With EVL we have a framework to deal with single visits to arbitrarily small sets. In fact, even in the presence of clustering EVL allows us to predict a single bulk/cluster of such visits.

Now, assume we are interested in keeping record of multiple visits taking place as time goes by. We require a more powerful tool kit which can be found in the theory of point processes.

In [FFT10], the primal version of the Exceedances Point Process/Rare Events Point Process (EPP/REPP) was introduced. Let

$$N_n(t) = N_n([0,t)) = \sum_{j=0}^{\lfloor nt \rfloor -1} \mathbb{1}_{X_j > u_n} \tag{1.3.1}$$

be the process which, at stage n, counts the number of exceedances of the level u_n by the first $\lfloor nt \rfloor$ random variables in the process $(X_n)_{n\in\mathbb{N}_0}$.

In the absence of clustering, N_n converges to a simple Poisson Process with intensity τ, for $(u_n)_{n\in\mathbb{N}} = (u_n(\tau))_{n\in\mathbb{N}}$ where $n\mu(X_0 > u_n(\tau)) \to \tau$ and provided the point process versions of the conditions D and D' hold (see [FFT10] for details). In the presence of clustering, N_n converges to a compound Poisson Process with intensity $\theta\tau$ and a multiplicity distribution, for $(u_n)_{n\in\mathbb{N}} = (u_n(\tau))_{n\in\mathbb{N}}$ where $n\mu(X_0 > u_n(\tau)) \to \theta\tau$ and provided the point process and clustering versions of the conditions D and D' hold (see [FFT13] for details). We remark that in the latter case the multiplicity distribution holds quantitative information on the size of the clusters which is more informative than the mean cluster size usually equal to θ^{-1} (recall the discussion by the end of Section 1.3.1). In particular, when clustering is associated to periodicity the multiplicity distribution is a geometric distribution of parameter θ (see [FFT13]) but different multiplicity distributions whose mean is still θ^{-1} arise with fake periodicity (see [AFFR16]).

In [FFM20], two-dimensional point processes were considered allowing for the recording of both times and magnitudes of exceedances.

In the notation used in [FFM20], the one-dimensional point process in (1.3.1) can be rewritten

$$N_n = \sum_{j=0}^{\infty} \delta_{\frac{j}{n}} \mathbb{1}_{X_j > u_n} \tag{1.3.2}$$

so that at stage n a mass is assigned to j/n whenever $X_j > u_n$.

In the absence of clustering, $N_n \to N$ where

$$N = \sum_{j=1}^{\infty} \delta_{T_j} \tag{1.3.3}$$

for $T_j = \sum_{l=1}^{j} \overline{T}_l$ with $\overline{T}_l \sim \exp(\tau)$ for all $l = 1, \dots, j$, *i.e.* each $\overline{T}_l$ has an exponential distribution of parameter τ. Thus, N is a simple Poisson process with intensity τ.

Recall the requirement that $n\mathbb{P}(X_0 > u_n(\tau)) \to \tau$ which means the average number of exceedances of the level u_n by the first n random variables in the process $(X_n)_{n\in\mathbb{N}_0}$ is asymptotically equal to τ. Note that N_n places all such exceedances on $[0,1]$. Therefore, we expect, asymptotically, τ exceedances on the time interval $[0,1]$. This is compatible with interarrival times being, on average, $1/\tau$ as is the case for an exponential distribution of parameter τ.

In the presence of clustering, $N_n \to N$ where

$$N = \sum_{j=1}^{\infty} D_j \delta_{T_j} \tag{1.3.4}$$

for $T_j = \sum_{l=1}^{j} \overline{T}_l$ with $\overline{T}_l \sim \exp(\theta\tau)$ for all $l = 1, \ldots, j$, and $(D_j)_{j\in\mathbb{N}}$, an iid sequence of positive integer valued random variables independent of $(T_j)_{j\in\mathbb{N}}$. Thus, N is a compound Poisson process with intensity $\theta\tau$ and multiplicity distribution that of the random variable D_1.

Here the process is keeping track of the arrivals of the clusters of exceedances and the masses assigned are enlarged by the size of the clusters.

Now, the two-dimensional point process presented in [FFM20] is written

$$N_n = \sum_{j=0}^{\infty} \delta_{\left(\frac{j}{n}, u_n^{-1}(X_j)\right)}. \tag{1.3.5}$$

In the absence of clustering, according to [HT19], $N_n \to N$ where

$$N = \sum_{i,j=1}^{\infty} \delta_{(T_{i,j}, U_{i,j})} \tag{1.3.6}$$

for $T_{i,j} = \sum_{l=1}^{j} \overline{T}_{i,l}$ with $\overline{T}_{i,l} \sim \exp(1)$ for all $l = 1, \ldots, j$ and for all i, and $U_{i,j} \sim U_{(i-1,i]}$ for all i and for all j, *i.e.* each $U_{i,j}$ has a uniform distribution on the interval $(i-1, i]$. The sequences $(T_{i,j})_{i,j}$ and $(U_{i,j})_{i,j}$ are mutually independent. Thus, N is a two-dimensional simple Poisson process with intensity measure the two-dimensional Lebesgue measure, *i.e.* $Leb \times Leb$.

Observe that N_n places all the first n observations of the stochastic process $(X_n)_{n\in\mathbb{N}_0}$ on the vertical strip $[0,1]\times[0,\infty)$ of the plane. Moreover, to each square $[0,1]\times(i-1,i]$ belong the observations which are exceedances of the threshold $u_n(i)$ but not of the threshold $u_n(i-1)$: since $n\mathbb{P}(X_0 > u_n(i)) \to i$, there is, on average, $i-(i-1) = 1$ observation recorded on $[0,1]\times(i-1,i]$. Indeed, such observation is recorded anywhere on $[0,1]\times(i-1,i]$ which is compatible with the two-dimensional Lebesgue measure being the limiting measure. Finally, recall that the exceedances interarrival times are exponential so the (expected) unique observation on $[0,1]\times(i-1,i]$ takes $\exp(1)$ amount of time to occur after the previous observation has occurred.

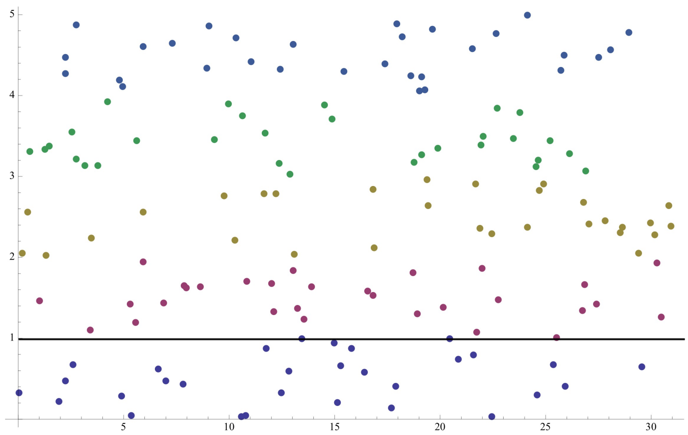

Figure 1.1: Plot of a finite sample simulation of $u_n^{-1}(X_0), \ldots, u_n^{-1}(X_{n-1})$ in the absence of clustering of extreme observations.

We remark that despite our focus being processes arising from dynamical systems, in particular, referring to [FFT10], [FFT13] and [HT19] where the convergence results are derived in the dynamical setting, the conclusions so far hold for general stationary stochastic processes.

In [FFM20], the authors present a limit to the two-dimensional point process which holds in the presence of clustering and for $(X_n)_{n\in\mathbb{N}_0}$ arising from a dynamical system. In fact, the clustering phenomena, recorded in the second component, is tied to the dynamics.

Assume that μ is absolutely continuous with respect to the Lebesgue measure (*i.e.* acip) and that its density is sufficiently regular so that $\mu(B_\varepsilon(\zeta)) \sim C\varepsilon^d$, as $\varepsilon \to 0$, for some $C > 0$ and where d is the dimension of $\mathcal{X}$. Also, take ζ a repelling periodic point of prime period p such that Df_ζ^p expands uniformly in every direction at rate α.

Then,

$$N = \sum_{i,j=1}^{\infty} \sum_{l=0}^{\infty} \delta_{(T_{i,j}, \alpha^{ld} U_{i,j})} \tag{1.3.7}$$

for $T_{i,j} = \sum_{l=1}^{j} \overline{T}_{i,l}$ with $\overline{T}_{i,l} \sim \exp(\theta)$ for all $l = 1, \dots, j$ and for all i, $U_{i,j} \sim U_{(i-1,i]}$ for all i and for all j, and $\theta = 1 - \alpha^{-d}$, where the sequences $(T_{i,j})_{i,j}$ and $(U_{i,j})_{i,j}$ are mutually independent. Thus, N is a two-dimensional compound Poisson process with intensity measure $\theta Leb \times \nu$ where ν is described by an outer measure.

What changes compared to the non-clustering situation is that now point masses pile up in the vertical direction. That is because an exceedance of a certain threshold is followed, not long after, by another exceedance (of another threshold), and so on; due to time compression in the limit, all such exceedances are placed on the same vertical line. To be more specific, in the dynamical context given, an exceedance with asymptotic frequency in $(i-1, i]$ is followed by one with asymptotic frequency in $\alpha^d(i-1, i]$, and by another one with asymptotic frequency in $\alpha^{2d}(i-1, i]$, and so on, due to the returns to neighbourhoods of ζ at times which are multiples of p (*cf.* the discussion on the extremal index and periodicity in Section 1.3.1). Note that, as was the case for the one-dimensional process in the presence of clustering, the clusters of exceedances are marked in the same time point which means that more time shall pass between subsequent clusters' arrivals (in fact, on average, an amount of time equal to $1/\theta$).

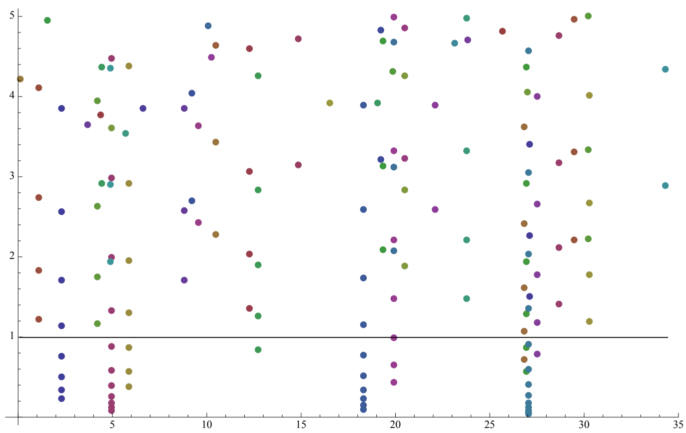

Figure 1.2: Plot of a finite sample simulation of $u_n^{-1}(X_0), \dots, u_n^{-1}(X_{n-1})$ in the presence of clustering of extreme observations.

The theory in [FFM20] extends further to deal with different rates of expansion of the dynamical system in the d possible directions. That leads to multidimensional point processes. We don't discuss those structures here as the theory in [FFT20], which we very briefly touch in the next Section and more comprehensively recall in Chapter 2, is well suited to general multidimensional systems.

1.3.3 Enriched Functional Limit Theorems

Recently, in [FFT20], a new structure called the *piling process*[1] was introduced giving rise to a multi-dimensional point process distinct from the one in [FFM20]. Essentially, while the version in [FFM20] looks into observations, one at a time, the point process proposed in [FFT20] deals with blocks of observations. The blocks are built such that at most one cluster of exceedances is expected inside each block. The piling process, keeping track of the observations inside a block and respective ordering (this was not accomplished by the point process in [FFM20]), is then recording the clustering pattern.

We will provide an overview of the theory developed in [FFT20] in Chapter 2, still we proceed here with a compact sketch.

The set-up in [FFT20] holds greater generality than that of previous works. That is due to the use of vector valued observables. More precisely, in a d-dimensional compact manifold, $\mathcal{X}$, with a norm, the observable $\Psi : \mathcal{X} \to \mathbb{R}^d$ is defined such that it is, on each component, a decreasing function of the distance to a certain maximal set, $\mathcal{M}$. For example, when $\mathcal{M} = \{\zeta\}$ where $\zeta \in \mathcal{X}$ is some hyperbolic point, we may write

$$\Psi(x) = g(\text{dist}(x, \zeta)) \frac{\Phi_\zeta^{-1}(x)}{\|\Phi_\zeta^{-1}(x)\|} \mathbb{1}_W(x) \tag{1.3.8}$$

where g is as in Section 1.3.1 and $\Phi_\zeta : V \subset T_\zeta\mathcal{X} \to W \subset \mathcal{X}$ is a local diffeomorphism defined in a neighbourhood of $\Phi^{-1}(\zeta)$ ($T_\zeta\mathcal{X}$ denotes the tangent space at $\zeta \in \mathcal{X}$).

For $\mathbf{X}_n = \Psi \circ f^n$, where $n \in \mathbb{N}_0$, for all $s < t \in \mathbb{Z}$ let

$$\mathbb{X}_n^{s,t} = \left(u_n^{-1}(\|\mathbf{X}_s\|) \frac{\mathbf{X}_s}{\|\mathbf{X}_s\|}, \ldots, u_n^{-1}(\|\mathbf{X}_t\|) \frac{\mathbf{X}_t}{\|\mathbf{X}_t\|} \right) \tag{1.3.9}$$

store the asymptotic frequencies of the observations at times $s, ..., t$ projected on the respective (tangent) directions.

At stage n, consider k_n blocks each with r_n observations. The point process

$$N_n = \sum_{i=1}^{\infty} \delta_{\left(i/k_n, \tilde{\pi}(\mathbb{X}_n^{r_n(i-1), r_n.i-1})\right)} \tag{1.3.10}$$

essentially places the asymptotic frequencies of the observations in a block on a vertical pile bound to that block.

It is proved in [FFT20] that $N_n \to N$ where

$$N = \sum_{i=1}^{\infty} \delta_{\left(T_i, U_i \tilde{\mathbf{Q}}_i\right)} \tag{1.3.11}$$

is a Poisson process with intensity measure $Leb \times \eta$ where η is given by the distribution of the piling process.

[1] Name changed to *anchored tail process* in the most recent version of the paper but we keep the designation piling process for the sake of this thesis.

The REPP from [FFT20] is valuable on its own, providing a neat description of the clustering phenomena by means of the piling process. On top of that, it allows for an *enriched functional limit theorem* for sums of heavy-tailed random variables.

Functional limit theorems for sums of dynamically defined random variables are statements about the convergence of

$$S_n(t) = \sum_{i=0}^{\lfloor nt \rfloor - 1} \frac{1}{a_n}\mathbf{X}_i - tc_n,\ t \in [0,1], \tag{1.3.12}$$

in a suitable space of functions, when the process $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ is dynamically defined by evaluating an observable function along the orbits of the dynamical system ($(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are appropriate scaling sequences).

A heavy-tailed random variable has an infinite second moment. In light of the very brief discussion in Section 1.2, the partial sums process, under appropriate scaling, may still admit a functional limit known as an α-stable Lévy process. Such processes are discontinuous and live on the space $D([0,1])$ of càdlàg functions on $[0,1]$. In fact, a value which is far away from the mean dominates a sum leading up to a jump in the limit. If the random variables in $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ have sufficiently regular tails then an α-stable Lévy process shows up in the limit.

[TK10] provides us with α-stable Lévy processes as functional limits for sums of heavy-tailed dynamically defined random variables, where the convergence holds in Skorohod's J_1 topology (see also references in [TK10] for other examples of work on the subject). In such cases, there is no clustering of extreme observations of the underlying stochastic process.

In the presence of clustering, complexity is added to the study. That is because clustering implies multiple jumps, close in time, for the partial sums process. Then, due to time compression, the latter end up in the limit as a sequence of points on the same vertical line which is not J_1-equivalent to any element of $D([0,1])$. Opting for a weaker topology, such as Skorohod's M_1 or M_2 topologies, although possibly allowing for limits, implies the loss of the clustering patterns. In other words, once the limit is established, it is impossible to retrieve the data corresponding to the intermediate jumps that took place inside the cluster. For a discussion of J_1, M_1 and M_2 topologies see [Whi02].

The distinctive feature of [FFT20, Theorem 2.4] is allowing for the functional convergence to hold without loss of the clustering patterns. A key role is played by the functional space F' defined in [FFT20] related to Whitt's space F ([Whi02]). Very briefly, F' is made of excursion triples $(V, S^V, (e_V^s)_{s\in S^V})$, where $V \in D([0,1])$, S^V is an at most countable set containing the discontinuities of V, and e_V^s is the excursion at $s \in S^V$, which lives in a quotient space of $D([0,1])$, and enjoys the essential property that $e_V^s(0) = V(s^-)$ and $e_V^s(1) = V(s)$, meaning that the information regarding the cluster associated to a discontinuity s of the càdlàg function V is recorded in e_V^s. Indeed, [FFT20, Theorem 2.4] is a statement about convergence in F' of heavy-tailed dynamical sums, where (1.3.12) converges to V an α-stable Lévy process.

[FFT20, Theorem 2.4] essentially follows from the weak convergence of the REPP. Once the piling process is well defined and usual dependence requirements are met by the stochastic sequence, the REPP converges to a Poisson process. Then, if the stochastic sequence has an α-regularly varying tail, the functional limit in F' is $(V, S^V, (e_V^s)_{s\in S^V})$ where V is an α-stable Lévy process whose Lévy-measure is associated to the intensity measure of the limiting Poisson process to the REPP (visit [FFT20] for details). The piling process plays a crucial role in [FFT20, Theorem 2.4] allowing for the characterisation of not only the exceedances in a given cluster (*i.e.* handling the clustering component on the REPP) but also the excursions $(e_V^s)_{s\in S^V}$ (*i.e.* recording the clustering pattern aside of the functional limit).

Chapter 2

Background

This chapter is dedicated to an overview of the theory developed in [FFT20] which we illustrate with some original applications and worked out examples in Chapters 3 and 4 to follow.

As mentioned at the beginning of Section 1.3, let $(\mathcal{X}, \mathcal{B}_{\mathcal{X}}, \mu, f)$ be a probability preserving system where $\mathcal{X}$ is a d-dimensional compact manifold with a norm $\|\cdot\|$, $\mathcal{B}_{\mathcal{X}}$ is the corresponding Borel σ-algebra and μ is a f-invariant probability measure.

Consider an observable $\Psi : \mathcal{X} \to \mathbb{R}^d$ such that there exists a maximal set, $\mathcal{M}$, for $\|\Psi(\cdot)\|$ whose high values correspond to entries in small neighbourhoods of $\mathcal{M}$. It is only demanded that $\mathcal{M}$ has zero measure, so $\mathcal{M}$ consisting of a finite set of points, a countable set of points, a submanifold or a fractal set are all permitted. Specifically, assume that on a neighbourhood of $\mathcal{M}$,

$$\|\Psi(x)\| = g(\mathrm{dist}(x, \mathcal{M}))$$

where $\mathrm{dist}(x, \mathcal{M}) = \inf\{\mathrm{dist}(x, \zeta) : \zeta \in \mathcal{M}\}$ and $g : [0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ has the following properties:

(i) 0 is a global maximum ($g(0) = +\infty$ is allowed);

(ii) g is a strictly decreasing bijection in a neighbourhood of 0;

(iii) g has one of three types of behaviour (see [FFT20]) which relate to the three classical types of EVL (*cf.* Section 1.3.1).

For our work we need only consider g of type g_2 that is $g_2(0) = +\infty$ and there exists $\alpha > 0$ such that for all $y > 0$, $\displaystyle\lim_{s \to +\infty} \frac{g_2^{-1}(sy)}{g_2^{-1}(s)} = y^{-\alpha}$. We remark that in our applications we will restrict to $g(x) = cx^{-\frac{1}{\alpha}}$ where $\alpha \in (0, 1)$ and $c > 0$.

Letting

$$\mathbf{X}_n = \Psi \circ f^n, \; n \in \mathbb{N}_0$$

we have $(\mathbf{X}_n)_{n \in \mathbb{N}_0}$ a stationary stochastic process.

2.1 Rare Events Point Process

In this section we aim to provide some intuition into the form of the REPP given by (1.3.10) and respective convergence to (1.3.11).

2.1.1 Threshold functions

Let $(u_n)_{n\in\mathbb{N}} : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+ = (0, +\infty)$, be such that:

(1) for each n, u_n is non-increasing, left continuous and such that

$$\lim_{\tau_1\to 0, \tau_2\to\infty} \mathbb{P}(u_n(\tau_2) < \|\mathbf{X}_0\| < u_n(\tau_1)) = 1;$$

(2) for each $\tau \in \mathbb{R}^+$,

$$\lim_{n\to\infty} n\mathbb{P}(\|\mathbf{X}_0\| > u_n(\tau)) = \tau. \tag{2.1.1}$$

$(u_n)_{n\in\mathbb{N}}$ is a normalising sequence of threshold functions with the crucial property expressed in (2.1.1) that the asymptotic frequency of exceedances of a threshold (that depends on τ) is constant (and equal to τ). The use of such normalising sequences dates as far back as the classical probabilistic setting of Extreme Value Theory and is crucial for obtaining limits (*cf.* Section 1.1).

Taking generalised inverses of the functions u_n, we are able to recover an asymptotic frequency: for every $z \in \mathbb{R}^+$, define

$$u_n^{-1}(z) = \sup\{\tau > 0 : z \le u_n(\tau)\}. \tag{2.1.2}$$

Observe that (2.1.2) gives $z > u_n(\tau) \implies u_n^{-1}(z) \le \tau$.

2.1.2 Extremal Index

For a sequence $(q_n)_{n\in\mathbb{N}}$ of positive integers (we discuss the importance of this sequence in the next section), let

$$U_n(\tau) = \{\|\mathbf{X}_0\| > u_n(\tau)\} \tag{2.1.3}$$

and

$$U_n^{(q_n)}(\tau) = \{\|\mathbf{X}_0\| > u_n(\tau), \|\mathbf{X}_1\| \le u_n(\tau), \dots, \|\mathbf{X}_{q_n}\| \le u_n(\tau)\} \tag{2.1.4}$$

where $(u_n)_{n\in\mathbb{N}}$ is as defined in Section 2.1.1 just above. Then, the extremal index, θ, is defined as

$$\theta = \lim_{n\to\infty} \frac{\mu(U_n^{(q_n)}(\tau))}{\mu(U_n(\tau))} \tag{2.1.5}$$

provided the limit exists.

The extremal index is a parameter $\theta \in [0, 1]$ which quantifies the intensity of clustering of extreme observations, that is how close in the time line arbitrarily high observations appear. θ equal to 1 means absence of clustering of extreme observations which becomes more intense as θ approaches 0. Recall the interesting link between the extremal index and the underlying dynamics that was mentioned in Section 1.3.1.

2.1.3 Dependence structure

From the onset of the theory of extremes for dynamical systems, back in [FFT10], dependence conditions proved necessary for the limit laws/processes to exist. We loosely refer to such conditions as D and D', which are tailored to each context (*i.e.* EVL, point processes, clustering, non-clustering) and, respectively, mimic some long-term and short-term sorts of independence.

In [FFT20], being able to keep record of the information associated to the clusters of exceedances by means of the piling process is unprecedented. Therefore, the conditions D and D' must be tailored to such piles of exceedances which compose the clusters. In order to include here the conditions $Д_{q_n}$ and $Д'_{q_n}$ introduced in [FFT20], we need some formalism from sections 3, 3.1 and 3.2 of the same paper that we summarise next.

First, we need a method to isolate the clusters.

A possible approach is given by the blocking method. At stage n, divide the observations into $k_n \in \mathbb{N}$ blocks of size $r_n := \lfloor n/k_n \rfloor$. In addition, take time gaps of size $t_n \in \mathbb{N}$ between consecutive blocks. This produces sequences $(k_n)_{n\in\mathbb{N}}$, $(r_n)_{n\in\mathbb{N}}$, and $(t_n)_{n\in\mathbb{N}}$ which we further require to be such that

$$k_n, r_n, t_n \underset{n\to\infty}{\longrightarrow} \infty \text{ and } k_n t_n = o(n). \tag{2.1.6}$$

We determine that exceedances in different blocks belong to different clusters. A good tuning of the size of the blocks is key to a separation of the actual clusters without splitting each of them apart. Also, the time gaps must be large enough to provide some independence between the blocks but small enough so that not too much information is disregarded.

Another option is to consider the runs declustering method. At stage n, take runs made up of $q_n \in \mathbb{N}$ time steps and determine that exceedances separated by less than q_n units of time belong to the same cluster or, in other words, a cluster ends when no exceedances are registered in q_n successive time instants. The sequence $(q_n)_{n\in\mathbb{N}}$ must satisfy

$$q_n = o(r_n) \tag{2.1.7}$$

and be such that $Д_{q_n}$ and $Д'_{q_n}$ (and $\tilde{Д}'_{q_n}$) below hold. We note that $q_n = q$ for all $n \in \mathbb{N}$ is allowed which suits period q periodicity scenarios.

In order to get to the usual dependence requirements for the stochastic sequence $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ so that the REPP converges, we proceed with some definitions and notation from [FFT20].

Let $\mathcal{V} = \mathbb{R}^d$ (with the Euclidean norm), and $\mathcal{V}^{\mathbb{N}_0}/\mathcal{V}^{\mathbb{Z}}$ be the spaces of one-sided/two-sided $\mathcal{V}$-valued sequences. Take the (one-sided/two-sided) shift map $\sigma : \mathcal{V}^{\mathbb{N}_0,\mathbb{Z}} \to \mathcal{V}^{\mathbb{N}_0,\mathbb{Z}}$. We identify our stationary sequence $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$, which takes values in $\mathbb{R}^d$, with the coordinate variable process in $(\mathcal{V}^{\mathbb{N}_0}, \mathcal{B}^{\mathbb{N}_0}, \mathbb{P})$ given by Kolmogorov's extension theorem where $\mathcal{B}^{\mathbb{N}_0}$ denotes the product σ-algebra, in other words, $\mathcal{B}^{\mathbb{N}_0}$ is the σ-algebra generated by the coordinate functions $Z_n : \mathcal{V}^{\mathbb{N}_0} \to \mathcal{V}$ such that $Z_n(x_0, x_1, \dots) = x_n$, for all $n \in \mathbb{N}_0$. Observe that $Z_i = \sigma \circ Z_{i-1}$, for all $i \in \mathbb{N}$, and that the stationarity of $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ results in σ-invariance of $\mathbb{P}$.

Let

$$\mathscr{F} = \left\{ \{(x_j)_j \in \mathcal{V}^{\mathbb{N}_0,\mathbb{Z}} : x_i \in H_i, i = 0, \dots, m\} : H_i \in \mathcal{F}_{\mathcal{V}}, i = 0, \dots, m,\ m \in \mathbb{N} \right\} \tag{2.1.8}$$

where $\mathcal{F}_{\mathcal{V}}$ denotes the field generated by the rectangles of $\mathcal{V}$ of the form $[e_1, f_1) \times \cdots \times [e_d, f_d)$ (in particular, $\mathscr{F}$ is a field).

For each $l = 1, \dots, m$, $m \in \mathbb{N}$, assume that $A_l \in \mathscr{F}$ and define

$$A_{n,l} = \left\{ \left(u_n^{-1}(\|\mathbf{X}_j\|) \frac{\mathbf{X}_j}{\|\mathbf{X}_j\|} \right)_j \in \mathcal{V}^{\mathbb{N}_0} : \left(u_n^{-1}(\|\mathbf{X}_j\|) \frac{\mathbf{X}_j}{\|\mathbf{X}_j\|} \right)_j \in A_l \right\} \tag{2.1.9}$$

where u_n^{-1} is as in (2.1.2). Observe that, in particular,

$$\left\| u_n^{-1}(\|\mathbf{X}_j\|) \frac{\mathbf{X}_j}{\|\mathbf{X}_j\|} \right\| < \tau \iff \|\mathbf{X}_j\| > u_n(\tau). \tag{2.1.10}$$

We will be particularly interested in the events

$$A_{n,l}^{(q_n)} = A_{n,l} \cap \bigcap_{j=1}^{q_n} \sigma^{-j}(A_{n,l})^c, \quad l = 1, \dots, m, \tag{2.1.11}$$

for q_n as defined above.

$A_{n,l}^{(q_n)}$ represents a set of observations with the requirement that a pattern which begins at time 0 is not to be repeated for the next q_n units of time.

Finally, let $J_l = [a_l, b_l)$ where $0 \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_m < b_m \le 1$ and, for each $n \in \mathbb{N}$, define

$$J_{n,l} = [(\lceil k_n a_l - 1 \rceil) r_n, (\lfloor k_n b_l + 1 \rfloor) r_n) \tag{2.1.12}$$

where k_n and r_n are as above. $J_{n,l}$ consists of the time frame for the k_n blocks and respective $r_n k_n$ observations which belong to the interval $J_l = [a_l, b_l)$ when the observations up to time n (and respective k_n blocks) are collapsed on the interval $[0, 1]$.

For an interval I contained in $[0, +\infty)$, we write

$$\mathscr{W}_I(A) = \bigcap_{j \in I \cap \mathbb{N}_0} \sigma^{-j}(A^c) \quad \text{and} \quad \mathscr{W}_I^c(A) = (\mathscr{W}_I(A))^c. \tag{2.1.13}$$

Condition $Д_{q_n}$. We say that $Д_{q_n}$ holds for the sequence $\mathbf{X}_0, \mathbf{X}_1, \dots$ if there exist sequences $(k_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as above, such that for every $m, t, n \in \mathbb{N}$ and every J_l and A_l, with $l = 1, \dots, m$, we have

$$\left| \mathbb{P}\left(A_{n,l}^{(q_n)} \cap \bigcap_{i=l}^{m} \mathscr{W}_{J_{n,i}}(A_{n,i}^{(q_n)}) \right) - \mathbb{P}(A_{n,l}^{(q_n)}) \mathbb{P}\left(\bigcap_{i=l}^{m} \mathscr{W}_{J_{n,i}}(A_{n,i}^{(q_n)}) \right) \right| \le \gamma(n, t)$$

where $\min\{J_{n,l} \cap \mathbb{N}_0\} \ge t$ and $\gamma(n, t)$ is decreasing in t for each n and $\lim_{n\to\infty} n\gamma(n, t_n) = 0$.

Condition $Д'_{q_n}$. We say that $Д'_{q_n}$ holds for the sequence $\mathbf{X}_0, \mathbf{X}_1, \dots$ if there exist sequences $(k_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as above, such that for every $A_1 \in \mathscr{F}$, we have

$$\lim_{n\to\infty} n\mathbb{P}\left(A_{n,1}^{(q_n)} \cap \mathscr{W}^c_{[q_n+1, r_n)}(A_{n,1}) \right) = 0.$$

We remark that Condition $Д_{q_n}$ essentially expresses that, if time has run for long enough, blocks that are sufficiently far apart are basically independent, and Condition $Д'_{q_n}$ imposes that not more than one cluster of exceedances is expected within the same block.

Condition $\tilde{Д}'_{q_n}$. We say that $\tilde{Д}'_{q_n}$ holds for the sequence $\mathbf{X}_0, \mathbf{X}_1, \dots$ if there exist sequences $(k_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as above, such that for every $\tau > 0$, we have

$$\lim_{n\to\infty} n\mathbb{P}\left(U_n(\tau) \cap \mathscr{W}^c_{[q_n+1, r_n)}(U_n(\tau)) \right) = 0.$$

Remark 2.1.1. Condition $\tilde{Д}'_{q_n}$ implies Condition $Д'_{q_n}$ and can be easier to check.

As pointed out in Section 4 of [FFT20], $Д_{q_n}$ and $Д'_{q_n}$ follow from decay of correlations for processes arising from dynamical systems.

We note that the formalism so far is suited for general stationary sequences. For a dynamically defined $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ the role of σ is played by f.

Definition 2.1.2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be Banach spaces of $\mathbb{R}$-valued measurable functions defined on $\mathcal{X}$. Let the *correlation* between non-zero functions $\phi \in \mathcal{C}_1$ and $\psi \in \mathcal{C}_2$ with respect to μ at time $n \in \mathbb{N}$ be defined as

$$\mathrm{Cor}_\mu(\phi, \psi, n) := \frac{1}{\|\phi\|_{\mathcal{C}_1}\|\psi\|_{\mathcal{C}_2}} \left| \int \phi(\psi \circ f^n) d\mu - \int \phi d\mu \int \psi d\mu \right|.$$

The system is said to have *decay of correlations*, with respect to μ, for observables in $\mathcal{C}_1$ *against* observables in $\mathcal{C}_2$ if there exists a rate function $\rho : \mathbb{N} \to [0, +\infty)$ with $\lim_{n\to\infty} \rho(n) = 0$ and such that for all $\phi \in \mathcal{C}_1$ and for all $\psi \in \mathcal{C}_2$ it holds $\mathrm{Cor}_\mu(\phi, \psi, n) \le \rho(n)$.

The following lemma gives the common shortcut to check that $Д_{q_n}$ and $Д'_{q_n}$ hold. It has analogous counterparts for weaker versions of the conditions $Д_{q_n}$ and $Д'_{q_n}$ that have already been established (for example, the versions compatible with extreme value laws or rare events point processes).

Lemma 2.1.3. *If the system has decay of correlations against L^1 and*

(1) $\lim_{n\to\infty} \|\mathbb{1}_{A^{(q_n)}_{n,l}}\|_{\mathcal{C}_1} n\rho(t_n) = 0$ *for some sequence* $(t_n)_n$ *with* $t_n = o(n)$*;*

(2) $\lim_{n\to\infty} \|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1} \sum_{j=q_n}^{n} \rho(j) = 0$*;*

are satisfied, then $Д_{q_n}$ and $Д'_{q_n}$ hold.

Proof. We first check that (1) implies $Д_{q_n}$. Taking $\phi = \mathbb{1}_{A^{(q_n)}_{n,l}}$ and $\psi = \mathbb{1}_{f^{-t}\left(\bigcap_{i=l}^{m} \mathscr{W}_{J_{n,i}}(A^{(q_n)}_{n,i})\right)}$, we have

$$\left| \mathbb{P}\left(A^{(q_n)}_{n,l} \cap \bigcap_{i=l}^{m} \mathscr{W}_{J_{n,i}}(A^{(q_n)}_{n,i}) \right) - \mathbb{P}(A^{(q_n)}_{n,l}) \mathbb{P}\left(\bigcap_{i=l}^{m} \mathscr{W}_{J_{n,i}}(A^{(q_n)}_{n,i}) \right) \right|$$
$$= \left| \int \phi(\psi \circ f^t) d\mu - \int \phi d\mu \int \psi d\mu \right| \le \|\mathbb{1}_{A^{(q_n)}_{n,l}}\|_{\mathcal{C}_1} \rho(t)$$

so that, by (1), $Д_{q_n}$ holds with $\gamma(n,t) = \|\mathbb{1}_{A^{(q_n)}_{n,l}}\|_{\mathcal{C}_1}\rho(t)$. In turn, taking $\phi = \psi = \mathbb{1}_{U_n(\tau)}$, and since

$$\mathbb{P}\left(U_n(\tau) \cap f^{-j}(U_n(\tau))\right) = \int \phi(\phi \circ f^j) d\mu \le (\mu(U_n(\tau)))^2 + \|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1} \mu(U_n(\tau))\rho(j)$$

which implies that

$$n \sum_{j=q_n+1}^{r_n-1} \mathbb{P}\left(U_n(\tau) \cap f^{-j}(U_n(\tau))\right) \le n(r_n - 1)(\mu(U_n(\tau)))^2 + n.\|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1} \mu(U_n(\tau)) \sum_{j=q_n+1}^{r_n-1} \rho(j)$$
$$\le \frac{n^2(\mu(U_n(\tau)))^2}{k_n} + n\|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1} \mu(U_n(\tau)) \sum_{j=q_n}^{n} \rho(j)$$
$$\le \frac{\tau^2}{k_n} + \tau\|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1} \sum_{j=q_n}^{n} \rho(j) \underset{n\to\infty}{\longrightarrow} 0$$

using (2). Therefore, we have that condition $\tilde{Д}'_{q_n}$ is satisfied, which implies $Д'_{q_n}$. □

2.1.4 Piling Process

We finally present the piling process: the new structure introduced in [FFT20] which allows for the most thorough depiction of the clusters of exceedances. For an extensive treatment visit sections 3.3.3, 3.3.4 and 3.3.5 of [FFT20].

For $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$, for all $s < t \in \mathbb{Z}$, let

$$\mathbb{X}_n^{s,t} = \left(u_n^{-1}(\|\mathbf{X}_s\|)\frac{\mathbf{X}_s}{\|\mathbf{X}_s\|}, \dots, u_n^{-1}(\|\mathbf{X}_t\|)\frac{\mathbf{X}_t}{\|\mathbf{X}_t\|} \right) \tag{2.1.14}$$

so that, for all $j = s, ..., t$, the asymptotic frequency of the observation $\mathbf{X}_j$ is projected on the (tangent) direction of the same observation.

The piling process is defined on the space

$$l_\infty = \{\mathbf{x} = (x_j)_{j\in\mathbb{Z}} \in (\overline{\mathbb{R}^d} \setminus \{0\})^{\mathbb{Z}} : \lim_{|j|\to\infty} \|x_j\| = \infty\}$$

being, therefore, a bi-infinite sequence (where $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$).

Definition 2.1.4. Given a process $(Y_j)_{j\in\mathbb{Z}}$ such that:

(1) $\mathcal{L}\left(\frac{1}{\tau}\mathbb{X}_n^{r_n+s, r_n+t} \;\middle|\; \|\mathbf{X}_{r_n}\| > u_n(\tau)\right) \xrightarrow[n\to\infty]{} \mathcal{L}((Y_j)_{j=s,\dots,t})$, for all $s < t \in \mathbb{Z}$ and all $\tau > 0$ (where $\mathcal{L}$ implies convergence in distribution);

(2) the process $(\Theta_j)_{j\in\mathbb{Z}}$ given by $\Theta_j = \frac{Y_j}{\|Y_0\|}$ is independent of $\|Y_0\|$;

(3) $\lim_{|j|\to\infty} \|Y_j\| = \infty$ (a.s.);

(4) $\mathbb{P}\left(\inf_{j\leq -1} \|Y_j\| \geq 1\right) > 0$;

the process $(Z_j)_{j\in\mathbb{Z}}$ defined by

$$\mathcal{L}((Z_j)_{j\in\mathbb{Z}}) = \mathcal{L}\left((Y_j)_{j\in\mathbb{Z}} \;\middle|\; \inf_{j\leq -1} \|Y_j\| \geq 1\right)$$

is called the *piling process*.

In words, (1) states that $(Y_j)_{j\in\mathbb{Z}}$ must be such that, for all $s < t \in \mathbb{Z}$ and all $\tau > 0$, $(Y_s, \dots, Y_t)$ has asymptotically the same distribution as $\left(\frac{1}{\tau}\mathbb{X}_n^{r_n+s, r_n+t} \;\middle|\; \|\mathbf{X}_{r_n}\| > u_n(\tau)\right)$ which is, for $j = s, ..., t$, the joint distribution of the asymptotic frequencies of the observations $\mathbf{X}_{r_n+j}$ compared with τ (projected on the (tangent) direction of $\mathbf{X}_{r_n+j}$), given that the asymptotic frequency of the observation $\mathbf{X}_{r_n}$ is at most τ. Then, $(Z_j)_{j\in\mathbb{Z}}$ distributes as $(Y_j)_{j\in\mathbb{Z}}$ conditional on $u_n^{-1}(\|\mathbf{X}_{r_n+j}\|) \geq \tau$ for all negative j which translates as the observation at time r_n being the highest so far.

The piling process provides us with quantitative information on the magnitudes of the observations in a cluster of exceedances. Moreover, being a sequence, it keeps the ordering at which the observations show up which leads to the clustering pattern being recorded with the most efficiency.

Let $\tilde{l}_\infty = l_\infty/\sim$ where, for all $\mathbf{x}, \mathbf{y} \in l_\infty$,

$$\mathbf{x} \sim \mathbf{y} \iff \mathbf{y} = \sigma^k(\mathbf{x})$$

for some $k \in \mathbb{Z}$, where σ is the left-shift map and $\tilde{\pi} : l_\infty \to \tilde{l}_\infty$ is the canonical projection. Define

$$L_Z = \inf_{j\in\mathbb{Z}} \|Z_j\|, \tag{2.1.15}$$

and, for all $j \in \mathbb{Z}$,

$$Q_j = \frac{Z_j}{L_Z}. \tag{2.1.16}$$

Finally,

$$\tilde{\mathbf{Q}} = \tilde{\pi}((Q_j)_{j\in\mathbb{Z}}). \tag{2.1.17}$$

Let

$$l_0 = \{\mathbf{x} = (x_j)_{j\in\mathbb{Z}} \in (\mathbb{R}^d)^{\mathbb{Z}} : \lim_{|j|\to\infty} \|x_j\| = 0\}.$$

Let

$$\begin{array}{rccl} p\colon & \overline{\mathbb{R}^d} \setminus \{0\} & \to & \mathbb{R}^d \\ & x & \mapsto & \begin{cases} \frac{x}{\|x\|^2}, & x \neq \infty \\ 0, & \text{otherwise} \end{cases} \end{array}$$

and $P : l_\infty \to l_0$ defined as $P((x_j)_{j\in\mathbb{Z}}) = (p(x_j))_{j\in\mathbb{Z}}$.
For $\tilde{l}_0$ defined in the same way as $\tilde{l}_\infty$ above, then $\tilde{P} : \tilde{l}_\infty \to \tilde{l}_0$ is such that $\tilde{P}(\tilde{\pi}(\mathbf{x})) = \tilde{\pi}(P(\mathbf{x}))$.
Let $\mathbb{S} = \{\tilde{\mathbf{x}} \in \tilde{l}_\infty : \|\tilde{P}(\tilde{\mathbf{x}})\|_\infty = 1\}$. Define

$$\begin{array}{rccl} \psi\colon & \tilde{l}_\infty \setminus \{\tilde{\infty}\} & \to & \mathbb{R}^+ \times \mathbb{S} \\ & \tilde{\mathbf{x}} & \mapsto & \left(\frac{1}{\|\tilde{P}(\tilde{\mathbf{x}})\|_\infty}, \frac{\tilde{\mathbf{x}}}{\|\tilde{P}(\tilde{\mathbf{x}})\|_\infty^{-1}} \right) \end{array}. \tag{2.1.18}$$

Remark 2.1.5. Notice that $\psi(\tilde{\pi}((Z_j)_{j\in\mathbb{Z}})) = (L_Z, \tilde{\mathbf{Q}})$.

Let $\mathbb{X}_{n,i}$ denote $\mathbb{X}_n^{r_n(i-1), r_n.i-1}$, *i.e.*

$$\mathbb{X}_{n,i} = \left(u_n^{-1}(\|\mathbf{X}_{r_n(i-1)}\|) \frac{\mathbf{X}_{r_n(i-1)}}{\|\mathbf{X}_{r_n(i-1)}\|}, \ldots, u_n^{-1}(\|\mathbf{X}_{r_n.i-1}\|) \frac{\mathbf{X}_{r_n.i-1}}{\|\mathbf{X}_{r_n.i-1}\|} \right). \tag{2.1.19}$$

Remark 2.1.6. Observe that there is a natural identification between $\mathbb{X}_{n,i}$ and an element of l_∞ simply by considering, in l_∞, the bi-infinite sequence with all entries to the left of $u_n^{-1}(\|\mathbf{X}_{r_n(i-1)}\|) \frac{\mathbf{X}_{r_n(i-1)}}{\|\mathbf{X}_{r_n(i-1)}\|}$ and all entries to the right of $u_n^{-1}(\|\mathbf{X}_{r_n.i-1}\|) \frac{\mathbf{X}_{r_n.i-1}}{\|\mathbf{X}_{r_n.i-1}\|}$ equal to ∞ (and $\mathbb{X}_{n,i}$ in between).

For $A \in \mathscr{F}$ (as in (2.1.8)), define

$$\tilde{A} = \{\tilde{\mathbf{x}} \in \tilde{l}_\infty : \tilde{\pi}^{-1}(\tilde{\mathbf{x}}) \cap A \neq \emptyset\} \tag{2.1.20}$$

and

$$\tilde{\mathscr{J}} = \{\tilde{A} : A \in \mathscr{F}\}. \tag{2.1.21}$$

Let

$$\mathbb{A} = \{\mathbf{x} \in l_\infty : \tilde{\pi}(\mathbf{x}) \in \tilde{A}\} = \left\{ \mathbf{x} \in l_\infty : \mathbf{x} \in \bigcup_{j\in\mathbb{Z}} \sigma^{-j}(A) \right\}. \tag{2.1.22}$$

We list some properties of the piling process which were proved in [FFT20]. For future reference, we organise the statements in a proposition.

Proposition 2.1.7. *Properties of the piling process:*

1. $\|Y_0\|$ *is uniformly distributed on* $[0,1]$.
2. $\lim_{n\to\infty} k_n \mathbb{P}(\mathscr{W}^c_{r_n}(U_n(\tau))) = \theta\tau$ *and* $\lim_{n\to\infty} \frac{\mathbb{P}(\mathscr{W}^c_{r_n}(U_n(\tau)))}{r_n \mathbb{P}(U_n(\tau))} = \theta$.
3. $\theta = \mathbb{P}\left(\inf_{j\geq 1}\|Y_j\| \geq 1\right) = \mathbb{P}\left(\inf_{j\leq -1}\|Y_j\| \geq 1\right)$.
4. *Under the assumptions used to define the piling process and* $Д'_{q_n}$*, for every* $\tau > 0$,
$$\mathcal{L}\left(\tilde{\pi}\left(\frac{\mathbb{X}_{n,1}}{\tau}\right) \in \tilde{A} \mid \mathbb{X}_{n,1} \in \mathscr{W}^c_{r_n}(U(\tau))\right) \to \mathcal{L}\left(\tilde{\pi}((Z_j)_{j\in\mathbb{Z}})\right)$$
where $U(\tau) = \{(x_j)_j \in \mathcal{V}^{\mathbb{N}_0,\mathbb{Z}} : x_0 \in B_\tau(0)\}$.
5. L_Z *is uniformly distributed on* $[0,1]$ *and independent of* $\tilde{\mathbf{Q}}$.
6. $\eta_n = k_n\mathbb{P}(\tilde{\pi}(\mathbb{X}_{n,1}) \in \cdot) \overset{w^\#}{\to} \eta = \theta(Leb \times \mathbb{P}_{\tilde{\mathbf{Q}}}) \circ \psi$ *where* $\mathbb{P}_{\tilde{\mathbf{Q}}}$ *is the distribution of* $\tilde{\mathbf{Q}}$.

2.1.5 Complete convergence of the REPP

We summarise the theory of point processes and respective convergence with focus on the simple point processes which are of interest to our context (see appendices B and C of [FFT20] and references therein).

Let $\mathcal{X}$ be a complete separable metric space.

A Borel measure on $\mathcal{X}$ which is finite on all bounded Borel sets is called a *boundedly finite measure.*

Let $\mathcal{M}^\#_{\mathcal{X}}$ denote the space of boundedly finite point measures on $\mathcal{X}$, with weak$^\#$ topology, and let $\mathcal{B}(\mathcal{M}^\#_{\mathcal{X}})$ denote the corresponding Borel σ-algebra. The weak$^\#$ topology is metrizable and convergence in the weak$^\#$ topology coincides with *weak*$^\#$ *convergence.* The following conditions are equivalent to *weak*$^\#$ *convergence*, denoted $\mu_n \overset{w^\#}{\to} \mu$:

(i) $\lim_{n\to+\infty}\int f d\mu_n = \int f d\mu$ for all bounded continuous functions f defined on $\mathcal{X}$ and vanishing outside a bounded set.

(ii) There exists an increasing sequence of bounded open sets B_k converging to $\mathcal{X}$ such that if $\mu_n^{(k)}$ and $\mu^{(k)}$ denote the restrictions of the measures μ_n and μ to B_k, respectively, then $\mu_n^{(k)}$ converges weakly to $\mu^{(k)}$, as $n\to\infty$, for all $k\in\mathbb{N}$. One must pay attention to the fact that $\mu_n^{(k)}$ and $\mu^{(k)}$ are not necessarily probability measures when using the classical Portmanteau Theorem to prove their weak convergence.

(iii) $\lim_{n\to+\infty}\mu_n(A) = \mu(A)$ for all bounded Borel set A with $\mu(\partial A) = 0$.

A *random measure* is a random element in $(\mathcal{M}^\#_{\mathcal{X}}, \mathcal{B}(\mathcal{M}^\#_{\mathcal{X}}))$, that is a measurable map defined on some probability space $(\Omega, \mathcal{B}, \mathbb{P})$ taking values in $(\mathcal{M}^\#_{\mathcal{X}}, \mathcal{B}(\mathcal{M}^\#_{\mathcal{X}}))$.

Definition 2.1.8. Let $\mathcal{N}^\#_{\mathcal{X}}$ denote the space of boundedly finite integer valued point measures on $\mathcal{X}$ with weak$^\#$ topology. A *point process* is an integer valued random measure, that is a measurable map $N : (\Omega, \mathcal{B}, \mathbb{P}) \to (\mathcal{N}^\#_{\mathcal{X}}, \mathcal{B}(\mathcal{N}^\#_{\mathcal{X}}))$, and it is said to be *simple* if $\mathbb{P}(N(\{x\}) > 1) = 0$ for all $x \in \Omega$.

Any $\mu \in \mathcal{N}_{\mathcal{X}}^{\#}$ can be written as

$$\mu = \sum_{i \in \mathbb{N}} k_i \delta_{x_i} \tag{2.1.23}$$

where $k_i \in \mathbb{N}$ for all $i \in \mathbb{N}$ and δ_{x_i} is the Dirac measure at $x_i \in \mathcal{X}$. Observe that N is simple if and only if $k_i = 1$ for all $i \in \mathbb{N}$.

Definition 2.1.9. A sequence $(N_n)_{n \in \mathbb{N}}$ of point processes is said to *converge weak-#* to a point process N (defined in the same state space), written $N_n \stackrel{w^{\#}}{\to} N$, when the respective distributions $P_n(A) := \mathbb{P}(N_n \in A)$ converge weakly (in the sense of weak convergence of probability measures on the metric space $\mathcal{N}_{\mathcal{X}}^{\#}$) to $P(A) := \mathbb{P}(N \in A)$ for all $A \in \mathcal{B}(\mathcal{N}_{\mathcal{X}}^{\#})$.

Define

$$\mathcal{I} = \left\{ \bigcup_{l=1}^{m} J_l \times \tilde{A}_l : m \in \mathbb{N}, J_l = [a_l, b_l), \tilde{A}_l \in \tilde{\mathscr{J}} \right\} \tag{2.1.24}$$

where $\tilde{A}_l$ and $\tilde{\mathscr{J}}$ are as given by (2.1.20) and (2.1.21), respectively.

Proposition 2.1.10. *Let $\mathcal{X} = \mathbb{R}_0^+ \times \tilde{l}_\infty \setminus \{\tilde{\infty}\}$. If, for all bounded $A \in \mathcal{I}$ ($\mathcal{I}$ as in (2.1.24)),*

(I) $\mathbb{P}(N_n(A) = 0) = \mathbb{P}(N(A) = 0)$,

(II) $\mathbb{E}(N_n(A)) = \mathbb{E}(N(A))$,

then $N_n \stackrel{w^{\#}}{\to} N$, provided N is a simple point process.

As explained in Section 1.3.3, the REPP considered in [FFT20] is written

$$N_n = \sum_{i=1}^{\infty} \delta_{(i/k_n, \tilde{\pi}(\mathbb{X}_{n,i}))}. \tag{2.1.25}$$

Theorem 2.1.11 ([FFT20, Theorem 3.17])**.** *If a normalising sequence $(u_n)_{n \in \mathbb{N}}$ as in Section 2.1.1 exists, the piling process is well defined and conditions $Д_{q_n}$ and $Д'_{q_n}$ are satisfied, then N_n converges weakly (in the space of boundedly finite point measures on $\mathbb{R}_0^+ \times \tilde{l}_\infty \setminus \{\tilde{\infty}\}$ with weak$^{\#}$ topology) to*

$$N = \sum_{i=1}^{\infty} \delta_{(T_i, U_i \tilde{\mathbf{Q}}_i)} \tag{2.1.26}$$

which is a Poisson process with intensity measure $Leb \times \eta$, where $\eta = \theta(Leb \times \mathbb{P}_{\tilde{\mathbf{Q}}}) \circ \psi$.

We sketch the proof of Theorem 2.1.11 which is detailed in [FFT20] and makes use of Proposition 2.1.10 as well as some results from previous work (*eg.* [FFM20]).

Let $B \in \mathcal{I}$ be a bounded set, that is $B = \bigcup_{l=1}^{m} J_l \times \tilde{A}_l$, for some $m \in \mathbb{N}$.

To obtain (I) in Proposition 2.1.10, note that

$$\mathbb{P}(N_n(B) = 0) = \mathbb{P}\left(\bigcap_{l=1}^{m} \left\{ N_n(J_l \times \tilde{A}_l) = 0 \right\} \right) = \mathbb{P}\left(\bigcap_{l=1}^{m} \mathscr{W}_{J_{n,l}}(A_{n,l}) \right) = \prod_{l=1}^{m} e^{-\nu(A_l)|J_l|}$$

where the second to last equality is by definition and the last equality follows from the asymptotic independence of disjoint time pieces and some convergence in distribution

([FFT20, Proposition 3.18]), where ν is the outer measure describing the clustering component in the multidimensional point process from [FFM20]. Now,

$$\mathbb{P}(N(B)=0)=\prod_{l=1}^{m}e^{-\eta(\tilde{A}_l)|J_l|}=\prod_{l=1}^{m}e^{-\nu(A_l)|J_l|}$$

where the first equality is by definition and the second equality (fully justified in [FFT20]) is a consequence of the piling process describing the clustering component in the REPP from [FFT20] (and thus $\nu(A_l)=\eta(\tilde{A}_l)$).

For (II) in Proposition 2.1.10, write

$$\mathbb{E}(N_n(B))=\mathbb{E}\left(\sum_{l=1}^{m}\sum_{i=\lceil k_n a_l\rceil}^{\lceil k_n b_l\rceil-1}\mathbb{1}_{\{\tilde{\pi}(\mathbb{X}_{n,i})\in\tilde{A}_l\}}\right)\sim\sum_{l=1}^{m}|J_l|k_n\mathbb{P}(\tilde{\pi}(\mathbb{X}_{n,1})\in\tilde{A}_l)$$
$$\xrightarrow[n\to\infty]{}\sum_{l=1}^{m}|J_l|\eta(\tilde{A}_l)=\mathbb{E}(N(B))$$

where stationarity and 6. in Proposition 2.1.7 were used.

We conclude with some intuition to why the limiting process should be N, especially to why the limiting measure which characterises the second component and, as a result, is given by the distribution of the piling process, should be η.

First, recall that retrieving the clusters by the blocking method results in each block containing at most one cluster of exceedances (this relies on the good tuning of the sequences $(k_n)_{n\in\mathbb{N}},(r_n)_{n\in\mathbb{N}},(t_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ and Condition $Д'_{q_n}$ being verified). Thus, clusters' arrivals are traded for blocks' arrivals which, in turn, are uniformly distributed on $[0,1]$ (notice the division by k_n in the first component of N_n). The intensity measure describing the first component of N is then the Lebesgue measure on $[0,1]$.

Now, the sequence/pile $\mathbb{X}_{n,i}$ of r_n observations in block i which, in light of Remark 2.1.6, can be seen as a bi-infinite sequence, is placed on the vertical line through i/k_n. Besides, $n\mathbb{P}(\|\mathbf{X}_0\|>u_n(1))\to\theta$ gives us that, among the first n observations of the process $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$, exactly θ observations are expected to land inside $[0,1]\times[0,1]$ or, equivalently, exactly one observation is expected to land inside $[0,1]\times[0,1/\theta]$. But such an observation belongs to one of the k_n piles of observations, in other words, the one observation expected in $[0,1]\times[0,1/\theta]$ is attached to a pile of observations. This should provide some insight into η as given by 6. in Proposition 2.1.7 (*cf.* Remark 2.1.5).

As we already mentioned in Section 1.3.3, there is a significant difference in the way observations are perceived by means of the multidimensional point process from [FFM20] *vs* the REPP from [FFT20]. That is because the multidimensional point process from [FFM20] captures the observations individually. Then, in the presence of clustering, piles of points build up on the plane (due to the time compression in the limit) but they are not ordered piles as is the case for the REPP from [FFT20] via the piling process.

2.2 Enriched Functional Limit Theorem

In this section we explain the theorem from [FFT20] which motivates our work in the subsequent chapters. For that, we recall the functional space F', introduced in [FFT20], where the convergence of the partial sums process

$$S_n(t) = \sum_{i=0}^{\lfloor nt \rfloor - 1} \frac{1}{a_n}\mathbf{X}_i - tc_n,\ t \in [0,1], \tag{2.2.1}$$

when guaranteed, has no loss of information ($(a_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ are appropriate scaling sequences). Then, [FFT20, Theorem 2.4] establishes such convergence when the random variables $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ in (2.2.1) are heavy-tailed with sufficiently regular tails.

2.2.1 Functional space

As briefly discussed in Section 1.3.3, the limits with discontinuous sample paths which occur when the random variables $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ in (2.2.1) are heavy-tailed may or may not exist in $D([0,1])$ endowed with the Skorohod's J_1, M_1 or M_2 topologies.

In fact, in the presence of clustering there is no limit to (2.2.1) in J_1. That is because two elements are close in J_1 if their jumps are close, up to some time deformation (and for the continuous parts they are uniformly close). Thus, the several intermediate jumps caused by the clustering effect which occur in (2.2.1) and get collapsed into the same time point in the limit (due to time compression) are unmatched.

Unmatched jumps is not an issue for either M_1 or M_2 topologies, where closeness of elements is determined by closeness of their completed graphs. Still, in either M_1 or M_2 overshooting is not allowed, that is the existence of an intermediate jump which is higher than the jumps in the limit (that aggregate several intermediate jumps).

The space $E = E([0,1])$ proposed by Whitt ([Whi02]) is the space of excursion triples $(x, S^x, \{I(s)\}_{s\in S^x})$ where $x \in D([0,1])$, $S^x \subset [0,1]$ is an at most countable set containing the discontinuities of x and, for each $s \in S^x$, $I(s)$ is a connected subset of $\mathbb{R}^d$ containing at least $x(s^-)$ and $x(s)$. Observe that the decoration of s by $I(s)$ displays the minimum and maximum values of the intermediate jumps (which in the limit are collapsed into the time point s).

We include some definitions and notation from Section 2.3 of [FFT20].

Identify each element of E with the set-valued function

$$\hat{x}(t) = \begin{cases} I(t),\ t \in S^x \\ \{x(t)\},\ \text{otherwise} \end{cases}, \tag{2.2.2}$$

and respective graph $\Gamma_{\hat{x}} = \{(t,z) \in [0,1] \times \mathbb{R}^d : z \in \hat{x}(t)\}$. Let $p_l : \mathbb{R}^d \to \mathbb{R}$ denote the projection onto the l-th coordinate, for $l = 1, ..., d$, and define $\hat{x}^l(t) = p_l(\hat{x}(t))$ and $\Gamma^l_{\hat{x}} = \{(t,z) \in [0,1] \times \mathbb{R}^d : z \in \hat{x}^l(t)\}$.

The topology in E is inspired by the M_2 topology given by the Hausdorff distance between compact sets.

Recall that for compact sets $A, B \subset \mathbb{R}^d$, the Hausdorff distance between A and B is

$$m(A,B) = \max\left\{\sup_{x\in A}\left\{\inf_{y\in B}\|x-y\|\right\}, \sup_{y\in B}\left\{\inf_{x\in A}\|x-y\|\right\}\right\}. \tag{2.2.3}$$

For $A \subset \mathbb{R}^d$, the diameter of A is $d(A) = \sup_{x,y\in A}\{\|x-y\|\}$.

Assume that the elements of E are such that for all $\varepsilon > 0$ there exist finitely many $s \in S^x$ such that $d(I(s)) > \varepsilon$. Thus, for each $\hat{x} \in E$ we have that $\Gamma_{\hat{x}}$ is a compact set.

Now, we endow E with the Hausdorff metric by setting

$$m_E(\hat{x}, \hat{y}) = \max_{l=1,\dots,d} m(\Gamma^l_{\hat{x}}, \Gamma^l_{\hat{y}}). \tag{2.2.4}$$

Alternatively, we may endow E with the uniform metric given by

$$m^*_E(\hat{x}, \hat{y}) = \max_{l=1,\dots,d} \sup_{t\in[0,1]} m(\hat{x}^l(t), \hat{y}^l(t)). \tag{2.2.5}$$

We note that E with m_E is separable but not complete while E with m^*_E is complete but not separable.

In E, having the graphs of S_n augmented by the decorations $I(s)$, for every $s \in S^{S_n}$, results in overshooting no longer being an obstacle to the convergence.

Still, the intermediate jumps inside a decoration can't be retrieved from the limit in E, only the maximal oscillations.

Whitt's space $F = F([0,1])$ captures the changing directions of the jumps (*i.e.* when a jump upwards is followed by a jump downwards, and vice-versa).

F is a quotient space of parametric representations of the graphs of the elements in E by setting that two parametrisations are equivalent if they visit the same points and in the same order. Clearly, F is larger than E as there are different parametrisations corresponding to the same augmented graph.

Still, the intermediate jumps in the same direction can't be retrieved from the limit in F.

The space $F' = F'([0,1])$ overcomes this limitation.

We follow closely Section 2.3.1 of [FFT20].

Let $\tilde{D} = \tilde{D}([0,1]) = D([0,1])/\sim$ where $x \sim y$ if there exists a continuous strictly increasing bijection $\lambda : [0,1] \to [0,1]$ such that $x \circ \lambda = y$. We call such λ a reparametrisation of $[0,1]$.

Denote by $[x]$ the equivalence class of x. Observe that $y \in [x]$ must be such that x and y have the same number of discontinuities.

Let

$$d_{\tilde{D}}([x],[y]) = \inf_{\lambda\in\Lambda} \|x \circ \lambda - y\|, \tag{2.2.6}$$

where Λ is the set of reparametrisations of $[0,1]$.

Define

$$F' = \{\underline{x} = (x, S^x, \{e^s_x\}_{s\in S^x})\} \tag{2.2.7}$$

where $x \in D([0,1])$, $S^x \subset [0,1]$ is an at most countable set containing the discontinuities of x and, for each $s \in S^x$, $e^s_x \in \tilde{D}([0,1])$ is the excursion at s which is such that $e^s_x(0) = x(s^-)$ and $e^s_x(1) = x(s)$.

The reason why F' is suited to functional limits of partial sums of heavy-tailed random variables without loss of the clustering patterns now becomes clear. Indeed, the excursion e^s_x, at a discontinuity s in the limiting Lévy process x, allows for a representation of all the intermediate jumps which are collapsed into the time point s by a càdlàg function. Thus, magnitudes and ordering are both preserved.

As an example, we bring the following figures from [FFT20].

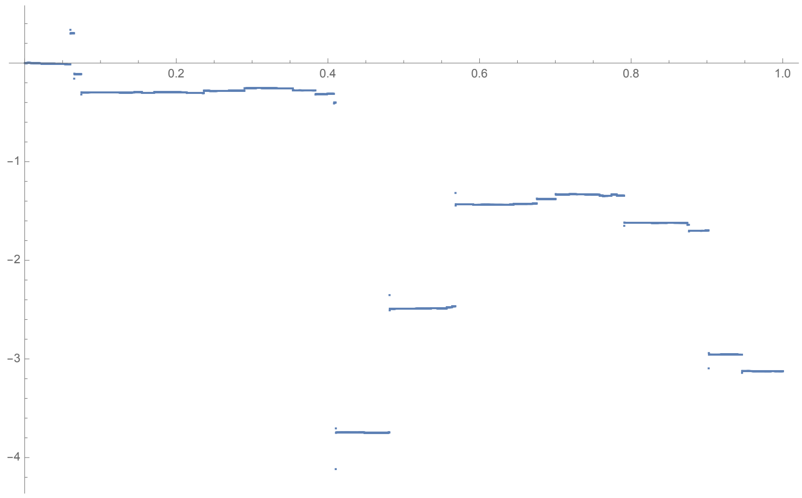

Figure 2.1: Plot of a finite sample simulation of $S_n(t)$ with $n = 5000$, where $X_j = \psi \circ f^j(x)$, where $f(x) = 3x \mod 1$, $\psi(x) = |x - 1/8|^{-2} - |x - 3/8|^{-2}$.

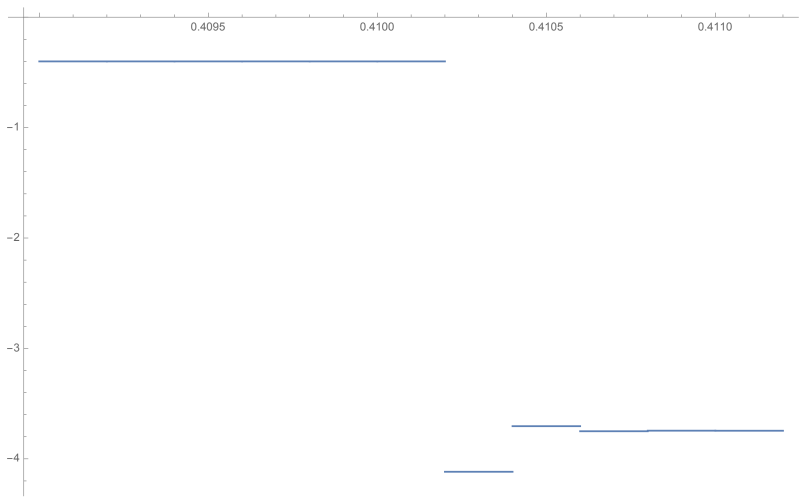

Figure 2.2: Blowup of the previous graph at the jump observed near 0.4: asymptotically the four jumps seen here happen instantaneously, necessitating an appropriate space for convergence.

We project F' into E and into $\tilde{D}$ as follows.

Let $\pi_E(\underline{x}) = \underline{x}^E = (x, S^x, \{I(s)\}_{s \in S^x})$ where, for all $s \in S^x$,

$$I(s) = \left[\inf_{t \in [0,1]} e_x^{s,1}(t),\ \sup_{t \in [0,1]} e_x^{s,1}(t) \right] \times \cdots \times \left[\inf_{t \in [0,1]} e_x^{s,d}(t),\ \sup_{t \in [0,1]} e_x^{s,d}(t) \right], \tag{2.2.8}$$

with $e_x^{s,l}(t) = p_l(e_x^s(t))$.

Now, assume that S^x is countable and write $S^x = \{s_i\}_{i=1}^{\infty}$. Let $0 = a_1 < a_2 < \cdots < 1$ be such that $a_i \to 1$ as $i \to \infty$. The interval $[a_i, a_{i+1}]$ is inserted at s_i in the following way. For all $i \in \mathbb{N}$, let

$$\overline{a}_i = \sum_{s_j \le s_i} (a_{j+1} - a_j),\ c_i = s_i + \overline{a}_i - (a_{i+1} - a_i),\ d_i = s_i + \overline{a}_i,\ \overline{t} = \sup\{\overline{a}_i : s_i < t\}.$$

Thus, $[c_i, d_i]$ is the domain, in $\tilde{D}$, for the excursion $e_x^{s_i}$. Note that the time line length increased to 2.

We define a representative of $[\tilde{\pi}(\underline{x})]$ by

$$\underline{x}^{\tilde{D}} = \begin{cases} x(2t-\bar{t}), \ 2t \notin \cup_i [c_i, d_i] \\ e_x^{s_i}\left(\dfrac{2t-c_i}{d_i-c_i}\right), \ 2t \in [c_i, d_i] \end{cases}. \tag{2.2.9}$$

Finally, we may define

$$d_{F'}(\underline{x}, \underline{y}) = d_E(\underline{x}^E, \underline{y}^E) + d_{\tilde{D}}(\underline{x}^{\tilde{D}}, \underline{y}^{\tilde{D}}) \tag{2.2.10}$$

where d_E denotes either m_E or m_E^* (as in (2.2.4) or (2.2.5), respectively) and $d_{\tilde{D}}$ is as in (2.2.6).

F' with m_E on the E component is separable but not complete while with m_E^* it is complete but not separable (see Appendix A of [FFT20]).

2.2.2 α-regular variation

Definition 2.2.1. $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ has *α-regularly varying tails*, $\alpha \in (0, 2)$, if there exists a sequence $(a_n)_{n\in\mathbb{N}}$ of positive real numbers such that

$$\lim_{n\to\infty} n\mathbb{P}(\|\mathbf{X}_0\| > ya_n) = y^{-\alpha}. \tag{2.2.11}$$

Remark 2.2.2. (2.2.11) implies (2.1.1): letting $\tau = y^{-\alpha}$ we have $u_n(\tau) = \tau^{-\frac{1}{\alpha}} a_n$. Moreover, $u_n^{-1}(z) = \left(\dfrac{z}{a_n}\right)^{-\alpha}$.

A stronger requirement than α-regular variation is joint α-regular variation.

Definition 2.2.3. A k-dimensional random vector $\mathbf{X}$ is *jointly α-regularly varying*, $\alpha > 0$, if there exists a sequence of constants $(a_n)_{n\in\mathbb{N}}$ and a random vector $\mathbf{\Theta}$ with $\mathbb{P}(\|\mathbf{\Theta}\| = 1) = 1$ such that

$$n\mathbb{P}(\|\mathbf{X}\| > xa_n, \mathbf{X}/\|\mathbf{X}\| \in \cdot) \xrightarrow{w} x^{-\alpha}\mathbb{P}(\mathbf{\Theta} \in \cdot) \tag{2.2.12}$$

where we are considering weak convergence of measures on $\mathbb{S}^{k-1}$, the unit sphere in $\mathbb{R}^k$. An $\mathbb{R}^d$-valued sequence $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ is *jointly α-regularly varying*, $\alpha > 0$, if all the finite dimensional vectors $(\mathbf{X}_j, \dots, \mathbf{X}_l)$, $j \le l \in \mathbb{N}_0$, are *jointly α-regularly varying.*

Let

$$\begin{aligned} \xi\colon \ \overline{\mathbb{R}^d} \setminus \{0\} &\to \mathbb{R}^d \\ x &\mapsto \begin{cases} (\|x\|)^{-\frac{1}{\alpha}} \frac{x}{\|x\|}, \ x \neq \infty \\ 0, \ \text{otherwise} \end{cases} \end{aligned} \tag{2.2.13}$$

and

$$\begin{aligned} \Xi\colon \quad l_\infty &\to l_0 \\ (x_j)_{j\in\mathbb{Z}} &\mapsto (\xi(x_j))_{j\in\mathbb{Z}} \end{aligned}. \tag{2.2.14}$$

Let $\tilde{\Xi} : \tilde{l}_\infty \to \tilde{l}_0$ be such that $\tilde{\Xi}(\tilde{\pi}(\mathbf{x})) = \tilde{\pi}(\Xi(\mathbf{x}))$.

Observe that

$$\xi\left(u_n^{-1}(\|\mathbf{X}_j\|)\frac{\mathbf{X}_j}{\|\mathbf{X}_j\|}\right) = \frac{\mathbf{X}_j}{a_n}. \tag{2.2.15}$$

Theorem 2.2.4. *Let* $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ *be a stationary jointly* α*-regularly varying sequence. Then, the piling process is well defined ([BS09]). If* $Д_{q_n}$ *and* $Д'_{q_n}$ *are verified then*

$$N_n^{'} = \sum_{i=1}^{\infty} \delta_{\left(i/k_n, \Xi^{\#}(\tilde{\pi}(\mathbb{X}_{n,i}))\right)} \tag{2.2.16}$$

converges weakly (in the space of boundedly finite point measures on $\mathbb{R}_0^+ \times \tilde{l}_0 \setminus \{\mathbf{0}\}$ *with weak*$^{\#}$ *topology) to*

$$N' = \sum_{i=1}^{\infty} \delta_{(T_i, U_i^{-\frac{1}{\alpha}} \tilde{\Xi}(\tilde{\mathbf{Q}}_i))} \tag{2.2.17}$$

where T_i*,* U_i *and* $\tilde{\mathbf{Q}}_i$ *are as in* N *in Theorem 2.1.11.*

Remark 2.2.5. Theorem 2.2.4 follows from Theorem 2.1.11 by use of the Continuous Mapping Theorem for $\Xi^{\#}$.

2.2.3 Main Theorem

Under the assumption that the piling process is well defined, let $(\mathcal{Q}_j)_{j\in\mathbb{Z}}$ where, for all $j \in \mathbb{Z}$,

$$\mathcal{Q}_j = \xi(Q_j) \tag{2.2.18}$$

for Q_j as in (2.1.16).

Theorem 2.2.6 ([FFT20, Theorem 2.4])**.** *Let* $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ *be as described in the beginning of the chapter with* g *of type* g_2*, and such that (2.2.11) holds. Assume that the piling process as given by Definition 2.1.4 is well defined. For* $\alpha \in [1,2)$ *assume that, for all* $\delta > 0$*,*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\left(\max_{1\le k\le n} \left\| \sum_{j=1}^{k} \left(\mathbf{X}_j \mathbb{1}_{\|\mathbf{X}_j\|\le \varepsilon a_n}\right) - \mathbb{E}\left(\mathbf{X}_j \mathbb{1}_{\|\mathbf{X}_j\|\le\varepsilon a_n}\right) \right\| \ge \delta a_n \right) = 0.$$

Additionally, assume that

$$\mathbb{E}\left(\left(\sum_{j\in\mathbb{Z}} \|\mathcal{Q}_j\| \right)^{\alpha} \right) < \infty,$$

when $\alpha \in (1,2)$*, or that*

$$\mathbb{E}\left(\sum_{j\in\mathbb{Z}} \|\mathcal{Q}_j\| \log\left(\|\mathcal{Q}_j\|^{-1} \sum_{i\in\mathbb{Z}} \|\mathcal{Q}_i\| \right) \right) < \infty,$$

when $\alpha = 1$*. Then,*

$$S_n(t) = \sum_{i=0}^{\lfloor nt\rfloor -1} \frac{1}{a_n} \mathbf{X}_i - t c_n, \ t \in [0,1],$$

converges in F' *to* $\underline{V} = (V, disc(V), (e_V^s)_{s\in disc(V)})$*, where* V *is an* α*-stable Lévy process on* $[0,1]$ *which, for* $\alpha \in (0,1)$*, can be written*

$$V(t) = \sum_{T_i \le t} \sum_{j\in\mathbb{Z}} U_i^{-\frac{1}{\alpha}} \mathcal{Q}_{i,j}$$

and for $\alpha \in [1,2)$,

$$V(t) = \lim_{\varepsilon\to 0}\left(\sum_{T_i\le t}\sum_{j\in\mathbb{Z}} U_i^{-\frac{1}{\alpha}}\mathcal{Q}_{i,j}\mathbb{1}_{\{\|U_i^{-\frac{1}{\alpha}}\mathcal{Q}_{i,j}\|>\varepsilon\}} - t\theta\int_0^{+\infty}\mathbb{E}\left(y\sum_{j\in\mathbb{Z}}\mathcal{Q}_j\mathbb{1}_{\{\varepsilon<y\|\mathcal{Q}_j\|\le -1\}}\right)d(-y^{-\alpha})\right),$$

and the excursions are given by

$$e_V^{T_i}(t) = V(T_i^-) + U_i^{-\frac{1}{\alpha}}\sum_{0\le j\le\lfloor\tan(\pi(t-\frac{1}{2}))\rfloor}\mathcal{Q}_{i,j},\ t\in[0,1]$$

where T_i *and* U_i *are as in* N *in Theorem 2.1.11 and* $\mathcal{Q}_{i,j} = \xi(Q_{i,j})$ *for* $Q_{i,j}$ *as in* N *in Theorem 2.1.11 and* ξ *as in (2.2.13).*

Remark 2.2.7. In our applications we are going to restrict to $\alpha \in (0,1)$ but we opted to write down the most general statement of [FFT20, Theorem 2.4]. In particular, for $\alpha \in (0,1)$ we have $c_n = 0$ for all $n \in \mathbb{N}$.

Example 2.2.8. Let $f(x) = 2x \mod 1$, $x \in [0,1]$, and μ = Lebesgue measure on $[0,1]$ (invariant for f). Take $\zeta = 0$ (fixed point), and define the observable ψ as

$$\psi(x) := |x|^{-2}$$

i.e. $\psi(x) = g(\mathrm{dist}(x,0))$ where $g(y) = y^{-\frac{1}{\alpha}}$ for $\alpha = \frac{1}{2}$. Let $X_n = \psi\circ f^n$ for all $n \in \mathbb{N}_0$.

Equation (2.2.11) holds with $a_n = n^2$.

The piling process is the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$

$$(\ldots,\infty,U,U.2,U.2^2,U.2^3,\ldots)$$

where U is uniformly distributed on $[0,1]$, *i.e.* $Z_j = U.2^j$ for all $j\in\mathbb{N}_0$ and $Z_j = \infty$ otherwise.

Since f has exponential decay of correlations for observables in BV against L^1, conditions $Д_{q_n}$ and $Д'_{q_n}$ hold. So, by Theorem 2.1.11,

$$N_n = \sum_{i=1}^{\infty}\delta_{(i/k_n,\tilde{\pi}(\mathbb{X}_{n,i}))} \xrightarrow{w^\#} N = \sum_{i=1}^{\infty}\delta_{(T_i,U_i\tilde{\mathbf{Q}}_i)}$$

where N is a Poisson process with intensity measure $Leb\times\eta$, where $\eta = \dfrac{1}{2}(Leb\times\mathbb{P}_{\tilde{\mathbf{Q}}})\circ\psi$ and $\tilde{Q}$ is (a.s.) the bi-infinite sequence

$$(\ldots,\infty,1,2,2^2,2^3,\ldots).$$

By Theorem 2.2.6,

$$S_n(t) = \sum_{i=0}^{\lfloor nt\rfloor-1}\frac{X_i}{n^2},\ t\in[0,1],$$

converges in F' to $\underline{V} = (V,\mathrm{disc}(V),(e_V^s)_{s\in\mathrm{disc}(V)})$, where V is an α-stable Lévy process on $[0,1]$ which can be written

$$V(t) = \sum_{T_i\le t}\sum_{j\in\mathbb{Z}}U_i^{-2}\mathcal{Q}_{i,j}$$

and the excursions are given by

$$e_V^{T_i}(t) = V(T_i^-) + U_i^{-2} \sum_{0 \le j \le \lfloor \tan(\pi(t-\frac{1}{2})) \rfloor} \mathcal{Q}_{i,j}, \ t \in [0,1]$$

where T_i and U_i are as in N above and $\mathcal{Q}_{i,j} = \xi(Q_{i,j})$ where $\xi(Q_j) = 2^{-2j}$.

We comment on the proof of Theorem 2.2.6 (which is detailed in Section 4.1 of [FFT20]).

The convergence of the REPP given by Theorem 2.1.11 is the crucial preliminary step in the proof of Theorem 2.2.6. In fact, by use of the Continuous Mapping Theorem (CMT) for $\Xi^{\#}$ we obtain $N_n^{'} \to N^{'}$ weakly (in the space of boundedly finite point measures on $\mathbb{R}_0^+ \times \tilde{l}_0 \setminus \{\mathbf{0}\}$ with weak$^{\#}$ topology) as given by Theorem 2.2.4. This implies that a point process convergence derived in the dynamical setting (*i.e.* $N_n \to N$ weakly) results in a point process convergence in the pure probabilistic setting of [BPS18] (*i.e.* $N_n^{'} \to N^{'}$ weakly).

We observe that, as long as the dynamically defined stochastic sequence $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ has α-regularly varying tails, the intensity measure of the limiting Poisson point process N' (obtained from N by use of the CMT) provides us with a Lévy measure. An α-stable Lévy process then shows up as the limit of a Poisson integral of the Poisson point process. For more on this see the beginning of Section 2.5 of [FFT20] and references therein.

Now, our goal is to prove the convergence of S_n to $\underline{V} \in F'$. Recall from the end of Section 2.2.1 that convergence in F' results from the convergence of the respective projections into E and into $\tilde{D}$. The convergence in E follows from [BPS18] and the convergence in $\tilde{D}$ is shown in [FFT20, Proposition 4.4].

Chapter 3

Correlated Maximal Sets

We apply the theory outlined in Chapter 2 to the context of correlated maximal sets, that is when the maximal set is made up of a finite or countable set of points belonging to the same dynamical orbit.

This set up was first introduced in [AFFR16] and [AFFR17] where the finite and countable $\mathcal{M}$, respectively, were investigated from the perspective of EVL and one-dimensional REPP. The conclusion is that a correlated maximal set mimics periodicity, in the sense that the clustering patterns observed have similarities to those obtained when $\mathcal{M}$ consisted of a single periodic point. Thus, correlated maximal sets are responsible for what has often been called 'fake periodicity'. And, in fact, there is more flexibility (for the clustering patterns) than with true periodicity. We elaborate a bit more on this.

Consider an exceedance of a high threshold by a dynamically defined stochastic process which, as usual, corresponds to a visit to a small neighbourhood of $\mathcal{M}$. To begin with, notice that such neighbourhood is no longer a ball centred at a certain $\zeta \in \mathcal{X}$ (as was the case in all the previously studied scenarios where $\mathcal{M} = \{\zeta\}$) but rather a union of balls centred at points which belong to the orbit of ζ, say ζ, $f(\zeta)$ and $f^3(\zeta)$. Now, if the referred exceedance means a visit to a ball centred at ζ, then a visit to a ball centred at $f(\zeta)$ might occur in one time step and, two time steps after that, there might be a visit to a ball centred at $f^3(\zeta)$ which ends the cluster. But it might also be the case that the first exceedance corresponds to a visit to a neighbourhood of $f(\zeta)$ which is followed, in two time steps, by a visit to a neighbourhood of $f^3(\zeta)$ which ends the cluster, or even that the first exceedance corresponds to a visit to a neighbourhood of $f^3(\zeta)$ and we are done. These are clearly three different clustering patterns, for $\mathcal{M} = \{\zeta, f(\zeta), f^3(\zeta)\}$, which can be recorded, in light of [FFT20], via the piling process.

If we think of fake periodicity, as in the case just described accounting for the three possible fake periodic phenomenon, then the formulas derived for the extremal index, θ, in [AFFR16] can been interpreted as an average of the fake periodic behaviours. In fact, if, for example, $\mathcal{M} = \{\zeta, f(\zeta), f^3(\zeta)\}$ with ζ periodic, then the extremal index is not the same as when $\mathcal{M} = \{\zeta\}$ with ζ periodic. The formal statements (which we are going to apply to our examples in the next sections) can be seen in corollaries 4.5 and 5.4 of [AFFR16]. Besides, the one-dimensional REPP converges to a compound Poisson process whose multiplicity distribution is no longer geometric (recall that for $\mathcal{M} = \{\zeta\}$ with ζ periodic and repelling the multiplicity distribution is geometric with parameter θ) which again validates the argument that clustering patterns obtained in the setting of [AFFR16] and [AFFR17] bring more variety than the ones seen before.

Furthermore, when $\mathcal{M} = \{\zeta\}$ with ζ a repelling periodic point, a cluster of observations consists of a bulk of strictly decreasing observations due to the repeated entries (at times multiple of the period) in the successively larger neighbourhoods of ζ (that are due to the original neighbourhood expanding as a result of the underlying dynamics). This isn't necessarily the case when $\mathcal{M}$ is a correlated maximal set where the successive observations due to the successive entries in the neighbourhoods of the points in a certain orbit may increase or decrease depending on the observable (which may itself change along the orbit). This results in an expanding dynamics at every point in a certain orbit being, for a correlated maximal set, compatible with clusters of strictly increasing observations.

We fix some notation.

For $\mathcal{I}$ a finite or countable set of indices, let $\mathcal{M} = \{\xi_i\}_{i\in\mathcal{I}}$ such that there exists m_i with $\xi_i = f^{m_i}(\zeta)$, where $\zeta \in \mathcal{X}$, and we take $m_1 = 0$ (*i.e.* $\xi_1 = \zeta$).

We require the following for f and μ:

(R1) along the orbit of ζ, f is C^1 and locally invertible;

(R2) μ is an *acip* with density ρ satisfying

$$\lim_{\varepsilon\to 0}\frac{\mu(B_\varepsilon(x))}{\mathrm{Leb}(B_\varepsilon(x))} = \rho(x)$$

which is finite on $\mathcal{M}$; for all $\xi_i \in \mathcal{M}$ let $D_i \equiv \rho(\xi_i)$.

Let Ψ be defined as

$$\Psi(x) = \sum_{i\in\mathcal{I}} h_i(\mathrm{dist}(x,\xi_i))\frac{\Phi_{\xi_i}^{-1}(x)}{\|\Phi_{\xi_i}^{-1}(x)\|}\mathbb{1}_{W_i}(x) \tag{3.0.1}$$

in the union of some neighbourhoods of each of the ξ_i, $i \in \mathcal{I}$, and equal to zero outside of it, where, for $\alpha \in (0,2)$ and for all $i \in \mathcal{I}$,

$$h_i(x) = c_i x^{-\frac{1}{\alpha}} \tag{3.0.2}$$

with $c_i > 0$ and $\Phi_{\xi_i} : V_i \to W_i$ denotes a diffeomorphism defined on an open ball, V_i, around ξ_i in $T_{\xi_i}\mathcal{X}$ (the tangent space at ξ_i) onto a neighbourhood, W_i, of ξ_i in $\mathcal{X}$ such that $\Phi_{\xi_i}(E^{s,u} \cap V_i) = W_{\xi_i}^{s,u} \cap W_i$.

The presence of $\dfrac{\Phi_{\xi_i}^{-1}(x)}{\|\Phi_{\xi_i}^{-1}(x)\|}$ in the definition of Ψ allows for labelling the observation $\Psi(x)$ with a direction, corresponding to the projection of x in $T_{\xi_i}\mathcal{X}$. In particular, when $\mathcal{X}$ is a subset of $\mathbb{R}$ then Ψ reduces to $\psi(x) = \sum_{i\in\mathcal{I}} h_i(|x-\xi_i|)\mathbb{1}_{V_i}(x)$ where V_i is an open interval around ξ_i.

In general, h_i is assumed to have one of three types of behaviour, g_1, g_2 or g_3, which correspond to the three classical EVL but since here we are restricting to the context of [FFT20, Theorem 2.4] only g_2 should be considered. In fact, it is interesting to observe the results obtained in terms of the clustering patterns when, for example, h_1 is of type g_1 and h_2 is of type g_2 (see Section 7 of [AFFR16]) but that extends beyond the focus of this work.

Notice that we are even restricting within type g_2 as we are taking h_i as defined above but that still is a rather general assumption. In particular, the choice of different c_i is what allows for the appearance of clustering patterns which no longer restrict to the strictly decreasing ones associated to periodicity, as shown in our examples ahead.

In the following sections we will prove general statements for the piling processes in the setting of correlated maximal sets: for a finite correlated maximal set, consisting of a finite number of points belonging to the same orbit which may be non-periodic or periodic, and for a countable correlated maximal set. We illustrate our statements with concrete examples. Then, we prove that the dependence requirements (*i.e.* the conditions $Д_{q_n}$ and $Д'_{q_n}$ from Section 2.1.3) hold. Hence, despite not having written them down here, the enriched FLT follow directly from our work.

3.1 A finite number of points in the same orbit

Here $\mathcal{M} = \{\xi_1, \dots, \xi_N\}$ such that there exist $m_1, \dots, m_N$ with $\xi_i = f^{m_i}(\zeta)$, where $\zeta \in \mathcal{X}$, and we take $m_1 = 0$ (*i.e.* $\xi_1 = \zeta$). The piling processes are fundamentally different when ζ is not/is periodic, as can be seen in Theorem 3.1.2 and in Theorem 3.1.14, respectively. We illustrate the use of the same theorems with examples inspired in those in sections 4.3 and 5.3 of [AFFR16].

Before presenting the statements and proofs of Theorems 3.1.2 and 3.1.14 we provide some heuristics and fix notation.

Let an exceedance of the level $u_n(\tau)$, at time r_n, be due to a hit to a specified small neighbourhood of ξ_i for a certain $i \in \{1, \dots, N\}$ that we now fix. Then, at times $r_n + j$ where $j = m_l - m_i$ for all $l = 1, \dots, i-1, i+1, \dots, N$, there are hits to neighbourhoods of $f^j(\xi_i)$, which correspond to exceedances of levels $u_n(\tau_1), \dots, u_n(\tau_{i-1}), u_n(\tau_{i+1}), \dots, u_n(\tau_N)$. The piling process stores, at position $j = m_l - m_i$ for all $l = 1, \dots, i-1, i+1, \dots, N$, (the asymptotic behaviour of) the ratio $\dfrac{\tau_l}{\tau}$ projected on the direction of the point whose observation exceeds $u_n(\tau_l)$. As we prove further on, this is (asymptotically) tied to the derivative of f and the constants c_i appearing in the h_i (recall (3.0.2)).

In addition, the definition of the piling process requires that all negatively indexed entries have norms greater than or equal to 1 (*cf.* Definition 2.1.4). When the c_i are all equal and f is expanding, then all negatively indexed entries must be equal to ∞. However, when the c_i differ, whether it is possible to have negatively indexed entries with norms greater than or equal to 1 (and that are not ∞) depends on the balance between $\left(\dfrac{c_i}{c_l}\right)^\alpha$ and the norm of $Df_{\xi_i}^{m_l - m_i}$, for all $l = 1, \dots, i-1$. One can think in terms of how much the factor $\left(\dfrac{c_i}{c_l}\right)^\alpha$ makes a contraction look like an expansion - we use the expression 'fake expansion' to refer to this.

We are going to restrict to the cases where fake expansion, if it exists, holds in every direction. We remark that we can always have that for a suitable choice of the constants c_i (depending on the derivative of f). Although we miss full generality with such an assumption, we do it to avoid an otherwise very technical statement.

The set $A^{(i)}$, that we define next, stores the indices $j = m_l - m_i$, for all $l \in \{1, \dots, i-1\}$, compatible with entries different from ∞ that appear left of index $j = 0$ (indices compatible with fake expansion).

Let $\mathcal{I} = \{1, \dots, N\}$ (recall from Section 1 that $\mathcal{M} = \{\xi_1, \dots, \xi_N\}$ corresponds to $\mathcal{I} = \{1, \dots, N\}$).

We split into two cases according to ζ (*i.e.* ξ_1) being non-periodic or periodic, respectively.

(A1) Assume ζ is non-periodic. Let $i \in \mathcal{I}$. If $i > 1$, for all $l = 1, \ldots, i-1$, let

$$\lambda_{i,l}^{min} = \min\{\|Df_{\xi_i}^{m_l - m_i}(w)\| : \|Df_{\xi_i}^{m_l - m_i}(w)\| < 1, w \in \mathbb{S}^{d-1}\},$$

$$\lambda_{i,l}^{max} = \max\{\|Df_{\xi_i}^{m_l - m_i}(w)\| : \|Df_{\xi_i}^{m_l - m_i}(w)\| < 1, w \in \mathbb{S}^{d-1}\},$$

and define $u_{i,l}^{min} = \left(\frac{c_l}{c_i}\right)^{\alpha} \frac{1}{\lambda_{i,l}^{min}}$ and $u_{i,l}^{max} = \left(\frac{c_l}{c_i}\right)^{\alpha} \frac{1}{\lambda_{i,l}^{max}}$. Let $A^{(1)} := \emptyset$ and, for all $i > 1$, $A^{(i)} := \{m_l - m_i : u_{i,l}^{min} < 1\}$.

(A2) Assume ζ is periodic of prime period q. Let $i \in \mathcal{I}$. For all $i, l \in \mathcal{I}$, if $s \in \mathbb{N}_0$ is such that $m_l - m_i - qs < 0$, let

$$\lambda_{i,l,s}^{min} = \min\{\|Df_{\xi_i}^{m_l - m_i - qs}(w)\| : \|Df_{\xi_i}^{m_l - m_i - qs}(w)\| < 1, w \in \mathbb{S}^{d-1}\},$$

$$\lambda_{i,l,s}^{max} = \max\{\|Df_{\xi_i}^{m_l - m_i - qs}(w)\| : \|Df_{\xi_i}^{m_l - m_i - qs}(w)\| < 1, w \in \mathbb{S}^{d-1}\},$$

and define $u_{i,l,s}^{min} = \left(\frac{c_l}{c_i}\right)^{\alpha} \frac{1}{\lambda_{i,l,s}^{min}}$ and $u_{i,l,s}^{max} = \left(\frac{c_l}{c_i}\right)^{\alpha} \frac{1}{\lambda_{i,l,s}^{max}}$.

Let $A^{(i)} := \{m_l - m_i - qs : u_{i,l,s}^{min} < 1\}$.

We give a visual interpretation of how $A^{(i)}$ is obtained in case (A1) in Figure 3.1. In case (A2) the reasoning is analogous accounting for the successive hits to neighbourhoods of the points in $\mathcal{M}$ at every integer multiple of the period q.

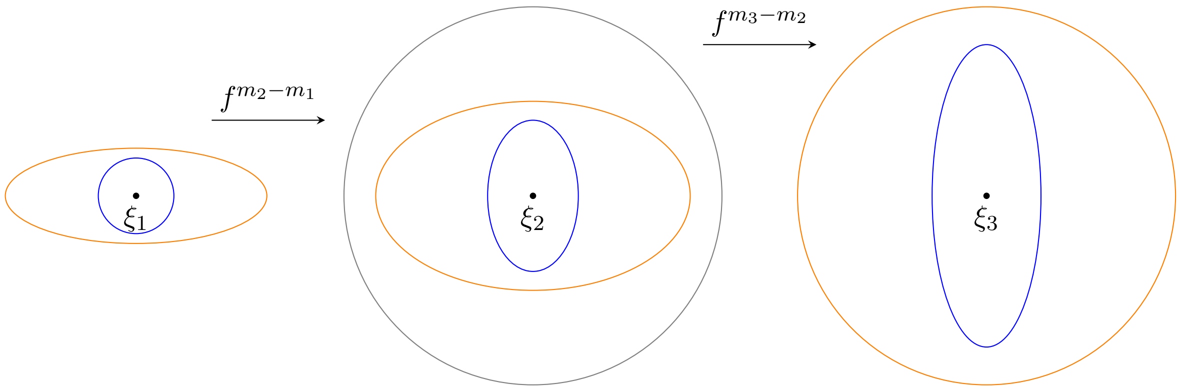

Figure 3.1: Example with $\mathcal{I} = \{1, 2, 3\}$, $i = 3$ and $A^{(3)} = \{m_1 - m_3\}$, for a 2-dimensional uniformly expanding f and observable Ψ as in (3.0.1) with c_1 sufficiently smaller than c_3 so that fake expansion holds at $j = m_1 - m_3$, and $c_2 = c_3$ (*cf.* (3.0.2)). The boundaries of the balls of radius $h_i^{-1}(u_n(\tau))$ centred at ξ_i for $i = 1, 2, 3$ are the blue, grey and orange circles, respectively. We may say blue, grey or orange ball to refer to, respectively, the ball around ξ_1, ξ_2 or ξ_3 whose boundary is the blue, grey or orange circle. We condition on an exceedance of the threshold $u_n(\tau)$ being due to a hit (at time $j = 0$) to the orange ball. Then, at times $j = m_2 - m_3$ and $j = m_1 - m_3$ there may have been hits to some neighbourhoods of ξ_2 and ξ_1, respectively. Because the connected component of the $(m_3 - m_2)$-th pre-image of the orange ball which intersects the grey ball is strictly contained in the grey ball, we have $u_{3,2}^{max} > 1$ (which implies $u_{3,2}^{min} > 1$). On the other hand, the connected component of the $(m_3 - m_1)$-th pre-image of the orange ball which

intersects the blue ball strictly contains the blue ball which is compatible with $u_{3,1}^{min} < 1$. This leads to $A^{(3)} = \{m_1 - m_3\}$. Thus, the piling process has (a.s.) an infinity entry at position $j = m_2 - m_3$; however, there exists a region around ξ_3 (the annulus delimited by the blue ellipse and the orange circle) for which the piling process has non-infinity entries at position $j = m_1 - m_3$. We note that the piling process still has infinity entries at position $j = m_1 - m_3$ if the hit to the orange ball around ξ_3 belongs to the interior of the blue ellipse, or if it is preceded by a hit (at time $m_1 - m_3$) to a neighbourhood of a pre-image of ξ_3 which is not ξ_1.

Remark 3.1.1. The condition $u_{i,l}^{min} < 1$, where $i \in \mathcal{I}$ and $l \in \{1, \ldots, i-1\}$, expresses the geometrical requirement that, for all $n \in \mathbb{N}$, the image under $f^{m_i - m_l}$ of the ball of radius $h_l^{-1}(u_n(\tau))$ around ξ_l is strictly contained in the ball of radius $h_i^{-1}(u_n(\tau))$ around ξ_i. In Figure 3.1, the blue ball around ξ_1 is mapped by $f^{m_3 - m_1}$ to an ellipse which is strictly contained in the orange ball around ξ_1.

Moreover, the balance between $\left(\frac{c_i}{c_l}\right)^{\alpha}$ and the norm of $Df_{\xi_i}^{m_l - m_i}$, for all $l = 1, \ldots, i-1$, also determines the probabilities with which the different sequences corresponding to the indices $m_l - m_i \in A^{(i)}$ appear - due to the $u_{i,l}^{min}, u_{i,l}^{max}$ determining ranges for each sequence. This is formalised in the statements of our theorems.

3.1.1 Main theorem and examples in the non-periodic case

Theorem 3.1.2. *Let f be a probability preserving system which preserves μ. Additionally, let f and μ be such that (R1) and (R2) hold. Let Ψ be as given by (3.0.1) for $\mathcal{M}$ as in Section 3.1, where ζ is a non-periodic point. Assume that $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ has an α-regularly varying tail, where $\alpha \in (0,1)$. For all $i \in \mathcal{I}$, define $p_i = \frac{D_i c_i^{\alpha d}}{\sum_{k=1}^{N} D_k c_k^{\alpha d}}$. Let $A^{(i)}$ be as defined in (A1). If $A^{(i)} = \emptyset$, the piling process is*

(0) with probability p_i the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

- *(i) entry $U.\Theta$ at $j = 0$;*
- *(ii) entries $U.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ at $j = m_l - m_i$ for all $l = i+1, \ldots, N$;*
- *(iii) ∞ for all other positive indices j;*
- *(iv) ∞ for all negative indices j;*

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

If $A^{(i)} \neq \emptyset$, assume there exists an increasing ordering of the $u_{i,l}^{min,max}$ such that $u_{i,l_p}^{min} \leq u_{i,l_{p+1}}^{max}$ for all $p \in \{1, \ldots, \#A^{(i)} - 1\}$. Then, the piling process is

(I) with probability $p_i u_{i,l_1}^{max}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

- *(i) entry $U_0.\Theta$ at $j = 0$;*
- *(ii) entries $U_0.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ at $j = m_l - m_i$ for all $l = i+1, \ldots, N$;*
- *(iii) ∞ for all other positive indices j;*
- *(iv) ∞ for all negative indices j;*

where U_0 is uniformly distributed on $[0, u^{max}_{i,l_1})$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U_0 and Θ are independent;

(II) with probability $p_i(u^{min}_{i,l_p} - u^{max}_{i,l_p})$, where $p \in \{1, \ldots, \#A^{(i)}\}$, the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry $U_{p'}.\Theta$ at $j = 0$;

(ii) entries $U_{p'}.Df^{m_l-m_i}_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l = i+1, \ldots, N$;

(iii) ∞ for all other positive indices j;

(iv) entries $U_{p'}.Df^{m_l-m_i}_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l \in \{l_1, \ldots, l_p\}$;

(v) ∞ for all other negative indices j;

where $U_{p'}$ is uniformly distributed on $[u^{max}_{i,l_p}, u^{min}_{i,l_p})$ and $\Theta \mid \{U_{p'} = u\}$ is uniformly distributed on $\left\{w \in \mathbb{S}^{d-1} : \|Df^{m_{l_p}-m_i}_{\xi_i}(w)\| \geq \frac{1}{u}\left(\frac{c_{l_p}}{c_i}\right)^\alpha\right\}$;

(III) with probability $p_i(u^{max}_{i,l_{p+1}} - u^{min}_{i,l_p})$, where $p \in \{1, \ldots, \#A^{(i)}-1\}$, the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry $U_p.\Theta$ at $j = 0$;

(ii) entries $U_p.Df^{m_l-m_i}_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l = i+1, \ldots, N$;

(iii) ∞ for all other positive indices j;

(iv) entries $U_p.Df^{m_l-m_i}_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l \in \{l_1, \ldots, l_p\}$;

(v) ∞ for all other negative indices j;

where U_p is uniformly distributed on $[u^{min}_{i,l_p}, u^{max}_{i,l_{p+1}})$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U_p and Θ are independent;

(IV) with probability $p_i(1 - u^{min}_{i,l_{\#A^{(i)}}})$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry $U_{\#A^{(i)}}.\Theta$ at $j = 0$;

(ii) entries $U_{\#A^{(i)}}.Df^{m_l-m_i}_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l = i+1, \ldots, N$;

(iii) ∞ for all other positive indices j;

(iv) entries $U_{\#A^{(i)}}.Df^{m_l-m_i}_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l \in A^{(i)}$;

(v) ∞ for all other negative indices j;

where $U_{\#A^{(i)}}$ is uniformly distributed on $[u^{min}_{i,l_{\#A^{(i)}}}, 1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and $U_{\#A^{(i)}}$ and Θ are independent.

Remark 3.1.3. Assume f is expanding and, for all $i \in \mathcal{I}$, $c_i = c$. Then, for any $i \in \mathcal{I}$, $\|Df^{m_l-m_i}_{\xi_i}(w)\| < 1$ for all $l = 1, \ldots, i-1$ and for all $w \in \mathbb{S}^{d-1}$. It follows that $A^{(i)} = \emptyset$ for all $i \in \mathcal{I}$. In particular, the piling process will be of the simplest form given by case (0) in Theorem 3.1.2 (where, in (ii), $\left(\frac{c_i}{c_l}\right)^\alpha = 1$ implies the reduction to $U.Df^{m_l-m_i}_{\xi_i}(\Theta)$).

Remark 3.1.4. If f is 1-dimensional then $u_{i,l}^{min} = u_{i,l}^{max}$ and case (II) in Theorem 3.1.2 doesn't occur.

Remark 3.1.5. When $A^{(i)} \neq \emptyset$, the requirement for an increasing ordering of the $u_{i,l}^{min,max}$ such that $u_{i,l_p}^{min} \leq u_{i,l_{p+1}}^{max}$ for all $p \in \{1, \ldots, \#A^{(i)} - 1\}$ means a picture like Figure 3.2. We observe that our choice of c_i for h_i (*cf.* (3.0.2)), besides being tied to fake expansion, determines the existence of such increasing ordering. When f is 1-dimensional it reduces to $u_{i,l_p} \leq u_{i,l_{p+1}}$, for all $p \in \{1, \ldots, \#A^{(i)} - 1\}$, which is trivially satisfied.

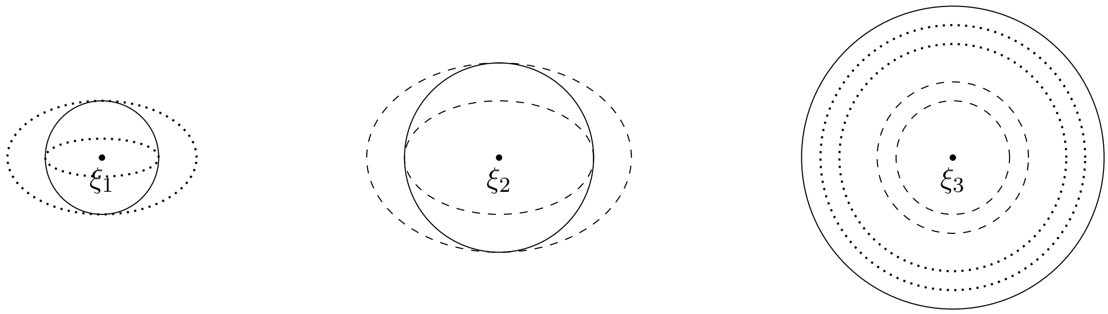

Figure 3.2: Example with $\mathcal{I} = 3$, $i = 3$ and $A^{(3)} = \{1, 2\}$, for a 2-dimensional uniformly expanding f and observable Ψ as in (3.0.1) with c_1 sufficiently smaller than c_2 and c_2 sufficiently smaller than c_3 so that fake expansion holds at both $j = m_1 - m_3$ and $j = m_2 - m_3$ and, additionally, $u_{3,l_1}^{min} \leq u_{3,l_2}^{max}$ with $l_1, l_2 \in \{1, 2\}$. The boundaries of the balls of radius $h_i^{-1}(u_n(\tau))$ centred at ξ_i for $i = 1, 2, 3$ are the three black circles. We condition on an exceedance of the threshold $u_n(\tau)$ being due to a hit (at time $j = 0$) to the ball of radius $h_3^{-1}(u_n(\tau))$ around ξ_3. The dotted annulus around ξ_1 (*i.e.* the annulus delimited by the dotted ellipses around ξ_1) is mapped by $f^{m_3 - m_1}$ to the dotted annulus around ξ_3 (*i.e.* the annulus delimited by the dotted circles around ξ_3). Analogous statement for the dashed annuli around ξ_2 and ξ_3 and $f^{m_3 - m_2}$. The interior of the smaller dotted circle around ξ_3 is compatible with infinity entries, in every direction, for the piling process at position $m_1 - m_3$, while the exterior of the bigger dotted circle around ξ_3 is compatible with non-infinity entries, in every direction, for the piling process at position $j = m_1 - m_3$ (*cf.* Figure 3.1). Analogous statement holds for the dashed annuli and entries at position $j = m_2 - m_3$. The requirement $u_{3,l_1}^{min} \leq u_{3,l_2}^{max}$ means that the dotted and dashed annuli around ξ_3 are disjoint. Then, a non-infinity entry in any chosen direction at position $j = m_1 - m_3$ implies a non-infinity entry in every direction at position $j = m_2 - m_3$. In particular, $l_1 = 2$ and $l_2 = 1$.

We illustrate simple cases of Theorem 3.1.2 by Examples 3.1.6 and 3.1.8. After these, we provide an example of application of the general version of the theorem.

Example 3.1.6. Let $f(x) = 2x \mod 1$, $x \in [0, 1]$, and $\mu =$ Lebesgue measure on $[0, 1]$ (invariant for f). Take $\zeta = \dfrac{\sqrt{2}}{16}$ (non-periodic), and define the observable ψ as

$$\psi(x) := \begin{cases} |x - \zeta|^{-2}, \ x \in B_{\varepsilon_1}(\zeta) \\ |x - f(\zeta)|^{-2}, \ x \in B_{\varepsilon_2}(f(\zeta)) \\ |x - f^3(\zeta)|^{-2}, \ x \in B_{\varepsilon_3}(f^3(\zeta)) \\ 0, \ \text{otherwise} \end{cases}$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. Observe that, presented as in (3.0.1),

$$\psi(x) = \sum_{i=1}^{3} h_i(|x - \xi_i|)\mathbb{1}_{B_{\varepsilon_i}(\xi_i)}(x)$$

for $\xi_1 = \zeta$, $\xi_2 = f(\zeta)$, $\xi_3 = f^3(\zeta)$, and $h_i(t) = t^{-2}$ for $i = 1, 2, 3$ (so that $\alpha = 1/2$). In particular, $\mathcal{M} = \{\zeta, f(\zeta), f^3(\zeta)\}$ and equation (2.2.11) holds with $a_n = 36n^2$.

Since μ = Lebesgue, we have that $D_i = 1$ for $i = 1, 2, 3$. Also, $d = 1$ and $c_i = 1$ for $i = 1, 2, 3$. Thus, $p_1 = p_2 = p_3 = \dfrac{1}{3}$. We have $f'(x) = 2$ for all $x \in [0, 1]$, so that $(f^{-j})'(\xi_i) = \dfrac{1}{2^j}$ when $j < 0$ for $i = 1, 2, 3$ leading to $A^{(i)} = \emptyset$ for $i = 1, 2, 3$. Applying Theorem 3.1.2, we conclude that the piling process is any of the bi-infinite sequences $(Z_j)_{j\in\mathbb{Z}}$

$$\big(\dots, \infty, U, U.2, \infty, U.2^3, \infty, \dots\big)$$
$$\big(\dots, \infty, U, \infty, U.2^2, \infty, \dots\big)$$
$$\big(\dots, \infty, U, \infty, \dots\big)$$

each with probability $\dfrac{1}{3}$, where U is uniformly distributed on $[0, 1]$.

We clarify the notation used by stressing that, in all the three sequences presented, the entries equal to U correspond to index $j = 0$ and the entries that are visibly different from ∞ are the only such entries, as is imposed by the statement of Theorem 3.1.2.

Remark 3.1.7. For Example 3.1.6, we have

$$\mu(U_n(\tau)) = \sum_{i=1}^{3} \mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i)) = 3.2u_n(\tau)^{-\frac{1}{2}}.$$

Let $q_n = m_3 - m_1 = 3$. It follows from [AFFR16, Corollary 4.5],

$$\begin{aligned}\mu(U_n^{(q_n)}(\tau)) &= \mu(B_{h_1^{-1}(u_n(\tau))}(\xi_1)) - \frac{1}{2}\mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2)) \\ &\quad + \mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2)) - \frac{1}{2^2}\mu(B_{h_3^{-1}(u_n(\tau))}(\xi_3)) \\ &\quad + \mu(B_{h_3^{-1}(u_n(\tau))}(\xi_3)) \\ &= 2u_n(\tau)^{-\frac{1}{2}}(1 - \frac{1}{2} + 1 - \frac{1}{2^2} + 1).\end{aligned}$$

Thus, the extremal index is

$$\vartheta = \lim_{n\to\infty} \frac{\mu(U_n^{(q_n)}(\tau))}{\mu(U_n(\tau))} = \lim_{n\to\infty} \frac{2u_n(\tau)^{-\frac{1}{2}}(1 - \frac{1}{2} + 1 - \frac{1}{2^2} + 1)}{3.2u_n(\tau)^{-\frac{1}{2}}} = \frac{3}{4}.$$

Example 3.1.8. Consider the same f and ζ as in Example 3.1.6, but take ψ to be

$$\psi(x) := \begin{cases} |x - \zeta|^{-2},\ x \in B_{\varepsilon_1}(\zeta) \\ 9|x - f(\zeta)|^{-2},\ x \in B_{\varepsilon_2}(f(\zeta)) \\ |x - f^3(\zeta)|^{-2},\ x \in B_{\varepsilon_3}(f^3(\zeta)) \\ 0,\ \text{otherwise} \end{cases}$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. Observe that, presented as in (3.0.1),

$$\psi(x) = \sum_{i=1}^{3} h_i(|x-\xi_i|)\mathbb{1}_{B_{\varepsilon_i}(\xi_i)}(x)$$

for $\xi_1 = \zeta$, $\xi_2 = f(\zeta)$, $\xi_3 = f^3(\zeta)$, $h_1(t) = h_3(t) = t^{-2}$ and $h_2(t) = 9t^{-2}$ (so that $\alpha = 1/2$). In particular, $\mathcal{M} = \{\zeta, f(\zeta), f^3(\zeta)\}$ and equation (2.2.11) holds with $a_n = 100n^2$.

Again, $D_i = 1$ for $i = 1,2,3$ and $d = 1$, but now $c_1 = c_3 = 1$ and $c_2 = 9$. Thus, $p_1 = p_3 = \dfrac{1^{\frac{1}{2}}}{1^{\frac{1}{2}} + 9^{\frac{1}{2}} + 1^{\frac{1}{2}}} = \dfrac{1}{5}$ and $p_2 = \dfrac{9^{\frac{1}{2}}}{1^{\frac{1}{2}} + 2^{\frac{1}{2}} + 1^{\frac{1}{2}}} = \dfrac{3}{5}$. We have $f'(x) = 2$ for all $x \in [0,1]$, so that $(f^{-j})'(\xi_i) = \dfrac{1}{2^j}$ when $j < 0$ for $i = 1,2,3$.

If $i = 2$, then $\lambda_{2,1}^{min} = \lambda_{2,1}^{max} = \dfrac{1}{2}$, giving $u_{2,1}^{min} = u_{2,1}^{max} = \left(\dfrac{1}{9}\right)^{\frac{1}{2}} \dfrac{1}{\left(\frac{1}{2}\right)} = \dfrac{2}{3}$, leading to $A^{(1)} = \{1\}$.

If $i = 3$, then $\lambda_{3,1}^{min} = \lambda_{3,1}^{max} = \dfrac{1}{2^3}$, giving $u_{3,1}^{min} = u_{3,1}^{max} = \left(\dfrac{1}{1}\right)^{\frac{1}{2}} \dfrac{1}{\left(\frac{1}{2^3}\right)} = 8$, and $\lambda_{3,2}^{min} = \lambda_{3,2}^{max} = \dfrac{1}{2^2}$, giving $u_{3,2}^{min} = u_{3,2}^{max} = \left(\dfrac{9}{1}\right)^{\frac{1}{2}} \dfrac{1}{\left(\frac{1}{2^2}\right)} = 12$, leading to $A^{(2)} = \emptyset$.

Applying Theorem 3.1.2, we conclude that the piling process is one of the bi-infinite sequences $(Z_j)_{j\in\mathbb{Z}}$

$$\left(\dots, \infty, U, U.2\left(\frac{1}{2}\right)^{\frac{1}{9}}, \infty, U.2^3, \infty, \dots\right)$$

$$(\dots, \infty, U, \infty, \dots)$$

each with probability $\dfrac{1}{5}$, where U is uniformly distributed on $[0,1]$; with probability $\dfrac{3}{5}.\dfrac{2}{3} = \dfrac{2}{5}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$

$$\left(\dots, \infty, U_0, \infty, U_0.2^2.9^{\frac{1}{2}}, \infty, \dots\right)$$

where U_0 is uniformly distributed on $[0, 2/3]$; and with probability $\dfrac{3}{5}.\left(1 - \dfrac{2}{3}\right) = \dfrac{1}{5}$

$$\left(\dots, U_1.\frac{1}{2}.9^{\frac{1}{2}}, U_1, \infty, U_1.2^2.9^{\frac{1}{2}}, \infty, \dots\right)$$

where U_1 is uniformly distributed on $[2/3, 1]$.

Again, in all the sequences, U (resp. U_0, U_1) is at index $j = 0$ and the entries that are visibly different from ∞ are the only such entries.

Remark 3.1.9. For Example 3.1.8, we have

$$\mu(U_n(\tau)) = \sum_{i=1}^{3} \mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i)) = 2u_n(\tau)^{-\frac{1}{2}} + 2.3u_n(\tau)^{-\frac{1}{2}} + 2u_n(\tau)^{-\frac{1}{2}} = 5.2u_n(\tau)^{-\frac{1}{2}}.$$

Let $q_n = m_3 - m_1 = 3$. From [AFFR16, Corollary 4.5],

$$\begin{aligned}\mu(U_n^{(q_n)}(\tau)) &= \mu(B_{h_1^{-1}(u_n(\tau))}(\xi_1)) - \frac{1}{2^3}\mu(B_{h_3^{-1}(u_n(\tau))}(\xi_3)) \\ &+ \mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2)) - \frac{1}{2^2}\mu(B_{h_3^{-1}(u_n(\tau))}(\xi_3)) \\ &+ \mu(B_{h_3^{-1}(u_n(\tau))}(\xi_3)) \\ &= 2u_n(\tau)^{-\frac{1}{2}}(1 - \frac{1}{2^3} + 3 - \frac{1}{2^2} + 1).\end{aligned}$$

Thus, the extremal index is

$$\vartheta = \lim_{n\to\infty} \frac{\mu(U_n^{(q_n)}(\tau))}{\mu(U_n(\tau))} = \lim_{n\to\infty} \frac{2u_n(\tau)^{-\frac{1}{2}}(1 - \frac{1}{2^3} + 3 - \frac{1}{2^2} + 1)}{5.2u_n(\tau)^{-\frac{1}{2}}} = \frac{37}{40}.$$

Example 3.1.10. Let $f(x, y) = (2x \mod 1, 3y \mod 1)$, $(x, y) \in [0, 1]^2$, and $\mu =$ Lebesgue measure on $[0, 1]^2$ (invariant for f). Take $\zeta = (\zeta_x, \zeta_y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (non-periodic), and define the observable Ψ as

$$\Psi(x, y) := \begin{cases} \|(x, y) - \zeta\|^{-4}\dfrac{(x - \zeta_x, y - \zeta_y)}{\|(x, y) - \zeta\|}, \ (x, y) \in B_{\varepsilon_1}(\zeta) \\ 256\|(x, y) - f(\zeta)\|^{-4}\dfrac{(x - f(\zeta)_x, y - f(\zeta)_y)}{\|(x, y) - f(\zeta)\|}, \ (x, y) \in B_{\varepsilon_2}(f(\zeta)) \\ 0, \ \text{otherwise} \end{cases}$$

for some $\varepsilon_1, \varepsilon_2 > 0$, where $f(\zeta) = (f(\zeta)_x, f(\zeta)_y)$. Observe that, presented as in (3.0.1),

$$\Psi(x, y) = \sum_{i=1}^{2} h_i(\text{dist}((x, y), \xi_i)) \frac{\Phi_{\xi_i}^{-1}((x, y))}{\|\Phi_{\xi_i}^{-1}((x, y))\|} \mathbb{1}_{B_{\varepsilon_i}(\xi_i)}((x, y))$$

for $\xi_1 = \zeta$, $\xi_2 = f(\zeta)$, $h_1(t) = t^{-4}$ and $h_2(t) = 256t^{-4}$ (so that $\alpha = 1/4$), and $\Phi_{\xi_i}^{-1} : B_{\varepsilon_i}(\xi_i) \to B_{\varepsilon_i}(0)$ being the translation by $-\xi_i$ for $i = 1, 2$. In particular, $\mathcal{M} = \{\zeta, f(\zeta)\}$ and equation (2.2.11) holds with $a_n = \frac{289}{256}n^2$.

Since $\mu =$ Lebesgue, we have $D_1 = D_2 = 1$. Also, $d = 2$ and $c_1 = 1$ and $c_2 = 256$. Thus, $p_1 = \frac{1^{\frac{1}{2}}}{1^{\frac{1}{2}} + 256^{\frac{1}{2}}} = \frac{1}{17}$ and $p_2 = \frac{256^{\frac{1}{2}}}{1^{\frac{1}{2}} + 256^{\frac{1}{2}}} = \frac{16}{17}$.

If $i = 2$, then $\lambda_{2,1}^{\min} = \frac{1}{3}$ and $\lambda_{2,1}^{\max} = \frac{1}{2}$, giving $u_{2,1}^{\min} = \left(\frac{1}{256}\right)^{\frac{1}{4}} \cdot \frac{1}{\left(\frac{1}{3}\right)} = \frac{3}{4}$ and $u_{2,1}^{\max} = \left(\frac{1}{256}\right)^{\frac{1}{4}} \cdot \frac{1}{\left(\frac{1}{2}\right)} = \frac{1}{2}$, leading to $A^{(2)} = \{1\}$. Also, we have the increasing order $0 \le u_{2,1}^{\max} \le u_{2,1}^{\min} \le 1$.

Notice that $(Df_\zeta)^j(\theta_x, \theta_y) = (Df_{f(\zeta)})^j(\theta_x, \theta_y) = (2^j\theta_x, 3^j\theta_y)$, where $j \in \mathbb{Z}$.

Applying Theorem 3.1.2, we conclude that the piling process is

(0) with probability $\frac{1}{17}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

- (i) entry $U.(\Theta_x, \Theta_y)$ at $j = 0$
- (ii) entry $U.(2\Theta_x, 3\Theta_y) \cdot \frac{1}{4}$ at $j = 1$

(iii) ∞ for all other positive indices j

(iv) ∞ for all negative indices j

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^1$, and U and Θ are independent;

(I) with probability $\frac{16}{17} \cdot \frac{1}{2} = \frac{8}{17}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry $U_0.(\Theta_x, \Theta_y)$ at $j = 0$

(ii) ∞ for all positive indices j

(iii) ∞ for all negative indices j

where U_0 is uniformly distributed on $[0, 1/2)$, Θ is uniformly distributed on $\mathbb{S}^1$, and U_0 and Θ are independent;

(II) with probability $\frac{16}{17}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{4}{17}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry $U_{1'}.(\Theta_x, \Theta_y)$ at $j = 0$

(ii) ∞ for all positive indices j

(iii) entry $U_{1'}.\left(\frac{1}{2}\Theta_x, \frac{1}{3}\Theta_y\right) \cdot 4$ at $j = -1$

(iv) ∞ for all other negative indices j

where $U_{1'}$ is uniformly distributed on $[1/2, 3/4)$ and $\Theta \mid \{U_{1'} = z\}$ is uniformly distributed on $\left\{(\theta_x, \theta_y) \in \mathbb{S}^1 : \left\|\left(\frac{1}{2}\theta_x, \frac{1}{3}\theta_y\right)\right\| \geq \frac{1}{4z}\right\}$;

(III) with probability $\frac{16}{17}\left(1 - \frac{3}{4}\right) = \frac{4}{17}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry $U_1.(\Theta_x, \Theta_y)$ at $j = 0$

(ii) ∞ for all positive indices j

(iii) entry $U_1.\left(\frac{1}{2}\Theta_x, \frac{1}{3}\Theta_y\right) \cdot 4$ at $j = -1$

(iv) ∞ for all other negative indices j

where U_1 is uniformly distributed on $[3/4, 1]$, Θ is uniformly distributed on $\mathbb{S}^1$, and U_1 and Θ are independent.

Remark 3.1.11. For Example 3.1.10, we have

$$\mu(U_n(\tau)) = \sum_{i=1}^{2} \mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i)) = u_n(\tau)^{-\frac{1}{2}} + 16u_n(\tau)^{-\frac{1}{2}} = 17u_n(\tau)^{-\frac{1}{2}}.$$

Let $q_n = m_2 - m_1 = 1$. From [AFFR16, Corollary 4.5],

$$\mu(U_n^{(q_n)}(\tau)) = \mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2)) = 16u_n(\tau)^{-\frac{1}{2}}.$$

Thus, the extremal index is

$$\vartheta = \lim_{n\to\infty} \frac{\mu(U_n^{(q_n)}(\tau))}{\mu(U_n(\tau))} = \lim_{n\to\infty} \frac{16u_n(\tau)^{-\frac{1}{2}}}{17u_n(\tau)^{-\frac{1}{2}}} = \frac{16}{17}.$$

Remark 3.1.12. Case (II) expresses that a range of thresholds $\mathcal{U} \subseteq [u_{2,1}^{max}, u_{2,1}^{min})$ determines a range of unit vectors $\mathcal{T} \subseteq \mathbb{S}^1$ for which $u.\|Df_{f(\zeta)}^{-1}(\theta)\|.4 \geq 1$ whenever $u \in \mathcal{U}$ and $\theta \in \mathcal{T}$. For instance, let $\mathcal{U} = [1/2, 3/5)$. Then, $u = 1/2$ gives us

$$\|Df_{f(\zeta)}^{-1}(\theta)\| \geq \frac{1}{2} \iff \left\|\left(\frac{1}{2}\theta_x, \frac{1}{3}\theta_y\right)\right\| \geq \frac{1}{2}.$$

In turn, $u = 3/5$ leads to

$$\|Df_{f(\zeta)}^{-1}(\theta)\| \geq \frac{5}{12} \iff \left\|\left(\frac{1}{2}\theta_x, \frac{1}{3}\theta_y\right)\right\| \geq \frac{5}{12}.$$

Since $\left\|\left(\frac{1}{2}\theta_x, \frac{1}{3}\theta_y\right)\right\| = \frac{5}{12}$ has solution $(\theta_x, \theta_y) = \left(\frac{3}{2\sqrt{5}}, \frac{\sqrt{11}}{2\sqrt{5}}\right)$ when $\theta_x, \theta_y > 0$, our knowledge of the geometry of $Df_{f(\zeta)}^{-1} : T_{f(\zeta)}[0,1]^2 \to T_\zeta[0,1]^2$ leads to the conclusion that

$$\theta \in \left[0, \tan^{-1}\left(\frac{\sqrt{11}}{3}\right)\right] \cup \left[\pi - \tan^{-1}\left(\frac{\sqrt{11}}{3}\right), \pi\right] \cup \left[\pi, \pi + \tan^{-1}\left(\frac{\sqrt{11}}{3}\right)\right]$$
$$\cup \left[2\pi - \tan^{-1}\left(\frac{\sqrt{11}}{3}\right), 2\pi\right].$$

3.1.2 Proof of the main theorem in the non-periodic case

Proof of Theorem 3.1.2. Our aim is to obtain the distribution of $(Z_j)_{j\in\mathbb{Z}}$ which is the same as the distribution of $(Y_j)_{j\in\mathbb{Z}}$ conditional on $\inf_{j\leq -1}\|Y_j\| \geq 1$ (see Definition 2.1.4). We take two main steps the first of which is further split into a couple of sub-steps.

Step 1 We check that the process $(Y_j)_{j\in\mathbb{Z}}$ is, with probability p_i, the bi-infinite sequence with:

(i) entry $U.\Theta$ at $j = 0$;

(ii) entries $U.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for all $l = i+1, \ldots, N$;

(iii) ∞ for all other positive indices j;

(iv) ∞ for all negative indices j except, possibly, $U.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for $l = 1, \ldots, i-1$;

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

Verifying that conditions (2)-(4) in Definition 2.1.4 are satisfied for $(Y_j)_{j\in\mathbb{Z}}$ as just described is straightforward, so we check that condition (1) holds.

Sub-step 1.1 p_i is the probability that an exceedance of the threshold $u_n(\tau)$ by $\|\mathbf{X}_{r_n}\|$ is due to a hit (at time r_n) to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$.

Observe that

$$\{x \in \mathcal{X} : \|\mathbf{X}_{r_n}(x)\| > u_n(\tau)\} = \left\{x \in \mathcal{X} : f^{r_n}(x) \in \bigcup_{i=1}^N B_{h_i^{-1}(u_n(\tau))}(\xi_i)\right\}.$$

We may assume that the union $\bigcup_{i=1}^N B_{h_i^{-1}(u_n(\tau))}(\xi_i)$ is disjoint as indeed it is for a sufficiently large n. Now, a hit to the union $\bigcup_{i=1}^N B_{h_i^{-1}(u_n(\tau))}(\xi_i)$ is indeed a hit to the ball $B_{h_i^{-1}(u_n(\tau))}(\xi_i)$, where $i \in \{1, \dots, N\}$, with probability

$$p_i = \frac{\mu(f^{-r_n}(B_{h_i^{-1}(u_n(\tau))}(\xi_i)))}{\sum_{k=1}^N \mu(f^{-r_n}(B_{h_k^{-1}(u_n(\tau))}(\xi_k)))} = \frac{\mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i))}{\sum_{k=1}^N \mu(B_{h_k^{-1}(u_n(\tau))}(\xi_k))} \sim \frac{D_i.\mathrm{Leb}(B_{h_i^{-1}(u_n(\tau))}(\xi_i))}{\sum_{k=1}^N D_k.\mathrm{Leb}(B_{h_k^{-1}(u_n(\tau))}(\xi_k))}$$

where the second equality follows by f-invariance of μ and the asymptotic equivalence is derived from (R2). In fact, we may further write

$$p_i \sim \frac{D_i.(h_i^{-1}(u_n(\tau)))^d}{\sum_{k=1}^N D_k.(h_k^{-1}(u_n(\tau)))^d} = \frac{D_i\left(\frac{u_n(\tau)}{c_i}\right)^{-\alpha d}}{\sum_{k=1}^N D_k\left(\frac{u_n(\tau)}{c_k}\right)^{-\alpha d}} = \frac{D_i c_i^{\alpha d}}{\sum_{k=1}^N D_k c_k^{\alpha d}}. \tag{3.1.1}$$

Sub-step 1.2 Assume a hit at time r_n to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$. Then, $(Y_j)_{j\in\mathbb{Z}}$ is as described by (i)-(iv) in Step 1.

Since $\Psi(x) = \sum_{i=1}^N h_i(\mathrm{dist}(x,\xi_i))\frac{\Phi_{\xi_i}^{-1}(x)}{\|\Phi_{\xi_i}^{-1}(x)\|}\mathbb{1}_{W_i}(x)$ and $f^{r_n}(x) \in W_i$,

$$\frac{\mathbf{X}_{r_n}(x)}{\|\mathbf{X}_{r_n}(x)\|} = \frac{\Psi(f^{r_n}(x))}{\|\Psi(f^{r_n}(x))\|} = \frac{\Phi_{\xi_i}^{-1}(f^{r_n}(x))}{\|\Phi_{\xi_i}^{-1}(f^{r_n}(x))\|} = w.$$

Dropping the dependence on x, we have that $\frac{\mathbf{X}_{r_n}}{\|\mathbf{X}_{r_n}\|} = \Theta$. As, in a sufficiently small neighbourhood of $\mathcal{M}$, μ looks like the Lebesgue measure (recall (R2)) it follows that Θ is uniformly distributed on $\mathbb{S}^{d-1}$. As $Y_0 \sim \frac{u_n^{-1}(\|\mathbf{X}_{r_n}\|)}{\tau}\frac{\mathbf{X}_{r_n}}{\|\mathbf{X}_{r_n}\|}$ and, from [FFT20, Lemma 3.9], $\|Y_0\|$ is uniformly distributed on $[0,1]$, we must have $\frac{u_n^{-1}(\|\mathbf{X}_{r_n}\|)}{\tau} \sim U$ where U is uniformly distributed on $[0,1]$. Thus,

$$Y_0 \sim U.\Theta$$

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$ and U and Θ are independent. Therefore, (i) for $(Y_j)_{j\in\mathbb{Z}}$ is satisfied.

Observe that continuity of f at ξ_i guarantees that $f^{r_n+j}(x)$ belongs to a small neighbourhood of $f^j(\xi_i)$ as long as $f^{r_n}(x)$ belongs to a sufficiently small neighbourhood of ξ_i. We have

$$\mathrm{dist}(f^{r_n+j}(x), f^j(\xi_i)) \sim \|Df^j_{\xi_i}(w)\|.\mathrm{dist}(f^{r_n}(x),\xi_i) \tag{3.1.2}$$

where $w = \frac{\Phi_{\xi_i}^{-1}(f^{r_n}(x))}{\|\Phi_{\xi_i}^{-1}(f^{r_n}(x))\|}$.

Now, since the sequence $\mathbf{X}_0, \mathbf{X}_1, \ldots$ has an α-regularly varying tail we use the formula $u_n^{-1}(z) = \left(\dfrac{z}{a_n}\right)^{-\alpha}$.

Notice that $f^{m_l - m_i}(\xi_i) = \xi_l$. Thus, $f^{r_n + m_l - m_i}(x)$ belongs to a small neighbourhood of ξ_l and we may write

$$\|\mathbf{X}_{r_n+m_l-m_i}(x)\| = h_l(\mathrm{dist}(f^{r_n+m_l-m_i}(x), \xi_l)) = h_l(\mathrm{dist}(f^{r_n+m_l-m_i}(x), f^{m_l-m_i}(\xi_i))).$$

So, for all $l = i+1, \ldots, N$,

$$\begin{aligned} u_n^{-1}(\|\mathbf{X}_{r_n+m_l-m_i}(x)\|) &= u_n^{-1}(h_l(\mathrm{dist}(f^{r_n+m_l-m_i}(x), f^{m_l-m_i}(\xi_i)))) \\ &= \left(\frac{h_l(\mathrm{dist}(f^{r_n+m_l-m_i}(x), f^{m_l-m_i}(\xi_i)))}{a_n}\right)^{-\alpha} \end{aligned}$$

which, by (3.1.2), can be rewritten as

$$\begin{aligned} u_n^{-1}(\|\mathbf{X}_{r_n+m_l-m_i}(x)\|) &\sim \left(\frac{h_l(\|Df_{\xi_i}^{m_l-m_i}(w)\|.\mathrm{dist}(f^{r_n}(x), \xi_i))}{a_n}\right)^{-\alpha} \\ &= \frac{c_l^{-\alpha}\|Df_{\xi_i}^{m_l-m_i}(w)\|.\mathrm{dist}(f^{r_n}(x), \xi_i)}{a_n^{-\alpha}}. \end{aligned} \tag{3.1.3}$$

In fact, $f^{r_n}(x) \in B_{h_i^{-1}(u_n(\tau))}(\xi_i)$ corresponds to

$$h_i(\mathrm{dist}(f^{r_n}(x), \xi_i)) > u_n(\tau)$$

which together with (2.1.2) implies

$$\tau \geq u_n^{-1}(h_i(\mathrm{dist}(f^{r_n}(x), \xi_i))) \iff \tau = \frac{u_n^{-1}(h_i(\mathrm{dist}(f^{r_n}(x), \xi_i)))}{v}$$

where $v \in [0,1]$. Using again the formula for u_n^{-1}, it follows

$$\tau = \frac{1}{v}\left(\frac{h_i(\mathrm{dist}(f^{r_n}(x), \xi_i))}{a_n}\right)^{-\alpha} = \frac{1}{v}\frac{c_i^{-\alpha}\mathrm{dist}(f^{r_n}(x), \xi_i)}{a_n^{-\alpha}}. \tag{3.1.4}$$

Since $\Psi(x) = \displaystyle\sum_{i=1}^{N} h_i(\mathrm{dist}(x, \xi_i))\frac{\Phi_{\xi_i}^{-1}(x)}{\|\Phi_{\xi_i}^{-1}(x)\|}\mathbb{1}_{W_i}(x)$, $f^{r_n}(x) \in W_i$ and $f^{r_n+m_l-m_i}(x) \in W_l$,

$$\begin{aligned} \frac{\mathbf{X}_{r_n+m_l-m_i}(x)}{\|\mathbf{X}_{r_n+m_l-m_i}(x)\|} = \frac{\Psi(f^{r_n+m_l-m_i}(x))}{\|\Psi(f^{r_n+m_l-m_i}(x))\|} &= \frac{\Phi_{\xi_l}^{-1}(f^{r_n+m_l-m_i}(x))}{\|\Phi_{\xi_l}^{-1}(f^{r_n+m_l-m_i}(x))\|} \\ &= \frac{Df_{\xi_i}^{m_l-m_i}(\Phi_{\xi_i}^{-1}(f^{r_n}(x))}{\|Df_{\xi_i}^{m_l-m_i}(\Phi_{\xi_i}^{-1}(f^{r_n}(x))\|} \\ &= \frac{Df_{\xi_i}^{m_l-m_i}(w)}{\|Df_{\xi_i}^{m_l-m_i}(w)\|}. \end{aligned} \tag{3.1.5}$$

Putting together (3.1.3), (3.1.4) and (3.1.5), we conclude that, for all $l = i+1, \ldots, N$,

$$\frac{u_n^{-1}(\|\mathbf{X}_{r_n+m_l-m_i}(x)\|)}{\tau}\frac{\mathbf{X}_{r_n+m_l-m_i}(x)}{\|\mathbf{X}_{r_n+m_l-m_i}(x)\|} \sim v\left(\frac{c_i}{c_l}\right)^{\alpha} Df_{\xi_i}^{m_l-m_i}(w)$$

where $v \in [0,1]$ and $w \in \mathbb{S}^{d-1}$.

Finally, v and w are attached to a particular observation, labelling with a magnitude and a direction the hit at time r_n to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$. Dropping the dependence on x, we have

$$\frac{u_n^{-1}(\|\mathbf{X}_{r_n+m_l-m_i}\|)}{\tau} \frac{\mathbf{X}_{r_n+m_l-m_i}}{\|\mathbf{X}_{r_n+m_l-m_i}\|} \sim U.Df_{\xi_i}^{m_l-m_i}(\Theta) \left(\frac{c_i}{c_l}\right)^{\alpha}$$

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent. So, (ii) for $(Y_j)_{j\in\mathbb{Z}}$ is satisfied.

For all positive indices $j \notin \{m_l - m_i : l = i+1, \ldots, N\}$, we claim that

$$\lim_{n\to\infty} \frac{u_n^{-1}(\|\mathbf{X}_{r_n+j}\|)}{\tau} \frac{\mathbf{X}_{r_n+j}}{\|\mathbf{X}_{r_n+j}\|} = \infty.$$

Suppose otherwise, so that $\lim\limits_{n\to\infty} \dfrac{u_n^{-1}(\|\mathbf{X}_{r_n+j}\|)}{\tau} = c \in \mathbb{R}$ for some positive index $j \notin \{m_l - m_i : l = i+1, \ldots, N\}$. Then, at time $r_n + j$ there is a hit to $B_{u_n(\tau')}(\xi_k)$ for some $\tau' \in (0, +\infty)$ and $k \in \{1, \ldots, N\}$. As $u_n(\tau) \underset{n\to\infty}{\longrightarrow} +\infty$, for any $\tau \in (0,+\infty)$, then $f^{r_n}(x) \underset{n\to\infty}{\longrightarrow} \xi_i$ and $f^{r_n+j}(x) \underset{n\to\infty}{\longrightarrow} \xi_k$. By continuity of f at ξ_i, $f^{r_n+j}(x) \underset{n\to\infty}{\longrightarrow} f^j(\xi_i)$ and so $f^j(\xi_i) = \xi_k$. However, by definition, $\xi_k = f^{m_k - m_i}(\xi_i)$ where $k \in \{1, \ldots, N\}$. It follows that ξ_i is periodic, thus ζ is periodic contrary to our assumption. So, (iii) for $(Y_j)_{j\in\mathbb{Z}}$ is satisfied.

We are left to check (iv): Y_j is equal to ∞ except, possibly, for a finite number of negative indices $j = m_1 - m_i, \ldots, m_{i-1} - m_i$. In fact, if the visit to a neighbourhood of ξ_i (at time r_n) is preceded by a visit to a neighbourhood of ξ_{i-1} (at time $r_n - (m_i - m_{i-1})$) then, arguing as in the justification of (ii) above,

$$\frac{u_n^{-1}(\|\mathbf{X}_{r_n+m_{i-1}-m_i}\|)}{\tau} \frac{\mathbf{X}_{r_n+m_{i-1}-m_i}}{\|\mathbf{X}_{r_n+m_{i-1}-m_i}\|} \sim U.Df_{\xi_i}^{m_{i-1}-m_i}(\Theta) \left(\frac{c_i}{c_{i-1}}\right)^{\alpha}.$$

On the other hand, if the visit to a neighbourhood of ξ_i (at time r_n) is preceded by a visit to a neighbourhood of $f^{-(m_i-m_{i-1})}(\xi_i) \setminus \{\xi_{i-1}\}$ (at time $r_n - (m_i - m_{i-1})$) then

$$\lim_{n\to\infty} \frac{u_n^{-1}(\|\mathbf{X}_{r_n+m_{i-1}-m_i}\|)}{\tau} \frac{\mathbf{X}_{r_n+m_{i-1}-m_i}}{\|\mathbf{X}_{r_n+m_{i-1}-m_i}\|} = \infty$$

by definition of the observable Ψ (*i.e.* in a neighbourhood of a $(m_i - m_{i-1})$-th pre-image of ξ_i which is not ξ_{i-1} then $\Psi \equiv 0$). Thus, inductively, we see that among the indices $j = m_1 - m_i, \ldots, m_{i-1} - m_i$ we can have $\lim\limits_{n\to\infty} \dfrac{u_n^{-1}(\|\mathbf{X}_{r_n+j}\|)}{\tau} \dfrac{\mathbf{X}_{r_n+j}}{\|\mathbf{X}_{r_n+j}\|}$ different or equal to ∞. By analogous reasoning to what was used to claim (iii) just above, we see that the negatively indexed entries corresponding to indices $j \neq m_1 - m_i, \ldots, m_{i-1} - m_i$ must all be ∞.

Step 2 The distribution of $(Z_j)_{j\in\mathbb{Z}}$ is given by (0)-(IV).

We look at the norms of the negatively indexed entries in $(Y_j)_{j\in\mathbb{Z}}$, more specifically for the entries corresponding to $j = m_l - m_i$ for all $l = 1, \ldots, i-1$ (as $Y_j = \infty$ for all the other negative indices j as determined by (iv) we have just shown).

First, assume $\lambda_{i,l}^{max}\left(\frac{c_i}{c_l}\right)^{\alpha}<1$ for all $l=1,\dots,i-1$. Then, for any $u\in[0,1]$, $u.\|Df_{\xi_i}^{m_l-m_i}(\Theta)\|\left(\frac{c_i}{c_l}\right)^{\alpha}<1$ (a.s.) for all $l=1,\dots,i-1$. Since $\lambda_{i,l}^{max}\left(\frac{c_i}{c_l}\right)^{\alpha}<1$ is equivalent to $u_{i,l}^{max}>1$ (which implies that $u_{i,l}^{min}>1$) then $A^{(i)}=\emptyset$ and $\|Y_j\|\geq 1$ when $j=m_l-m_i$ and $l=1,\dots,i-1$ can only be so if $Y_j=\infty$. We have dealt with case (0) in the statement of the Proposition.

On the other hand, assume that $\lambda_{i,l}^{min}\left(\frac{c_i}{c_l}\right)^{\alpha}\geq 1$ for some $l=1,\dots,i-1$ that we now fix. Then, for some $u\in[0,1]$, $u.\|Df_{\xi_i}^{m_l-m_i}(\Theta)\|\left(\frac{c_i}{c_l}\right)^{\alpha}\geq 1$ (a.s.). In particular, $u_{i,l}^{min}\leq 1$, as in the case where $A^{(i)}\neq\emptyset$. We consider $\#A^{(i)}=2$ as the general case follows analogously. So, we have $u_{i,l_1}^{max}\leq u_{i,l_1}^{min}\leq u_{i,l_2}^{max}\leq u_{i,l_2}^{min}\leq 1$ for $l_1,l_2\in\{1,\dots,i-1\}$. If $u\in[0,1]$ is such that $u<u_{i,l_1}^{max}$ it follows that $u.\|Df_{\xi_i}^{m_l-m_i}(\Theta)\|\left(\frac{c_i}{c_l}\right)^{\alpha}<1$ (a.s.) at $j=m_l-m_i$ for both $l=l_1$ and $l=l_2$ and, therefore, the entries corresponding to indices $j=m_l-m_i$ for both $l=l_1$ and $l=l_2$ must be ∞; since $u<u_{i,l_1}^{max}$ has probability $\mathbb{P}(U<u_{i,l_1}^{max})=u_{i,l_1}^{max}$ (U is uniformly distributed on $[0,1]$), we are in case (I).

If $u\in[0,1]$ is such that $u_{i,l_1}^{min}\leq u<u_{i,l_2}^{max}$ it follows that $u.\|Df_{\xi_i}^{m_l-m_i}(\Theta)\|\left(\frac{c_i}{c_l}\right)^{\alpha}\geq 1$ (a.s.) at $j=m_l-m_i$ for $l=l_1$ but not for $l=l_2$, so that only the entry corresponding to index $j=m_l-m_i$ for $l=l_1$ can be different from ∞ (and indeed equal to $u.Df_{\xi_i}^{m_l-m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ where u is chosen equally likely among the elements of $[u_{i,l_1}^{min},u_{i,l_2}^{max})$); since $u_{i,l_1}^{min}\leq u<u_{i,l_2}^{max}$ has probability $\mathbb{P}(u_{i,l_1}^{min}\leq U<u_{i,l_2}^{max})=u_{i,l_2}^{max}-u_{i,l_1}^{min}$, we are in case (III).

If $u\in[0,1]$ is such that $u\geq u_{i,l_2}^{min}$, then $u.\|Df_{\xi_i}^{m_l-m_i}(\Theta)\|\left(\frac{c_i}{c_l}\right)^{\alpha}\geq 1$ at $j=m_l-m_i$ for both $l=l_1$ and $l=l_2$ leading to the entries corresponding to indices $j=m_l-m_i$ for $l=l_1$ and $l=l_2$ both being different from ∞ (and indeed equal to $u.Df_{\xi_i}^{m_l-m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ where u is chosen equally likely among the elements of $[u_{i,l_2}^{min},1)$); since $u\geq u_{i,l_2}^{min}$ has probability $\mathbb{P}(U\geq u_{i,l_2}^{min})=1-u_{i,l_2}^{min}$, we are in case (IV).

Finally, if $u\in[0,1]$ is such that $u_{i,l_1}^{max}\leq u<u_{i,l_1}^{min}$, which occurs with probability $u_{i,l_1}^{min}-u_{i,l_1}^{max}$, then $u.\|Df_{\xi_i}^{m_l-m_i}(w)\|\left(\frac{c_i}{c_l}\right)^{\alpha}\geq 1$ for all $w\in\mathbb{S}^{d-1}$ such that $\|Df_{\xi_i}^{m_l-m_i}(w)\|\geq\frac{1}{u}\left(\frac{c_l}{c_i}\right)^{\alpha}$; thus, $\Theta\mid\{U_{1'}=u\}$ is uniformly distributed on $\left\{w\in\mathbb{S}^{d-1}:\|Df_{\xi_i}^{m_l-m_i}(w)\|\geq\frac{1}{u}\left(\frac{c_l}{c_i}\right)^{\alpha}\right\}$, for $U_{1'}$ uniformly distributed on $[u_{i,l_1}^{max},u_{i,l_1}^{min})$, and we are in case (II) (the other situation where $u_{i,l_2}^{max}\leq u<u_{i,l_2}^{min}$ is entirely analogous). □

The following corollary justifies why we imposed from the beginning that all the h_i in Ψ have the same index α.

Corollary 3.1.13. *Let f be a probability preserving system which preserves μ. Additionally, let f and μ be such that (R1) and (R2) hold. Let Ψ be as given by (3.0.1) for $\mathcal{M}$ as in Section 3.1, where ζ is a non-periodic point. If the h_i (as in (3.0.2)) were allowed not all with the same α then $\mathbb{P}(S=\xi_i)=p_i$ would define a degenerate random variable S.*

Proof. Let $\mathcal{M} = \{\xi_1, \xi_2\} = \{\zeta, f(\zeta)\}$, with $h_1(x) = c_1 x^{-\frac{1}{\alpha_1}}$ and $h_2(x) = c_2 x^{-\frac{1}{\alpha_2}}$ where $\alpha_1 < \alpha_2$. Assume $d = 1$. Then, making use of (3.1.1),

$$\mathbb{P}(S = \xi_1) = p_1 \sim \frac{D_1 \left(\frac{u_n(\tau)}{c_1}\right)^{-\alpha_1}}{D_1 \left(\frac{u_n(\tau)}{c_1}\right)^{-\alpha_1} + D_2 \left(\frac{u_n(\tau)}{c_2}\right)^{-\alpha_2}} \sim 1;$$

$$\mathbb{P}(S = \xi_2) = p_2 \sim \frac{D_2 \left(\frac{u_n(\tau)}{c_2}\right)^{-\alpha_2}}{D_1 \left(\frac{u_n(\tau)}{c_1}\right)^{-\alpha_1} + D_2 \left(\frac{u_n(\tau)}{c_2}\right)^{-\alpha_2}} \sim 0.$$

The general case follows analogously. □

3.1.3 Periodic case

Theorem 3.1.14. *Let f be a probability preserving system which preserves μ. Additionally, let f and μ be such that (R1) and (R2) hold. Let Ψ be as given by (3.0.1) for $\mathcal{M}$ as in Section 3.1, where ζ is a periodic point of prime period q. Assume that $(\mathbf{X}_n)_{n \in \mathbb{N}_0}$ has an α-regularly varying tail, where $\alpha \in (0, 1)$. Additionally, assume that f is uniformly expanding along the orbit of ζ. For all $i \in \mathcal{I}$, define $p_i = \dfrac{D_i c_i^{\alpha d}}{\sum_{k=1}^N D_k c_k^{\alpha d}}$. Let $A^{(i)}$ be as defined in (A2). If $A^{(i)} = \emptyset$, the piling process is*

(0) with probability p_i the bi-infinite sequence $(Z_j)_{j \in \mathbb{Z}}$ with:

- *(i) entry $U.\Theta$ at $j = 0$;*
- *(ii) entries $U.Df^j_{\xi_i}(\Theta) \left(\frac{c_i}{c_l}\right)^{\alpha}$ at $j = m_l - m_i + qs$ for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_i + qs > 0$;*
- *(iii) ∞ for all other positive indices j;*
- *(iv) ∞ for all negative indices j;*

where U is uniformly distributed on $[0, 1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

If $A^{(i)} \neq \emptyset$, assume there exists an increasing order

$$0 \le u^{max}_{i,l_1} \le u^{min}_{i,l_1} \le u^{max}_{i,l_2} \le u^{min}_{i,l_2} \le \cdots \le u^{max}_{i,l_{\#A^{(i)}}} \le u^{min}_{i,l_{\#A^{(i)}}} \le 1$$

and for $u^{min}_{i,l_p} = u^{min}_{i,l,s}$ (resp. $u^{max}_{i,l_p} = u^{max}_{i,l,s}$), $p \in \{1, \ldots, \#A^{(i)}\}$, let $\rho(u^{min}_{i,l_p}) := m_l - m_i - qs$ and $\rho(u^{max}_{i,l_p}) := m_l - m_i - qs$, so that we abbreviate to $\rho(u_{i,l_p}) := m_l - m_i - qs$. Then, the piling process is

(I) with probability $p_i u^{max}_{i,l_1}$ the bi-infinite sequence $(Z_j)_{j \in \mathbb{Z}}$ with:

- *(i) entry $U_0.\Theta$ at $j = 0$;*
- *(ii) entries $U_0.Df^j_{\xi_i}(\Theta) \left(\frac{c_i}{c_l}\right)^{\alpha}$ at $j = m_l - m_i + qs$ for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_i + qs > 0$;*
- *(iii) ∞ for all other positive indices j;*

(iv) ∞ *for all negative indices* j*;*

where U_0 *is uniformly distributed on* $[0, u^{max}_{i,l_1})$*,* Θ *is uniformly distributed on* $\mathbb{S}^{d-1}$*, and* U_0 *and* Θ *are independent;*

(II) with probability $p_i(u^{min}_{i,l_p} - u^{max}_{i,l_p})$*, where* $p \in \{1, \ldots, \#A^{(i)}\}$*, the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) entry $U_{p'}.\Theta$ *at* $j = 0$*;*

(ii) entries $U_{p'}.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ *at* $j = m_l - m_i + qs$ *for all* $l \in \mathcal{I}$ *and* $s \in \mathbb{N}_0$ *such that* $m_l - m_i + qs > 0$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) entries $U_{p'}.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ *at* $j = \rho(u_{i,l_1}), \ldots, \rho(u_{i,l_p})$*;*

(v) ∞ *for all other negative indices* j*;*

where $U_{p'}$ *is uniformly distributed on* $[u^{max}_{i,l_p}, u^{min}_{i,l_p})$ *and* $\Theta \mid \{U_{p'} = u\}$ *is uniformly distributed on* $\left\{w \in \mathbb{S}^{d-1} : \|Df^{\rho(u_{i,l_p})}_{\xi_i}(w)\| \geq \frac{1}{u}\left(\frac{c_{l_p}}{c_i}\right)^\alpha\right\}$*;*

(III) with probability $p_i(u^{max}_{i,l_{p+1}} - u^{min}_{i,l_p})$*, where* $p \in \{1, \ldots, \#A^{(i)} - 1\}$*, the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) entry $U_p.\Theta$ *at* $j = 0$*;*

(ii) entries $U_p.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ *at* $j = m_l - m_i + qs$ *for all* $l \in \mathcal{I}$ *and* $s \in \mathbb{N}_0$ *such that* $m_l - m_i + qs > 0$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) entries $U_p.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ *at* $j = \rho(u_{i,l_1}), \ldots, \rho(u_{i,l_p})$*;*

(v) ∞ *for all other negative indices* j*;*

where U_p *is uniformly distributed on* $[u^{min}_{i,l_p}, u^{max}_{i,l_{p+1}})$*,* Θ *is uniformly distributed on* $\mathbb{S}^{d-1}$*, and* U_p *and* Θ *are independent;*

(IV) with probability $p_i.(1 - u^{min}_{i,l_{\#A^{(i)}}})$ *the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) entry $U_{\#A^{(i)}}.\Theta$ *at* $j = 0$*;*

(ii) entries $U_{\#A^{(i)}}.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ *at* $j = m_l - m_i + qs$ *for all* $l \in \mathcal{I}$ *and* $s \in \mathbb{N}_0$ *such that* $m_l - m_i + qs > 0$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) entries $U_{\#A^{(i)}}.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ *at* j *for all* $j \in A^{(i)}$*;*

(v) ∞ *for all other negative indices* j*;*

where $U_{\#A^{(i)}}$ *is uniformly distributed on* $[u^{min}_{i,l_{\#A^{(i)}}}, 1]$*,* Θ *is uniformly distributed on* $\mathbb{S}^{d-1}$*, and* $U_{\#A^{(i)}}$ *and* Θ *are independent.*

Proof. Recall that $(Z_j)_{j\in\mathbb{Z}}$ has the distribution of $(Y_j)_{j\in\mathbb{Z}}$ conditional on $\inf_{j\le -1}\|Y_j\| \ge 1$ (see Definition 2.1.4). The proof follows analogously to the proof of Theorem 3.1.2 so we write down the same steps highlighting the differences that arise from the fact that ζ is now periodic (of prime period q).

Step 1 We check that the process $(Y_j)_{j\in\mathbb{Z}}$ is, with probability p_i, the bi-infinite sequence with:

(i) entry $U.\Theta$ at $j=0$;

(ii) entries $U.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i + qs$, for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_i + qs > 0$;

(iii) ∞ for all other positive indices j;

(iv) ∞ for all negative indices j except, possibly, $U.Df^j_{\xi_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i - qs$, for $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_i - qs < 0$;

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

Verifying that condition (2) in Definition 2.1.4 is satisfied for $(Y_j)_{j\in\mathbb{Z}}$ as just described is straightforward. The (a.s.) positive exponential growth of $\|Df^j_{\xi_i}(\Theta)\|$ for positive j as well as the (a.s.) negative exponential growth of $\|Df^j_{\xi_i}(\Theta)\|$ for negative j lead to (3) being satisfied. Again, the (a.s.) negative exponential growth of $\|Df^j_{\xi_i}(\Theta)\|$ for negative j results in $\|Y_j\| \le 1$ for all but a finite number of negative indices j so that (4) is satisfied.

We are left to check that condition (1) in Definition 2.1.4 holds.

Sub-step 1.1 p_i is the probability that an exceedance of the threshold $u_n(\tau)$ by $\|\mathbf{X}_{r_n}\|$ is due to a hit (at time r_n) to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$.

From the exact same reasoning as in Sub-step 1.1 in the proof of Theorem 3.1.2: what matters is how much a neighbourhood around ξ_i (corresponding to the exceedance of a threshold $u_n(\tau)$) weighs in a neighbourhood of $\mathcal{M}$ (corresponding to the exceedance of the same $u_n(\tau)$), which is not dependent on ζ being or not periodic.

Sub-step 1.2 Assume a hit, at time r_n, to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$. Then, $(Y_j)_{j\in\mathbb{Z}}$ is as described by (i)-(iv) in Step 1.

In fact, if ζ is periodic of prime period q and a hit to a neighbourhood of a certain ξ_i, corresponding to the exceedance of a threshold $u_n(\tau)$, occurs at time r_n, then hits to neighbourhoods of the same ξ_i will occur at times $r_n + qs$ and at times $r_n - qs$, for all $s \in \mathbb{N}_0$. The rest follows analogously to Sub-step 1.2 in the proof of Theorem 3.1.2 - we point out that in (iii) the contradiction is with the minimality of q.

Step 2 The distribution of $(Z_j)_{j\in\mathbb{Z}}$ is given by (0)-(IV).

This is analogous to Theorem 3.1.2, accounting for the changes in $(Y_j)_{j\in\mathbb{Z}}$ as discussed in Step 1. □

Example 3.1.15. Let $f(x,y) = (2x \mod 1, 3y \mod 1)$, $(x,y)\in[0,1]^2$, and $\mu =$ Lebesgue measure on $[0,1]^2$ (invariant for f). Take $\zeta = (\zeta_x, \zeta_y) = \left(\frac{1}{7}, 0\right)$ (periodic of period 3), and define the observable Ψ as

$$\Psi(x,y) := \begin{cases} \|(x,y)-\zeta\|^{-4}\dfrac{(x-\zeta_x, y-\zeta_y)}{\|(x,y)-\zeta\|}, & (x,y)\in B_{\varepsilon_1}(\zeta) \\ 256\|(x,y)-f(\zeta)\|^{-4}\dfrac{(x-f(\zeta)_x, y-f(\zeta)_y)}{\|(x,y)-f(\zeta)\|}, & (x,y)\in B_{\varepsilon_2}(f(\zeta)) \\ 0, & \text{otherwise} \end{cases}$$

for some $\varepsilon_1, \varepsilon_2 > 0$, where $f(\zeta) = (f(\zeta)_x, f(\zeta)_y)$. Observe that, presented as in (3.0.1),

$$\Psi(x,y) = \sum_{i=1}^{2} h_i(\text{dist}((x,y),\xi_i))\frac{\Phi_{\xi_i}^{-1}((x,y))}{\|\Phi_{\xi_i}^{-1}((x,y))\|}\mathbb{1}_{B_{\varepsilon_i}(\xi_i)}((x,y))$$

for $\xi_1 = \zeta$, $\xi_2 = f(\zeta)$, $h_1(t) = t^{-4}$ and $h_2(t) = 256t^{-4}$ (so that $\alpha = 1/4$), and $\Phi_{\xi_i}^{-1} : B_{\varepsilon_i}(\xi_i) \to B_{\varepsilon_i}(0)$ being the translation by $-\xi_i$ for $i = 1, 2$. In particular, $\mathcal{M} = \{\zeta, f(\zeta)\}$ and equation (2.2.11) holds with $a_n = \dfrac{289}{256}n^2$.

Since μ = Lebesgue, we have $D_1 = D_2 = 1$. Also, $d = 2$ and $c_1 = 1$ and $c_2 = 256$. Thus, $p_1 = \dfrac{1^{\frac{1}{2}}}{1^{\frac{1}{2}} + 256^{\frac{1}{2}}} = \dfrac{1}{17}$ and $p_2 = \dfrac{256^{\frac{1}{2}}}{1^{\frac{1}{2}} + 256^{\frac{1}{2}}} = \dfrac{16}{17}$.

If $i = 1$ then, for all $s \in \mathbb{N}$, $\lambda_{1,2,s}^{\min} = \dfrac{1}{3^{2s}}$, $\lambda_{1,2,s}^{\max} = \dfrac{1}{2^{2s}}$ and $\lambda_{1,1,s}^{\min} = \dfrac{1}{3^{3s}}$ $\lambda_{1,1,s}^{\max} = \dfrac{1}{2^{3s}}$, giving $u_{1,2,s}^{\min} = \left(\dfrac{256}{1}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{3^{2s}}\right)} = 4.3^{2s}$, $u_{1,2,s}^{\max} = \left(\dfrac{256}{1}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{2^{2s}}\right)} = 4.2^{2s}$ and $u_{1,1,s}^{\min} = \left(\dfrac{1}{1}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{3^{3s}}\right)} = 4.3^{3s}$, $u_{1,1,s}^{\max} = \left(\dfrac{1}{1}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{2^{3s}}\right)} = 4.2^{3s}$, leading to $A^{(1)} = \emptyset$.

If $i = 2$ then, $\lambda_{2,1,0}^{\min} = \dfrac{1}{3}$ and $\lambda_{2,1,0}^{\max} = \dfrac{1}{2}$, giving $u_{2,1,0}^{\min} = \left(\dfrac{1}{256}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{3}\right)} = \dfrac{3}{4}$ and $u_{2,1,0}^{\max} = \left(\dfrac{1}{256}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{2}\right)} = \dfrac{1}{2}$. Also, for all $s \in \mathbb{N}$, $\lambda_{2,2,s}^{\min} = \dfrac{1}{3^{3s}}$, $\lambda_{2,2,s}^{\max} = \dfrac{1}{2^{3s}}$ and $\lambda_{2,1,s}^{\min} = \dfrac{1}{3^{3s+1}}$, $\lambda_{2,1,s}^{\max} = \dfrac{1}{2^{3s+1}}$, giving $u_{2,2,s}^{\min} = \left(\dfrac{256}{256}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{3^{3s}}\right)} = 3^{3s}$, $u_{2,2,s}^{\max} = \left(\dfrac{256}{256}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{2^{3s}}\right)} = 2^{3s}$ and $u_{2,1,s}^{\min} = \left(\dfrac{1}{256}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{3^{3s+1}}\right)} = \dfrac{3^{3s+1}}{4}$, $u_{2,1,s}^{\max} = \left(\dfrac{1}{256}\right)^{\frac{1}{4}}\dfrac{1}{\left(\frac{1}{2^{3s+1}}\right)} = \dfrac{2^{3s+1}}{4}$. Thus, $A^{(2)} = \{-1\}$, and we have the increasing order $0 \le u_{2,1,0}^{\max} \le u_{2,1,0}^{\min} \le 1$.

Notice that $(Df_\zeta)^j(\theta_x, \theta_y) = (Df_{f(\zeta)})^j(\theta_x, \theta_y) = (2^j\theta_x, 3^j\theta_y)$, where $j \in \mathbb{Z}$.

Applying Theorem 3.1.14, we conclude that the piling process is

(0) with probability $\dfrac{1}{17}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

- (i) entry $U.(\Theta_x, \Theta_y)$ at $j = 0$
- (ii) entries $U.(2^j\Theta_x, 3^j\Theta_y)\cdot\left(\dfrac{1}{c_l}\right)^{\alpha}$ at $j = m_l + qs$ for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l + qs > 0$
- (iii) ∞ for all other positive indices j
- (iv) ∞ for all negative indices j

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^1$, and U and Θ are independent;

(I) with probability $\frac{16}{17} \cdot \frac{1}{2} = \frac{8}{17}$ the bi-infinite sequence $(Z_j)_{j \in \mathbb{Z}}$ with:

(i) entry $U_0.(\Theta_x, \Theta_y)$ at $j = 0$

(ii) entries $U_0.(2^j\Theta_x, 3^j\Theta_y) \cdot \left(\frac{256}{c_l}\right)^\alpha$ at $j = m_l - m_2 + qs$ for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_2 + qs > 0$

(iii) ∞ for all other positive indices j

(iv) ∞ for all negative indices j

where U_0 is uniformly distributed on $[0, 1/2)$, Θ is uniformly distributed on $\mathbb{S}^1$, and U_0 and Θ are independent;

(II) with probability $\frac{16}{17}\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{4}{17}$ the bi-infinite sequence $(Z_j)_{j \in \mathbb{Z}}$ with:

(i) entry $U_{1'}.(\Theta_x, \Theta_y)$ at $j = 0$

(ii) entries $U_{1'}.(2^j\Theta_x, 3^j\Theta_y) \cdot \left(\frac{256}{c_l}\right)^\alpha$ at $j = m_l - m_2 + qs$ for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_2 + qs > 0$

(iii) entry $U_{1'}.\left(\frac{1}{2}\Theta_x, \frac{1}{3}\Theta_y\right) \cdot 4$ at $j = -1$

(iv) ∞ for all other negative indices j

where $U_{1'}$ is uniformly distributed on $[1/2, 3/4)$ and $\Theta \mid \{U_{1'} = z\}$ is uniformly distributed on $\left\{(\theta_x, \theta_y) \in \mathbb{S}^1 : \left\|\left(\frac{1}{2}\theta_x, \frac{1}{3}\theta_y\right)\right\| \geq \frac{1}{4z}\right\}$;

(III) with probability $\frac{16}{17}\left(1 - \frac{3}{4}\right) = \frac{4}{17}$ the bi-infinite sequence $(Z_j)_{j \in \mathbb{Z}}$ with:

(i) entry $U_1.(\Theta_x, \Theta_y)$ at $j = 0$

(ii) entries $U_1.(2^j\Theta_x, 3^j\Theta_y) \cdot \left(\frac{256}{c_l}\right)^\alpha$ at $j = m_l - m_2 + qs$ for all $l \in \mathcal{I}$ and $s \in \mathbb{N}_0$ such that $m_l - m_2 + qs > 0$;

(iii) entry $U_1.\left(\frac{1}{2}\Theta_x, \frac{1}{3}\Theta_y\right) \cdot 4$ at $j = -1$

(iv) ∞ for all other negative indices j

where U_1 is uniformly distributed on $[3/4, 1]$, Θ is uniformly distributed on $\mathbb{S}^1$, and U_1 and Θ are independent.

Remark 3.1.16. For Example 3.1.15, we have

$$\mu(U_n(\tau)) = \sum_{i=1}^{2} \mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i)) = u_n(\tau)^{-\frac{1}{2}} + 16u_n(\tau)^{-\frac{1}{2}} = 17u_n(\tau)^{-\frac{1}{2}}.$$

Let $q_n = m_2 + 3 - m_1 = 4$. From [AFFR16, Corollary 5.4],

$$\begin{aligned}\mu(U_n^{(q_n)}(\tau)) &= \mu(B_{h_1^{-1}(u_n(\tau))}(\xi_1)) - \frac{1}{\det(Df_{\xi_1}^{-4})}\mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2)) \\ &\quad + \mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2) - \frac{1}{\det(Df_{\xi_2}^{-3})}\mu(B_{h_2^{-1}(u_n(\tau))}(\xi_2)) \\ &= u_n(\tau)^{-\frac{1}{2}} - \frac{1}{16\times 81}16u_n(\tau)^{-\frac{1}{2}} + 16u_n(\tau)^{-\frac{1}{2}} - \frac{1}{8\times 27}16u_n(\tau)^{-\frac{1}{2}}.\end{aligned}$$

Thus, the extremal index is

$$\begin{aligned}\vartheta &= \lim_{n\to\infty}\frac{\mu(U_n^{(q_n)}(\tau))}{\mu(U_n(\tau))} \\ &= \lim_{n\to\infty}\frac{u_n(\tau)^{-\frac{1}{2}} - \frac{1}{16\times 81}16u_n(\tau)^{-\frac{1}{2}} + 16u_n(\tau)^{-\frac{1}{2}} - \frac{1}{8\times 27}16u_n(\tau)^{-\frac{1}{2}}}{u_n(\tau)^{-\frac{1}{2}} + 16u_n(\tau)^{-\frac{1}{2}}} = \frac{1370}{1377}.\end{aligned}$$

3.2 A countable number of points in the same orbit

Now we consider $\mathcal{M} = \{\xi_1, \xi_2, \dots\} = \{\xi_i\}_{i\in\mathbb{N}}$ such that there exist m_i, $i \in \mathbb{N}$, with $\xi_i = f^{m_i}(\zeta)$, where $\zeta \in \mathcal{X}$, and $\xi_0 = \lim_{i\to\infty}\xi_i$. Again, we take $m_1 = 0$ (*i.e.* $\xi_1 = \zeta$). We explore the case in which a sufficiently small neighbourhood of $\mathcal{M}$ (corresponding to the exceedance of a sufficiently high threshold $u_n(\tau)$) is a countable union of non-overlapping balls centred at each of the ξ_i, $i \in \mathbb{N}$. As an application of Theorem 3.2.1, we compute the piling process for a modified version of Example 4.5 of [AFFR17].

Theorem 3.2.1. *Let f be a probability preserving system which preserves μ. Additionally, let f and μ be such that (R1) and (R2) hold. Let Ψ be as given by (3.0.1) for $\mathcal{M}$ as in Section 3.2. Assume that $(\mathbf{X}_n)_{n\in\mathbb{N}_0}$ has an α-regularly varying tail, where $\alpha \in (0,1)$. Additionally, assume that f is uniformly expanding along the orbit of ζ. Let $U_n(\tau) = \bigcup_{i=1}^{\infty} B_{h_i^{-1}(u_n(\tau))}(\xi_i)$. Assume that there exists $(N(n))_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} N(n) = +\infty$ with $N(n) = o(n)$ and $\lim_{n\to\infty}\frac{\mu(U_n(\tau)\setminus\tilde{U}_n(\tau))}{\mu(U_n(\tau))} = 0$, where $\tilde{U}_n(\tau) = \bigcup_{i=1}^{N(n)} B_{h_i^{-1}(u_n(\tau))}(\xi_i)$. For all $i \in \mathcal{I}$, define $p_i = \frac{D_i c_i^{\alpha d}}{\sum_{k=1}^{\infty} D_k c_k^{\alpha d}}$. Let $A^{(i)}$ be as defined in (A1). If $A^{(i)} = \emptyset$, the piling process is*

(0) with probability p_i the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:
- *(i) entry $U.\Theta$ at $j = 0$;*
- *(ii) entries $U.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ at $j = m_l - m_i$ for all $l \geq i+1$;*
- *(iii) ∞ for all other positive indices j;*
- *(iv) ∞ for all negative indices j;*

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

If $A^{(i)} \neq \emptyset$, assume there exists an increasing ordering of the $u_{i,l}^{min,max}$ such that $u_{i,l_p}^{min} \leq u_{i,l_{p+1}}^{max}$ for all $p \in \{1, \dots, \#A^{(i)} - 1\}$. Then, the piling process is

(I) *with probability* $p_i u_{i,l_1}^{max}$ *the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) *entry* $U_0.\Theta$ *at* $j = 0$*;*

(ii) *entries* $U_0.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \geq i+1$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) ∞ *for all negative indices* j*;*

where U_0 *is uniformly distributed on* $[0, u_{i,l_1}^{max})$*,* Θ *is uniformly distributed on* $\mathbb{S}^{d-1}$*, and* U_0 *and* Θ *are independent;*

(II) *with probability* $p_i(u_{i,l_p}^{min} - u_{i,l_p}^{max})$*, where* $p \in \{1, \ldots, \#A^{(i)}\}$*, the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) *entry* $U_{p'}.\Theta$ *at* $j = 0$*;*

(ii) *entries* $U_{p'}.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \geq i+1$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) *entries* $U_{p'}.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \in \{l_1, \ldots, l_p\}$*;*

(v) ∞ *for all other negative indices* j*;*

where $U_{p'}$ *is uniformly distributed on* $[u_{i,l_p}^{max}, u_{i,l_p}^{min})$ *and* $\Theta \mid \{U_{p'} = u\}$ *is uniformly distributed on* $\left\{w \in \mathbb{S}^{d-1} : \|Df_{\xi_i}^{m_{l_p} - m_i}(w)\| \geq \frac{1}{u}\left(\frac{c_{l_p}}{c_i}\right)^{\alpha}\right\}$*;*

(III) *with probability* $p_i(u_{i,l_{p+1}}^{max} - u_{i,l_p}^{min})$*, where* $p \in \{1, \ldots, \#A^{(i)} - 1\}$*, the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) *entry* $U_p.\Theta$ *at* $j = 0$*;*

(ii) *entries* $U_p.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \geq i+1$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) *entries* $U_p.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \in \{l_1, \ldots, l_p\}$*;*

(v) ∞ *for all other negative indices* j*;*

where U_p *is uniformly distributed on* $[u_{i,l_p}^{min}, u_{i,l_{p+1}}^{max})$*,* Θ *is uniformly distributed on* $\mathbb{S}^{d-1}$*, and* U_p *and* Θ *are independent;*

(IV) *with probability* $p_i.(1 - u_{i,l_{\#A^{(i)}}}^{min})$ *the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with:*

(i) *entry* $U_{\#A^{(i)}}.\Theta$ *at* $j = 0$*;*

(ii) *entries* $U_{\#A^{(i)}}.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \geq i+1$*;*

(iii) ∞ *for all other positive indices* j*;*

(iv) *entries* $U_{\#A^{(i)}}.Df_{\xi_i}^{m_l - m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^{\alpha}$ *at* $j = m_l - m_i$ *for all* $l \in A^{(i)}$*;*

(v) ∞ *for all other negative indices* j*;*

where $U_{\#A^{(i)}}$ *is uniformly distributed on* $[u_{i,l_{\#A^{(i)}}}^{min}, 1]$*,* Θ *is uniformly distributed on* $\mathbb{S}^{d-1}$*, and* $U_{\#A^{(i)}}$ *and* Θ *are independent.*

Proof. Recall that $(Z_j)_{j\in\mathbb{Z}}$ has the distribution of $(Y_j)_{j\in\mathbb{Z}}$ conditional on $\inf_{j\le -1}\|Y_j\| \ge 1$ (see Definition 2.1.4). The result is a consequence of Theorem 3.1.2 in the sense that the use of the sequence $(N(n))_{n\in\mathbb{N}}$ implies that, at each n, $\mathcal{M}$ is as in Section 3.1 (for $\mathcal{I} = \{1,\dots,N(n)\}$).

Step 1 We check that the process $(Y_j)_{j\in\mathbb{Z}}$ is, with probability p_i, the bi-infinite sequence with:

(i) entry $U.\Theta$ at $j=0$;

(ii) entries $U.Df_{\xi_i}^{m_l-m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$, for all $l \ge i+1$;

(iii) ∞ for all other positive indices j;

(iv) ∞ for all negative indices j except, possibly, $U.Df_{\xi_i}^{m_l-m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for $l = 1,\dots,i-1$;

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

Verifying that condition (2) in Definition 2.1.4 is satisfied for $(Y_j)_{j\in\mathbb{Z}}$ as just described is straightforward. The (a.s.) positive exponential growth of $\|Df_{\xi_i}^j(\Theta)\|$ for positive j leads to (3) being satisfied. Since $p_1 > 0$, with positive probability all negatively indexed entries are equal to ∞ so (4) is satisfied.

We are left to check that condition (1) in Definition 2.1.4 holds.

Sub-step 1.1 p_i is the probability that an exceedance of the threshold $u_n(\tau)$ by $\|\mathbf{X}_{r_n}\|$ is due to a hit (at time r_n) to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$.

Analogous argument to that in Sub-step 1.1 in the proof of Theorem 3.1.2: how much does a neighbourhood around ξ_i (corresponding to the exceedance of a threshold $u_n(\tau)$) weights in a neighbourhood of $\mathcal{M}$ (corresponding to the exceedance of the same $u_n(\tau)$). Observe that

$$\lim_{n\to\infty}\sum_{i=N(n)+1}^{\infty} p_i = \lim_{n\to\infty}\frac{\mu(U_n(\tau)\setminus \tilde{U}_n(\tau))}{\mu(U_n(\tau))} = 0$$

so that $\sum_{i=1}^{\infty} p_i < \infty$ (and, in particular, equal to 1).

Sub-step 1.2 Assume a hit at time r_n to the ball around ξ_i of radius $h_i^{-1}(u_n(\tau))$. Then, $(Y_j)_{j\in\mathbb{Z}}$ is as described by (i)-(iv) in Step 1.

Let $\{\|\mathbf{X}_{r_n}\| > u_n(\tau)\} = f^{-r_n}(\tilde{U}_n(\tau))$. Then, by Theorem 3.1.2, we have that $(Y_j)_{j\in\mathbb{Z}}$ is, with probability p_i, the bi-infinite sequence with:

(i) entry $U.\Theta$ at $j=0$;

(ii) entries $U.Df_{\xi_i}^{m_l-m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$, for all $l = i+1,\dots,N(n)$;

(iii) ∞ for all other positive indices j;

(iv) ∞ for all negative indices j except, possibly, $U.Df_{\xi_i}^{m_l-m_i}(\Theta)\left(\frac{c_i}{c_l}\right)^\alpha$ at $j = m_l - m_i$ for $l = 1,\dots,i-1$;

where U is uniformly distributed on $[0,1]$, Θ is uniformly distributed on $\mathbb{S}^{d-1}$, and U and Θ are independent.

Since $\tilde{U}_n(\tau) \sim U_n(\tau)$ (a.s.), by letting $N(n) \to \infty$, we conclude that $(Y_j)_{j\in\mathbb{Z}}$ is as described by (i)-(iv) in Step 1.

<u>**Step 2**</u> The distribution of $(Z_j)_{j\in\mathbb{Z}}$ is given by (0)-(IV).

Analogous to Theorem 3.1.2, accounting for the changes in $(Y_j)_{j\in\mathbb{Z}}$ that have already been discussed. □

Example 3.2.2. Let $f(x) = 3x \mod 1$, $x \in [0,1]$, and $\mu =$ Lebesgue measure on $[0,1]$ (invariant for f). Take $\zeta = \sum_{j=1}^{+\infty} \left(\frac{1}{3}\right)^{3^j}$, and define the observable ψ as

$$\psi(x) := \begin{cases} |x-\zeta|^{-2}, x \in B_{\varepsilon_1}(\zeta) \\ \frac{1}{2^{i-1}}|x - f^{3^{i-1}}(\zeta)|^{-2},\ x \in B_{\varepsilon_i}(f^{3^{i-1}}),\ i = 2,3,\dots \\ 0,\ \text{otherwise} \end{cases}$$

where $\varepsilon_i > 0$ for all $i \in \mathbb{N}$. Observe that, presented as in (3.0.1),

$$\psi(x) = \sum_{i=1}^{\infty} h_i(|x-\xi_i|)\mathbb{1}_{B_{\varepsilon_i}(\xi_i)}(x)$$

for $\xi_1 = \zeta$, $\xi_i = f^{3^{i-1}}(\zeta)$ for $i = 2,3,\dots$, $\xi_0 = \lim_{i\to\infty} \xi_i = 0$, and $h_i(t) = \frac{1}{2^{i-1}}t^{-2}$ for all $i \in \mathbb{N}$ (so that $\alpha = 1/2$). In particular, $\mathcal{M} = \{\xi_i\}_{i\in\mathbb{N}}$ and equation (2.2.11) holds with $a_n = (24 + 16\sqrt{2})n^2$.

Since $\mu =$ Lebesgue, we have that $D_i = 1$ for all $i \in \mathbb{N}$. Also, $d = 1$ and $c_i = \frac{1}{2^{i-1}}$ for all $i \in \mathbb{N}$. Thus, $p_i = \dfrac{\left(\frac{1}{2^{i-1}}\right)^{\frac{1}{2}}}{\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}}\right)^{\frac{1}{2}}} = \dfrac{\left(\frac{1}{\sqrt{2}}\right)^{i-1}}{\sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^{k-1}} = \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^{i-1}$ for all $i \in \mathbb{N}$.

Because

$$\lim_{n\to\infty} \frac{\mu(U_n(\tau) \setminus \tilde{U}_n(\tau))}{\mu(U_n(\tau))} = \lim_{n\to\infty} \sum_{i=N(n)+1}^{\infty} p_i = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^{N(n)} = 0$$

for any $N(n)$ such that $\lim_{n\to\infty} N(n) = +\infty$ with $N(n) = o(n)$, then $(N(n))_{n\in\mathbb{N}}$ as in the statement of Theorem 3.2.1 exists. For example, let $N(n) = \log(n)$.

We have $f'(x) = 3$ for all $x \in [0,1]$, so that $(f^{-j})'(\xi_i) = \frac{1}{3^j}$ when $j < 0$ and, if $i > 1$, $\left(\frac{c_l}{c_i}\right)^{\alpha} \geq \sqrt{2}$ for all $l = 1,\dots,i-1$. Therefore, $A^{(i)} = \emptyset$ for all $i \in \mathbb{N}$. Applying Theorem 3.2.1, we conclude that the piling process is with probability $p_i = \left(1 - \frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)^{i-1}$ the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with:

(i) entry U at $j = 0$;

(ii) entries $U.3^{m_l-m_i}.\left(\frac{c_i}{c_l}\right)^{\frac{1}{2}}$ at $j=m_l-m_i$ for all $l\geq i+1$;

(iii) ∞ for all other positive indices j;

(iv) ∞ for all negative indices j;

where U is uniformly distributed on $[0,1]$, $i\in\mathbb{N}$.

Remark 3.2.3. For Example 3.2.2, we have

$$\mu(\tilde{U}_n(\tau))=\sum_{i=1}^{N(n)}\mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i))=\sum_{i=1}^{N(n)}2.\left(\frac{1}{\sqrt{2}}\right)^{i-1}u_n(\tau)^{-\frac{1}{2}}=2u_n(\tau)^{-\frac{1}{2}}\frac{1-\left(\frac{1}{\sqrt{2}}\right)^{N(n)}}{1-\frac{1}{\sqrt{2}}}.$$

Let $q_n=N(n)$. From [AFFR16, Corollary 4.5],

$$\begin{aligned}\mu(\tilde{U}_n^{(q_n)}(\tau))&=\sum_{i=1}^{N(n)-1}\left(\mu(B_{h_i^{-1}(u_n(\tau))}(\xi_i))-\frac{1}{3^3.3^{3^{i-1}}}\mu(B_{h_{i+1}^{-1}(u_n(\tau))}(\xi_{i+1}))\right)\\&\quad+\mu(B_{h_{N(n)}^{-1}(u_n(\tau))}(\xi_{N(n)}))\\&=\sum_{i=1}^{N(n)-1}\left(2.\left(\frac{1}{\sqrt{2}}\right)^{i-1}u_n(\tau)^{-\frac{1}{2}}-\frac{1}{3^3.3^{3^{i-1}}}2.\left(\frac{1}{\sqrt{2}}\right)^{i}u_n(\tau)^{-\frac{1}{2}}\right)\\&\quad+2.\left(\frac{1}{\sqrt{2}}\right)^{N(n)-1}u_n(\tau)^{-\frac{1}{2}}\\&=2u_n(\tau)^{-\frac{1}{2}}\left[\sum_{i=1}^{N(n)-1}\left(\left(\frac{1}{\sqrt{2}}\right)^{i-1}-\frac{1}{3^33^{3^{i-1}}}\left(\frac{1}{\sqrt{2}}\right)^{i}\right)+\left(\frac{1}{\sqrt{2}}\right)^{N(n)-1}\right].\end{aligned}$$

Thus, the extremal index is

$$\vartheta=\lim_{n\to\infty}\frac{\mu(\tilde{U}_n^{(q_n)}(\tau))}{\mu(\tilde{U}_n(\tau))}\approx 0.997242$$

round off after stabilisation of decimal places (using numerical computation).

3.3 Dependence requirements for the examples in Sections 3.1 and 3.2

We check that the Examples presented in Sections 3.1 and 3.2 meet the dependence requirements $Д_{q_n}$ and $Д'_{q_n}$. For that matter, we clearly are interested in proving that (1) and (2) of Lemma 2.1.3 hold for the same examples. In fact, it is enough that the system has summable decay of correlations against $L^1(\mu)$ and that there exists $C>0$ such that, for all $n\in\mathbb{N}$, $\mathbb{1}_{A_{n,l}^{(q_n)}},\mathbb{1}_{U_n(\tau)}\in\mathcal{C}_1$ and $\|\mathbb{1}_{A_{n,l}^{(q_n)}}\|_{\mathcal{C}_1},\|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1}\leq C$ for (1) and (2) of Lemma 2.1.3 to follow. For all the one dimensional examples illustrating both the finite and countable maximal set scenarios, we indeed have exponential decay of correlations for BV against L^1 observables. For completeness of the exposition, we recall the definition of the Banach space BV.

Definition 3.3.1. Let $\phi : I \to \mathbb{R}$ be a measurable function, where $I \subset \mathbb{R}$ is an interval. Let the *variation* of ϕ be defined as

$$Var(\phi) := \sup \left\{ \sum_{i=0}^{n-1} |\phi(x_{i+1}) - \phi(x_i)| \right\}$$

where the supremum is taken over all ordered sequences $(x_i)_{i=0}^{n}$ in I. The BV*-norm* is defined as $\|\phi\|_{BV} := \sup|\phi| + Var(\phi)$ and the space $BV := \{\phi : I \to \mathbb{R} : \|\phi\|_{BV} < \infty\}$ (equipped with the BV-norm) is a Banach space, the space of functions with *bounded variation.*

It is immediate to notice that the BV-norm of $\mathbb{1}_A$ is bounded above by

$$1 + 2\#\{\text{connected components of A}\}$$

for any measurable $A \subset I$. In particular, when the maximal set is finite, since the sets $A_{n,l}^{(q_n)}$ and $U_n(\tau)$ correspond, respectively, to a finite number of annuli or balls around the maximal points then $\|\mathbb{1}_{A_{n,l}^{(q_n)}}\|_{BV}, \|\mathbb{1}_{U_n(\tau)}\|_{BV} \le C$ for some $C > 0$ that doesn't depend on $n \in \mathbb{N}$. Therefore, in case $\mathcal{M}$ is finite we are done with proving that (1) and (2) of Lemma 2.1.3 hold.

In the countable setting, however, we do not expect a uniform bound on the BV-norms of the relevant indicator functions. Still, we can prove what we had wished for.

Lemma 3.3.2. *Let f, μ, $\mathcal{M}$ and ψ be as in Example 3.2.2. Then*

(1) $\lim_{n\to\infty} \|\mathbb{1}_{\tilde{A}_{n,l}^{(q_n)}}\|_{\mathcal{C}_1} n\rho(t_n) = 0$ *for some sequence $(t_n)_n$ with $t_n = o(n)$;*

(2) $\lim_{n\to\infty} \|\mathbb{1}_{\tilde{U}_n(\tau)}\|_{\mathcal{C}_1} \sum_{j=q_n}^{n} \rho(j) = 0.$

are satisfied, where $\tilde{U}_n(\tau) = \tilde{U}_n$ as in Example 3.2.2 and the sets $\tilde{A}_{n,l}^{(q_n)}$ are to be characterised in the proof.

Proof. Here $\mathcal{C}_1 = BV$ and $\rho(t) = \left(\frac{1}{3}\right)^t$. Setting $q_n = N(n)$ then (2) rewrites

$$\lim_{n\to\infty} \|\mathbb{1}_{\tilde{U}_n(\tau)}\|_{BV} \left(\frac{1}{3}\right)^{N(n)} \le \lim_{n\to\infty} (1 + 2N(n)) \left(\frac{1}{3}\right)^{N(n)} = 0.$$

As for (1), we take $A_l = [0, \tau_0) \times \cdots \times [0, \tau_m)$ where $\tau_0 \le \tau_1 \le \cdots \le \tau_m$ (*i.e.* $H_j = [0, \tau_j)$ for all $j = 0, \dots, m$). Then, $\tilde{A}_{n,l}^{(q_n)} := \bigcap_{j=0}^{l} f^{-j}(\tilde{U}_n(\tau_j)) \cap \bigcap_{j=1}^{q_n} \bigcup_{k=j}^{l} f^{-k}(\tilde{U}_n^c(\tau_{k-j}))$. We claim that our choice of A_l does provide us with the biggest possible number of connected components for $\tilde{A}_{n,l}^{(q_n)}$. Now, $\bigcap_{j=0}^{l} f^{-j}(\tilde{U}_n(\tau_j))$ consists of $N(n)$ balls, each centred at ξ_i, $i = 1, \dots, N(n)$, that are the result of intersecting l nested balls at each ξ_i, $i = 1, \dots, N(n)$. In turn, $\bigcap_{j=1}^{q_n} \bigcup_{k=j}^{l} f^{-k}(\tilde{U}_n^c(\tau_{k-j}))$ determines that, for each $j = 1, \dots, q_n$, one must take the complementary of at least one (so exactly one) nested ball around some ξ_i, $i = 1, \dots, N(n)$.

Thus, $\tilde{A}_{n,l}^{(q_n)}$ consists of a union of at most $N(n)-1$ balls with an annulus each centred at some ξ_i, $i=1,\dots,N(n)$. In particular, it can be the case that $\tilde{A}_{n,l}^{(q_n)}$ is made of $N(n)$ annuli centred at each ξ_i, $i=1,\dots,N(n)$, which gives the biggest possible number of connected components for $\tilde{A}_{n,l}^{(q_n)}$, that is $2N(n)$ connected components. Setting $t_n = N(n)$, we conclude

$$\lim_{n\to\infty} \|\mathbb{1}_{\tilde{A}_{n,l}^{(q_n)}}\|_{BV} n \left(\frac{1}{3}\right)^{N(n)} \leq \lim_{n\to\infty} (4N(n)+1)n\left(\frac{1}{3}\right)^{N(n)} < \lim_{n\to\infty} 2^{N(n)} n \left(\frac{1}{3}\right)^{N(n)} = 0.$$

We are left to justify our claim. Without loss of generality, let $l=1$. First, we consider the case where $A_1 = [0,\tau_0)\times[\tau_1,+\infty)$, which means that $\bigcap_{j=0}^{1} f^{-j}(\tilde{U}_n(\tau_j))$ already consists of $N(n)$ annuli. Thus, the number of connected components of $\tilde{A}_{n,l}^{(q_n)}$ can't be bigger than if $\mathcal{I}_0 = (0,\tau_0)$ and $\mathcal{I}_1 = (0,\tau_1)$ were to be considered. The same reasoning allows us to discard the case where $A_l = [\tau_0,+\infty)\times[\tau_1,+\infty)$, where, in fact, the number of connected components of $\tilde{A}_{n,l}^{(q_n)}$ is necessarily smaller than with $A_l = [0,\tau_0)\times[0,\tau_1)$. Finally, $A_l = [\tau_0,+\infty)\times[0,\tau_1)$ doesn't make sense. □

For higher dimensional expanding systems, Saussol's space will play a similar role to BV.

Definition 3.3.3. Let $\phi : \mathcal{X}\to\mathbb{R}$ be an integrable function where $\mathcal{X}$ is a compact subset of $\mathbb{R}^n$ (*i.e.* $\phi \in L^1(\text{Leb})$). Given a Borel set $\Gamma\subset\mathcal{X}$, let the *oscillation* of ϕ over Γ be defined as

$$osc(\phi,\Gamma) := \operatorname*{ess\,sup}_{\Gamma}\phi - \operatorname*{ess\,inf}_{\Gamma}\phi.$$

Given real numbers $0<\alpha\leq 1$ and $\varepsilon_0>0$, define the α-seminorm of ϕ as

$$|\phi|_\alpha := \sup_{0<\varepsilon\leq\varepsilon_0} \varepsilon^{-\alpha}\int_{\mathbb{R}^n} osc(\phi, B_\varepsilon(x))d\text{Leb}(x).$$

The space of functions with *bounded α-seminorm* $V_\alpha := \{\phi\in L^1(\text{Leb}) : |\phi|_\alpha<\infty\}$ equipped with the norm $\|\phi\|_\alpha := \|\phi\|_1 + |\phi|_\alpha$ is a Banach space.

Remark 3.3.4. The definition of the space V_α (and corresponding norm) is independent of the choice of ε_0.

The system in Examples 3.1.10 and 3.1.15 has exponential decay of correlations for observables in V_α, for some $\alpha\in(0,1]$, against L^1. We conclude with checking that (1) and (2) of Lemma 2.1.3 hold for the same examples.

Lemma 3.3.5. *Let f, μ, $\mathcal{M}$ and Ψ be as in Example 3.1.15. Then*

(1) $\lim_{n\to\infty}\|\mathbb{1}_{A_{n,l}^{(q_n)}}\|_{\mathcal{C}_1} n\rho(t_n) = 0$ *for some sequence $(t_n)_n$ with $t_n = o(n)$;*

(2) $\lim_{n\to\infty}\|\mathbb{1}_{U_n(\tau)}\|_{\mathcal{C}_1}\sum_{j=q_n}^{n}\rho(j) = 0.$

are satisfied.

Proof. Here $\mathcal{C}_1 = V_\alpha$, where $\alpha\in(0,1]$. Therefore,

$$\|\mathbb{1}_{A_{n,l}^{(q_n)}}\|_{V_\alpha} = \|\mathbb{1}_{A_{n,l}^{(q_n)}}\|_1 + |\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha = \mu(A_{n,l}^{(q_n)}) + |\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha$$

so we want to bound the term $|\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha$. In words, $|\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha$ looks into the supremum, where $\varepsilon \in (0, \varepsilon_0]$, of the areas of ε-neighbourhoods of the boundary of the set $A_{n,l}^{(q_n)}$ multiplied by the factor $\varepsilon^{-\alpha}$. We observe that, in general, $A_{n,l}^{(q_n)}$ is made up of annuli or balls, more precisely of an amount of $|\mathcal{M}|$ of annuli or balls centred at each one of the elements of $\mathcal{M}$. Since the outer circumference of $A_{n,l}^{(q_n)}$ has radius equal to $u_n(\tau_0)^{-2}$ (because $A_{n,l}^{(q_n)} \subset U_n(\tau_0)$, by definition) then, as long as $\varepsilon < u_n(\tau_0)^{-2}$ for all $\varepsilon \in (0, \varepsilon_0]$, any ε-neighbourhood of the boundary of $A_{n,l}^{(q_n)}$ can't have area bigger than $2|\mathcal{M}|\varepsilon^{-\alpha}\varepsilon.2\pi u_n(\tau_0)^{-2}$ (when $A_{n,l}^{(q_n)}$ consists of annulus around each point in $\mathcal{M}$). Since $2|\mathcal{M}|\varepsilon^{-\alpha}\varepsilon.2\pi u_n(\tau_0)^{-2}$ attains the supremum at ε_0, we have

$$|\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha \leq 2|\mathcal{M}|\varepsilon_0^{-\alpha+1}.2\pi u_n(\tau_0)^{-2}.$$

However, our assumption that $\varepsilon < u_n(\tau_0)^{-2}$ for all $\varepsilon \in (0, \varepsilon_0]$ does not hold if n is large. As a consequence, any ε-neighbourhood of the boundary of $A_{n,l}^{(q_n)}$, for large enough n, gets to be a union of $|\mathcal{M}|$ balls of radius ε. Thus,

$$|\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha \leq |\mathcal{M}|\varepsilon_0^{-\alpha}.\pi\varepsilon_0^2 = |\mathcal{M}|\pi\varepsilon_0^{2-\alpha}.$$

We conclude that

$$|\mathbb{1}_{A_{n,l}^{(q_n)}}|_\alpha \leq |\mathcal{M}|\pi\varepsilon_0^{2-\alpha}.$$

As mentioned above, $A_{n,l}^{(q_n)} \subset U_n(\tau_0)$ so that $\mu(A_{n,l}^{(q_n)}) \leq \mu(U_n(\tau_0)) = \pi(u_n(\tau_0)^{-2})^2$. Finally,

$$\begin{aligned}\|\mathbb{1}_{A_{n,l}^{(q_n)}}\|_{V_\alpha} &\leq \pi(u_n(\tau_0)^{-2})^2 + 2\pi\varepsilon_0^{2-\alpha} \\ &\leq \pi(u_1(\tau_0)^{-2})^2 + 2\pi\varepsilon_0^{2-\alpha}.\end{aligned}$$

Since ρ decays exponentially we can choose, for example, $t_n = \log(n)$ and (1) follows. We obtain the exact same estimates for $\|\mathbb{1}_{U_n(\tau)}\|_{V_\alpha}$ so that (2) is now trivially satisfied. □

Chapter 4

A Fractal Maximal Set

We take $\mathcal{M}$ equal to the middle-$\frac{1}{3}$ Cantor set.

The application of the theory of extremes for dynamical systems (in the sense brought about by [FFT10]) to fractal maximal sets was first considered in [FFRS20]. The authors were inspired by the experimental work in [MP16] where computational evidence suggested that EVL can be obtained and, moreover, no clustering of exceedances was noticed in the studies carried out.

In [FFRS20] it was established that the existence of clustering phenomena relies on how well the dynamics relates to the geometry of the fractal maximal set. More precisely, despite the intersections of the maximal set with its small neighbourhoods (corresponding to exceedances of high levels by the dynamically generated stochastic process) all having zero measure, it is the fractal dimension of such intersections, measuring the amount of overlap of sets with a fine geometrical structure, which determines the existence of an extremal index strictly less than 1 (*i.e.* the presence of clustering).

4.1 Setting

The middle-$\frac{1}{3}$ Cantor set, $\mathcal{C}$, is dynamically defined by

$$\begin{array}{rccl} f\colon & [0,1] & \to & [0,1] \\ & x & \mapsto & 3x \mod 1 \end{array} \tag{4.1.1}$$

in the following sense.

Let f_1, f_2, f_3 denote each of the three branches of $f(x) = 3x \mod 1$, more precisely, for $i = 1, 2, 3$, if $I_i = \left[\frac{i-1}{3}, \frac{i}{3}\right]$ then $f_i : I_i \to [0,1]$ is linear and surjective.

Let $\mathcal{C}_0 = [0,1]$ and, for all $n \in \mathbb{N}$, define

$$\mathcal{C}_n = f_1^{-1}(\mathcal{C}_{n-1}) \cup f_3^{-1}(\mathcal{C}_{n-1}). \tag{4.1.2}$$

Then, $\mathcal{C} = \bigcap_{n\in\mathbb{N}} \mathcal{C}_n$.

Recall that $([0,1], \mathcal{B}_{[0,1]}, m, f)$, where m denotes the Lebesgue measure on $[0,1]$, is a probability preserving dynamical system.

We want an observable which suits the theory from [FFT20] and whose maximal set is precisely $\mathcal{C}$. For that, we define a distance on $[0,1]$ as follows.

Let $\Sigma_3 = \{0, 1, 2\}$ and let $\sigma : \Sigma_3^{\mathbb{N}} \to \Sigma_3^{\mathbb{N}}$ be the left-shift map.

For $x \in [0, 1]$, let $\underline{x}$ denote the projection of x onto $\Sigma_3^{\mathbb{N}}$, that is if $x = \sum_{i \in \mathbb{N}} a_i 3^{-i}$ then $\underline{x} = a_1 a_2 \cdots \in \Sigma_3^{\mathbb{N}}$ (we drop the subset of $[0, 1]$ of points whose base 3 representation is finite, which constitutes a set of zero measure).

Observe that $x \in \mathcal{C} \iff \underline{x} = a_1 a_2 \cdots$ where $a_i \in \{0, 2\}$ for all $i \in \mathbb{N}$.

Definition 4.1.1. Let $\underline{x} \in \Sigma_3^{\mathbb{N}}$. Let $\mathcal{I}_x = \{i \in \mathbb{N} : a_i = 1\}$. Let

$$\psi(x) = \sum_{i \in \mathcal{I}_x} 2^{-i}. \tag{4.1.3}$$

ψ provides us with a notion of distance to $\mathcal{C}$ such that the earlier a point exits the dynamical construction of $\mathcal{C}$ (in the sense of (4.1.2)) the bigger the corresponding distance to $\mathcal{C}$.

Remark 4.1.2. Properties of ψ:

(i) $\psi(x) = 0 \iff x \in \mathcal{C}$

(ii) $\psi(x) < 2^{-n} \iff x \in \mathcal{C}_n \setminus \{p_1, \dots, p_{2^n}\}$, where $\{p_1, \dots, p_{2^n}\} = \{x \in [0, 1] : x = \sum_{i=1}^{n} a_i 3^{-i} + \sum_{i=n+1}^{\infty} 3^{-i} : a_i \in \{0, 2\}$ for all $i = 1, ..., n, n \in \mathbb{N}\}$

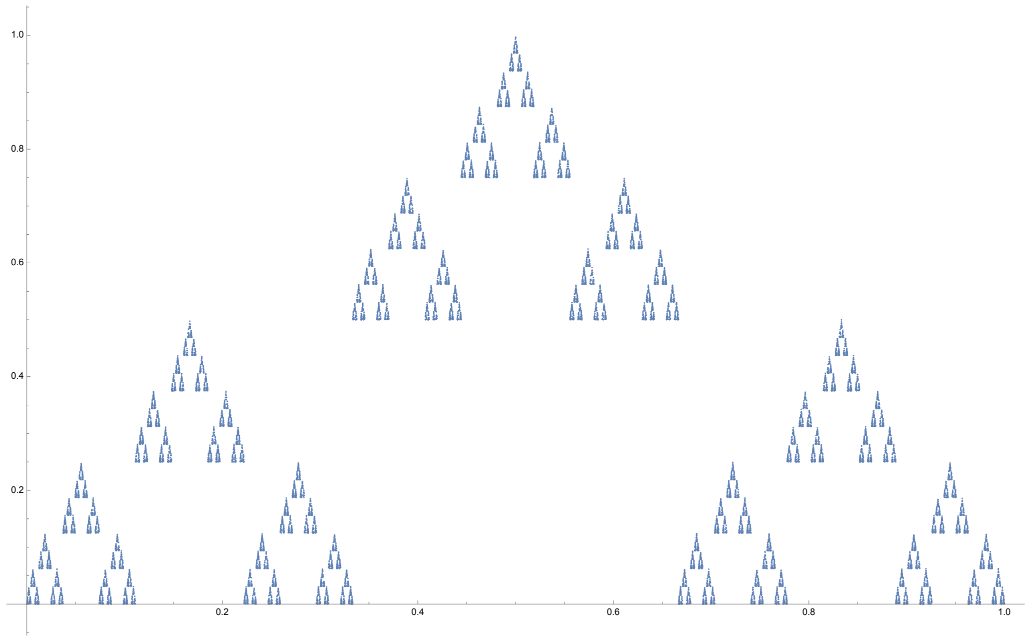

Figure 4.1: Graph of ψ.

We may now define our observable.

Definition 4.1.3. For $\alpha \in (0, 2)$, let

$$\varphi_\alpha(x) = (F \circ \psi(x))^{-\frac{1}{\alpha}}. \tag{4.1.4}$$

where $F : [0, 1] \to [0, 1]$ is the distribution function of ψ.

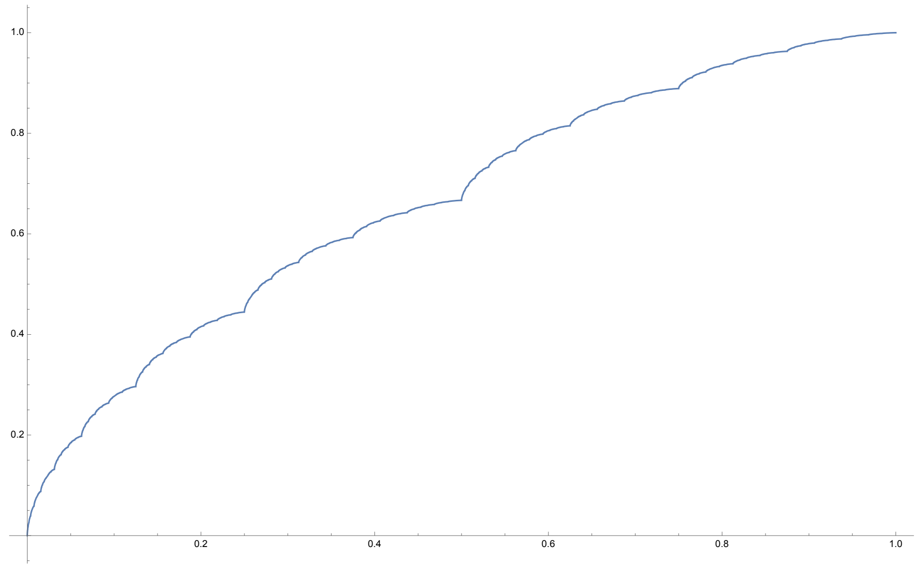

Figure 4.2: Graph of F.

Remark 4.1.4. Let $a_i = a_i(y) \in \{0, 1\}$ be defined so that

$$y = \sum_{i \in \mathbb{N}} a_i 2^{-i} = \sum_{j_i \in \mathcal{J}_y} 2^{-j_i} \tag{4.1.5}$$

where $\mathcal{J}_y = \{i \in \mathbb{N} : a_i = 1\} = \{j_i : j_i < j_{i+1} \text{ for all } i \in \mathbb{N}\}$.

Then,

$$\begin{aligned} F(y) &= \mathbb{P}(\psi \leq y) = m(\{x \in [0,1] : \psi(x) \leq y\}) \\ &= \left(\frac{2}{3}\right)^{j_1} + \frac{1}{2}\left(\frac{2}{3}\right)^{j_1}\left(\frac{2}{3}\right)^{j_2 - j_1} + \frac{1}{2}\frac{1}{2}\left(\frac{2}{3}\right)^{j_1}\left(\frac{2}{3}\right)^{j_2 - j_1}\left(\frac{2}{3}\right)^{j_3 - j_2} + \dots \\ &= \left(\frac{2}{3}\right)^{j_1} + \frac{1}{2}\left(\frac{2}{3}\right)^{j_2} + \left(\frac{1}{2}\right)^{2}\left(\frac{2}{3}\right)^{j_3} + \dots \\ &= \sum_{i \in \mathbb{N}_0, j_i \in \mathcal{J}_{n,y}} \left(\frac{1}{2}\right)^{i-1}\left(\frac{2}{3}\right)^{j_i}. \end{aligned} \tag{4.1.6}$$

Remark 4.1.5. Note that, whatever the choice of $\alpha \in (0, 2)$, φ_α takes finite values precisely on the elements of $[0, 1]$ that do not belong to $\mathcal{C}$; in particular, $\varphi_\alpha(\mathcal{C}) = \infty$ and $\mathcal{C}$ is the maximal set for φ_α. Also, φ_α strictly increases (up to a finite set of points) with depth in the dynamical construction of $\mathcal{C}$ (*i.e.* regarding $\mathcal{C}_n$ as the n-th step in the dynamical construction of $\mathcal{C}$ we have that $\varphi_\alpha(\mathcal{C}_n \setminus \{p_1, \dots, p_{2^n}\}) < \varphi_\alpha(\mathcal{C}_{n+1})$ for all $n \in \mathbb{N}$, where $\{p_1, \dots, p_{2^n}\}$ is as above).

We build the (stationary) stochastic process

$$X_n = \varphi_\alpha \circ f^n \tag{4.1.7}$$

for all $n \in \mathbb{N}_0$.

Observe that

$$\begin{aligned} \{x \in [0,1] : X_n(x) > u\} &= \{x \in [0,1] : \varphi_\alpha(f^n(x)) > u\} \\ &= \{x \in [0,1] : F(\psi(f^n(x))) < u^{-\alpha}\} \\ &= \{x \in [0,1] : \psi(f^n(x)) < F^{-1}(u^{-\alpha})\}. \end{aligned} \tag{4.1.8}$$

Our main goal is to deduce an enriched functional limit theorem for the sums of $(X_n)_{n\in\mathbb{N}_0}$. For that, a well defined piling process is key, and is given in the following theorem.

Theorem. *Let $f(x) = 3x \mod 1$, $x \in [0,1]$. Let φ_α be as in Definition 4.1.3. Then, the piling process (see Definition 2.1.4) is (a.s.) the bi-infinite sequence $(Z_j)_{j\in\mathbb{Z}}$ with entry $U.(1-\theta)^{-j}$ at $j \in \mathbb{N}_0$ and ∞ otherwise, where U is uniformly distributed on $[0,1]$ and θ is as defined in (4.3.5) (see also Proposition 4.3.1).*

We prove the theorem in Section 4.5, from which the convergence of the REPP as in Section 2.1.5 as well as the enriched FLT as in Section 2.2.3 follow (written down in sections 4.6 and 4.7, respectively).

4.2 Threshold functions

Because ultimately we are interested in an enriched functional limit theorem we are going to check the α-regular variation of the tails of the process $(X_n)_{n\in\mathbb{N}_0}$ (recall Definition 2.2.1 and Remark 2.2.2).

Lemma 4.2.1. *For $f(x) = 3x \mod 1$, $x \in [0,1]$, and observable φ_α as given in Definition 4.1.3, the random variables in the process $(X_n)_{n\in\mathbb{N}_0}$ (as in (4.1.7)) have α-regularly varying tails.*

Proof. We want to find $(a_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} n\mathbb{P}(X_0 > ya_n) = y^{-\alpha}. \tag{4.2.1}$$

We have

$$\{x \in [0,1] : X_0(x) > ya_n\} = \{x \in [0,1] : F(\psi(x)) < y^{-\alpha}a_n^{-\alpha}\}. \tag{4.2.2}$$

Observe that

$$\mathbb{P}(F \circ \psi \le z) = \mathbb{P}(\psi \le F^{-1}(z)) = F(F^{-1}(z)) = z \tag{4.2.3}$$

($i.e.$ $F \circ \psi$ is a uniformly distributed random variable).

It follows

$$\mathbb{P}(X_0 > ya_n) = \mathbb{P}(F \circ \psi < y^{-\alpha}a_n^{-\alpha}) = y^{-\alpha}a_n^{-\alpha}. \tag{4.2.4}$$

We conclude that $a_n = n^{\frac{1}{\alpha}}$ for all $n \in \mathbb{N}$. □

4.3 Extremal Index

Let $a_i = a_i(n,\tau) \in \{0,1\}$ be defined so that

$$F^{-1}((u_n(\tau))^{-\alpha}) = \sum_{i\in\mathbb{N}} a_i 2^{-i} = \sum_{j\in\mathcal{J}_{n,\tau}} 2^{-j} =: u_{n,\tau} \tag{4.3.1}$$

where $\mathcal{J}_{n,\tau} = \{i \in \mathbb{N} : a_i = 1\}$.

Let $j_{n,\tau} = \min\{j : j \in \mathcal{J}_{n,\tau}\}$. Then,

$$\begin{aligned} U_n(\tau) = \{x \in [0,1] : X_0(x) > u_n(\tau)\} &= \{x \in [0,1] : \psi(x) < F^{-1}((u_n(\tau))^{-\alpha})\} \\ &= \{x \in [0,1] : \psi(x) < u_{n,\tau}\} \\ &= \mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})} \end{aligned} \tag{4.3.2}$$

where $H^{(j_{n,\tau})}$ denotes a union of cylinders contained in $\mathcal{C}_{j_{n,\tau}-1} \setminus \mathcal{C}_{j_{n,\tau}}$ which are, at most, $(j_{n,\tau})$-cylinders. Recall (ii) in Remark 4.1.2 to justify the appearance of $\mathcal{C}_{j_{n,\tau}}$ and observe that $u_{n,\tau} = 2^{-j_{n,\tau}} + k$ with $k \leq 2^{-j_{n,\tau}}$, so $H^{(j_{n,\tau})}$ also makes sense.

Also,

$$\begin{aligned} U_n^{(q_n)}(\tau) &= \{x \in [0,1] : X_0(x) > u_n(\tau), X_1(x) \leq u_n(\tau), \dots, X_{q_n}(x) \leq u_n(\tau)\} \\ &= \{x \in [0,1] : \psi(x) < u_{n,\tau}, \psi(f(x)) \geq u_{n,\tau}, \dots, \psi(f^{q_n}(x)) \geq u_{n,\tau}\} \end{aligned} \tag{4.3.3}$$

which, for n sufficiently large, can be rewritten

$$\begin{aligned} U_n^{(q_n)}(\tau) &= \{x \in [0,1] : \psi(x) < u_{n,\tau}, \psi(f(x)) \geq u_{n,\tau}\} \\ &= \mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})} \cap f^{-1}([0,1] \setminus (\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})})) \\ &= H^{(j_{n,\tau})} \cup (\mathcal{C}_{j_{n,\tau}} \cap f^{-1}((\mathcal{C}_{j_{n,\tau}-1} \setminus \mathcal{C}_{j_{n,\tau}}) \setminus H^{(j_{n,\tau})})) \end{aligned} \tag{4.3.4}$$

so, in particular, $q_n = 1$ for all $n \in \mathbb{N}$.

The extremal index, θ, is defined as

$$\theta = \lim_{n\to\infty} \frac{m(U_n^{(1)}(\tau))}{m(U_n(\tau))} \tag{4.3.5}$$

provided the limit exists.

In light of [FFRS20], for $\mathcal{M}$ equal to the middle-$\frac{1}{3}$ Cantor set then $\theta < 1$ if and only if, among the class of maps $mx \mod 1$, $m \geq 2$, we restrict to $m = 3^k$, $k \in \mathbb{N}$. Hence, an extremal index strictly less than 1 is obtained if and only if the dynamics is capable of generating the middle-$\frac{1}{3}$ Cantor set. We proceed with the computation of the extremal index since our observable is different from the one used in [FFRS20].

Proposition 4.3.1. *For $f(x) = 3x \mod 1$, $x \in [0,1]$, and observable φ_α as given in Definition 4.1.3, the extremal index is $\theta = \dfrac{1}{3}$.*

Proof. Let $u_{n,\tau} \in [2^{-j}, 2^{-j+1})$. Then, $j_{n,\tau} = j$ and, by (4.3.2), $U_n(\tau) = \mathcal{C}_j \cup H^{(j)}$ so that

$$m(U_n(\tau)) = \left(\frac{2}{3}\right)^j + \lambda 2^{j-1}\left(\frac{1}{3}\right)^j \tag{4.3.6}$$

where $\lambda \in [0,1]$ (note that $H^{(j)}$ is a fraction of $\mathcal{C}_{j-1} \setminus \mathcal{C}_j$, thus $m(H^{(j)}) = \lambda \dfrac{1}{2}\left(\dfrac{2}{3}\right)^j$ for some $\lambda \in [0,1]$).

Now, by (4.3.4),

$$U_n^{(1)}(\tau) = H^{(j)} \cup (\mathcal{C}_j \cap f^{-1}((\mathcal{C}_{j-1} \setminus \mathcal{C}_j) \setminus H^{(j)})) \tag{4.3.7}$$

and we have

$$m(U_n^{(1)}(\tau)) = \lambda 2^{j-1}\left(\frac{1}{3}\right)^j + (1-\lambda)\frac{1}{3}\left(\frac{2}{3}\right)^j \tag{4.3.8}$$

where λ is the same as in (4.3.6) (in particular, observe that the second summand reflects the fact that only a fraction of $\mathcal{C}_j \setminus \mathcal{C}_{j+1}$, whose measure is $(1-\lambda)m(\mathcal{C}_j \setminus \mathcal{C}_{j+1})$, is iterated in one time step to the complement of $H^{(j)}$ in $\mathcal{C}_{j-1} \setminus \mathcal{C}_j$).

Finally,

$$\theta_n = \frac{m(U_n^{(1)}(\tau))}{m(U_n(\tau))} = \frac{\lambda 2^{j-1}\left(\frac{1}{3}\right)^j + (1-\lambda)\frac{1}{3}\left(\frac{2}{3}\right)^j}{\left(\frac{2}{3}\right)^j + \lambda 2^{j-1}\left(\frac{1}{3}\right)^j} = \frac{\lambda 2^{j-1}\left(\frac{1}{3}\right)^j + \frac{2^j}{3^{j+1}} - \lambda\frac{2^j}{3^{j+1}}}{\left(\frac{2}{3}\right)^j + \lambda\frac{2^{j-1}}{3^j}}$$
$$= \frac{\frac{2^j}{3^{j+1}} + \lambda\frac{2^{j-1}}{3^j}\left(1-\frac{2}{3}\right)}{\left(\frac{2}{3}\right)^j + \lambda\frac{2^{j-1}}{3^j}} = \frac{\frac{2}{3}+\frac{1}{3}\lambda}{2+\lambda} = \frac{1}{3} \tag{4.3.9}$$

Therefore, $\theta = \lim_{n\to\infty} \theta_n = \frac{1}{3}$. □

4.4 Dependence requirements

In order to prove that the conditions $Д_{q_n}$ and $Д'_{q_n}$ from Section 2.1.3 hold in our setting we use a different strategy to the one used when $\mathcal{M}$ was a finite or countable set of points. That is due to the set $A_{n,l}^{(q_n)}$ possibly having a very large number of connected components and, consequently, (1) and (2) of Lemma 2.1.3 failing to be true.

We begin by proving that $A_{n,l}^{(q_n)}$ can be well approximated by cylinders of some fixed depth. Recall that

$$\{x \in [0,1] : u_n^{-1}(X_0(x)) \in [\tau'', \tau')\} = \{x \in [0,1] : X_0(x) \in (u_n(\tau'), u_n(\tau'')]\}.$$

Now,

$$\begin{aligned} X_0(x) \in (u_n(\tau'), u_n(\tau'')] &\implies X_1(x) \in (F(2^{-\frac{1}{\alpha}})u_n(\tau'), F(2^{-\frac{1}{\alpha}})u_n(\tau'')] \\ &\vdots \\ &\implies X_m(x) \in (F(2^{-\frac{m}{\alpha}})u_n(\tau'), F(2^{-\frac{m}{\alpha}})u_n(\tau'')]. \end{aligned}$$

We conclude that the choice of H_0 in the definition of A_l completely determines $A_{n,l}$.

We take $H_0 = [\tau'', \tau')$, as the class of half-open intervals is a $\mathcal{F}_\mathcal{V}$-generating class (see Section 2.1.3), so

$$\begin{aligned} A_{n,l} &= (\mathcal{C}_{j_{n,\tau'}} \cup H^{(j_{n,\tau'})}) \setminus (\mathcal{C}_{j_{n,\tau''}} \cup H^{(j_{n,\tau''})}) \\ &= H^{(j_{n,\tau'})} \cup (\mathcal{C}_{j_{n,\tau'}} \setminus (\mathcal{C}_{j_{n,\tau''}} \cup H^{(j_{n,\tau''})})) \end{aligned} \tag{4.4.1}$$

that is, $A_{n,l}$ is made up of holes in the dynamical construction of $\mathcal{C}$ (*i.e.* subsets of the $\mathcal{C}_k \setminus \mathcal{C}_{k+1}$ where $k \in \mathbb{N}$), and $j_{n,\tau'}$ is as defined in the beginning of Section 4.3.

Finally, for n sufficiently large,

$$\begin{aligned} A_{n,l}^{(q_n)} &= A_{n,l} \cap f^{-1}(A_{n,l})^c \cap \cdots \cap f^{-q_n}(A_{n,l})^c \\ &= A_{n,l} \cap f^{-1}(A_{n,l})^c \end{aligned} \tag{4.4.2}$$

so, in particular, $q_n = 1$ for all $n \in \mathbb{N}$ (*cf.* $U_n^{(q_n)}$ as given by (4.3.4)).

Combining (4.4.1) and (4.4.2), we obtain

$$A_{n,l}^{(1)} = H^{(j_{n,\tau'})} \bigcup ((\mathcal{C}_{j_{n,\tau'}} \setminus \mathcal{C}_{j_{n,\tau''}}) \setminus H^{(j_{n,\tau''})}) \cap f^{-1}((\mathcal{C}_{j_{n,\tau'}-1} \setminus \mathcal{C}_{j_{n,\tau'}}) \setminus H^{(j_{n,\tau'})}). \quad (4.4.3)$$

Lemma 4.4.1. *There exist Λ_n^+ and Λ_n^- unions of, at most, $(2j_{n,\tau'})$-cylinders which approximate $A_{n,l}^{(1)}$ from above and from below, respectively, and such that*

$$\frac{m(\Lambda_n^+ \setminus \Lambda_n^-)}{m(A_{n,l}^{(1)})} \leq \rho_n \quad (4.4.4)$$

where $\lim_{n\to\infty} \rho_n = 0$.

Proof. We may write

$$A_{n,l}^{(1)} = H^{(j_{n,\tau'})} \cup K \quad (4.4.5)$$

where $K \subset \mathcal{C}_{j_{n,\tau'}} \setminus \mathcal{C}_{j_{n,\tau'}+1}$. Therefore,

$$m(A_{n,l}^{(1)}) = \frac{\beta}{2}\left(\frac{2}{3}\right)^{j_{n,\tau'}} + \frac{\gamma}{3}\left(\frac{2}{3}\right)^{j_{n,\tau'}} = \left(\frac{3\beta+2\gamma}{6}\right)\left(\frac{2}{3}\right)^{j_{n,\tau'}} \quad (4.4.6)$$

where $\beta, \gamma \in [0,1]$ (as $H^{(j_{n,\tau'})}$ is a fraction of $\mathcal{C}_{j_{n,\tau'}-1} \setminus \mathcal{C}_{j_{n,\tau'}}$ while K is a fraction of $\mathcal{C}_{j_{n,\tau'}} \setminus \mathcal{C}_{j_{n,\tau'}+1}$).

If $A_{n,l}^{(1)}$ can itself be written as a union of cylinders which are, at most, $(2j_{n,\tau'})$-cylinders, then (4.4.4) is trivially satisfied. Otherwise, we separate the cases where $H^{(j_{n,\tau'})}$ is too small or too large.

First, assume that $H^{(j_{n,\tau'})}$ is too small so that its elements (which correspond to the points right above $y = 2^{-j_{n,\tau'}}$ in the graph of ψ) are approximated from above by $(2j_{n,\tau'})$-cylinders. Let $C_{2j_{n,\tau'}}^+$ denote the union of $(2j_{n,\tau'})$-cylinders which produces the finest such approximation.

Since a very small $H^{(j_{n,\tau'})}$ implies a large K we may take

$$\Lambda_n^+ = C_{2j_{n,\tau'}}^+ \cup (\mathcal{C}_{j_{n,\tau'}} \setminus \mathcal{C}_{j_{n,\tau'}+1})$$

and

$$\Lambda_n^- = \mathcal{C}_{j_{n,\tau'}} \setminus \mathcal{C}_{j_{n,\tau'}+1}.$$

Then, $\Lambda_n^+ \setminus \Lambda_n^- = C_{2j_{n,\tau'}}^+$ so that

$$m(\Lambda_n^+ \setminus \Lambda_n^-) = 2^{j_{n,\tau'}-1}.2^{j_{n,\tau'}}.\left(\frac{1}{3}\right)^{2j_{n,\tau'}} = \frac{1}{2}\left(\frac{2}{3}\right)^{2j_{n,\tau'}} \quad (4.4.7)$$

(as, for each connected component of $\mathcal{C}_{j_{n,\tau'}-1} \setminus \mathcal{C}_{j_{n,\tau'}}$, $C_{2j_{n,\tau'}}^+$ consists of $2^{j_{n,\tau'}}$ cylinders which are $(2j_{n,\tau'})$-cylinders).

Therefore,

$$\frac{m(\Lambda_n^+ \setminus \Lambda_n^-)}{m(A_{n,l}^{(1)})} = \frac{\frac{1}{2}\left(\frac{2}{3}\right)^{2j_{n,\tau'}}}{\frac{3\beta+2\gamma}{6}\left(\frac{2}{3}\right)^{j_{n,\tau'}}} = \frac{3}{3\beta+2\gamma}\left(\frac{2}{3}\right)^{j_{n,\tau'}}. \quad (4.4.8)$$

Now, assume that K is too small so that its elements (which correspond to the points right below $y = 2^{-j_{n,\tau'}}$ in the graph of ψ) are approximated from above by $(2j_{n,\tau'})$-cylinders. Let $C^-_{2j_{n,\tau'}}$ denote the union of $(2j_{n,\tau'})$-cylinders which produces the finest such approximation (in fact, $C^-_{2j_{n,\tau'}}$ is made up of only two $(2j_{n,\tau'})$-cylinders).

Since a very small K implies a large $H^{(j_{n,\tau'})}$ we may take

$$\Lambda_n^+ = (\mathcal{C}_{j_{n,\tau'}-1} \setminus \mathcal{C}_{j_{n,\tau'}}) \cup C^-_{2j_{n,\tau'}}$$

and

$$\Lambda_n^- = \mathcal{C}_{j_{n,\tau'}-1} \setminus \mathcal{C}_{j_{n,\tau'}}.$$

Then, $\Lambda_n^+ \setminus \Lambda_n^- = C^-_{2j_{n,\tau'}}$ so that

$$m(\Lambda_n^+ \setminus \Lambda_n^-) = 2\left(\frac{1}{3}\right)^{2j_{n,\tau'}}. \tag{4.4.9}$$

Therefore,

$$\frac{m(\Lambda_n^+ \setminus \Lambda_n^-)}{m(A_{n,l}^{(1)})} = \frac{2\left(\frac{1}{3}\right)^{2j_{n,\tau'}}}{\frac{3\beta+2\gamma}{6}\left(\frac{2}{3}\right)^{j_{n,\tau'}}} = \frac{6}{(3\beta+2\gamma).2^{2j_{n,\tau'}-1}}\left(\frac{2}{3}\right)^{j_{n,\tau'}}. \tag{4.4.10}$$

□

It is a direct consequence of Lemma 4.4.1 that the REPP counting the number of hits to $A_{n,l}^{(1)}$ converges if and only if the REPP counting the number of hits to either Λ_n^+ or Λ_n^- converges, and the limits are the same.

As a result, we may check that the conditions $Д_{q_n}$ and $Д'_{q_n}$ are verified for the cylinder approximating sets.

Condition $Д^*_{q_n}$. We say that $Д^*_{q_n}$ holds for the sequence $\mathbf{X}_0, \mathbf{X}_1, \dots$ if there exist sequences $(k_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as defined in Section 2.1.3, such that for every $m, t, n \in \mathbb{N}$ and every J_l and A_l, with $l = 1, \dots, m$, we have

$$\left|\mathbb{P}\left(\Lambda_n^+ \cap \bigcap_{i=l}^m \mathscr{W}_{J_{n,i}}(\Lambda_n^+)\right) - \mathbb{P}(\Lambda_n^+)\mathbb{P}\left(\bigcap_{i=l}^m \mathscr{W}_{J_{n,i}}(\Lambda_n^+)\right)\right| \le \gamma(n,t)$$

where $\min\{J_{n,l} \cap \mathbb{N}_0\} \ge t$ and $\gamma(n,t)$ is decreasing in t for each n and $\lim_{n\to\infty} n\gamma(n,t_n) = 0$.

Condition $Д'^*_{q_n}$. We say that $Д'^*_{q_n}$ holds for the sequence $\mathbf{X}_0, \mathbf{X}_1, \dots$ if there exist sequences $(k_n)_{n\in\mathbb{N}}, (r_n)_{n\in\mathbb{N}}, (t_n)_{n\in\mathbb{N}}$ and $(q_n)_{n\in\mathbb{N}}$ as defined in Section 2.1.3, such that for every $A_1 \in \mathscr{F}$, we have

$$\lim_{n\to\infty} n\mathbb{P}\left(\Lambda_n^+ \cap \mathscr{W}^c_{[q_n+1,r_n)}(A_{n,1})\right) = 0.$$

Both $Д^*_{q_n}$ and $Д'^*_{q_n}$ follow from the fact that f is exponential ϕ-mixing.

Definition 4.4.2. Let $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \mu, f)$ be a probability preserving system. Let ω_1 denote a measurable partition of $\mathcal{X}$ made up of 1-cylinders for f and let $\mathcal{F}_{1,k}$ denote the σ-algebra generated by $\omega_1, \dots, \omega_k$. $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \mu, f)$ is *exponential ϕ-mixing* if there exist $C > 0$ and $\lambda \in (0,1)$ such that for every $H \in \mathcal{F}_{1,k}$ and every $A \in \mathcal{B}_\mathcal{X}$

$$|\mu(H \cap f^{-(t+k)}(A)) - \mu(H)\mu(A)| \le C\lambda^t\mu(A). \tag{4.4.11}$$

Proof of Condition $Д^*_{q_n}$. Write (4.4.11) with $H = \Lambda_n^+$ and $f^{-(t+k)}(A) = \bigcap_{i=l}^m \mathscr{W}_{J_{n,i}}(\Lambda_n^+)$ to obtain $\gamma(n,t) = C\lambda^t \mathbb{P}(\bigcap_{i=l}^m \mathscr{W}_{J_{n,i}}(\Lambda_n^+))$. Here $k = 2j_{n,\tau'}$ (see Lemma 4.4.1). □

Proof of Condition $Д'^*_{q_n}$. We have

$$\begin{aligned}
\lim_{n\to\infty} n\mathbb{P}\left(\Lambda_n^+ \cap \mathscr{W}^c_{[q_n+1,r_n)}(A_{n,1})\right) &= \lim_{n\to\infty} n \sum_{j=q_n+1}^{r_n-1} \mathbb{P}(\Lambda_n^+ \cap f^{-j}(A_{n,1})) \\
&\leq \lim_{n\to\infty} n \sum_{j=q_n+1}^{r_n-1} \mathbb{P}(\Lambda_n^+)\mathbb{P}(A_{n,1}) + \lim_{n\to\infty} n \sum_{j=q_n+1}^{r_n-1} C\lambda^j \mathbb{P}(A_{n,1}) \\
&\leq \lim_{n\to\infty} nr_n \mathbb{P}(\Lambda_n^+)\mathbb{P}(A_{n,1}) + \lim_{n\to\infty} nC\mathbb{P}(A_{n,1}) \sum_{j=q_n+1}^{+\infty} \lambda^j \\
&\leq \lim_{n\to\infty} \frac{\tau^2}{k_n} + \lim_{n\to\infty} C\tau \sum_{j=q_n+1}^{+\infty} \lambda^j = 0
\end{aligned} \tag{4.4.12}$$

where we used (4.4.11) with $H = \Lambda_n^+$, $f^{-k}(A) = A_{n,1}$ and $k = 2j_{n,\tau'}$ (see Lemma 4.4.1) to derive the first inequality. □

4.5 Piling Process

We will see that for our f and φ_α the middle-$\frac{1}{3}$ Cantor set behaves like a fixed point.

Theorem 4.5.1. *Let* $f(x) = 3x \mod 1$, $x \in [0,1]$. *Let* φ_α *be as in Definition 4.1.3. Then, the piling process (see Definition 2.1.4) is (a.s.) the bi-infinite sequence* $(Z_j)_{j\in\mathbb{Z}}$ *with entry* $U.(1-\theta)^{-j}$ *at* $j \in \mathbb{N}_0$ *and* ∞ *otherwise, where* U *is uniformly distributed on* $[0,1]$ *and* θ *is as defined in (4.3.5) (see also Proposition 4.3.1).*

Proof. We check that the process $(Y_j)_{j\in\mathbb{Z}}$ is (a.s.) the bi-infinite sequence with:

(i) entry U at $j = 0$;

(ii) entries $U.(1-\theta)^{-j}$ for all positive indices j;

(iii) ∞ for all negative indices j except, possibly, $U.(1-\theta)^{-j}$ at $j \geq -m$ for some $m \in \mathbb{N}$;

where U is uniformly distributed on $[0,1]$.

Observe that $U.(1-\theta)^{-j} < 1$ for all $j < 0$. Therefore, if $(Y_j)_{j\in\mathbb{Z}}$ is as described by (i)-(iii) then $(Z_j)_{j\in\mathbb{Z}}$ is as in the statement of the theorem.

We must check that conditions (1)-(4) in Definition 2.1.4 are satisfied for $(Y_j)_{j\in\mathbb{Z}}$ as given by (i)-(iii).

Conditions (2) and (3) are straightforward.

As for condition (4), $Y_j \geq 1$ for all $j \leq -1$ implies that the hit to $\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ at time r_n is the first hit to that same neighbourhood of $\mathcal{C}$ (*i.e.* there is no hit to $\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ before time r_n given that there is a hit to $\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ at time r_n - recall, from (4.3.2), that $\{X_{r_n} > u_n(\tau)\} = f^{-r_n}(\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})})$). Since

$$\frac{m(f^{-r_n}(\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}) \cap f^{-(r_n-1)}([0,1] \setminus (\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})})))}{m(f^{-r_n}(\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}))} > 0$$

we have that condition (4) is satisfied.

So, we check that condition (1) holds. Recall (see Remark 2.2.2) that $u_n^{-1}(X_j) = \left(\frac{X_j}{a_n}\right)^{-\alpha}$. Therefore,

$$\left\{\frac{1}{\tau}u_n^{-1}(X_{r_n+j}) \mid X_{r_n} > u_n(\tau)\right\} = \left\{\frac{1}{\tau}\left(\frac{X_{r_n+j}}{a_n}\right)^{-\alpha} \mid X_{r_n} > u_n(\tau)\right\}. \tag{4.5.1}$$

Now, $\{x \in [0,1] : X_{r_n}(x) > u_n(\tau)\} = f^{-r_n}(\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})})$.

Let $j \geq 0$. If $f^{r_n}(x) \in \mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ then $f^{r_n+j}(x) \in \mathcal{C}_{j_{n,\tau}-j} \cup H^{(j_{n,\tau}-j)}$ provided $j_{n,\tau} - j \geq 0$, in which case

$$\begin{aligned}\frac{F(\psi(f^{r_n+j}(x)))}{F(\psi(f^{r_n}(x)))} = \frac{\mathbb{P}(\psi \leq \psi(f^{r_n+j}(x)))}{\mathbb{P}(\psi \leq \psi(f^{r_n}(x)))} &= \frac{m(\mathcal{C}_{j_{n,\tau}-j} \cup H^{(j_{n,\tau}-j)})}{m(\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})})} \\ &= \left(\frac{2}{3}\right)^{-j} = (1-\theta)^{-j}.\end{aligned} \tag{4.5.2}$$

We have

$$\lim_{n\to\infty} \mathbb{P}\left(\left\{\frac{u_n^{-1}(X_{r_n+j})}{u_n^{-1}(X_{r_n})} = (1-\theta)^{-j} \mid X_{r_n} > u_n(\tau)\right\}\right) = 1 \tag{4.5.3}$$

(since $j_{n,\tau} - j \geq 0$ when $n \to \infty$).

We may write

$$\begin{aligned}&\lim_{n\to\infty} \mathbb{P}\left\{\frac{1}{\tau}u_n^{-1}(X_{r_n+j}) = s \mid X_{r_n} > u_n(\tau)\right\} \\ &\quad = \lim_{n\to\infty} \mathbb{P}\left\{\frac{u_n^{-1}(X_{r_n+j})}{u_n^{-1}(X_{r_n})}\frac{u_n^{-1}(X_{r_n})}{\tau} = s \mid X_{r_n} > u_n(\tau)\right\}\end{aligned} \tag{4.5.4}$$

which, because (2) in Definition 2.1.4 holds, gives $\mathcal{L}(Y_j) = \mathcal{L}(\Theta_j Y_0) = (1-\theta)^{-j}.U$ (a.s.).

Let $j < 0$. If $f^{r_n}(x) \in \mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ then $f^{r_n+j}(x) \in \mathcal{C}_{j_{n,\tau}-j} \cup H^{(j_{n,\tau}-j)}$ for an at most finite number of indices $j \geq -m$ where $m \in \mathbb{N}$. In words, a hit, at time r_n, to the approximation (of $\mathcal{C}$) $\mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ can only be preceded by a finite number of hits to the finer approximations $\mathcal{C}_{j_{n,\tau}-j} \cup H^{(j_{n,\tau}-j)}$, as, otherwise, the point would be in $\mathcal{C}$. This leads to $Y_j = U.(1-\theta)^{-j}$ for an at most finite number of indices $j \geq -m$ where $m \in \mathbb{N}$.

If $f^{r_n}(x) \in \mathcal{C}_{j_{n,\tau}} \cup H^{(j_{n,\tau})}$ is such that $f^{r_n+k}(x) \in \left[\frac{1}{3}, \frac{2}{3}\right]$ for some $k \in \mathcal{K} \subseteq \{1, \ldots, -j\}$, we have

$$\frac{\psi(f^{r_n+j}(x))}{\psi(f^{r_n}(x))} = \frac{\sum_{k\in\mathcal{K}} 2^{-k} + 2^j\psi(f^{r_n}(x))}{\psi(f^{r_n}(x))}. \tag{4.5.5}$$

This leads to

$$\lim_{n\to\infty} \mathbb{P}\left\{\frac{u_n^{-1}(X_{r_n+j})}{u_n^{-1}(X_{r_n})} = \infty \mid X_{r_n} > u_n(\tau)\right\} = 1. \tag{4.5.6}$$

So, $Y_j = \infty$. □

4.6 Complete convergence of the REPP

The REPP is written

$$N_n = \sum_{i=1}^{\infty} \delta_{(i/k_n, \tilde{\pi}(\mathbb{X}_{n,i}))}. \tag{4.6.1}$$

The complete convergence of the REPP follows as a direct application of Theorem 2.1.11.

Theorem 4.6.1. *N_n converges weakly (in the space of boundedly finite point measures on $\mathbb{R}_0^+ \times \tilde{l}_\infty \setminus \{\tilde{\infty}\}$ with weak$^\#$ topology) to*

$$N = \sum_{i=1}^{\infty} \delta_{(T_i, U_i \tilde{\mathbf{Q}}_i)} \tag{4.6.2}$$

which is a Poisson process with intensity measure $Leb \times \eta$, where $\eta = \theta(Leb \times \mathbb{P}_{\tilde{\mathbf{Q}}}) \circ \psi$ and where $\tilde{\mathbf{Q}}$ is (a.s.) the bi-infinite sequence with entry $(1-\theta)^{-j}$ at $j \in \mathbb{N}_0$ and ∞ otherwise.

4.7 Enriched FLT

As a direct application of Theorem 2.2.6, we obtain an enriched FLT for sums of heavy-tailed random variables maximised at the middle-$\frac{1}{3}$ Cantor set.

Theorem 4.7.1. *Let $\alpha \in (0,1)$. Then,*

$$S_n(t) = \sum_{i=0}^{\lfloor nt \rfloor - 1} \frac{1}{n^{\frac{1}{\alpha}}} X_i, \ t \in [0,1],$$

converges in F' to $(V, disc(V), (e_V^s)_{s \in disc(V)})$, where V is an α-stable Lévy process on $[0,1]$

$$V(t) = \sum_{T_i \le t} \sum_{j \in \mathbb{Z}} U_i^{-\frac{1}{\alpha}} \mathcal{Q}_{i,j}$$

and the excursions are given by

$$e_V^{T_i}(t) = V(T_i^-) + U_i^{-\frac{1}{\alpha}} \sum_{0 \le j \le \lfloor \tan(\pi(t - \frac{1}{2})) \rfloor} \mathcal{Q}_{i,j}, \ t \in [0,1]$$

where T_i and U_i are as in N in Theorem 4.6.1, $\mathcal{Q}_{i,j} = \xi(\tilde{\mathbf{Q}}_{i,j})$ for $\tilde{\mathbf{Q}}_{i,j}$ as in N in Theorem 4.6.1 and ξ as in (2.2.13).

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