Viana, V., Nagy, D., Xavier, J., Neiva, A., Ginoulhiac, M., Mateus, L. & Varela, P. (Eds.). (2022). Symmetry: Art and Science | 12th SIS-Symmetry Congress [Special Issue]. *Symmetry: Art and Science*. International Society for the Interdisciplinary Study of Symmetry. 210-217. https://doi.org/10.24840/1447-607X/2022/12-26-210

SYMMETRY IN MOTION: *STELLAE OCTANGULAE*, AND EQUIFACED POLYHEDRA

GUNTER WEISS

Name: Gunter Weiss (b. St. Veit / Glan, Austria 1946)
Profession: retired. Prof.
Fields of interest: Geometry
Affiliation: Universities of Technology Dresden, Germany, and Vienna, Austria
E-mail: weissgunter@gmx.at
Homepage: https://tu-dresden.de/mn/math/geometrie/das-institut/beschaeftigte/prof-weiss
Major publications and/or exhibitions:
Weiss, G., Pech, P.: A Quadratic Mapping Related to Frégier's Theorem and Some Generalisations. *JGG 25* (2021), No. 1, p.127-137.
Stavric, M., Wiltsche, A., Weiss, G.: Polyhedrons the Faces of which are Special Quadric Patches. *KoG-25* (2021), p.45-52.
Gfrerrer A., Weiss, G.: Osculating Conic Biarcs. *Computer Aided Geometric Design*. 81. 101904, DOI: 10.1016/j.cagd.2020.101904.
Weiss, G.: The Three Reflections Theorem Revisited. *KoG-22* (2018), p. 41-48.

Abstract: *We discuss three seemingly independent topics concerning polyhedra, which finally show some relations. The first topic concerns generalisations of the classical Stella Octangula, whereby the pair of indirect congruent tetrahedra, which generates such a Stella, allows forced motions of one tetrahedron along the other. Symmetry arguments are used to identify pairs of right triangular pyramids and pairs of indirect congruent tetrahedra as candidates for such "movable" Stellae Octangulae. The second topic discusses equifaced octahedra, as they occur as the common body of equifaced Stellae Octangulae. It turns out that, in general, there exist four octahedra with the same acute face triangles. They differ in the way their diagonals intersect and in the number of symmetries and are, in this paper, distinguished "type A-octahedra" and "type B-octahedra". For obtuse face triangles the is no type A-octahedron. For general right triangles there are two type B-octahedra, which for isosceles right triangles coincide and become the single Rodrigues-octahedron. The third topic concerns polyhedra with congruent isosceles right triangular faces, thus generalising the Rodrigues-octahedron. This chapter also the aims at providing material for educational purposes.*

Keywords: Polyhedron; *Stella Octangula*; Rodrigues Octahedron.

INTRODUCTION

In 1988, H. Stachel (Stachel, 1988, p. 65-75) showed that the two congruent tetrahedra of a classical *Stella Octangula*, if materialized by their edges, allow a two-parameter set of motions of one tetrahedron relative to the other. Thereby all edges of the moving tetrahedron T_2 slide along the edges of the fixed one T_1. We pick up Stachel's discovery and look for "generalised" *Stellae Octangulae* possessing the afore mentioned property. As somehow natural generalisations we shall consider

Stellae Octangulae consisting of a pair of indirect congruent tetrahedra T_1 and T_2. For movability of T_2 against T_1 it is necessary that their faces are acute triangles. Here we shall consider only those cases of T_1 and T_2, which can be treated by symmetry arguments and geometric reasoning alone.

In the start position of T_1 and T_2 their edges intersect in the midpoints of the edges, and the solids T_1, T_2 intersect in an equifaced octahedron. This connects generalised *Stellae Octangulae* with the question to find *all* types of equifaced octahedra and their symmetries. One example is mentioned by D. Wyllie Rodrigues (Rodrigues, 2017, e-mail), who designed an octahedron, the faces of which are isosceles right triangles, and he connected the ratio of its diagonals with the Golden Mean. It turns out that, in general, there are up to three different equifaced octahedra, which can be built with eight acute triangles. Adding equifaced tetrahedra to the faces of such an octahedron one obtains an equifaced *Stella Octangula* as mentioned above.

Furthermore, one might extend Rodrigues' original idea of using isosceles right triangular faces for the octahedron also for other closed and open polyhedra with congruent set square faces. Besides polyhedra with the symmetry groups of Platonic solids one can construct also irregular polyhedra, cylindrical and even helical ones. For example, with $2n + 1$ such triangles, $(n \geq 3)$ it is already possible to make Möbius stripes (Weiss, 2020, p.20-26).

As far as it is possible, we treat these three topics by geometric reasoning. This makes it possible to omit lengthy calculations and gives better insight, why a supposed property is true.

STELLAE OCTANGULAE IN MOTION

H. Stachel's discovery and description of the motions of one regular tetrahedron T_1 relative to the other T_2 of a classical *Stella Octangula* such that all edges of T_2 slide along the edges of T_1 see (Stachel, 1988, p. 65-75) connects symmetry properties with the kinematics of forced motions. It seems that the question, whether the regular *Stella Octangula* is the only one with movable tetrahedral parts, or whether there are more such objects, still is unanswered. Here we pick up Stachel's discovery and consider generalised *Stellae Octangulae* consisting of indirect congruent tetrahedra T_1 and T_2. We demand that the two tetrahedra T_1 and T_2, if materialized by their edges, allow at least a one-parameter set of motions of T_2 relative to T_1. Thereby all edges of the moving tetrahedron T_2 slide along the edges of the fixed one T_1. It turns out that each tetrahedron T_1, the faces of which are acute triangles, can be completed to a *Stella Octangula* by an indirect congruent, "symmetric" version T_2 of that tetrahedron T_1, and there always will exist, at least a one-parameter set of forced motions of T_2 along T_1. For example, if T_1 is a right three-sided pyramid, there is a one parameter set of helical motions from the extremal start position of T_2, Figure 1 (left), where T_1 and T_2

are centrally symmetric to another position of T_2, Figure 1 (right). Note that one vertex of T_2 moves on the screw axis, the other vertices trace a curve on a coaxial cylinder, but not a helix, Figure 2.

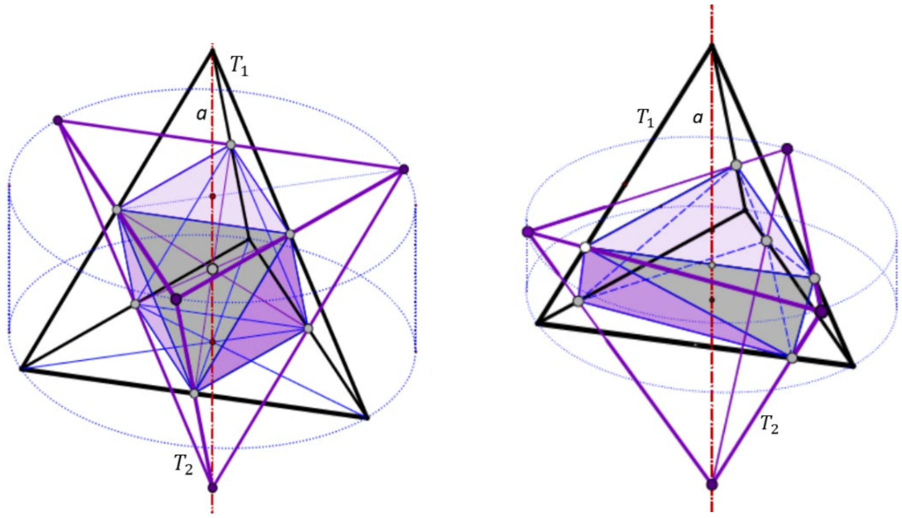

Figure 1 Stella Octangula consisting of two right tetrahedra T_1 and T_2 in extremal start position (left), and in an intermediate position (right). At the start position the intersection $T_1 \cap T_2$ is an octahedral antiprism with a regular face triangle at the top and the bottom and six congruent isosceles triangles in between (left). At the intermediate position $T_1 \cap T_2$ still is an antiprism with regular top and bottom, but with irregular face triangles between (right).

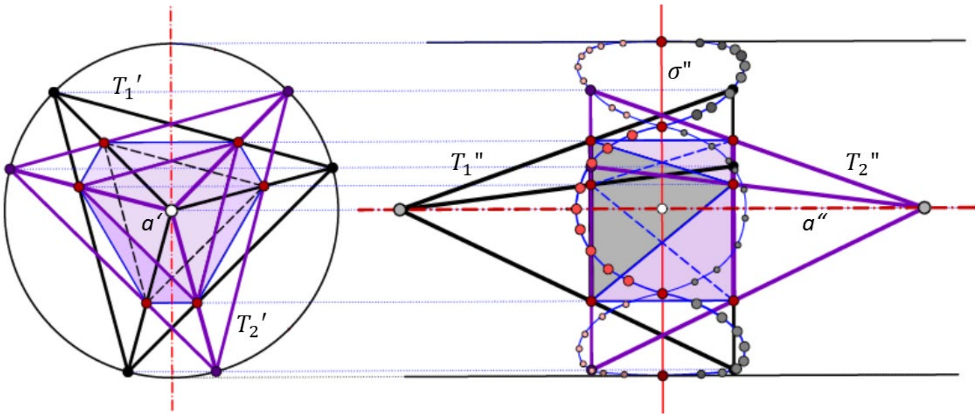

Figure 2 Top- and Side-view of the right tetrahedra T_1 and T_2 from Figure 1. Here both, T_1 and T_2, translate symmetric to a plane σ orthogonal to their common altitude a, while rotating in opposite directions at a. Furthermore, the traces of the vertices of T_1 and T_2 are shown as a formally closed curve on a cylinder with axis a (grey: paths of black vertices, pink: paths of purple vertices).

As next cases we consider pairs of indirect congruent equifaced tetrahedra T_1 and T_2. In the standard position, when their edges intersect in midpoints, their common inner part $T_1 \cap T_2$ is an equifaced octahedron $\mathcal{O}$. (These octahedra are subject to the next topic.) The convex hull of T_1, T_2 is a rectangular box $\mathcal{B}$ with edges parallel to the diagonals of $\mathcal{O}$, Figure 3 (left). We keep one diagonal a of $\mathcal{O}$ fixed and interpret those edges of T_1, which intersect a, as diagonals of a skew quadrilateral consisting of the remaining edges of T_1. Two opposite edges of this quadrilateral are generators of a hyperboloid of revolution Φ_1 with axis a, the other pair of edges defines a second hyperboloid of revolution Φ_2 coaxial with a, Figure 3 (right). Now we reflect T_1 at planes σ through a. Obviously, Φ_1 and $\Phi_2$remain fixed, and we receive symmetric versions T_2 of T_1 in positions, where all edges of T_2 must intersect those of T_1. Having chosen σ suitably all intersection points are indeed inner points of the edge segments. Thereby "suitably" means that the reflection planes σ through a must be chosen within a restricted angle-interval to ensure that the edge segments intersect in inner points (for example, in Figure 3 this angle interval has the size of $\sphericalangle PAQ$).

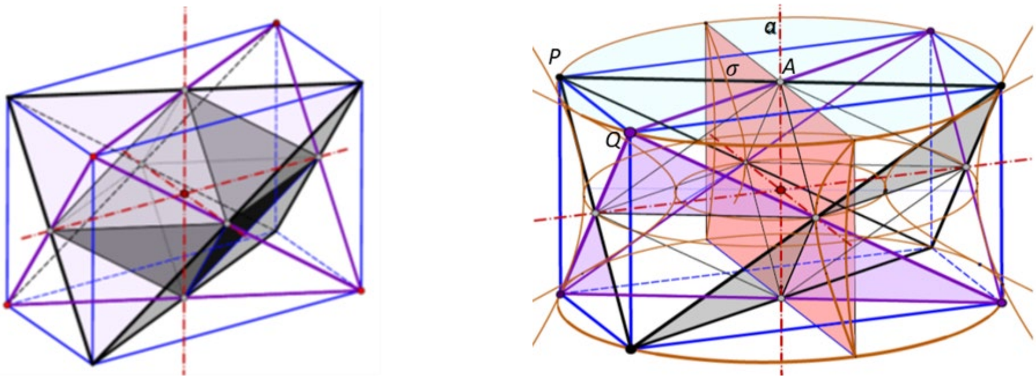

Figure 3 The *Stella Octangula* based on equifaced tetrahedra T_1 and T_2 has an equifaced octahedron in common, its convex hull is a right prism (*left*). The edges of T_1, considered as a skew quadrilateral together with its diagonals, are generators of two hyperboloids of revolution with common axis a. Reflecting T_1 at a plane σ through delivers T_2 with edges intersecting the edges of T_1 (*right*). The figure at right shows the reflection at a special plane σ delivering the standard position of T_2.

As we have three possibilities to choose axis a, there are three one-parameter sets of motions of T_2 "along" T_1. We collect this in

> *Theorem 1:* Besides the classical Stella Octangula based on regular tetrahedra T_1, T_2 also those Stellae based on equifaced tetrahedra T_1, T_2 allow three one-parameter sets of restricted motions of T_2 along T_1. Stellae Octangulae based on regular three-sided pyramids T_1, T_2 allow (at least) one such set of motions.

Finally, we show an edge model of an equifaced *Stella Octangula* in start position and in an intermediate position, Figure 4. We leave the discussion whether there are even more general cases of movable *Stellae Octangulae* to another occasion.

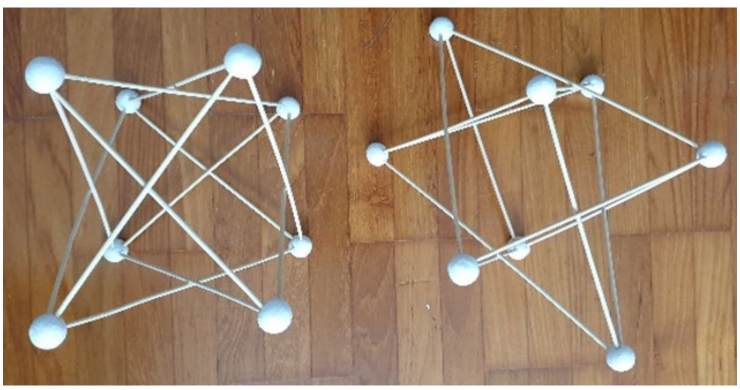

Figure 4 Edge model of an equifaced *Stella Octangula* in start position (left) and in an intermediate situation (right).

EQUIFACED OCTAHEDRA AND THEIR SYMMETRIES

Now we turn to equifaced octahedrons O, as they are basic also for the *Stellae Octangulae* mentioned above. We ask for all different types, which can be made of eight congruent and acute triangles. For example, with four congruent acute triangles with side lengths $|a| < |b| < |c|$ one can build six different four-sided pyramids, disregarding symmetric versions. Figure 5 shows two characteristic developments of such pyramids, which can occur at an octahedron.

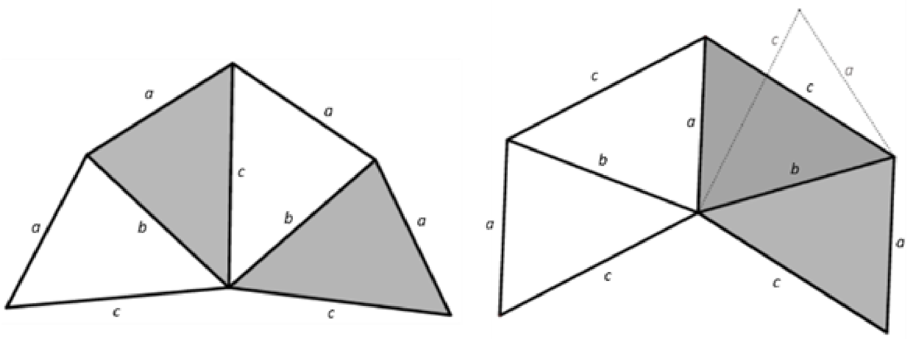

Figure 5 Two possibilities (out of six) to develop a vertex pyramid of an equifaced octahedron into the plane. Direct congruent face triangles are marked with the same colour.

An octahedron can be generated by gluing together a pair of symmetric pyramids along their (not necessarily planar) base polygons. It turns out that from these six pairs of symmetric pyramids one can obtain exactly four different octahedra. They have pairwise orthogonal diagonals. There is one octahedron, "type A", with co-punctal diagonals, Figure 6 (left), and three with a skew pair of diagonals, whereby the third is their common normal, Figure 6 (right). The latter ones will be referred to as "type B"-octahedra. That there are, in general, indeed three different octahedra of the second type can be read off from the top vertex of Figure 6 (right): There are three possibilities of edge combinations at this vertex, namely $abab, bcbc$ and $acac$.

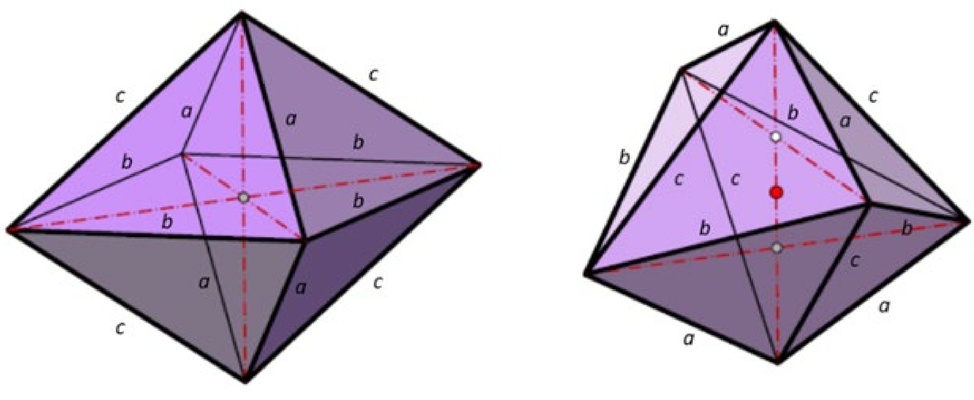

Figure 6 The equifaced type A-octahedron (left) has pairwise orthogonal diagonals intersecting in a common point, while equifaced type B-octahedra (right) have a pair of skew (but orthogonal) diagonals, while the third is the common normal of the other diagonals. Local symmetry of adjacent faces is also a global symmetry of equifaced octahedra.

If the eight (acute) triangles are isosceles, i.e., $|a| = |b| \neq |c|$, then two of the octahedrons of the types of Figure 6 (right) are identical. Obviously, there is only the single regular octahedron to equilateral triangles with $|a| = |b| = |c|$. Adding equifaced tetrahedrons to the faces of octahedra with acute triangular faces one obtains equifaced *Stellae Octangulae*, which only exist for acute triangles. Nevertheless, one might ask equifaced octahedra with obtuse face triangles, too. Obviously, there is no such octahedron with diagonals intersecting in one point, but there are still three other ones of type Figure 6 (right). We collect this as

> *Theorem 2:* From 8 congruent triangles one can build (in general) 4 different equifaced octahedra, whereof one, a type A-octahedron, has acute triangular faces and three co-punctal and pairwise orthogonal diagonals. Type A-octahedra are symmetric to three pairwise orthogonal planes and occur at equifaced Stellae Octangulae. The other three, type B-octahedra, have pairwise orthogonal diagonals, too, but two of them are skew. Type B-octahedra have only two symmetry planes. Less than four possibilities of equifaced octahedra are due to cases, when the face triangles are obtuse or isosceles or equilateral.

A single limit case with isosceles right triangles as faces is worthy to be mentioned here. It is due to Dr. Wyllie Rodrigues, who mentioned it in an e-mail 2017 (Rodrigues, 2017). He also found out that the ratio of its diagonals relates to the Golden Mean. Such a "Rodrigues-octahedron" is an equifaced type B-octahedron and can be made of set squares. Therefore, it is appropriate for teamwork in a classroom offside the usual treatment of Platonic Solids, Figure 7.

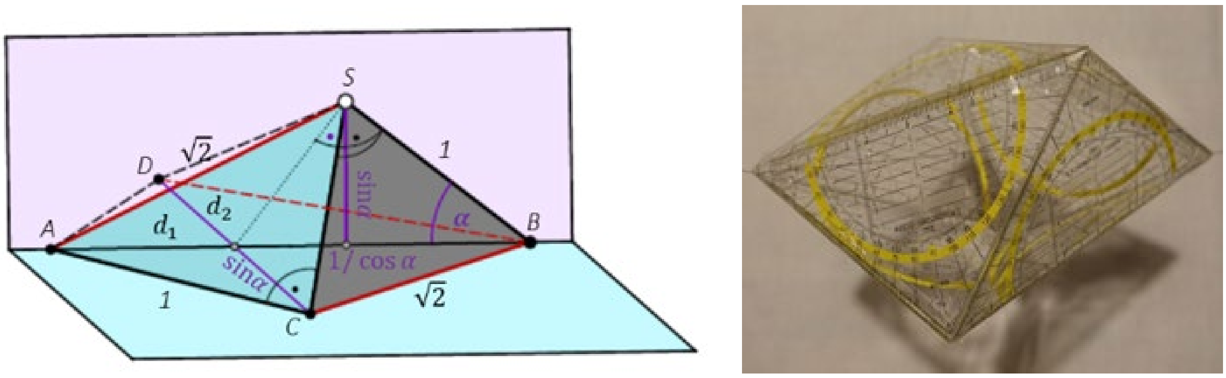

Figure 7 Vertex pyramid (left) of a Rodrigues-octahedron (left) and a model made of set squares (right).

EQUIFACED SET-SQUARE POLYHEDRA

Finally, one might extend Rodrigues' idea and use (congruent) isosceles right triangular faces also for other closed and open polyhedra and model them with congruent set squares, c.f. Weiss (2019, pp.95-96) and Weiss (2020, pp.20-26). Thereby one should exclude co-planar set squares as faces. Besides polyhedra with the symmetry groups of Platonic solids one can construct also irregular polyhedra and cylindrical or helical ones, the latter connecting the topic also to folded plates and Origami, Figures 8 to 10. The topic provides at least non-trivial material for geometry and maths courses at different levels. Even so the construction principle and its practical modelling is simple, the discussion of occurring symmetries could be of interest.

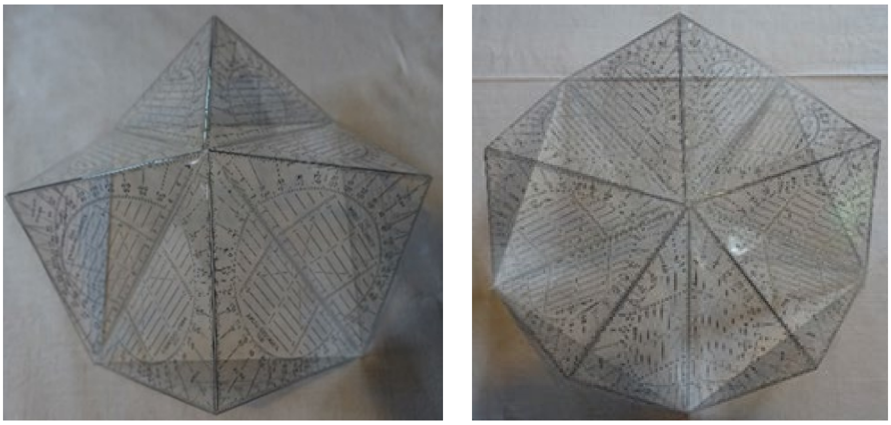

Figure 8 Cupolas over a square (left) and a regular pentagon (right).
They could replace the faces of a cube or a pentagonal dodecahedron or one can simply glue them together with a reflected cupola to form a closed polyhedron.

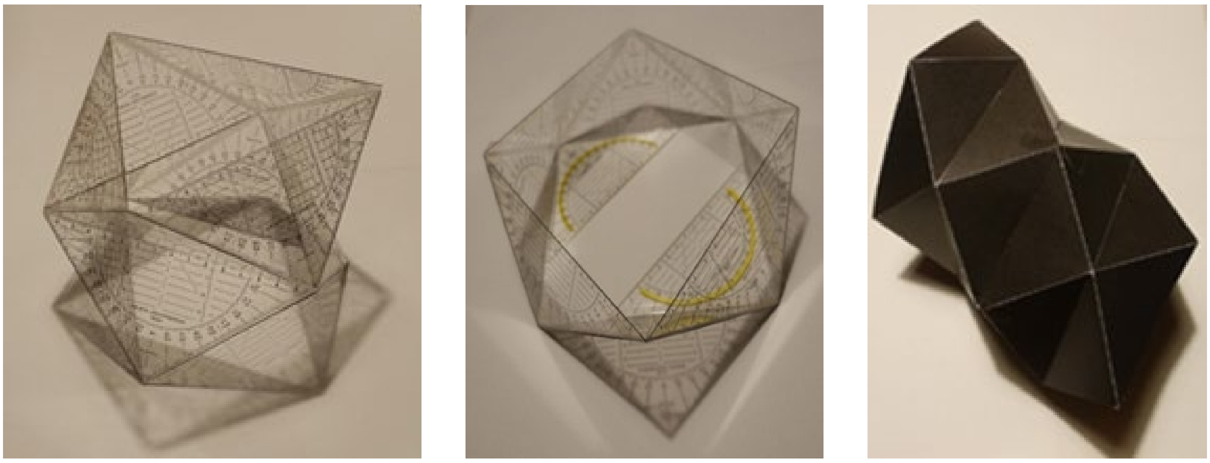

Figure 9 Antiprisms made of set squares (left and middle). They can be duplicated to form Cylindrical shapes and closed by suitable cupolas. The image at right shows a cardboard model of a helical polyhedron, the faces of which are congruent isosceles right triangles.

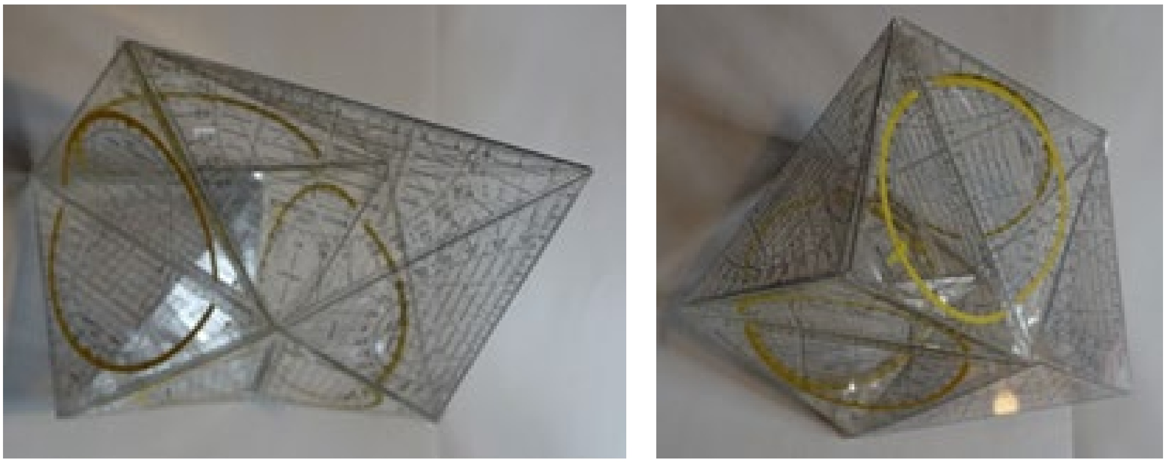

Figure 10 Two pphotographs of a closed polyhedron with 14 set square faces.
It is symmetric with respect to two orthogonal symmetry planes.

As an unexpected example we mention that it is possible to make even a Möbius stripe with a chain of $2n + 1$ such triangles, $(n \geq 3)$, Figure 11. Such polyhedral Möbius stripes are, within limits movable. In certain positions one can find local symmetries of three adjacent face triangles, but there is no global symmetry.

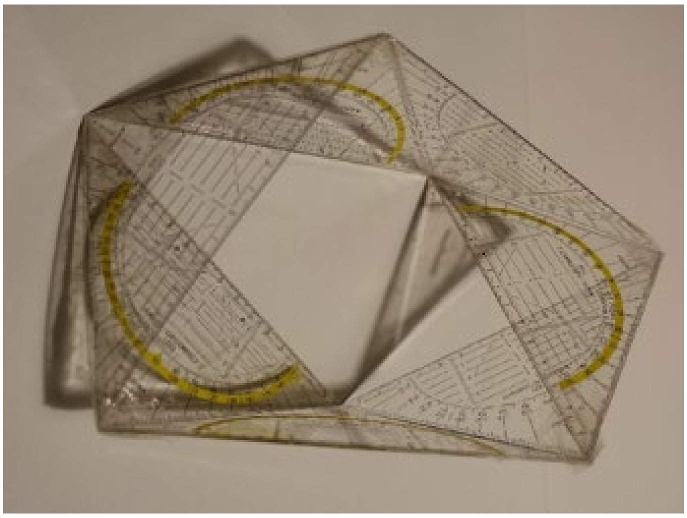

Figure 11 Photograph of a Möbius stripe consisting of 7 set squares; it is, within limits, movable.

CONCLUSION

At a first glance the presented geometric topics are independent, but they finally show some common features. One is of course symmetry in a wider sense, another is the way of deducing results via geometric reasoning. Thereby real models can give deeper insight and are an important stimulant for research. An educational aspect of this paper is to present polyhedra off the beaten track of Platonic and Archimedean solids. Finally, it shall be shown that even rather elementary topics can lead to many unexpected aha- moments.

REFERENCES

Rodrigues, W. D. (2017), private message via e-mail about an octahedron with set square faces.

Stachel, H. (1988), Ein bewegliches Tetraederpaar, (A movable pair of tetrahedrons), *Elemente d. Math*. 43(1988), 65-75

Weiss, G. (2019), *Equifaced Simplices and Polytopes*. *Geometrias 19: Book of Abstracts 2019*, 95-96. DOI:10.24840/978-989-98926-8-2.

Weiss, G. (2020), *Modellbau mit dem Geodreieck*, (Modelling with set squares), IBDG, Vol. 39(2), 2020, 20-2

Günter WEISS

Günter Weiss is a retired professor at the Institute for Geometry, Dresden University of Technology, Germany. He got his academic education in Austria. From 2000 until 2008 he also was president and vice-president of the International Society for Geometry and Graphics. His research fields are Advanced Elementary Geometry, Education in Mathematics and Geometry, (classical) Differential Geometry and Line Geometry, Bio-Geometry, Theory of Geometric Mappings.