QUIVER BUNDLES AND WALL CROSSING FOR CHAINS

P. B. GOTHEN AND A. NOZAD

Abstract

Holomorphic chains on a Riemann surface arise naturally as fixed points of the natural $\mathbb{C}^*$-action on the moduli space of Higgs bundles. In this paper we associate a new quiver bundle to the Hom-complex of two chains, and prove that stability of the chains implies stability of this new quiver bundle. Our approach uses the Hitchin-Kobayashi correspondence for quiver bundles. Moreover, we use our result to give a new proof of a key lemma on chains (due to Álvarez-Cónsul–García-Prada–Schmitt), which has been important in the study of Higgs bundle moduli; this proof relies on stability and thus avoids the direct use of the chain vortex equations.

1. Introduction

A *holomorphic* $(m+1)$*-chain* on a compact Riemann surface X of genus $g \geqslant 2$ is a diagram

$$C: \ E_m \xrightarrow{\phi_m} E_{m-1} \xrightarrow{\phi_{m-1}} \cdots \xrightarrow{\phi_2} E_1 \xrightarrow{\phi_1} E_0,$$

where each E_i is a holomorphic vector bundle and $\phi_i : E_i \longrightarrow E_{i-1}$ is a holomorphic map. Moduli spaces for holomorphic chains have been constructed by Schmitt [18] using GIT and, as is usual for decorated bundles, depend on a stability parameter $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_m)$, where $\alpha_i \in \mathbb{R}$.

One important application of holomorphic chains stems from the fact that, for a specific value of the stability parameter, their moduli can be identified with fixed loci for the natural $\mathbb{C}^*$-action on the moduli space of Higgs bundles. Thus, knowledge of moduli spaces of chains can be used to study the moduli space of Higgs bundles. The basic idea (in the case of rank 2 Higgs bundles) goes back to the seminal paper of Hitchin [13].

For higher rank Higgs bundles, knowledge of the moduli of chains becomes in itself difficult to come by, and a successful strategy for this has been to study the variation of the moduli of chains under changes in the parameter, using wall crossing arguments. This approach goes back to the work of Thaddeus [19] (used for rank 3 Higgs bundles in [10]). Recent important examples of the study of wall crossing of chains and applications to moduli of Higgs bundles include the work of García-Prada–Heinloth–Schmitt [9], García-Prada–Heinloth [8], and Heinloth (see also Bradlow–García-Prada–Gothen–Heinloth [7] for an application to $\mathrm{U}(p,q)$-Higgs bundles). We should mention here that recently alternative approaches to the study of the cohomology of Higgs bundle moduli have been highly succesful: see Schiffman [17], Mozgovoy–Schiffman [15] and Mellit [14]; also, Maulik–Pixton have announced a proof of a conjecture of Chuang–Diaconescu–Pan [4] which leads to a calculation of the motivic class of the moduli space of twisted Higgs bundles.

All the aforementioned results on chains rely on a key result of Álvarez-Cónsul–García-Prada–Schmitt [5, Proposition 4.14] which, in particular, is used in estimating codimensions of flip loci under wall crossing. The proof of this result is analytic in nature and relies on the solutions to the

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chain vortex equations, whose existence is guaranteed by the Hitchin–Kobayashi correspondence for holomorphic chains (see Álvarez-Cónsul–García-Prada [1, 2]).

In this paper, given a pair of chains, we associate to them a new quiver bundle which extends and refines the Hom-complex of the chains; we call it the *extended* Hom-*quiver*. Moreover, we show that polystability of the chains implies polystability of this extended Hom-quiver (see Theorem 3.3). We then use our result to give a new and simpler proof of the key result [5, Proposition 4.14] mentioned above (see Theorem 4.3). The main merit of our argument is that it is algebraic, in the sense that it only uses stability of the extended Hom-quiver and avoids direct use of the chain vortex equations. Thus, though our proof of Theorem 3.3 does ultimately rely on the Hitchin–Kobayashi correspondence (through Lemma 3.2), the roles of the correspondence and of stability are clarified. Our result can be viewed as a generalization of a result of [6] for length two chains (also known as triples), though in this case the extended Hom-quiver is itself a chain.

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2. Definitions and basic results

In this section we recall definitions and relevant facts on quiver bundles, from [11] and [2].

2.1. **Quiver bundles.** A *quiver* Q is a directed graph specified by a set of vertices Q_0, a set of arrows Q_1 and head and tail maps $h, t : Q_1 \to Q_0$. We shall assume that Q is finite.

Definition 2.1. A holomorphic *quiver bundle*, or simply a Q-*bundle*, is a pair $\mathcal{E} = (V, \varphi)$, where V is a collection of holomorphic vector bundles V_i on X, for each $i \in Q_0$, and φ is a collection of morphisms $\varphi_a : V_{ta} \to V_{ha}$, for each $a \in Q_1$.

The notions of Q-subbundles and quotient Q-bundles, as well as simple Q-bundles are defined in the obvious way. The subobjects $(0, 0)$ and $\mathcal{E}$ itself are called the *trivial subobjects*. The *type* of a Q-bundle $\mathcal{E} = (V, \varphi)$ is given by

$$t(\mathcal{E}) = (\mathrm{rk}(V_i); \deg(V_i))_{i \in Q_0},$$

where $\mathrm{rk}(V_i)$ and $\deg(V_i)$ are the rank and degree of V_i, respectively. We sometimes write $\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\bigoplus V_i)$ and call it the *rank of* $\mathcal{E}$. Note that the type is independent of φ.

2.2. **Stability.** Fix a tuple $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{R}^{|Q_0|}$ of real numbers. For a non-zero Q-bundle $\mathcal{E} = (V, \varphi)$, the associated $\boldsymbol{\alpha}$-*slope* is defined as

$$\mu_{\boldsymbol{\alpha}}(\mathcal{E}) = \frac{\sum_{i \in Q_0} \big(\alpha_i \,\mathrm{rk}(V_i) + \deg(V_i)\big)}{\sum_{i \in Q_0} \mathrm{rk}(V_i)}.$$

Definition 2.2. A Q-bundle $\mathcal{E} = (V, \varphi)$ is said to be $\boldsymbol{\alpha}$*-(semi)stable* if, for all non-trivial subobjects $\mathcal{F}$ of $\mathcal{E}$, $\mu_{\boldsymbol{\alpha}}(\mathcal{F}) < (\leqslant) \mu_{\boldsymbol{\alpha}}(\mathcal{E})$. An $\boldsymbol{\alpha}$*-polystable* Q-bundle is a finite direct sum of $\boldsymbol{\alpha}$-stable Q-bundles, all of them with the same $\boldsymbol{\alpha}$-slope.

A Q-bundle $\mathcal{E}$ is *strictly* $\boldsymbol{\alpha}$*-semistable* if there is a non-trivial subobject $\mathcal{F} \subset \mathcal{E}$ such that $\mu_{\boldsymbol{\alpha}}(\mathcal{F}) = \mu_{\boldsymbol{\alpha}}(\mathcal{E})$.

Remark 2.3. *In fact, the most general stability condition for quiver bundles involves additional parameters, see [5]. Since* $\boldsymbol{\alpha}$ *is the parameter which has been used in the literature for the study of moduli of chains via wall crossing, we confine ourselves to considering this parameter.*

2.3. **The gauge theory equations.** Let $\mathcal{E} = (V, \varphi)$ be a Q-bundle on X. A Hermitian metric on $\mathcal{E}$ is a collection H of Hermitian metrics H_i on V_i, for each $i \in Q_0$. To define the gauge equations on $\mathcal{E}$, we note that $\varphi_a : V_{ta} \to V_{ha}$ has a smooth adjoint morphism $\varphi_a^* : V_{ha} \to V_{ta}$ with respect to the Hermitian metrics H_{ta} on V_{ta} and H_{ha} on V_{ha}, for each $a \in Q_1$, so it makes sense to consider the compositions $\varphi_a \circ \varphi_a^*$ and $\varphi_a^* \circ \varphi_a$.

Let $\boldsymbol{\alpha}$ be the stability parameter. Define $\boldsymbol{\tau}$ to be the vector of real numbers τ_i given by

$$\tau_i = \mu_{\boldsymbol{\alpha}}(\mathcal{E}) - \alpha_i,\ i \in Q_0. \tag{2.1}$$

Since the stability condition does not change under a global translation $\boldsymbol{\alpha}$ can be recovered from $\boldsymbol{\tau}$ as follows

$$\alpha_i = \tau_0 - \tau_i,\ i \in Q_0.$$

Definition 2.4. A Hermitian metric H satisfies the *quiver $\boldsymbol{\tau}$-vortex equations* if

$$\sqrt{-1}\Lambda F(V_i) + \sum_{i=ha} \varphi_a \varphi_a^* - \sum_{i=ta} \varphi_a^* \varphi_a = \tau_i \mathrm{Id}_{V_i} \tag{2.2}$$

for each $i \in Q_0$, where $F(V_i)$ is the curvature of the Chern connection associated to the metric H_i on the holomorphic vector bundle V_i, and $\Lambda : \Omega^{i,j}(M) \to \Omega^{i-1,j-1}(M)$ is the contraction operator with respect to a fixed Kähler form ω on X.

The following is the *Hitchin–Kobayashi correspondence* between the twisted quiver vortex equations and the stability condition for holomorphic twisted quiver bundles, given by Álvarez-Cónsul and García-Prada [2, Theorem 3.1]:

Theorem 2.5. *A holomorphic Q-bundle $\mathcal{E}$ is $\boldsymbol{\alpha}$-polystable if and only if it admits a Hermitian metric H satisfying the quiver $\boldsymbol{\tau}$-vortex equations* (2.2)*, where $\boldsymbol{\alpha}$ and $\boldsymbol{\tau}$ are related by* (2.1)*.*

Note that the definitions and facts can be specialized for holomorphic chains.

2.4. **The** Hom**-complex for chains.** Fix two holomorphic chains C'' and C', given by

$$C' :\ E'_m \xrightarrow{\phi'_m} E'_{m-1} \xrightarrow{\phi'_{m-1}} \cdots \xrightarrow{\phi'_2} E'_1 \xrightarrow{\phi'_1} E'_0$$

$$C'' : E''_m \xrightarrow{\phi''_m} E''_{m-1} \xrightarrow{\phi''_{m-1}} \cdots \xrightarrow{\phi''_2} E''_1 \xrightarrow{\phi''_1} E''_0$$

Consider the following two terms complex of sheaves

$$\mathcal{H}^\bullet(C'', C') : \mathcal{H}^0 \xrightarrow{d} \mathcal{H}^1 \tag{2.3}$$

with terms

$$\mathcal{H}^0 = \bigoplus_{i-j=0} \mathrm{Hom}(E''_i, E'_j),\ \mathcal{H}^1 = \bigoplus_{i-j=1} \mathrm{Hom}(E''_i, E'_j),$$

and the map d is defined by

$$d(g_0, \ldots, g_m) = (g_{i-1} \circ \phi''_i - \phi'_i \circ g_i),\ \text{for } g_i \in \mathrm{Hom}(E''_i, E'_i).$$

The complex $\mathcal{H}^\bullet(C'', C')$ is called the Hom-*complex*. It governs the homological algebra of chains; in particular $\mathcal{H}^\bullet(C, C)$ is the deformation complex of a chain C.

3. The extended Hom-quiver

Here we introduce a Q-bundle, associated to two chains, and show that solutions to the vortex equations on the holomorphic chains produce a solution on the corresponding quiver bundle. The basic idea is the following: to the chains C'' and C' we associate the vector bundles E'' and E', obtained as the direct sum of the individual bundles in the chains. The quiver bundle structure on the chains then induces a natural quiver bundle structure on the bundle $\mathrm{Hom}(E'', E')$. Thus our construction can be seen as a kind of extension of structure group and it becomes natural to expect that a solution to the vortex equations on the chains should give a solution on the induced quiver bundle. Indeed, this is exactly the content of our Lemma 3.2 below. This in turn implies the main result of this section, Theorem 3.3, which says that $\tilde{\mathcal{H}}(C'', C')$ is (poly)stable for suitable values of the parameter.

We note that, since there are algebraic proofs of results saying that stability is preserved under extension of structure group (in the setting of principal bundles by Ramanan–Ramanathan [16] and for Hitchin pairs by Balaji–Parameswaran [3]) this might indicate the possibility of an algebraic proof of our result as well, though we do not pursue this possibility here.

We also point out that one might attempt to generalize our construction to more general quiver bundles than chains; see [12, Section 4.2] for the case of $\mathrm{U}(p, q)$-Higgs bundles.

Definition 3.1. Let C' and C'' be chains of length m. The *extended* Hom-*quiver* $\tilde{\mathcal{H}}(C'', C')$ is a quiver bundle defined as follows:

- For each (i, j) with $0 \leqslant i, j \leqslant m$, there is a vertex to which we associate the bundle $\mathrm{Hom}(E''_i, E'_j)$, of *weight* $k = i - j$.
- For each $\mathrm{Hom}(E''_i, E'_j)$, of weight $k = i - j$, there are maps

$$\begin{aligned} \delta^-_{ij} \colon \mathrm{Hom}(E''_i, E'_j) &\to \mathrm{Hom}(E''_i, E'_{j-1}), \\ f &\mapsto -\phi'_j \circ f, \end{aligned}$$

and

$$\begin{aligned} \delta^+_{ij} \colon \mathrm{Hom}(E''_i, E'_j) &\to \mathrm{Hom}(E''_{i+1}, E'_j), \\ f &\mapsto f \circ \phi''_{i+1}. \end{aligned}$$

In other words, $\tilde{\mathcal{H}}(C'', C')$ is defined by associating to $(E'' = \bigoplus E''_i, \phi'' = \sum_i \phi''_i)$ and $(E' = \bigoplus E'_i, \phi' = \sum_i \phi'_i)$ the bundle $\mathrm{Hom}(E'', E')$ and the map

$$\begin{aligned} \mathrm{Hom}(E'', E') &\to \mathrm{Hom}(E'', E'), \\ f &\mapsto f \circ \phi'' - \phi' \circ f, \end{aligned}$$

and then taking the quiver bundle induced from the splitting $\mathrm{Hom}(E'', E') = \bigoplus_{i,j} \mathrm{Hom}(E''_i, E'_j)$. We can picture this construction as follows:

(3.1)

$$
\begin{tikzcd}[column sep=tiny]
& & & & \mathrm{Hom}(E_0'',E_0') \arrow[dr] & & & & \\
& & & \mathrm{Hom}(E_0'',E_1') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_1'',E_0') \arrow[dr] & & & \\
& & \iddots \arrow[ur] \arrow[dr] & \vdots & \mathrm{Hom}(E_1'',E_1') \arrow[ur] & \vdots & \ddots \arrow[dr] & & \\
& \mathrm{Hom}(E_0'',E_{m-1}') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_{i-1}'',E_i') \arrow[dr] & \vdots & \mathrm{Hom}(E_i'',E_{i-1}') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_{m-1}'',E_0') \arrow[dr] & \\
\mathrm{Hom}(E_0'',E_m') \arrow[ur] \arrow[dr] & & \cdots & & \mathrm{Hom}(E_i'',E_i') \arrow[ur] \arrow[dr] & & \cdots & & \mathrm{Hom}(E_m'',E_0') \\
& \mathrm{Hom}(E_1'',E_m') \arrow[ur] \arrow[dr] & \vdots & \mathrm{Hom}(E_i'',E_{i+1}') \arrow[ur] & \vdots & \mathrm{Hom}(E_{i+1}'',E_i') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_m'',E_1') \arrow[ur] & \\
& & \ddots \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_{m-1}'',E_{m-1}') \arrow[dr] & & \iddots \arrow[ur] & & \\
& & & \mathrm{Hom}(E_{m-1}'',E_m') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_m'',E_{m-1}') \arrow[ur] & & & \\
& & & & \mathrm{Hom}(E_m'',E_m') \arrow[ur] & & & &
\end{tikzcd}
$$

Note that if we take the direct sums of the middle two columns

$$\bigoplus_{i-j=0} \mathrm{Hom}(E_i'',E_j') \xrightarrow{\delta^+ + \delta^-} \bigoplus_{i-j=1} \mathrm{Hom}(E_i'',E_j')$$

we obtain the Hom-complex of the chains C'' and C', defined in (2.3).

Lemma 3.2. *Let C' and C'' be holomorphic chains and suppose we have solutions to the $(\tau_0',\ldots,\tau_m')$-vortex equations on C' and the $(\tau_0'',\ldots,\tau_m'')$-vortex equations on C''. Then the induced Hermitian metric on the extended* Hom*-quiver $\widetilde{\mathcal{H}}(C'',C')$ pictured in* (3.1) *satisfies the quiver $\widetilde{\boldsymbol{\tau}}$-vortex equations, for $\widetilde{\boldsymbol{\tau}} = (\tilde{\tau}_{ij}) = (\tau_j' - \tau_i'')$.*

Proof. To show that the induced Hermitian metric satisfies the equation at $\mathrm{Hom}(E_i'',E_j')$ of weight k, for $-m \leqslant k \leqslant m$, first recall that we have the following identity of curvature operators:

$$F\big(\mathrm{Hom}(E_i'',E_j')\big)(f) = F(E_j') \circ f - f \circ F(E_i'').$$

Also, the vortex equations for C' and C'' are

$$\sqrt{-1}\Lambda F(E_i') + \phi_{i+1}'\phi_{i+1}'^{*} - \phi_i'^{*}\phi_i' = \tau_i' \,\mathrm{Id}_{E_i'},\ i = 0,\ldots,m$$

$$\sqrt{-1}\Lambda F(E_i'') + \phi_{i+1}''\phi_{i+1}''^{*} - \phi_i^{*''}\phi_i'' = \tau_i'' \,\mathrm{Id}_{E_i''},\ i = 0,\ldots,m.$$

Now, considering the quiver $\widetilde{\mathcal{H}}(C'',C')$ at $\mathrm{Hom}(E_i'',E_j')$ we have

$$
\begin{tikzcd}
\mathrm{Hom}(E_{i-1}'',E_j') \arrow[dr, "\delta_c"] & & \mathrm{Hom}(E_i'',E_{j-1}') \\
& \mathrm{Hom}(E_i'',E_j') \arrow[ur, "\delta_b"] \arrow[dr, "\delta_a"'] & \\
\mathrm{Hom}(E_i'',E_{j+1}') \arrow[ur, "\delta_d"'] & & \mathrm{Hom}(E_{i+1}'',E_j')
\end{tikzcd}
$$

where for ease of notation we have written

$$\begin{aligned}\delta_a(f) &= \delta^+_{ij}(f) = f\circ\phi''_{i+1}\\ \delta_b(f) &= -\delta^-_{ij}(f) = \phi'_j\circ f\\ \delta_c(g) &= \delta^+_{i-1,j}(g) = g\circ\phi''_i\\ \delta_d(h) &= -\delta^-_{i,j+1}(h) = \phi'_{j+1}\circ h\end{aligned}$$

A straightforward calculation gives the following

$$\begin{aligned}\delta^*_a(g) &= g\circ\phi''^*_{i+1}\\ \delta^*_b(h) &= \phi'^*_j\circ h\\ \delta^*_c(f) &= f\circ\phi''^*_i\\ \delta^*_d(f) &= \phi'^*_{j+1}\circ f\end{aligned}$$

therefore

$$\begin{aligned}(\delta_c\delta^*_c+\delta_d\delta^*_d-\delta^*_a\delta_a-\delta^*_b\delta_b)(f) &= \delta_c(f\circ\phi''^*_i)+\delta_d(\phi'^*_{j+1}\circ f)-\delta^*_a(f\circ\phi''_{i+1})-\delta^*_b(\phi'_j\circ f)\\ &= f\circ\phi''^*_i\circ\phi''_i+\phi'_{j+1}\circ\phi'^*_{j+1}\circ f-f\circ\phi''_{i+1}\phi''^*_{i+1}-\phi'^*_j\phi'_j\circ f\end{aligned}$$

Hence, using the vortex equations for C' and C'' and the above identity of curvature operators, we have for $f\in\mathrm{Hom}(E''_{i+k},E'_i)$:

$$\begin{aligned}&(\sqrt{-1}\Lambda F(\mathrm{Hom}(E''_i,E'_j))+\delta_c\delta^*_c+\delta_d\delta^*_d-\delta^*_a\delta_a-\delta^*_b\delta_b)(f)\\ &= \Big(\big(\sqrt{-1}\Lambda F(E'_j)+\phi''_{j+1}\phi''^*_{j+1}-\phi''^*_j\phi'_j\big)\circ f-f\circ\big(\sqrt{-1}\Lambda F(E''_i)-\phi'_i\phi'^*_i+\phi'^*_{i+1}\phi'_{i+1}\big)\Big)\\ &= (\tau'_j-\tau''_i)f.\end{aligned}$$

This finishes the proof. □

Theorem 3.3. *Let C' and C'' be $\boldsymbol{\alpha}'=(\alpha'_1,\dots,\alpha'_m)$ and $\boldsymbol{\alpha}''=(\alpha''_1,\dots,\alpha''_m)$-polystable holomorphic chains, respectively. Then the extended* Hom*-quiver $\tilde{\mathcal{H}}(C'',C')$, as in* (3.1)*, is $\widetilde{\boldsymbol{\alpha}}=(\widetilde{\alpha}_{ij})$-polystable for $\widetilde{\alpha}_{ij}=\alpha''_m+\alpha'_j-\alpha''_i$.*

Proof. Since the holomorphic chains C' and C'' are $\boldsymbol{\alpha}'$- and $\boldsymbol{\alpha}''$-polystable, it follows from Proposition 2.5 that both the $(\tau'_0,\dots,\tau'_m)$- and the $(\tau''_0,\dots,\tau''_m)$-vortex equations have a solution. Then, by Lemma 3.2 the extended Hom-quiver $\tilde{\mathcal{H}}(C'',C')$ satisfies the quiver $(\tau'_j-\tau''_i)$-vortex equations and therefore the Hitchin–Kobayashi correspondence implies that $\tilde{\mathcal{H}}(C'',C')$ is $\widetilde{\boldsymbol{\alpha}}$-polystable for

$$\widetilde{\alpha}_{ij} = \tau'_0-\tau''_m-(\tau'_j-\tau''_i)=\tau'_0-\tau'_j+\tau''_0-\tau''_m+\tau''_i-\tau''_0=\alpha''_m+\alpha'_j-\alpha''_i.$$

□

4. Application to wall crossing for chains

As an application of Theorem 3.3 we give a simplified and more conceptual proof of a result of Álvarez-Cónsul, García-Prada and Schmitt in [5], showing how it follows from stability of the quiver bundle (4.1). This result is a key ingredient in wall crossing arguments for holomorphic chains, which have had a number of important applications lately as explained in the introduction. First we state a particular case of our main theorem which will be used in the proof.

If we take $\boldsymbol{\alpha}=\boldsymbol{\alpha}'=\boldsymbol{\alpha}''$ in Theorem 3.3, then the stability parameter at every vertex in the middle column of (3.1) is $\tilde{\alpha}_{ii}=\alpha_m+\alpha_i-\alpha_i=\alpha_m$. Hence we can collapse the central column in

the quiver into a single vertex, to which we associate the direct sum of the corresponding bundles and obtain the following quiver bundle:
(4.1)

$$\begin{tikzcd}[column sep=small]
 & & & \mathrm{Hom}(E_0'',E_1') \arrow[dddr] & & \mathrm{Hom}(E_1'',E_0') \arrow[dr] & & & \\
 & & \cdots \arrow[ur] \arrow[dr] & \vdots & & \vdots & \cdots \arrow[dr] & & \\
 & \mathrm{Hom}(E_0'',E_{m-1}') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_{i-1}'',E_i') \arrow[dr] & & \mathrm{Hom}(E_i'',E_{i-1}') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_{m-1}'',E_0') \arrow[dr] & \\
\mathrm{Hom}(E_0'',E_m') \arrow[ur] \arrow[dr] & & \cdots & & \bigoplus\limits_{i-j=0} \mathrm{Hom}(E_i'',E_j') \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & \cdots & & \mathrm{Hom}(E_m'',E_0') \\
 & \mathrm{Hom}(E_1'',E_m') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_i'',E_{i+1}') \arrow[ur] & & \mathrm{Hom}(E_{i+1}'',E_i') \arrow[ur] \arrow[dr] & & \mathrm{Hom}(E_m'',E_1') \arrow[ur] & \\
 & & \cdots \arrow[ur] \arrow[dr] & & & & \cdots \arrow[ur] & & \\
 & & & \mathrm{Hom}(E_{m-1}'',E_m') \arrow[uuur] & & \mathrm{Hom}(E_m'',E_{m-1}') \arrow[ur] & & &
\end{tikzcd}$$

The next theorem says that this will be an $\bar{\boldsymbol{\alpha}}$-polystable quiver bundle for the corresponding collapsed stability parameter $\bar{\boldsymbol{\alpha}}$.

Theorem 4.1. *Let C' and C'' be $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$-polystable holomorphic chains. Then the quiver bundle pictured in* (4.1) *is $\bar{\boldsymbol{\alpha}}$-semistable, where the stability parameter $\bar{\boldsymbol{\alpha}}$ is defined by $\bar{\alpha}_{ij} = \alpha_m + \alpha_j - \alpha_i$ (at the central vertex we mean by this that the parameter is α_m).*

Proof. Any quiver subbundle F of (3.1) induces a quiver subbundle of (4.1) by collapsing the middle column and, by our assumption on the stability parameters, the $\bar{\boldsymbol{\alpha}}$-slope of the collapsed quiver bundle F equals the $\boldsymbol{\alpha}$-slope of the original quiver bundle. Thus quiver subbundles F of (4.1) obtained from quiver subbundles of (3.1) by collapsing the middle column satisfy the $\bar{\boldsymbol{\alpha}}$-semistability condition. This in fact suffices to prove the result by using a standard argument (see, e.g., [1, Proposition 3.11] or [10, Lemma 2.2]): the idea is to use that any quiver subbundle of (4.1) can be obtained by successive extensions of quiver subbundles of (3.1). □

Remark 4.2. *An alternative proof of Theorem 4.1 (allowing to conclude polystability rather than semistability) can be given using the Hitchin–Kobayashi correspondence. Simply note that by Lemma 3.2 a solution to the vortex equations on C' and C'' gives a solution on the quiver bundle* (3.1)*. Under the assumption on the parameters this, in turn, gives a solution on the collapsed quiver bundle* (4.1)*.*

Theorem 4.3 (Álvarez-Cónsul–García-Prada–Schmitt [5, Proposition 4.4])**.** *Let C' and C'' be $\boldsymbol{\alpha}$-polystable holomorphic chains and let $\alpha_i - \alpha_{i-1} \geqslant 2g-2$ for all $i = 1, \cdots, m$. Then the following inequalities hold*

$$\mu(\ker(d)) \leqslant \mu_{\boldsymbol{\alpha}}(C') - \mu_{\boldsymbol{\alpha}}(C''), \tag{4.2}$$

$$\mu(\operatorname{coker}(d)) \geqslant \mu_{\boldsymbol{\alpha}}(C') - \mu_{\boldsymbol{\alpha}}(C'') + 2g - 2 \tag{4.3}$$

where $d : \mathcal{H}^0 \longrightarrow \mathcal{H}^1$ is the morphism in the Hom*-complex $\mathcal{H}^\bullet(C'', C')$, defined in* (2.3)*.*

Proof. Denote the quiver bundle (4.1) by $\bar{\mathcal{E}}$. Using $\ker(d)$, define a subobject of $\bar{\mathcal{E}}$ as follows:

(4.4)
$$\begin{tikzcd}
 & & & 0 \arrow[dddr] & & 0 \arrow[dr] & & & \\
 & & \cdot^{\cdot^{\cdot}} \arrow[ur] \arrow[dr] & \vdots & & \vdots & \ddots \arrow[dr] & & \\
 & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & \cdots & & \ker(d) \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & \cdots & & 0 \\
 & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
 & & \ddots \arrow[ur] \arrow[dr] & & & & \cdot^{\cdot^{\cdot}} \arrow[ur] & & \\
 & & & 0 \arrow[uuur] & & 0 \arrow[ur] & & &
\end{tikzcd}$$

By Theorem 4.1 $\bar{\mathcal{E}}$ is $\bar{\boldsymbol{\alpha}}$-semistable, and a simple calculation shows that $\mu_{\bar{\boldsymbol{\alpha}}}(\bar{\mathcal{E}}) = \mu_{\boldsymbol{\alpha}}(C^{'}) - \mu_{\boldsymbol{\alpha}}(C^{''}) + \alpha_m$. Hence

$$\mu(\ker(d)) + \alpha_m \leqslant \mu_{\boldsymbol{\alpha}}(C^{'}) - \mu_{\boldsymbol{\alpha}}(C^{''}) + \alpha_m,$$

which is equivalent to (4.2).

To prove (4.3) consider the following quotient quiver bundle

(4.5)
$$\begin{tikzcd}
 & & & 0 \arrow[dddr] & & \operatorname{coker}(d_1) \arrow[dr] & & & \\
 & & \cdot^{\cdot^{\cdot}} \arrow[ur] \arrow[dr] & \vdots & & \vdots & \ddots \arrow[dr] & & \\
 & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & & \operatorname{coker}(d_i) \arrow[ur] \arrow[dr] & & 0 \arrow[dr] & \\
0 \arrow[ur] \arrow[dr] & & \cdots & & 0 \arrow[uuur] \arrow[ur] \arrow[dr] \arrow[dddr] & & \cdots & & 0 \\
 & 0 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & & \operatorname{coker}(d_{i+1}) \arrow[ur] \arrow[dr] & & 0 \arrow[ur] & \\
 & & \ddots \arrow[ur] \arrow[dr] & & & & \cdot^{\cdot^{\cdot}} \arrow[ur] & & \\
 & & & 0 \arrow[uuur] & & \operatorname{coker}(d_m) \arrow[ur] & & &
\end{tikzcd}$$

where d_i is defined as the composition:

$$d_i : \bigoplus_{i-j=0} \operatorname{Hom}(E_i'', E_j') \xrightarrow{\delta^+ + \delta^-} \bigoplus_{i-j=1} \operatorname{Hom}(E_i'', E_j') \to \operatorname{Hom}(E_i'', E_{i-1}').$$

By $\bar{\boldsymbol{\alpha}}$-semistability of $\bar{\mathcal{E}}$ the $\bar{\boldsymbol{\alpha}}$-slope of (4.5) is greater than or equal to $\mu_{\bar{\boldsymbol{\alpha}}}(\bar{\mathcal{E}})$, which means that

$$\mu(\operatorname{coker}(d)) + \alpha_m + \frac{\sum_{i=1}^{m} (\alpha_i - \alpha_{i-1}) \operatorname{rk}(\operatorname{coker}(d_i))}{\sum_{i=1}^{m} \operatorname{rk}(\operatorname{coker}(d_i)} \geqslant \mu_{\boldsymbol{\alpha}}(C^{'}) - \mu_{\boldsymbol{\alpha}}(C^{''}) + \alpha_m$$

and therefore

$$\begin{aligned}\mu(\operatorname{coker}(d)) &\geqslant \mu_{\boldsymbol{\alpha}}(C^{'}) - \mu_{\boldsymbol{\alpha}}(C^{''}) + \frac{\sum_{i=1}^{m}(\alpha_i - \alpha_{i-1})\operatorname{rk}(\operatorname{coker}(d_i))}{\sum_{i=1}^{m}\operatorname{rk}(\operatorname{coker}(d_i)}\\ &\geqslant \mu_{\boldsymbol{\alpha}}(C^{'}) - \mu_{\boldsymbol{\alpha}}(C^{''}) + 2g-2,\end{aligned}$$

which gives (4.3). □

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Centro de Matemática da Universidade do Porto, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre, s/n, 4169-007 Porto, Portugal
E-mail address: `pbgothen@fc.up.pt`

Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, Faculdade de Ciências da Universidade de Lisboa, Edf. C6, Campo Grande, 1749-016 Lisboa, Portugal
E-mail address: `anozad@fc.ul.pt`