Statistical Instability in Chaotic Dynamics

Muhammad Ali Khan

Tese de Doutoramento apresentada à
Faculdade de Ciências da Universidade do Porto,
Matemática

2018

Statistical Instability in Chaotic Dynamics

Muhammad Ali Khan

UC|UP Joint PhD Program in Mathematics

Programa Inter-Universitário de Doutoramento em Matemática

PhD Thesis | Tese de Doutoramento

December 2017

To my beloved parents, lovely wife and sweet daughter

Acknowledgements

I am feeling great honor in writing some words on the completion of my PhD thesis. First and foremost I humbly thanks to almighty God for his everlasting help in all the matters of my life, in particular during the period of my PhD.

I take immense pleasure to express my sincere and deep sense of gratitude to my PhD supervisor Prof. José Alves for setting the right direction of my thesis work and providing me his kind assistance, with his tremendous knowledge, and great support to accomplish this work. His outstanding supervision and continuos motivation are the key factors in carrying out this research and writing my thesis. In this regard, I would like to thank to Prof. Maria Carvalho for teaching me a productive course on Hyperbolic Dynamics and for her precious advice to work with the right person.

My sincere and heartfelt thanks also goes to Prof. Mujahid Abbas for his magnificent guidance and affection in the initial stage of my research career. I am grateful to the joint PhD program UC|UP for accepting me as a student and for providing excellent facilities to work for my thesis. My gratitude also extends to all of my teachers, friends and colleagues who have supported me along the way, particularly, I would like to mention my teacher Shoaib Ghoman. I also want to thank to all the nice and friendly staff of the Math department and the academic division.

At last but not the least, I would like to thank my family, specially my parents, for their patience, prayers and moral support throughout my PhD studies and my life in general. I would take this opportunity to appreciate my wife and thank her from the bottom of my heart for standing besides me to overcome every single situation, in particular, her incredible support for my PhD studies and specially during the final stage of my thesis writing.

This work was partially supported by CMUP (UID/MAT/00144/2013), the project PTDC/MAT-CAL/3884/2014 and the doctoral grant SFRH/BD/93856/2013, funded by Fundação para a Ciência e a Tecnologia (FCT) Portugal with national (MEC) and European structural funds through the program FEDER, under the partnership agreement PT2020.

Abstract

We consider a one parameter family of one-dimensional maps, introduced by Rovella, obtained through modifying the eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ of the geometric Lorenz attractor, replacing the expanding condition $\lambda_3 + \lambda_1 > 0$ by a contracting one $\lambda_3 + \lambda_1 < 0$. By referring the techniques of Benedicks-Carleson, Rovella proved that there exists a positive Lebesgue measure set of parameters, so called set of Rovella parameters, such that the derivatives of corresponding maps along critical orbits increase exponentially and the critical orbits have slow recurrence to the critical point.

Metzger proved the existence of unique absolutely continuous (with respect to Lebesgue) invariant probability measures (SRB) for the Rovella maps. Later on, Alves and Soufi showed that those maps are strongly statistically stable, i.e., the mapping which maps the parameters to the densities of the SRB measures is continuos (in the L^1-norm) on the set of Rovella parameters.

In this work, we show that there exist parameters such that the corresponding maps having super-stable periodic orbits and prove that Rovella maps are not statistically stable on an extended set of parameters, consists of Rovella parameters and super-stable parameters.

Resumo

Consideramos uma família a um parâmetro de transformações unidimensionais, introduzida por Rovella, obtida modificando os valores próprios $\lambda_2 < \lambda_3 < 0 < \lambda_1$ do atrator de Lorenz geométrico, substituindo a condição de expansão $\lambda_3 + \lambda_1 > 0$ pela de contração $\lambda_3 + \lambda_1 < 0$. Usando técnicas de Benedicks-Carleson, Rovella provou que existe um conjunto de parâmetros com medida de Lebesgue positiva, chamado conjunto de parâmetros de Rovella, para os quais as derivadas da respetiva transformação ao logo da órbita do valor crítico crescem exponencialmente e as órbitas críticas têm recorrência lenta ao ponto crítico.

Metzger provou a existência de uma única medida de probabilidade absolutamente contínua com respeito à medida de Lebesgue (medida de SRB) para essas transformações. Mais tarde, Alves e Soufi mostraram essa família de transformações é fortemente estatisticamente estável, i.e. a função que associa a cada parâmetro de Rovella a densidade da respetiva medida de SRB é contínua (na norma L^1).

Neste trabalho, mostramos que existem parâmetros cujas transformações correspondentes têm órbitas periódicas super-estáveis e provamos que as transformações de Rovella não são estatisticamente estáveis num conjunto de parâmetros estendido, constituído pelos parâmetros de Rovella mais os parâmetros com órbitas periódicas super-estáveis.

Table of contents

1 Introduction **1**

2 Geometric and Contracting Lorenz Attractors **9**

2.1 Lorenz Flow 10

2.2 Geometric Lorenz attractor 12

2.2.1 Robustness 15

2.3 Contracting Lorenz Attractor 16

2.3.1 Robustness 18

3 One-dimensional Maps Associated to the Contracting Lorenz Attractors **21**

3.1 Perturbations of the Initial Vector Field 21

3.2 Rovella Maps 22

3.3 SRB Measures for Rovella Maps 24

3.4 Statistical Stability for Rovella Maps 26

4 Construction of the Set of Rovella Parameters **31**

4.1 The Initial Interval 34

4.2 The Bound Periods 42

4.3 Basic Construction 52

5 Main Results **63**

5.1 The Extended Set of Parameters 64

5.1.1 Hyperbolic Periodic Repellers 66

5.1.2 Critically-stable and Post-critically Finite Maps near Rovella Maps . 67

5.1.3 The Extended Set 71
5.2 Accumulation of Critical Sum by Physical Measures 72
5.3 Statistical Instability of the Rovella Maps 77
5.3.1 Accumulation of Rovella Maps by Post-critically Finite Rovella Maps 78
5.3.2 Statistical Instability 79

6 Final Comments 85
6.1 Lifting of ACIP Measures 86
6.1.1 Physical Measure for the Poincaré Map 86
6.1.2 Physical Measure for the Flow 87
6.2 Lifting for the Measures supported on the Attracting Periodic Orbits 88
6.2.1 Physical Measure for the Poincaré Map 89
6.2.2 Physical Measure for the Flow 89
6.3 Inverse Procedure 90
6.3.1 Physical measure for the Poincaré map 90
6.3.2 Physical measure for the One-dimensional Map 91
6.4 Final Remark: work in progress 92

References 93

Chapter 1

Introduction

The theory of *Dynamical Systems* started in the work of Poincaré on the three-body problem of celestial mechanics studies processes which evolve in time. The description of these processes may be given by flows (continuous time) or iterations of maps (discrete time). An orbit is a collection of points related by the evolution function of the dynamical system. The main objectives of this theory are to describe the typical behavior of orbits when time approaches to infinity and to understand the changes in this behavior with the perturbations of the system or to which extent it is stable.

Ergodic Theory deals with the measure preserving processes in a measure space. In this approach, one tries in particular to illustrate the average time spent by typical orbits in different portions of the phase space. Birkhoff's Ergodic Theorem states that such times are well defined for almost all points, with respect to an invariant probability measure. Nevertheless, the notion of typical orbit usually meant with respect to volume (Lebesgue measure) which might not be an invariant measure in general.

It is a fundamental open problem to understand under which conditions the behavior of typical (positive Lebesgue measure) orbits is well defined from the statistical point of view. In chaotic dynamics this problem can be precisely expressed through *Sinai-Ruelle-Bowen (SRB) measures*, which were introduced by Sinai for Anosov diffeomorphisms [25] and later extended by Ruelle [24] for Axiom A diffeomorphisms and Bowen-Ruelle [11] for flows. Here we consider discrete time system given by a map f defined on a manifold M. An *f-invariant measure* μ is called *physical measure* for f if the *basin* of μ, i.e., the set of points

x such that

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x))=\int\varphi d\mu,\qquad \text{for any continuous map } \varphi: M\to\mathbb{R},$$

has a positive Lebesgue measure. It is to be noted that from Birkhoff's Ergodic Theorem it follows that any ergodic invariant probability measure which is absolutely continuous with respect to Lebesgue measure is physical measure. We shall refer to this special type of measure as an *SRB measure*. On the other extreme if a map f has an attracting periodic orbit $\{x_0, x_1, \cdots, x_{k-1}\}$ of period k, then the measure $\mu=\frac{1}{k}(\delta_{x_0}+\delta_{x_1}+\cdots+\delta_{x_{k-1}})$ given by the convex sum of the delta Dirac measures supported on the points in the periodic orbit is a physical measure for the map f. The *basin* of an invariant measure for the flow $(X^t)_{t\in\mathbb{R}}$ on M is the set of points $x\in M$, such that for any continuous map $\varphi: M\to\mathbb{R}$

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T\varphi(X^t(x))dt=\int\varphi d\mu.$$

The physical measure for the flow $(X^t)_{t\in\mathbb{R}}$ on M is defined in the similar way. Therefore the statistical behavior of orbits can be nicely characterize by physical measure in the sense that for a large (positive volume) set of points the time average of a physical observable of the system can be determined by space average.

While studying the persistence of the statistical properties of a dynamical system, Alves and Viana [5] proposed the notion of *statistical stability* which expresses the continuous variation of the physical measure, in the weak*-topology, as a function of dynamical system. Let $\mathcal{G}$ be a family of maps defined on a manifold M corresponding a unique physical measure, the map $f\in\mathcal{G}$ is said to be statistically stable if

$$g\longmapsto\mu_g$$

is continuous at f in the *weak**-topology, where μ_g is the physical measure corresponding to the map g. This kind of stability essentially states that the small perturbations of the system does not cause much effect on the evaluation of continuous maps along the orbits. *Strong statistical stability* refers as the continuous variation of the densities (if they exist) of the physical measures in the L^1-norm as a function of the dynamical system. There are certain situations which assure the existence of the densities of physical measure. For example if a map f admits physical measure μ as an SRB measure then the well-known Radon-Nikodym Theorem guarantees the existence of density function for the map f.

Lorenz [20] formulated an algebraic simple model of differential equation in $\mathbb{R}^3$ as a finite dimensional approximation of the evolution equation of atmospheric dynamics. He also showed numerically that it is highly dependent on initial conditions near an attractor. It was then a question of great interest to rigorously prove this experimental demonstration. By getting motivated through this problem, Guckenheimer and Williams [18] tried to write down the abstract properties of that attractor and produced a prototype so called *geometric Lorenz attractor* which we introduced in the next paragraph. This was the first example of a robust attractor with a hyperbolic singularity. It is given as 14th problem of Smale [26] that if the dynamics of the Lorenz system is same as that of the geometric model. Morales-Pacifico-Pujals [22] significantly moved in this direction by introducing the notion of *singular hyperbolicity*, i.e., a partially hyperbolic set with volume expanding or contracting central bundle and all of its singularities are hyperbolic. They also accomplished the fact that a robust attractor of a flow in $\mathbb{R}^3$ having a singularity is singular hyperbolic and has the properties of geometric Lorenz attractor. Then it was just remained to prove that the Lorenz system in fact corresponds to a sensitive robustly transitive non-hyperbolic attractor having a singularity. Later on Tucker [29] take on this problem during his PhD thesis and he produced a proof of it by using computer applications.

The geometric Lorenz attractor is a transitive maximal invariant set for a flow in $\mathbb{R}^3$ given by a vector field having a singularity at the origin 0 and the derivative of that vector field at singularity has real eigenvalues satisfying

$$0 < -\lambda_3 < \lambda_1 < -\lambda_2.$$

The vector field has a cross-section Σ intersecting the (two-dimensional) stable manifold of the hyperbolic singularity along a curve Γ. The Poincaré return map $P : \Sigma \setminus \Gamma \to \Sigma$ admits a stable smooth foliation $\mathcal{F}$ on Σ into curves, having Γ as a leaf, which are invariant and uniformly contracted by the forward iterates of the map P. The quotient space of the Poincaré section with stable leaves is diffeomorphic to the interval $I = [-1,1]$ and P induces a map on I which is uniformly expanding and having a singularity at 0 with derivative tends to infinity as one approaching to Γ.

Contracting Lorenz attractor, introduce by Rovella [23], is the maximal invariant set of a geometric flow whose construction is same as the geometric Lorenz attractor. The only difference is that the eigenvalue relation for the vector field, corresponds the contracting Lorenz attractor, is given as

$$0 < \lambda_1 < -\lambda_3 < -\lambda_2.$$

This attractor is robust in the measure theoretic sense, i.e., there exists a one parameter family of positive Lebesgue measure vector fields, C^3-close to the original one, having a strange attractor [23]. Similarly as in the case of geometric Lorenz attractor, the initial vector field has a global cross-section Σ and the one dimensional foliation is contracted by the first return map P_0 defined on the space $\Sigma \setminus \Gamma$. Therefore P_0 induces a map f_0 on the interval I which has a singularity at the origin and two critical values, unlike the map induced though the geometric Lorenz attractor. The reason that map f_0 has critical point is that the eigenvalues satisfy $\lambda_1 + \lambda_3 < 0$. In fact that one parameter family of vector fields induces a one parameter family $\{f_a\}_{a\geq 0}$ of interval maps, which we refer as *contracting Lorenz-like family*, such that each map in the contracting Lorenz-like family carries a singularity at 0 and two critical values.

In 1980's and early 1990's Benedicks and Carleson [8, 9] studied the dynamics near the well known Hénon attractor. In tuned out that first they need to understand some features of the dynamics of so called one parameter family of the quadratic maps $f_a = 1 - ax^2$ where parameter $a \in [0,2]$. They developed a technique to construct inductively a set of parameters, refer as Benedicks-Carleson set of parameters for quadratic family, with positive Lebesgue measure and having full density at point 2 such that the derivatives of corresponding maps along critical orbits increase exponentially and the critical orbits have slow recurrence to the critical point. Immediately after that, in 1992, Benedicks and Young [10] proved that each of the Benedicks and Carleson quadratic map admits a unique *SRB* measure.

By referring to the techniques of Benedicks and Carleson, Rovella in [23] also showed that for the contracting Lorenz-like family there exists a set of parameters with positive Lebesgue measure and having full density at 0 such that the derivatives of corresponding maps along critical orbits increase exponentially and those orbits have slow recurrence to the critical point. We denote that set of Rovella parameters by R and refer the maps associated with R as Rovella maps. Afterwards, Metzger [21], proved that each Rovella map admits an ergodic absolutely continuous invariant probability measure. However to prove the uniqueness of the *SRB* measure, he considered a slightly smaller class of parameters, inside the set of Rovella parameters, having full density at 0 and the associated maps admit a strong mixing property.

Based on the work given in [5], Alves [1] provided sufficient conditions for the strong statistical stability of the non-uniformly expanding maps. Those conditions have to deal with the volume decay of the set of points that deny either a non-uniformly expanding requirement or a slow recurrence, up to a given time. It was proved by Freitas [14–16], that the Benedicks-Carleson quadratic maps are non-uniformly expanding, have slow recurrence to the critical set, and the volume of their tail sets decays exponentially fast. As a consequence he obtained the strong statistical stability for those maps by restricting himself on the set of Benedicks-Carleson parameters. Later on, Alves and Soufi [3, 27] deduce that in fact all the maps in the Rovella family admit a unique *SRB* measure. They used the techniques developed by Freitas to conclude the strong statistical stability of Rovella maps in the set R.

On the other hand, in the beginning of this century, Thunberg [28] showed that in the neighbourhood of every Benedicks-Carleson parameter there are parameters whose associated maps have *super-attractors*: periodic orbits containing critical point. Then he proved that on a larger class, containing Benedicks-Carleson parameters and parameters associated to maps having super-attractors, the mapping $a \mapsto \mu_a$ is severely discontinuous in the weak*-topology at every Benedicks-Carleson parameter and hence Benedicks-Carleson maps are statistically unstable in a larger class.

Inspired by the work of Thunberg, first in this work we prove a result to discover some *super-stable parameters* for the contracting Lorenz-like family. We refer a map in the contracting Lorenz family as *critically-stable* if it admits a *super-attractor* and call the parameters linked with that map as super-stable parameters (see Definition 5.1.1 for precise definitions in our setting). Moreover we also obtain some parameters corresponding to maps whose critical orbits are pre-periodic to the points in the repelling periodic orbits. In the contracting Lorenz-like family, this type of map refers as *post-critically finite map* and the associated parameter as *post-critically finite parameter*. The above mentioned result is given as Lemma 5.1.2 and it is, indeed, a key product in our work since it guarantees the existence of critically-stable maps which admit physical measures supported on the super-attractors. Therefore those maps are distinct from the Rovella maps and it enables us to study the statistical stability of the Rovella maps on a larger class of maps consists of Rovella maps and the critically-stable maps associated to super-stable parameters. In order to present that lemma, we needed the precise demonstrations of the notions which occur in the basic construction of the set of Rovella parameters R. Moreover the proof of that lemma is also supported by some facts related to that basic construction. That is why, first we settled ourselves to unveil the complete details of the construction of set R to precisely introduce those notions and to workout the corresponding results. As referred by Rovella in [23], to construct the set R we followed the approach of Benedicks and Carleson given in [8, 9] for quadratic family. The precise and detailed construction of the set R is framed in chapter 4.

We denote by $\mathcal{E}$ the extended set of parameters consists of Rovella and super-stable parameters. Then the map f_a corresponding to each parameter $a \in \mathcal{E}$ admits a unique physical measure which we denote by μ_a. For a critical value $c = \pm 1$ of a map f_a in the contracting Lorenz-like family, we denote $\mu_a^n(c) := \frac{1}{n}\sum_{k=0}^{n-1} \delta_{f_a^k(c)}$ and if $\lim_{n\to\infty} \mu_a^n(c)$ exists in the weak*-topology we call the limit as *critical measure* for f_a and denote it by $\mu_a(c)$. One of our main result states that if the critical measure for a Rovella map exists then there exists a sequence of super-stable parameters such that the corresponding sequence of physical measures converges to the *critical measure* in the weak*-topology. The following is the precise statement of that result which is proved as Theorem 5.2.3.

Theorem A. For every $a \in R$, there exits a sequence of super-stable parameters $\{a_k\}_{k=1}^{\infty}$ such that if the critical measure $\mu_a(c)$ for the map f_a exits, then

$$\mu_{a_k} \overset{weak*}{\longrightarrow} \mu_a(c),\ k \to \infty.$$

The above result is obtained by using Lemma 5.1.2.

Finally as one of the main objective of this work we manage to study the statistical stability of Rovella maps on the extended set of parameters $\mathcal{E}$ and we conclude that Rovella maps are not statistically stable in the set $\mathcal{E}$.

Our main result Theorem 5.3.2 states as follows:

Theorem B. The map $\mathcal{E} \ni a \mapsto \mu_a$ is not continuous in the weak*-topology at any point in R.

One of the important result, which is used in the proof of the above theorem, simply states that any Rovella map is accumulated by post-critically finite Rovella maps. This result also obtained using Lemma 5.1.2 mentioned before. Then the proof of the above result is accomplished by showing that for every Rovella parameter a there exists a sequence of super-stable parameters converging to parameter a but the sequence of physical measures of the corresponding critically-stable maps is not converging to the *SRB* measure of the Rovella map f_a in the weak*-topology.

Chapter 2

Geometric and Contracting Lorenz Attractors

In this chapter we shall first present the Lorenz system of equations which is an example of a system having chaotic behavior near an attractor. Then we describe in detail the geometric model of flow associated to Lorenz system which illustrates the dynamics of that system. Afterwards, we explain the dynamics of, so called, the contracting Lorenz attractor which was introduced by Rovella for a flow in $\mathbb{R}^3$. The geometric model of Lorenz flow admits a robust attractor, which is known as geometric Lorenz attractor, whereas the contracting Lorenz attractor is not robust but it is persistent in measure.

Let M be a manifold and X be a smooth vector field on M and X^t denotes the flow of diffeomorphisms generated by X.

Definition 2.0.1. An *attractor* for the smooth flow X^t is a transitive (contains a dense orbit) set $\Lambda \subset M$, invariant under the flow, such that it has an open neighbourhood U with $X^t(\overline{U}) \subset U$ for all $t > 0$ and

$$\Lambda = \bigcap_{t \geq 0} X^t(U).$$

The *basin of attraction* of Λ is defined as

$$B(\Lambda) = \{x : \lim_{t \to +\infty} \text{dist}(X^t(x), \Lambda) = 0\}.$$

We say that Λ is *robust* if for any smooth vector field Y in a neighbourhood of X, $\bigcap_{t\geq 0} Y^t(U)$ is also an attractor.

2.1 Lorenz Flow

In the early 1960's, Lorenz [20] studied numerically the vector field X given in the form of differential equations

$$\begin{cases} \dot{x} = a(y-x), \\ \dot{y} = bx - y - xz, \\ \dot{z} = xy - cx, \end{cases}$$

for the parametric values $a = 10$, $b = 28$ and $c = 8/3$. Through experimental computations, he observed that the flow has sensitivity to the initial conditions, i.e., even a small initial error can lead to enormous differences in the outcome. It was then a question of great interest to rigorously prove this experimental demonstration. Later on Tucker [29], by the help of computer, showed that the original Lorenz system corresponds to a sensitive, non-hyperbolic and robustly transitive attractor having a singularity. Since the attractor is transitive so we may plot its trajectory starting form any point in the basin of attraction and it can be seen the picture of chaotic attractor which resembles a butterfly. An attractor formed by a chaotic system is also called *strange attractor*. The following properties are well known for the vector field X:

1. X has a singularity at the origin with eigenvalues

$$0 < 2.6 \approx \lambda_3 < \lambda_1 \approx 11.83 < -\lambda_2 \approx 22.83;$$

2. It has a *trapping region*, i.e., there is an open set U with $X^t(\overline{U}) \subseteq U$, for $t > 0$, such that $\Lambda = \bigcap_{t>0} X^t(U)$, the maximal invariant set, is an attractor and the origin is the unique singularity contained in U;

3. The divergence of X is negative

$$\text{div}X = \frac{\partial \dot{x}}{dx} + \frac{\partial \dot{y}}{dy} + \frac{\partial \dot{z}}{dz} = -(1+a+c) < 0.$$

Thus X is *strongly dissipative* and consequently it contracts volume: for initial volume V_0, from Liouville's formula, the volume at time t is given by $V(t) = V_0 e^{-(1+a+c)t}$. In particular Λ has zero volume.

The trajectory of a generic point in U starts spiraling around one of the singularities and suddenly jumps to other one and starts spiraling around it. This mechanism continues and the Lorenz attractor appears to be a sketch of butterfly, as shown in Figure 2.1. It rotates randomly around each singularity.

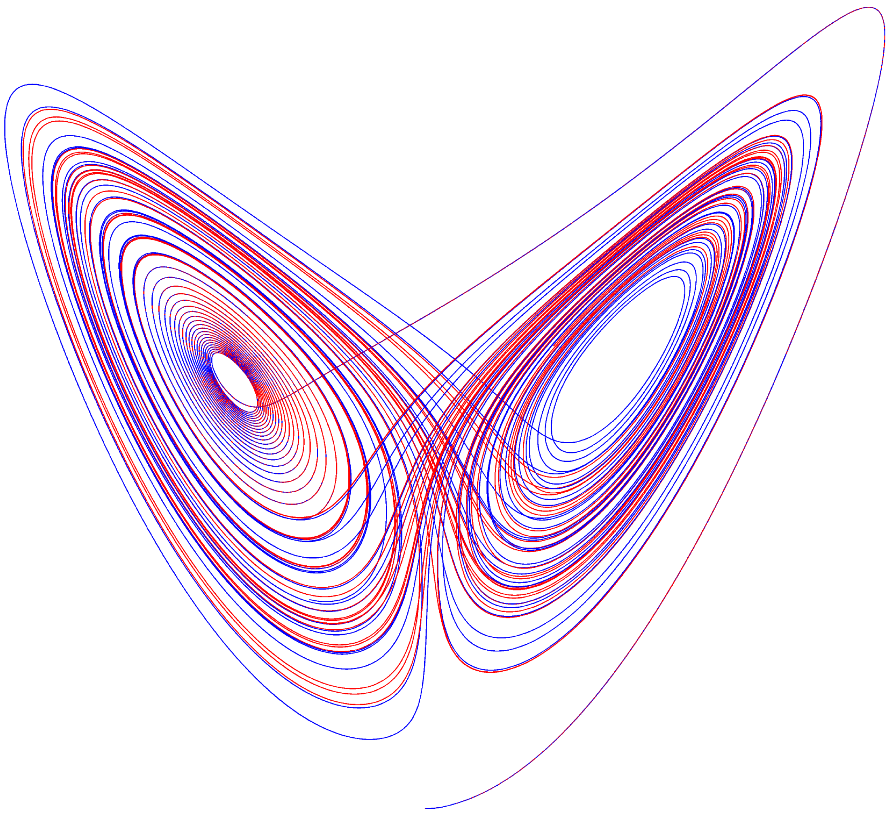

Fig. 2.1 Lorenz Attractor

2.2 Geometric Lorenz attractor

In the late 1970's, Guckenheimer and Williams [18] introduced the geometric description of a flow having similar dynamical behavior as that of Lorenz system, known as geometric Lorenz flow. This geometric model posses a trapping region containing a transitive attractor which has a singularity accumulated by the regular orbits preventing the attractor to be hyperbolic. In fact, if there is a hyperbolic invariant splitting of the tangent space then the continuity of splitting and the transitivity of attractor affirm that the dimensions of subspaces are equal. However the central direction of the singularity is zero dimensional since the vector field vanishes at singularity, consequently the dimension of either stable or unstable direction at singularity must be different from the dimension of transitive regular orbit inside of the attractor.

The construction of the geometric model is as follows: The vector field X has a singularity at $(0,0,0)$ and it is linear in a neighbourhood containing the cube $\{(x,y,z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$. The derivative of X at singularity admits three real eigenvalues λ_1, λ_2 and λ_3 satisfying $0 < -\lambda_3 < \lambda_1 < -\lambda_2$. We denote by Σ the roof $\{|x| \leq 1, |y| \leq 1, z = 1\}$ of the cube, intersecting the (2-dimensional) stable manifold of singularity along a curve Γ which divides Σ into two regions $\Sigma^+ = \{(x,y,1) \in \Sigma : x > 0\}$ and $\Sigma^- = \{(x,y,1) \in \Sigma : x < 0\}$; see Figure 2.2 below.

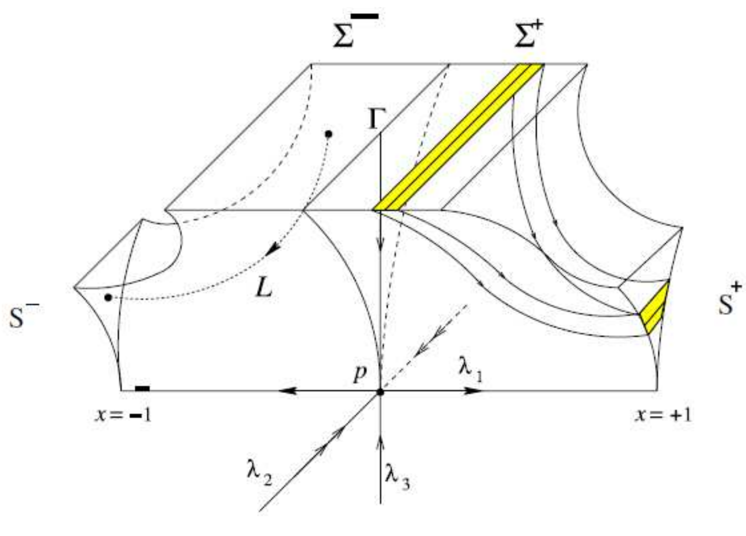

Fig. 2.2 Cross Section

The return map is given by

$$P : \Sigma^{\pm} \longrightarrow \{(\pm 1, y, z) : y, z \in \mathbb{R}\}$$
$$(x, y, 1) \longmapsto (sgn(x), y|x|^{r}, |x|^{s}),$$

where $r = \frac{-\lambda_2}{\lambda_1}$ and $s = \frac{-\lambda_3}{\lambda_1}$. The images of $\Sigma^{\pm}$ under P are the triangles $S^{\pm}$ except vertices $(\pm 1, 0, 0)$, and the line segments $\{x = constant\} \cap \Sigma$ are mapped to the segments $\{z = constant\} \cap S^{\pm}$. The time τ needed to go from $\Sigma^{\pm}$ to $S^{\pm}$ is given by

$$\tau(x, y, 1) = -\frac{1}{\lambda_1} \log|x|.$$

We assume that the flow smoothly carries the triangles back to Σ as in Figure 2.3.

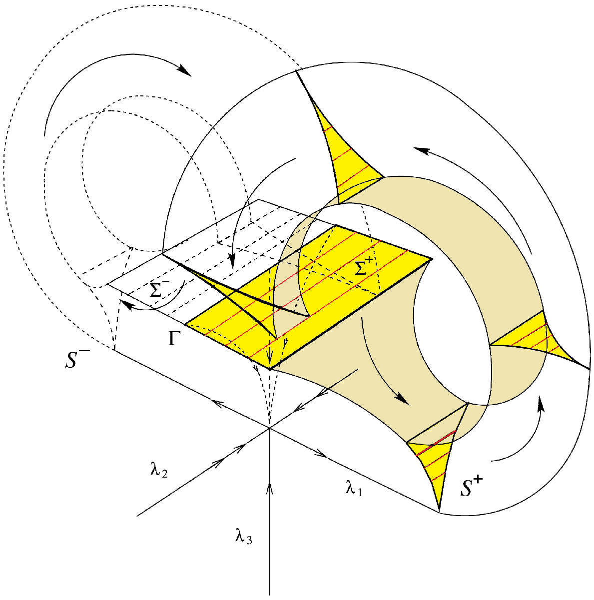

Fig. 2.3 The Return Map

The triangles stretched in the direction along x-axis and compressed in the other transversal direction. The dynamics in linearized region will dominate all estimates of contraction and expansion. To complete the geometric model, it is assumed that the flow from $S^{\pm}$ reaches Σ in finite time T. Hence the return time from Σ to itself is

$$\tau(x, y, 1) = -\frac{1}{\lambda_1} \log|x| + T.$$

Then we assume that the line segments $\{z = constant\} \cap S^{\pm}$ are mapped by the return map to the segments contained in $\{x = constant\} \cap \Sigma$. Consequently we obtain the following expression for Poincaré return map

$$P(x,y) = (f(x), g(x,y)),$$

for some maps $f : I_0 \setminus \{0\} \to I_0$ and $g : I_0 \setminus \{0\} \times I_0 \to I_0$, with $I_0 = [-\frac{1}{2}, \frac{1}{2}]$. The one dimensional map f is shown in the Figure 2.4 and has the following properties:

(1) f is discontinuous at $x = 0$, $\lim_{x \to 0^+} f(x) = -1/2$ and $\lim_{x \to 0^-} f(x) = 1/2$;

(2) f is differentiable on $I_0 \setminus \{0\}$ and $f'(x) > \sqrt{2}$ for all $x \in I_0 \setminus \{0\}$;

(3) $\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^-} f'(x) = +\infty$.

The map g satisfies $|\frac{\partial g}{\partial y}| < \kappa < \frac{1}{2}$, which implies that the foliation given by the segment $\Sigma \cap \{x = constant\}$ contracting uniformly, i.e., there exists a constant $C > 0$ such that for any leaf γ of the foliation and $p, q \in \gamma$, and for large enough $n \in \mathbb{N}$, we have

$$\text{dist}(P^n(p), P^n(q)) \le C\kappa^n \text{dist}(p,q).$$

The orbits of points in Σ will return back to itself by following first the linear vector field until the triangles $S^{\pm}$ and then X. The pair $(\bigcup_{t \in \mathbb{R}} X^t(\Sigma), X^t)$ denotes the geometric flow.

In order to obtain some important results, it is quite useful to reduce the study of flow to the study of 2-dimensional Poincaré map P, which further can be reduce to work on the one dimensional map f obtained through assigning to each point $x \in \Sigma$ the leaf containing it (since the orbits of any two points on a leaf lie in the same leaves and distance of their images tends to zero under the iterations). The map f is called to be the Lorenz map which can have continuous extension at 0, it can be assigned two values to f at 0 such that it is continuous on the intervals $[-\frac{1}{2}, 0]$ and $[0, \frac{1}{2}]$. Accordingly, one can consider P as a 2-valued map with the domain of definition as Σ. The map P is continuous when we restrict it on the closure of connected components of $\Sigma \setminus \Gamma$ and it maps the curve Γ down to a point. We set

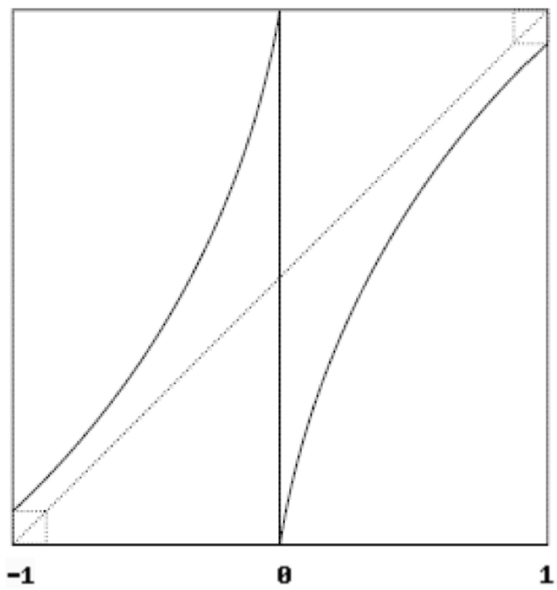

Fig. 2.4 Lorenz One-dimensional Map

$\Lambda_P = \bigcap_{n\geq 0} P^n(\Sigma)$ and the geometric Lorenz attractor Λ is given by the union of orbits of the points in Λ_P by the flow of X.

2.2.1 Robustness

One of the important fact about the geometric Lorenz attractor is robustness, i.e., the vector fields C^1-close to the one constructed above also admit strange attractors. There exists an open set $U \subset \mathbb{R}^3$, containing the geometric Lorenz attractor, and an open neighbourhood $\mathcal{U}$ of X in C^1 topology such that for any vector field $Y \in \mathcal{U}$ the maximal invariant set $\Lambda_Y = \bigcap_{t\geq 0} Y^t(U)$ is transitive and Y-invariant. This fact follows from the persistence of invariant contracting foliation on the cross section Σ.

Theorem 2.2.1. *[7, Theorem 3.10] Suppose X is a geometric Lorenz flow with an invariant contracting stable foliation $\mathcal{F}_X$ on the cross section Σ. Every vector field Y which is C^1 sufficiently close to X has an invariant contracting stable foliation $\mathcal{F}_Y$ on the cross section Σ.*

Note that X has a hyperbolic singularity and the cross section Σ is transversal to any flow C^1-close to X. Therefore it persists and the eigenvalues satisfying same relations for every

$Y \in \mathcal{U}$. Indeed, through C^1 change of coordinates the singularity of any $Y \subset \mathcal{U}$ stands on the origin and the derivative of Y at origin has eigenvectors in the direction of coordinate axis as before, whereas the stable manifold of singularity lies on the plane $x = 0$. Consequently Y has a Poincaré return map of the form $P_Y = J_Y \circ P'$, where J_Y is C^1-close to identity and P' has same properties as P. Then f_Y can be define as the one-dimensional quotient map corresponding to P_Y over the leaves of foliations $\mathcal{F}_Y$. Since $r - s > 1$ and foliation is continuous with C^1 leaves, therefore f_Y is C^1-close to f. Thus there esists $c_0 \in [-\frac{1}{2}, \frac{1}{2}]$ which plays the same role for f_Y as 0 for f and hence f_Y holds the same properties as that of f.

2.3 Contracting Lorenz Attractor

By considering a vector field almost identical to that used by Guckenheimer and Williams [18], Rovella [23] introduced a bit different kind of attractor Λ named as *contracting Lorenz attractor* which is not persistent. He showed that in a neighbourhood $\mathcal{U}$ of the initial vector field there is an open and dense subset for which the attractor breaks up into one or at most two attracting periodic orbits, a hyperbolic set, the singularity and wandering trajectories linking these objects. On the other hand the attractor Λ admits a local basin U.

The corresponding flow of this attractor has similar construction as that of geometric one with the initial vector field X_0 in $\mathbb{R}^3$ which has the following properties:

1. X_0 has a singularity at the origin and its derivative at singularity has three real eigenvalues λ_1, λ_2 and λ_3 satisfying:

   (i) $0 < \lambda_1 < -\lambda_3 < -\lambda_2$,

   (ii) $r > s + 3$, where $r = \frac{-\lambda_2}{\lambda_1}$ and $s = \frac{-\lambda_3}{\lambda_1}$;

2. There exists an open set $U \in \mathbb{R}^3$ which is positively invariant by the flow and it contains the cube $\{(x, y, z) : |x| \leq 1, |y| \leq 1, |z| \leq 1\}$. The top of cube Σ has a foliation by the stable line segments $\{x = constant\} \cap \Sigma$ which are invariant by the Poincaré return map P_0. As in the case of geometric Lorenz flow, the invariance of stable foliation on Poincaré

section gives rise to a one dimensional map $f_0 : I \setminus \{0\} \to I$ such that

$$f_0 \circ \pi = \pi \circ P_0,$$

where I denotes the interval $[-1,1]$ and it is obtained by the mean of canonical projection π which assigns to every point in I the leaf in Σ containing that point;

3. There is a sufficiently small $\rho > 0$ such that the contraction along the invariant foliation of the lines $x = constant$ in U is stronger than ρ.

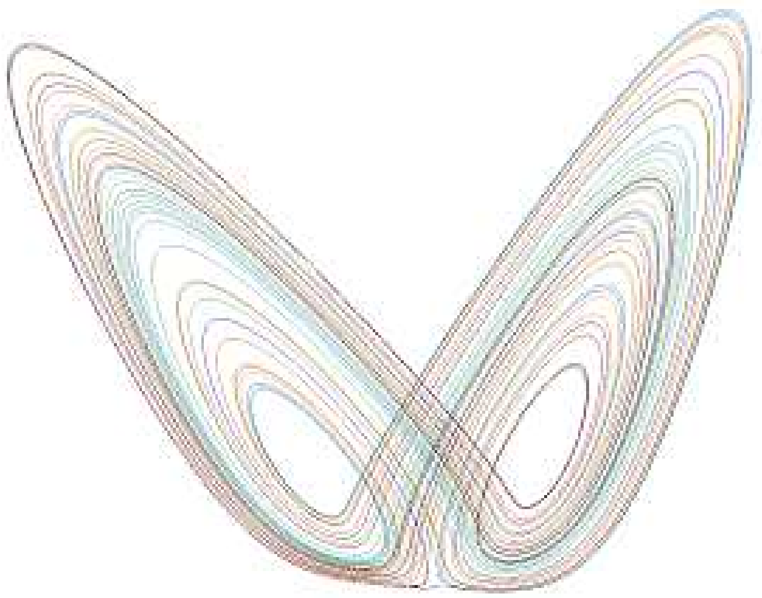

Fig. 2.5 Contracting Lorenz Attractor

The main idea adopted by Rovella was to replace the expanding condition $\lambda_1 + \lambda_3 > 0$ of the geometric flow by the contracting condition $\lambda_1 + \lambda_3 < 0$.

The map f_0 holds the following properties:

(1) $f_0'(x) > 0$ for $x \neq 0$, and the order of the derivative of f_0 at 0 is $s - 1 > 0$, i.e., $\lim_{x\to 0} \frac{f_0'(x)}{|x|^{s-1}}$ is finite and not equal to zero;

(2) f_0 has a discontinuity at 0, $f_0(0^+) = -1$, $f_0(0^-) = 1$, $\max_{x>0} f_0'(x) = f_0'(1)$, $\max_{x<0} f_0'(x) = f_0'(-1)$;

(3) The points ±1 pre-periodic repelling, i.e., there exit integers k^+, k^-, n^+, n^- such that

$$f_0^{k^+ + n^+}(1) = f_0^{k^+}(1), \qquad (f_0^{n^+})'(f_0^{k^+}(1)) > 1$$
$$f_0^{k^- + n^-}(-1) = f_0^{k^-}(-1), \quad (f_0^{n^-})'(f_0^{k^-}(-1)) > 1;$$

For the purpose of simplicity Rovella supposed that the points ±1 are fixed by f_0 which is given as property $V.4$ in [23].

(4) f_0 has negative Schwarzian derivative, i.e., there is $\chi < 0$ such that on $I \setminus \{0\}$

$$S(f_0) = \left(\frac{f_0''}{f_0'}\right)' - \frac{1}{2}\left(\frac{f_0''}{f_0'}\right)^2 < \chi.$$

The map f_0 can be seen in the Figure 2.6.

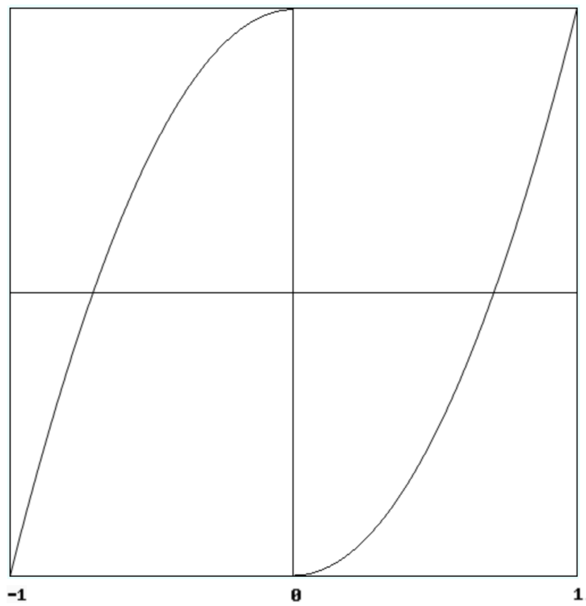

Fig. 2.6 Graph of the map f_0

2.3.1 Robustness

Unlike the geometric Lorenz attractor, the contracting one $\Lambda = \bigcap_{t \geq 0} X_0^t(U)$ is not robust. Whereas Rovella proved its robustness in a measure theoretical sense by proving the existence of a one parameter family of vector fields C^3-close to X_0, with positive Lebesgue measure, such that each vector field in that family has transitive non-hyperbolic attractor.

Theorem 2.3.1. *[23, Theorem] There exists a C^∞ vector field X_0 in $\mathbb{R}^3$ having an attractor Λ containing a singularity, and satisfying the following properties:*

(a) There exist a local basin U of Λ, a neighbourhood $\mathcal{U}$ of X_0, and an open and dense subset $\mathcal{U}_1$ of $\mathcal{U}$ such that for all $X \in \mathcal{U}_1$, $\Lambda_X = \bigcap_{t \geq 0} X^t(U)$ consists of the union of one or at most two attracting periodic orbits, a hyperbolic set of topological dimension one, a singularity, and wandering orbits linking them.

(b) Λ is 2-dimensionally almost persistent in the C^3 topology.

The term 2-dimensionally almost persistence means that Λ has a local basin U such that X_0 is a 2-dimensional full density point of the set of vector fields $\{Y : \Lambda_Y = \bigcap_{t \geq 0} Y^t(U) \text{ is an attractor } \}$.

Chapter 3

One-dimensional Maps Associated to the Contracting Lorenz Attractors

This chapter is devoted to briefly describe the properties and to state some of the interesting results for the one dimensional maps which comes from the geometric model of contracting Lorenz attractor.

3.1 Perturbations of the Initial Vector Field

There are some properties of the initial vector field X_0 which are valid for the C^3 perturbations. Consider a small neighbourhood $\mathcal{U}$ of X_0 such that each $X \in \mathcal{U}$ has a singularity near origin with eigenvalues $\lambda_1(X), \lambda_2(X), \lambda_3(X)$ satisfying $-\lambda_2(X) > -\lambda_3(X) > \lambda_1(X) > 0$ and $r_X > s_X + 3$, $r_X = -\frac{\lambda_2(X)}{\lambda_1(X)}$ and $s_X = -\frac{\lambda_3(X)}{\lambda_1(X)}$. Moreover, the trajectories contained in the stable manifold still intersect Σ. The sets $\mathcal{U}$ and U can be taken small enough so that U is positively invariant by the flow of every $X \in \mathcal{U}$. The existence of C^3 stable 1-dimensional foliations in U and their continuous variation with X was proved by Rovella [23].

For each $X \in \mathcal{U}$, we may take a square Σ_X close to Σ formed by line segments of the foliations so that the first return map P_X to Σ_X has an invariant foliation and we can choose the coordinates (x, y) in Σ_X so that the segment $x = 0$ corresponds to the stable manifold of singularity and

$$P_X(x,y) = (f_X(x), g_X(x,y)).$$

The map f_X is of class C^3 everywhere but at $x = 0$ where it has a discontinuity.

In order to prove his main result, Rovella considered a one parameter family $\{X_a \in \mathcal{U} : a \geq 0\}$ of vector fields and the corresponding family $\{f_a : I \setminus \{0\} \to I : a \geq 0\}$ of C^3 one dimensional maps which we will refer as *contracting Lorenz-like family* in the sequel. The maps in that family have the following properties:

(A0) $f_0(1) = 1$ and $f_0(-1) = -1$;

(A1) $f_a(0^+) = -1$ and $f_a(0^-) = 1$;

(A2) $f_a' > 0$, $f_a''|_{[-1,0)} < 0$ and $f_a''|_{(0,1]} > 0$;

(A3) there exist $K_1, K_2 > 0$ and $s > 1$ (independent of a) such that for all $x \in I \setminus \{0\}$

$$K_2|x|^{s-1} \leq f_a'(x) \leq K_1|x|^{s-1};$$

(A4) f_a has negative Schwarzian derivative: there is $\chi < 0$ such that for all $x \in I \setminus \{0\}$

$$S(f_a)(x) = \left(\frac{f_a''}{f_a'}\right)'(x) - \frac{1}{2}\left(\frac{f_a''}{f_a'}\right)^2(x) < \chi;$$

(A5) f_a depend continuously on a in the C^3 topology;

(A6) the functions $a \to f_a(\pm 1)$ have derivative 1 at $a = 0$.

It follows from (A0)-(A3) that $f_0'(\pm 1) > 2$.

3.2 Rovella Maps

Rovella [23] also introduced a set of parameters R, known as set of Rovella parameters, extracted from the contracting Lorenz-like family. The choice of those parameters was made according to some exclusion procedure. He mentioned there that the construction of set R

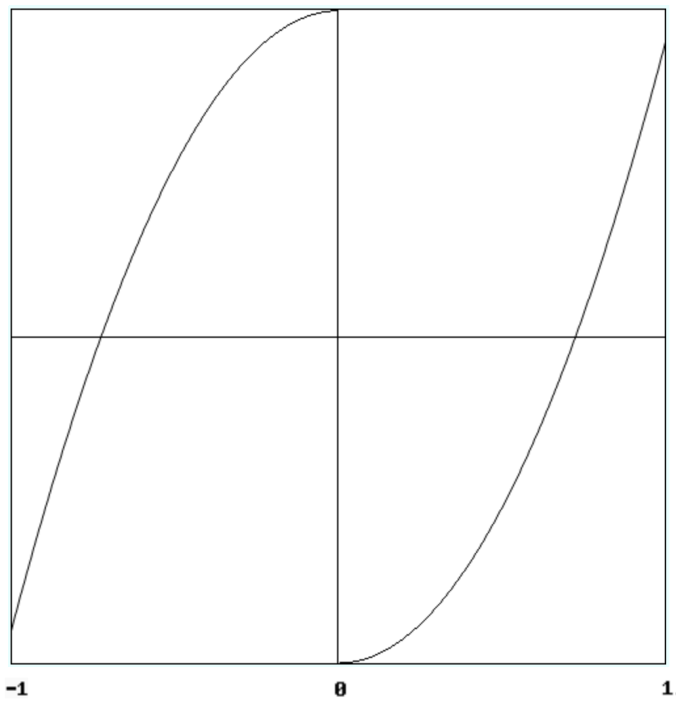

Fig. 3.1 Rovella Map

is based on the method of Benedicks and Carleson [8, 9] to obtain a set of parameters for *quadratic family*.

Rovella worked out that the maps associated with the set R admit stronger properties like Benedicks and Carleson quadratic maps and one of those properties forces the critical orbits to stay away from the critical point, which is given as:

(R1) There is a sufficiently small $\alpha > 0$ such that for every $a \in R$

$$|f_a^{n-1}(\pm 1)| \geq e^{-\alpha n}, \quad \text{for all } n > 0.$$

Following is one of the main results, given in [23], which describes the exponential growth of derivatives along the critical orbits for the Rovella family of maps and it also states that the critical orbits are dense in I for almost every Rovella parameter. Moreover this result provides a full density point of the set of Rovella parameters R.

Theorem 3.2.1. *[23, Theorem 2] The Rovella maps have the following properties:*

(R2) For every $a \in R$, the points ± 1 have positive Lyapunov exponent, i.e., there is $\lambda \geq 1$ such that

$$(f_a^n)'(\pm 1) > \lambda^n, \quad \textit{for all } n > 0.$$

(R3) The orbits of the points 1 and -1 under f_a are dense in I, for almost every $a \in R$.

(R4) The parameter 0 is a full density point for R, i.e.,

$$\lim_{a\to 0} \frac{m_1\big([0,a)\cap R\big)}{m_1\big([0,a)\big)} = 1,$$

where m_1 denotes the Lebesgue measure on the real line $\mathbb{R}$.

We will see the detailed arguments to construct the set R in the next chapter.

3.3 SRB Measures for Rovella Maps

Here we consider discrete time system given by a map f defined on the interval I. Recall that a measure μ on I is called:

- An *invariant measure* for f if for every measurable set $A \subset I$, $\mu(f^{-1}(A)) = \mu(A)$;
- An ergodic measure for f if for every measurable set $A \subset I$ with $f^{-1}(A) = A$, either $\mu(A) = 0$ or $\mu(A) = 1$.

And if ν is another measure on I such that $\mu(A) = 0$ for any measurable set $A \subset I$ with $\nu(A) = 0$, then μ is called absolutely continuous with respect to ν.

The physical measure associated to the map f is defined as follows.

Definition 3.3.1. An f-invariant measure μ is called a *physical measure* for f if the *basin* of μ, i.e., the set of points $x \in I$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu, \quad \text{for any continuous map } \varphi : I \to \mathbb{R},$$

has a positive Lebesgue measure.

The following are two important examples of physical measures:

1. From Birkhoff's Ergodic Theorem it follows that any ergodic invariant probability measure which is absolutely continuous with respect to Lebesgue measure is a physical measure. We shall refer to this special type of measure as an *SRB measure*.
2. If a map f has an attracting periodic orbit $\{x_0, x_1, \cdots, x_{k-1}\}$ of period k, then the measure $\mu = \frac{1}{k}(\delta_{x_0} + \delta_{x_1} + \cdots + \delta_{x_{k-1}})$ given by the convex sum of the delta Dirac measures supported on the points in the periodic orbit is a physical measure for f.

As in the case of Benedicks and Carleson [8] maps in the quadratic family, Rovella maps also grow exponentially along the critical orbits. It was therefore a natural question to address the existence of ergodic absolutely continuous invariant probability (*SRB*) measures for the Rovella maps, as Benedicks and Young [10] studied in the case of Benedicks and Carleson quadratic maps.

In the beginning of this century Metzger [21] positively answered that question by proving the existence of *SRB* measures associated with Rovella maps. For that purpose he used the properties *(A3)*, *(R1)* and *(R2)*. However to prove the uniqueness of the *SRB* measure he considered a smaller class of maps for which properties *(R1)* and *(R2)* imply the following strong mixing condition:

(M) For any interval $J \subset I$ there exists a number $n = n(J) > 0$ such that $[f_a(0^+), f_a(0^-)] \subset f_a^n(J)$.

The following lemma states that the set R can be chosen such that the corresponding maps also satisfy condition (M).

Lemma 3.3.2. *[21, Lemma A] Let the parameter a be in a small enough neighbourhood of the full density point* 0 *of the set of Rovella parameters R. If the corresponding map* f_a *satisfies* $(R1)$ *and* $(R2)$ *then it satisfies* (M).

Metzger mainly followed the techniques given by Viana in [30]. His fundamental strategy was to reduce the non-uniform hyperbolicity of the dynamics of Rovella maps to that of piecewise uniformly expanding maps. For that purpose he used the definition of tower extension given in [30] to transform the Rovella family to a family of uniformly expanding maps. Note that the Rovella maps are not continuous having two critical values which makes this case quite different than the one considered in [30]. In order to sort out this complication Metzger tried to define the tower to keep track of both the critical orbits which end up with a tower extension with two blocks (cf. [21]).

Following is one of the main results by Metzger given in that article.

Theorem 3.3.3. *[21, Theorem A] Under the conditions* (A3)*,* (R1)*,* (R2)*, and* (M) f_a *admits an absolutely continuous invariant measure. This measure is unique and ergodic.*

3.4 Statistical Stability for Rovella Maps

During the early period of this century, Alves and Viana [5] were studying the statistical properties of some dynamical systems and they purposed the notion of *statistical stability*. This particular type of stability studies the continuous variation of physical measures as a function of dynamical system. The precise definitions are as follows:

Definition 3.4.1. Let $\mathcal{G}_1$ be a family of maps defined on I corresponding unique physical measures. We say that $f \in \mathcal{G}_1$ is *statistically stable* if the function

$$g \longmapsto \mu_g$$

is continuous at f in the *weak** topology, where μ_g is the physical measure corresponding to map g.

Definition 3.4.2. Let $\mathcal{G}_2$ be a family of maps defined on I corresponding unique physical measures and those measures admit density functions. We say that $f \in \mathcal{G}_2$ is *strongly statistically stable* if the function

$$g \longmapsto h_g,$$

is continuous at f in L^1-norm, where h_g is density function for the physical measure μ_g corresponding to map g.

There are certain situations which assure the existence of densities of physical measures, e.g., if a map f admits physical measure μ as an SRB measure, i.e., μ is absolutely continuous with respect to Lebesgue measure, then the well known Radon-Nikodym Theorem guarantees the existence of density function for μ.

Keller [19] obtained the strong stability results for the piecewise expanding maps by proving the convergence of the densities of *SRB* measures in L^1-norm. Alves [1] presented sufficient conditions for the strong statistical stability for non-uniformly expanding maps. These conditions involve the volume decay of the tail set.

Definition 3.4.3. The map f is said to be *non-uniformly* expanding if there exists a constant c such that for Lebesgue almost every $x \in I$

$$\liminf_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log(f'(f^i(x))) > c.$$

Definition 3.4.4. The map f has *slow recurrence to the critical set* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for Lebesgue almost every $x \in I$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} -\log d_\delta((f^i(x),0) \le \varepsilon,$$

where d_δ is the delta truncated distance given as

$$d_\delta(x,y) = \begin{cases} |x-y|, & |x-y| \le \delta, \\ 1, & |x-y| > \delta. \end{cases}$$

The expansion time function is given by

$$\mathcal{H}(x) = \min\left\{N \ge 1 : \frac{1}{n} \sum_{i=0}^{n-1} \log(f'(f^i(x))) > c, \forall n \ge N\right\},$$

which is defined and finite almost everywhere in I provided f is non-uniformly expanding. By fixing $\varepsilon > 0$ and choosing a convenient $\delta > 0$, the *recurrence time function* is given by the expression

$$\mathcal{R}(x) = \min\left\{N \geq 1 : \frac{1}{n}\sum_{i=0}^{n-1} -\log d_\delta((f^i(x),0) \leq \varepsilon, \forall n \geq N\right\},$$

which is defined and finite almost everywhere in I provided f has slow recurrence to the critical set. Then the *tail set* at time n is the set of points that resist satisfying either a non-uniformly expanding condition or uniform slow recurrence at time n:

$$\mathcal{T}^n = \{x \in I : \mathcal{H}(x) > n \text{ or } \mathcal{R}(x) > n\}.$$

Freitas [14] proved that the Benedicks-Carleson quadratic maps are non-uniformly expanding. Moreover, he showed that those maps have slow recurrence to the critical set and the tail set loses volume exponentially fast. Therefore, applying the conditions given in [1], Freitas concluded that Benedicks-Carleson quadratic maps are strongly statistically stable by restricting himself on Benedicks-Carleson maps.

Recently, Alves and Soufi [3] studied the statistical stability for the Rovella maps. By following the techniques developed by Freitas [14], they established that the Rovella maps are non-uniformly expanding, have slow recurrence to the critical point and their tail set decay exponentially fast. Following is the main result given in [3].

Theorem 3.4.5. *[3, Theorem A] Each f_a, with $a \in R$, is non-uniformly expanding and has slow recurrence to the critical set. Moreover, there are $C > 0$ and $\tau > 0$ such that for all $a \in R$ and $n \in \mathbb{N}$,*

$$|\mathcal{T}_a^n| \leq Ce^{-\tau n}.$$

By making use of [2, Lemma 5.6], Alves and Soufi established the uniqueness of *SRB measure* for Rovella maps which is presented in their article as a corollary of the above theorem.

Corollary 3.4.6. *[3, Corollary B] For all $a \in R$, f_a has a unique ergodic absolutely continuous invariant probability measure μ_a.*

In the same article they also concluded the strong statistical stability of Rovella maps as a corollary of their main theorem.

Corollary 3.4.7. *[3, Corollary C] Let* $\frac{d\mu_a}{dm}$ *denotes the density of the measure* μ_a*. Then the function*

$$R \ni a \mapsto \frac{d\mu_a}{dm}$$

is continuos, if the L^1*-norm is considered in the space of densities, and the entropy of* μ_a *varies continuously with* $a \in R$*.*

Chapter 4

Construction of the Set of Rovella Parameters

In this chapter we consider the one parameter family $\{f_a : I \setminus \{0\} \to I : a \geq 0\}$ of maps, which was introduced by Rovella [23], arises through contracting Lorenz attractor and we refer this as contracting Lorenz-like family. For this family, Rovella briefly formed a set of parameters R, so called set of Rovella parameters, such that the derivatives of corresponding maps along the critical orbits increase exponentially and critical orbits have slow recurrence to the critical point. He indicated there that the idea of construction of the set R goes back to the work of Benedicks and Carleson in [8, 9]. In this chapter our aim is to construct that set in a more detailed and precise way.

Following the techniques of Benedicks and Carleson, we will construct through induction a nested sequence of sets of parameters $\{R_n\}_{n\in\mathbb{N}}$ such that *the derivative of each map associated with the set R_n has exponential growth along the critical orbits up to time n*, i.e., there exists some $\lambda > 1$ such that for every $a \in R_n$

$$D_j^{\pm}(a) := (f_a^j)'(\mp 1) \geq \lambda^j \quad \text{for } j = 1, \ldots, n. \tag{EG_n}$$

In addition, those parameters will satisfy so called *basic assumption*: for some $\alpha > 0$ sufficiently small

$$|\xi_j^{\pm}(a)| \geq e^{-\alpha j} \quad \text{for } j = 1, \ldots, n, \tag{BA_n}$$

where for any subset P of the set of parameters corresponding to the contracting Lorenz-like family the mappings $\xi_k^{\pm} : P \to I$ are defined as

$$\xi_k^{\pm}(a) = f_a^{k-1}(\mp 1) \quad \text{for all } k \geq 1.$$

Note that it is useful to impose (BA_n) to keep $\xi_n^{\pm}(a)$ away from the critical point which guarantees that $D_n^{\pm}(a)$ do not vanish for a parameter a satisfying (EG_{n-1}). By setting

$$R = \bigcap_{n=1}^{+\infty} R_n,$$

we obtain the set of Rovella parameters.

We will first try to find a parameter $a_0 > 0$ such that for a sufficiently large integer N_1, the conditions (BA_{N_1-1}) and (EG_{N_1-1}) are satisfied by f_a for every $a \in [0, a_0]$. Afterwards by setting $R_i = [0, a_0]$ for $i = 1, \ldots, N_1 - 1$, we assume that R_{n-1} satisfies (BA_{n-1}) for $n \geq N_1$. Then we exclude some parameters from R_{n-1} in order to obtain R_n such that every $a \in R_n$ satisfies (BA_n) and (EG_n) and we inductively construct the sequence $\{R_n\}_{n \in \mathbb{N}}$.

The parameter exclusion will be made in the following way: the sequences $\{\gamma_i\}_{i=0}^{\nu}$ and $\{p\}_{i=0}^{\nu}$, $\nu = \nu(n)$, can be associated to each $a \in R_{n-1}$ with $\gamma_0 = 1$, $p_0 = -1$ and $1 \leq \gamma_i + p_i + 1 < \gamma_{i+1} \leq n$ for $i = 0, \ldots, \nu - 1$. By setting

$$q_i = \gamma_{i+1} - (\gamma_i + p_i + 1) \quad \text{for } i = 0, \ldots, \nu - 1,$$

and

$$q_\nu = \begin{cases} 0 & \text{if } n \leq \gamma_\nu + p_\nu \\ n - (\gamma_\nu + p_\nu + 1) & \text{if } n > \gamma_\nu + p_\nu. \end{cases}$$

For some $B_1 > 0$, $c > 0$ and $\lambda_0 > 1$, we will have

$$(f_a^{q_\nu})'(\xi^{\pm}_{\gamma_\nu+p_\nu+1}(a)) \geq c\lambda_0^{q_\nu} \tag{4.0.1}$$
$$(f_a^{q_i})'(\xi^{\pm}_{\gamma_i+p_i+1}(a)) \geq \lambda_0^{q_i} \qquad \text{for } i = 0,\dots,\nu-1 \tag{4.0.2}$$
$$(f_a^{p_i+1})'(\xi^{\pm}_{\gamma_i}(a)) \geq 1 \qquad \text{for } i = 0,\dots,\nu-1 \tag{4.0.3}$$
$$(f_a^{k})'(\xi^{\pm}_{\gamma_i+1}(a)) \geq \frac{1}{B_1}\lambda^k \quad \text{for } k = 1,\dots,p_i, \text{ and for } i = 0,\dots,\nu. \tag{4.0.4}$$

Using chain rule, for $n > \gamma_\nu + p_\nu$

$$D_n^{\pm}(a) = \prod_{i=0}^{\nu}(f_a^{q_i})'(\xi^{\pm}_{\gamma_i+p_i+1}(a))\cdot(f_a^{p_i+1})'(\xi^{\pm}_{\gamma_i}(a)),$$

and for $n \leq \gamma_\nu + p_\nu$

$$D_n^{\pm}(a) = f_a'(\xi^{\pm}_{\nu}(a))(f_a^{n-\gamma_\nu})'(\xi^{\pm}_{\gamma_\nu+1}(a))\prod_{i=0}^{\nu-1}(f_a^{q_i})'(\xi^{\pm}_{\gamma_i+p_i+1}(a))\cdot(f_a^{p_i+1})'(\xi^{\pm}_{\gamma_i}(a)).$$

Then, defining $H_n(a) = q_0 + \ldots + q_\nu$, using inequalities (4.0.1)-(4.0.4) and property $(A3)$, we get

$$D_n^{\pm}(a) \geq c\lambda_0^{H_n(a)}, \quad \text{if } n > \gamma_\nu + p_\nu, \tag{4.0.5}$$

and

$$D_n^{\pm}(a) \geq \frac{K_2}{B_1}|\xi^{\pm}_{\gamma_\nu}(a)|^{s-1}\lambda^{(n-\gamma_\nu-p_\nu)}c\lambda_0^{H_n(a)}, \quad \text{if } n \leq \gamma_\nu + p_\nu. \tag{4.0.6}$$

We exclude parameters $a \in R_{n-1}$ such that $|\xi_n^{\pm}(a)| < e^{-\alpha n}$ or they do not satisfy

$$H_n(a) \geq (1-\alpha)n, \tag{H_n}$$

to obtain the set R_n. Since each $a \in R_n$ also satisfies (BA_n), from (4.0.5) and (4.0.6), we have

$$D_n^{\pm}(a) \geq c\lambda_0^{\alpha n}\cdot\lambda_0^{(1-2\alpha)n} \quad \text{if } n > \gamma_\nu + p_\nu, \tag{4.0.7}$$

and

$$\begin{aligned} D_n^{\pm}(a) &\geq \frac{K_2}{B_1} e^{-(s-1)\alpha n} \lambda^{(n-\gamma_v - p_v)} c \lambda_0^{(1-\alpha)n} \qquad \text{if } n \leq \gamma_v + p_v \\ &\geq \frac{K_2}{B_1} c \lambda_0^{-\frac{1}{\ln \lambda_0}(s-1)\alpha n} \lambda_0^{(1-\alpha)n} \\ &= \frac{K_2}{B_1} c \lambda_0^{\alpha n} \cdot \lambda_0^{[1-(2\alpha + \frac{1}{\ln \lambda_0}(s-1)\alpha)]n}. \end{aligned} \tag{4.0.8}$$

We may choose N_1 sufficiently large so that the first factors in both the inequalities (4.0.7) and (4.0.8) are greater than 1. Also α can be chosen small enough so that $2\alpha + \frac{1}{\ln \lambda_0}(s-1)\alpha < 1$, then by setting $c' = 1 - (2\alpha + \frac{1}{\ln \lambda_0}(s-1)\alpha)$ and $\lambda = \lambda_0^{c'} (> 1)$, we conclude that $D_n^{\pm}(a) \geq \lambda^n$, i.e., every $a \in R_n$ also satisfies (EG_n). Let us fix a sufficiently small $\alpha > 0$ such that $\alpha s < \ln \lambda$. This will be useful in order to establish some important results in the sequel.

The key idea is to split the orbit of a parameter a, $\{\xi_k^{\pm}(a),\ k \geq 1\}$, into pieces corresponding to the times: *returns* γ_i, *bound periods* $\{\gamma_i + 1, \ldots, \gamma_i + p_i\}$, and *free periods* $\{\gamma_i + p_i + 1, \ldots, \gamma_{i+1} - 1\}$ before the next returns γ_{i+1}. The returns corresponding to a parameter are the times when the orbit of that parameter visits a small neighbourhood of 0, the bound periods consist of times when orbit, after visiting that small neighbourhood, shadows an initial segment of one of the critical orbits closely, and the period of times when orbit stays outside that small neighbourhood as well as it is not in some bound period refer as the free periods. We shall precisely define all these notions later in this chapter.

4.1 The Initial Interval

In this section our goal is to acquire the initial interval of parameters in order to make the induction. First we remark that from now onwards by ω we refer an interval contained in the set of parameters corresponding to contracting Lorenz-like family. The following lemma by Alves and Soufi provides very useful properties for the dynamics of the maps f_0.

Lemma 4.1.1. *[3, Lemma 2.1] There is $\lambda_c > 1$ and a sufficiently large integer Δ_c such that: for any $\Delta \geq \Delta_c$ there are $a_0' > 0$ and $c > 0$, depending on Δ, such that given any $x \in I$ and $a \in [0, a_0']$,* *(1) If* $x, f_a(x), \ldots, f_a^{n-1}(x) \notin (-e^{-\Delta}, e^{-\Delta})$*, then* $(f_a^n)'(x) \geq c\lambda_c^n$*;*

(2) If $x, f_a(x), \ldots, f_a^{n-1}(x) \notin (-e^{-\Delta}, e^{-\Delta})$ *and* $f_a^n(x) \in (-e^{-\Delta}, e^{-\Delta})$*, then* $(f_a^n)'(x) \geq \lambda_c^n$*;*

(3) If $x, f_a(x), \ldots, f_a^{n-1}(x) \notin (-e^{-\Delta}, e^{-\Delta})$ *and* $f_a^n(x) \in (-e^{-1}, e^{-1})$*, then* $(f_a^n)'(x) \geq \frac{1}{e}\lambda_c^n$*.*

The following result is based on the fact that the maps $\xi_k^\pm$ are differentiable as long as they stay away from 0, and states that under strong growth of the derivatives of f_a at the critical values ± 1 the parameter and the space derivatives are comparable.

Proposition 4.1.2. *Given* $\lambda > 1$ *and* $\eta > 2$*, there is an integer* $N^\pm \geq 2$ *and* $A^\pm > 0$ *such that if a parameter* $a \geq 0$ *and* $n \geq N^\pm$ *satisfy both*

(1) $D_j^\pm(a) \geq \eta^j$ *for* $1 \leq j \leq N^\pm$*, and*

(2) $D_j^\pm(a) \geq \lambda^j$*, for* $1 \leq j \leq n-1$*,*

then

$$\frac{1}{A^\pm} \leq \frac{|(\xi_n^\pm)'(a)|}{D_{n-1}^\pm(a)} \leq A^\pm.$$

Proof. We consider the case of the critical value -1, the case of $+1$ is similar. Setting $f(a,x) = f_a(x)$ and using the chain rule for $k \geq 1$, we have

$$\begin{aligned} D_k^+(a) &= \frac{\partial f}{\partial x}(a, \xi_k^+(a)) \cdot D_{k-1}^+(a) \\ &= \prod_{i=1}^{k} \frac{\partial f}{\partial x}(a, \xi_i^+(a)). \end{aligned} \tag{4.1.1}$$

On the other hand

$$
\begin{aligned}
(\xi_{k+1}^{+})'(a) &= \frac{\partial f}{\partial x}(a,\xi_k^{+}(a))\cdot(\xi_k^{+})'(a)+\frac{\partial f}{\partial a}(a,\xi_k^{+}(a))\\
&= \frac{\partial f}{\partial x}(a,\xi_k^{+}(a))[\frac{\partial f}{\partial x}(a,\xi_{k-1}^{+}(a))\cdot(\xi_{k-1}^{+})'(a)\\
&\quad+\frac{\partial f}{\partial a}(a,\xi_{k-1}^{+}(a))]+\frac{\partial f}{\partial a}(a,\xi_k^{+}(a))\\
&= \frac{\partial f}{\partial x}(a,\xi_k^{+}(a))\frac{\partial f}{\partial x}(a,\xi_{k-1}^{+}(a)[\frac{\partial f}{\partial x}(a,\xi_{k-2}^{+}(a)\cdot(\xi_{k-2}^{+})'(a)\\
&\quad+\frac{\partial f}{\partial a}(a,\xi_{k-2}^{+}(a))]+\frac{\partial f}{\partial x}(a,\xi_k^{+}(a))\frac{\partial f}{\partial a}(a,\xi_{k-1}^{+}(a))+\ \frac{\partial f}{\partial a}(a,\xi_k^{+}(a))\\
&= \prod_{i=1}^{k}\frac{\partial f}{\partial x}(a,\xi_i^{+}(a))\cdot(\xi_1^{+})'(a)+\prod_{i=2}^{k}\frac{\partial f}{\partial x}(a,\xi_i^{+}(a))\frac{\partial f}{\partial a}(a,\xi_1^{+}(a))\\
&\quad+\ldots+\frac{\partial f}{\partial x}(a,\xi_k^{+}(a))\frac{\partial f}{\partial a}(a,\xi_{k-1}^{+}(a))+\frac{\partial f}{\partial a}(a,\xi_k^{+}(a)).
\end{aligned}
\tag{4.1.2}
$$

From (4.1.1) and (4.1.2), we have

$$
\frac{(\xi_{k+1}^{+})'(a)}{D_k^{+}(a)}-\frac{(\xi_k^{+})'(a)}{D_{k-1}^{+}(a)}=\frac{\frac{\partial f}{\partial a}(a,\xi_k^{+}(a))}{\prod_{i=1}^{k}\frac{\partial f}{\partial x}(a,\xi_i^{+}(a))}=\frac{\frac{\partial f}{\partial a}(a,\xi_k^{+}(a))}{D_k^{+}(a)}. \tag{4.1.3}
$$

After summing the both sides of (4.1.3) over $k=1,\ldots,n-1$, we get

$$
\frac{(\xi_n^{+})'(a)}{D_{n-1}^{+}(a)}-\frac{(\xi_1^{+})'(a)}{D_0^{+}(a)}=\sum_{k=1}^{n-1}\frac{\frac{\partial f}{\partial a}(a,\xi_k^{+}(a))}{D_k^{+}(a)}.
$$

We may assume that there exist $A_1, A_2>0$ such that for every parameter a,

$$
A_1<\sup_{x\in I}\left|\frac{\partial f}{\partial a}(a,x)\right|\le|(\xi_1^{+})'(a)|\le A_2.
$$

Since $D_0^+(a) = 1$, from the above equation, we get

$$\begin{aligned}
\Big| \big| \frac{(\xi_n^+)'(a)}{D_{n-1}^+(a)} \big| - |(\xi_1^+)'(a)| \Big| &\le \Big| \frac{(\xi_n^+)'(a)}{D_{n-1}^+(a)} - (\xi_1^+)'(a) \Big| \\
&= \Big| \sum_{k=1}^{n-1} \frac{\frac{\partial f}{\partial a}(a, \xi_k^+(a))}{D_k^+(a)} \Big| \\
&\le \sup_{x \in I} \big| \frac{\partial f}{\partial a}(a,x) \big| \sum_{k=1}^{n-1} \frac{1}{D_k^+(a)} \\
&\le |(\xi_1^+)'(a)| \sum_{k=1}^{n-1} \frac{1}{D_k^+(a)}.
\end{aligned}$$

And from the above inequality, we get

$$A_1 \Big(1 - \sum_{k=1}^{n-1} \frac{1}{D_k^+(a)} \Big) \le \frac{|(\xi_n^+)'(a)|}{D_{n-1}^+(a)} \le A_2 \Big(1 + \sum_{k=1}^{n-1} \frac{1}{D_k^+(a)} \Big). \tag{4.1.4}$$

On the other hand since $\eta > 2$ and $\lambda > 1$, therefore $\sum_{k=1}^{+\infty} \frac{1}{\eta^k} < 1$ and $\sum_{k}^{+\infty} \frac{1}{\lambda^k} \to 0$ as $k \to +\infty$. Thus we can choose an integer N_0^+ and a number $\varepsilon' > 0$ such that $\sum_{k=1}^{+\infty} \frac{1}{\eta^k} + \sum_{k=N_0^+ +1}^{+\infty} \frac{1}{\lambda^k} < 1 - \varepsilon'$. Then if $D_k^+(a) \ge \eta^k$ for every $k = 1, \ldots, N_0^+$, and $D_k^+(a) \ge \lambda^k$ for every $k = N_0^+ + 1, \ldots, n-1$, we obtain

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{1}{D_{k-1}^+(a)} &\le \sum_{k=1}^{N_0^+} \frac{1}{\eta^k} + \sum_{k=N_0^+ +1}^{n-1} \frac{1}{\lambda^k} \\
&\le \sum_{k=1}^{\infty} \frac{1}{\eta^k} + \sum_{k=N_0^+ +1}^{\infty} \frac{1}{\lambda^k} \\
&\le 1 - \varepsilon'.
\end{aligned}$$

The result follows from (4.1.4) with $A^+ \ge \max \Big\{ \frac{1}{\varepsilon' A_1}, A_2(2 - \varepsilon') \Big\}$. □

From here on we take

$$N = \max\{N^+, N^-\} \quad \text{and} \quad A = \max\{A^+, A^-\},$$

where $N^{\pm}$ and $A^{\pm}$ are provided by Proposition 4.1.2.

Remark 4.1.3. Observe that if the conditions (1), (3) of Proposition 4.1.2 are satisfied for some $n \geq N$ and for every a in some parameter interval ω then we have in particular $\xi_k^{\pm}(a) \neq 0$ for all $a \in \omega$ and $k = N, \cdots, n$, since $|(\xi_k^{\pm})'(a)| \geq \frac{1}{A} D_{k-1}^{\pm}(a)$. Then for any $N \leq k \leq n$, $\xi_k^{\pm}|_{\omega}$ are diffeomorphisms with the inverses defined as: for any $x^{\pm} \in \xi_k^{\pm}(\omega)$ with $\xi_k^{\pm}(a) = x^{\pm}$ for some $a \in \omega$, then

$$(\xi_k^{\pm})^{-1}(x^{\pm}) := \xi_{-k}^{\pm}(x^{\pm}) = a.$$

In fact $\xi_k^{\pm}|_{\overline{\omega}}$ are diffeomorphisms and this assertion plays an important part to inductively construct the set of Rovella parameters. Consequently for every $N \leq i \leq j \leq n$, we can define the following functions

$$\begin{aligned} \psi^{\pm}: \xi_i^{\pm}(\omega) &\longrightarrow \quad \xi_j^{\pm}(\omega) \\ x \quad &\mapsto \quad \xi_j^{\pm} \circ (\xi_i^{\pm})^{-1}(x), \end{aligned}$$

with the derivative given for $a \in \omega$ by

$$(\psi^{\pm})'(\xi_i^{\pm}(a)) = \frac{(\xi_j^{\pm})'(a)}{(\xi_i^{\pm})'(a)}, \ a \in \omega.$$

The functions $\psi^{\pm}$ will be useful in the proof of the next lemma which is useful in finding an estimate for the lengths of $\xi_n^{\pm}(\omega)$ at particular time n, where ω is a parameter interval.

Lemma 4.1.4. *Given $\lambda > 1$ and $\eta > 2$, consider a parameter interval ω such that every $a \in \omega$ and some $n \geq N$ hold both*

(1) $D_j^{\pm}(a) \geq \eta^j$ for $1 \leq j \leq N$, and

(2) $D_j^{\pm}(a) \geq \lambda^j$, for $1 \leq j \leq n-1$.

Then, for any $N \leq i \leq j \leq n$, there is $a^{\pm} \in \omega$ such that

$$\frac{1}{A^2}\left|(f_{a^{\pm}}^{j-i})'(\xi_i^{\pm}(a^{\pm}))\right| \leq \frac{|\xi_j^{\pm}(\omega)|}{|\xi_i^{\pm}(\omega)|} \leq A^2\left|(f_{a^{\pm}}^{j-i})'(\xi_i^{\pm}(a^{\pm}))\right|.$$

Proof. We are going to present the proof corresponding to critical value -1, the other case can be seen along similar lines. Since (1) and (2) hold for every $a \in \omega$, it follows from Proposition 4.1.2 that

$$\frac{1}{A^2} \cdot \frac{D^+_{j-1}(a)}{D^+_{i-1}(a)} \leq \frac{|(\xi^+_j)'(a)|}{|(\xi^+_i)'(a)|} \leq A^2 \cdot \frac{D^+_{j-1}(a)}{D^+_{i-1}(a)}. \tag{4.1.5}$$

On the other hand, by the Mean Value Theorem, for some $a^+ \in \omega$ we have

$$\frac{|\xi^+_j(\omega)|}{|\xi^+_i(\omega)|} = |(\xi^+_{j-i})'(\xi^+_i(a^+))| = |(\xi^+_j \circ \xi^+_{-i})'(\xi^+_i(a^+))| = |(\psi^+)'(\xi^+_i(a^+))|. \tag{4.1.6}$$

Also

$$\begin{aligned} D^+_{j-1}(a^+) = (f^{j-1}_{a^+})'(-1) &= (f^{j-i}_{a^+} \circ f^{i-1}_{a^+})'(-1) \\ &= (f^{j-i}_{a^+})'(f^{i-1}_{a^+}(-1))(f^{i-1}_{a^+})'(-1) \\ &= (f^{j-i}_{a^+})'(\xi^+_i(a^+))D^+_{i-1}(a^+), \end{aligned}$$

which gives

$$\frac{D^+_{j-1}(a^+)}{D^+_{i-1}(a^+)} = (f^{j-i}_{a^+})'(\xi^+_i(a^+)). \tag{4.1.7}$$

Now using (4.1.6) and (4.1.7) in (4.1.5), we get

$$\frac{1}{A^2}\left|(f^{j-i}_{a^+})'(\xi^+_i(a^+))\right| \leq \frac{|\xi^+_j(\omega)|}{|\xi^+_i(\omega)|} \leq A^2\left|(f^{j-i}_{a^+})'(\xi^+_i(a^+))\right|.$$

Hence the result follows. □

Since the points 1 and -1 are fixed by the map f_0, $f_0(0^+) = -1$, $f_0(0^-) = 1$ and f_0 is smooth in the intervals $[-1,0)$ and $(0,1]$, so we can find numbers $\eta_0 > 2$ and $\varepsilon_0 > 0$ with $\eta_0 - \varepsilon_0 > 2$ such that $f_0'(-1) = \eta_0$. We set $\eta_1 = \eta_0 - \varepsilon_0 > 2$, and denote $O^-(a) \in [-1,0)$ and $O^+(a) \in (0,1]$ the zeros of the map f_a on the left and right side of the origin, respectively, i.e., $f_a(O^\pm(a)) = 0$. Also since the point 1 is a critical value for f_0 with $f_0(0^-) = 1$, $O^-(0) \in (-1,0)$, $f_0(0^-) = 1$ and $f_0'(x) \leq f_0'(y)$ for $x, y \in [O^-(0), 0)$ with $x \geq y$, then we may choose $\varepsilon_0 > 0$ such that $f_0'(O^-(0)) \geq$

$1+\varepsilon_0$. Therefore we can take $\delta_0 > 0$ such that $O^-(0)+\delta_0 < 0$ and $f_0'(O^-(0)+\delta_0) \geq 1+\frac{\varepsilon_0}{2}$. Note that $f_0(x) \geq g(x)$ for every $x \in [O^-(0),0)$, where $g(x) = -\frac{x-O^-(0)}{O^-(0)}$ is the linear map passing through the points $(O^-(0),0)$ and $(0,1)$ with $g(O^-(0)+\delta_0) = -\frac{\delta_0}{O^-(0)}$. Thus we may choose a positive integer $\Delta_0 > -\log(\frac{-\delta_0}{O^-(0)})$ and set $x_0 = O^-(0)+\delta_0$ and $\lambda_0' = 1+\frac{\varepsilon_0}{2}$ such that $f_0'(x_0) \geq \lambda_0'$ and $f_0(x_0) > e^{-\Delta_0}$ with $x_0 \in (O^-(0),0)$. Let us fix a $\lambda_0 > 1$ and Δ with $\lambda_0 \leq \min\{\lambda_c, \lambda_0'\}$ and $\Delta \geq \max\{\Delta_c, \Delta_0\}$, where λ_c and Δ_c are provided by Lemma 4.1.1. Note that this λ_0 will work for Lemma 4.1.1. These notations will be useful in the next proposition which provides the initial interval of our later construction of the set of parameters.

Proposition 4.1.5. *Given any integer $N_0 \geq N$, there exist an integer $N_1 \geq N_0$ and a parameter $0 < a_0 \leq a_0'$ such that*

(i) $D_j^+(a) \geq \eta_1^j$ *for every* $a \in [0,a_0]$ *and* $1 \leq j \leq N_0 - 1$,

(ii) $D_j^+(a) \geq \lambda_0^j$ *for every* $a \in [0,a_0]$ *and* $1 \leq j \leq N_1 - 1$,

(iii) $\xi_j^+([0,a_0]) \cap (-e^{-\Delta}, e^{-\Delta}) = \phi$ *for every* $1 \leq j \leq N_1 - 1$,

(iv) $\xi_{N_1}^+([0,a_0]) \supset (-e^{-\Delta}, e^{-\Delta})$.

Proof. For $1 \leq n \leq N_0$, set

$$\begin{aligned} \Phi_n :\ [0,a_0'] &\longrightarrow [-1,1]\times[0,+\infty) \\ a &\longmapsto (\xi_{n+1}^+(a), D_n^+(a)). \end{aligned}$$

Since -1 is fixed by f_0, using the chain rule we get

$$D_n^+(0) = (f_0^n)'(-1) = \prod_{i=0}^{n-1} f_0'(f_0^i(-1)) = \prod_{i=0}^{n-1} f_0'(-1)$$

Recalling that $f_0'(-1) = \eta_0$, we have $\Phi_n(0) = (-1, \eta_0^n)$. Since Φ_k is continuous as long as ξ_k^+ mapped onto the origin, so, for $1 \leq n \leq N_0$ we have sequence of parameters $\{a_n : a_n \in [0,a_0']\}_{n=1}^{N_0}$ with $a_i \leq a_k$, for $i \geq k$, and

$$\Phi_n([0,a_n]) \subset [-1, O^-(0)] \times [\eta_1^n, +\infty).$$

That is for every $1 \le n \le N_0$ and every $a \in [0, a_{N_0}]$, $\xi^+_{n+1}(a) \le O^-(0)$ and

$$D^+_n(a) \ge \eta_1^n.$$

Thus any $a \in [0, a_{N_0}]$ satisfies (i). Since $f_0'(x_0) \ge \lambda_0$, then it is to be noted that if for some parameter a, $\xi^+_j(a) \in [-1, x_0]$ for every $j = 1, \ldots, k$, then

$$D^+_k(a) \ge \lambda_0^k.$$

Now as long as $\xi^+_i([0, a_{N_0}])$, $i \ge 1$, is contained in $[-1, x_0)$, any $a \in [0, a_{N_0}]$ satisfies the hypothesis of Proposition (4.1.2), thus by using mean value theorem, for some $a \in (0, a_{N_0})$, we have

$$\begin{aligned}
|\xi^+_{i+1}([0, a_{N_0}])| &= |(\xi^+_{i+1})'(a)| \cdot a_{N_0} \\
&\ge \frac{a_{N_0}}{A} D^+_i(a) \\
&\ge \frac{a_{N_0}}{A} \lambda_0^i.
\end{aligned}$$

The above inequality reveals that while $\xi^+_i([0, a_{N_0}])$, $i \ge 1$, remains inside the interval $[-1, x_0)$, we have exponential growth of $\xi^+_i([0, a_{N_0}])$, and then there exists an integer k such that $\xi^+_k([0, a_{N_0}]) \not\subset [-1, x_0)$. Let N_1' be the first integer to have the above situation, i.e.,

$$\xi^+_i([0, a_{N_0}]) \subset [-1, x_0) \quad \text{for every } 1 \le i < N_1',$$

and

$$\xi^+_{N_1'}([0, a_{N_0}]) \not\subset [-1, x_0).$$

Therefore we may chose $a_0 \in [0, a_{N_0}]$ such that $\xi^+_{N_1'}(a_0) = x_0$, since $f_{a_0}(x_0) \ge e^{-\Delta}$, then $\xi^+_{N_1'+1}([0, a_0]) \supset [-1, e^{-\Delta})$. Hence the result follows by taking $N_1 = N_1' + 1$. □

Remark 4.1.6. From the property (A0), we know that the points 1 and −1 are fixed by the map f_0, therefore by the definition of f_0, it can be seen that the connected components of the graph of f_0 in the intervals $[-1,0)$ and $(0,1]$ are symmetric about origin, i.e., $f_0(x) = -f_0(-x)$ for all $x \in I \setminus \{0\}$. Therefore for the sake of simplicity we may assume that for any parameter a corresponding to contracting Lorenz-like family, $f_a(x) = -f_a(-x)$ for all $x \in I \setminus \{0\}$. Thus the similar result as Proposition 4.1.5 can be obtain for ξ^- and D^- with the same integer N_1 and the parameter interval $[0, a_0]$. We also remark that the results can be proved in more general setting without the assumption of symmetry.

4.2 The Bound Periods

The periods of time occurring after the returns of critical orbits $\xi_k^{\pm}(a)$ to a small neighbourhood of 0 have a significant role and we call those periods as *bound periods*. In this section first we will precisely define those periods of time and then obtain some results which are used to get the exponential growth property (EG_n) under the assumptions (BA_n) and (H_n). In order to explicitly describe the closeness to 0, we set $\delta := e^{-\Delta}$, where Δ is the one which is used in Proposition 4.1.5, and consider the following neighbourhoods of 0 for $m \geq \Delta - 1$

$$U_m = (-e^{-m}, e^{-m}).$$

We also set for $m \geq \Delta - 1$

$$I_m = [e^{-(m+1)}, e^{-m}) \quad \text{and} \quad I_m^+ = I_{m-1} \cup I_m \cup I_{m+1}.$$

We extend the above definition, setting for $m \leq -(\Delta - 1)$

$$I_m = -I_{|m|} \quad \text{and} \quad I_m^+ = -I_{|m|}^+.$$

Since we will study the iterations of small parameter intervals, therefore the notions like, returns, bound periods and free periods must be constant in small parameter intervals. Here we fix some $\beta > 0$ such that $s\alpha \leq \beta$ and $\beta \dfrac{s+5}{\beta + \log \lambda} < 1$.

Definition 4.2.1. Let $x \in I_m^+$, we denote $p(a,m)$ to be the largest integer such that

$$|f_a^j(x) - \xi_j^+(a)| \leq e^{-\beta j} \quad if\ m > 0,$$

and

$$|f_a^j(x) - \xi_j^-(a)| \leq e^{-\beta j} \quad if\ m < 0,$$

for $j = 1,\dots,p(a,m)$. Then the time interval $1,\dots,p(a,m)$ is called the bound period for x.

Note that by the above definition

$$|f_a^{j-1}\big([-1, f_a(e^{-|m|+1})]\big)| \leq e^{-\beta j},$$

for all $1 \leq j \leq p(a,m)$. The above definition allows us to state our next result which essentially assures that the bound period $p(a,m)$ for $\xi_j^+(a) \in I_m^+$, satisfies the properties (4.0.3) and (4.0.4). First we mention that $R_n \subset [0,a_0]$ denotes a set satisfying (BA_n) and (EG_n). In fact we will encounter these sets later in the construction of set of Rovella parameters. It is also to be noted that if $a \in R_{n-1}$ and $\xi_n^+(a) \in I_m^+$ for some m with $|m| \geq \Delta$, then $\xi_n^-(a) \in I_{-m}^+$ and $p(a,m) = p(a,-m)$.

Lemma 4.2.2. *Assume that $a \in R_{n-1}$ and either $\xi_n^+(a)$ or $\xi_n^-(a)$ belongs to an interval I_m^+, for some $\Delta \leq |m| \leq [\alpha n] - 1$. Then*

(1) there exists $B_1 = B_1(\alpha,\beta)$ such that for every $k = 1,\dots,p(a,m)$

(a) $\dfrac{1}{B_1} \leq \dfrac{(f_a^k)'(y)}{D_k^+(a)} \leq B_1$ *if* $y \in \big[-1, f_a(e^{-|m|+1})\big]$,

(b) $\dfrac{1}{B_1} \leq \dfrac{(f_a^k)'(y)}{D_k^-(a)} \leq B_1$ *if* $y \in \big[f_a(-e^{-|m|+1}), 1\big]$;

(2) $p(a,m) \leq \dfrac{s+1}{\beta + \log\lambda}|m|$;

(3) letting $p = p(a,m)$ and $\kappa_1 = \beta\dfrac{s+2}{\beta+\log\lambda}$, *we have for all $x \in I_m^+$*

$$(f_a^{p+1})'(x) \geq e^{(1-\kappa_1)|m|}.$$

Proof. For obtaining (1) it is sufficient to prove the first item, since the second one can be obtained by following similar lines. We may assume that $\xi_n^+(a) \in I_m^+$. First using chain rule, for $k = 1, \dots, \min\{p, n\}$, we have

$$\begin{aligned}
\frac{(f_a^k)'(y)}{D_k^+(a)} = \frac{(f_a^k)'(y)}{(f_a^k)'(-1)} &= \prod_{j=0}^{k-1} \frac{f_a'(f_a^j(y))}{f_a'(\xi_{j+1}^+(a))} \\
&= \prod_{j=0}^{k-1} \Big(1 + \frac{f_a'(f_a^j(y)) - f_a'(\xi_{j+1}^+(a))}{f_a'(\xi_{j+1}^+(a))}\Big) \\
&\leq \exp\Big(\sum_{j=0}^{k-1} \Big|\frac{f_a'(f_a^j(y)) - f_a'(\xi_{j+1}^+(a))}{f_a'(\xi_{j+1}^+(a))}\Big|\Big).
\end{aligned}$$

Therefore we conclude the proof of this item by showing that

$$\sum_{j=0}^{k-1} \frac{|f_a'(f_a^j(y)) - f_a'(\xi_{j+1}^+(a))|}{f_a'(\xi_{j+1}^+(a))}$$

is uniformly bounded. Since 0 is not in $[\xi_j^+(a) - e^{-\beta j}, \xi_j^+(a) + e^{-\beta j}]$ and f_a has negative Schwarzian derivative inside this interval, as long as $f_a^j(y) \in [\xi_j^+(a) - e^{-\beta j}, \xi_j^+(a) + e^{-\beta j}]$,

$$\begin{aligned}
\frac{|f_a'(f_a^j(y)) - f_a'(\xi_{j+1}^+(a))|}{f_a'(\xi_{j+1}^+(a))} &\leq |f_a''(z)| \frac{|f_a^j(y) - \xi_{j+1}^+(a)|}{f_a'(\xi_{j+1}^+(a))} \\
&\leq C|z|^{s-2} \frac{|f_a^j(y) - \xi_{j+1}^+(a)|}{f_a'(\xi_{j+1}^+(a))}.
\end{aligned}$$

Now $k \leq n, p$ and a satisfies (BA_{n-1}), therefore from the above inequality, using the binding condition and property $(A3)$, we get

$$\sum_{j=0}^{k-1} \frac{|f_a'(f_a^j(y)) - f_a'(\xi_{j+1}^+(a))|}{f_a'(\xi_{j+1}^+(a))} \leq \frac{C}{K_2} \sum_{j=0}^{k-1} \frac{e^{-\beta j}}{e^{-\alpha(s-1)(j+1)}}.$$

The right side of the above inequality is uniformly bounded since $\beta \geq s\alpha$ with $s > 1$. Consequently to conclude the proof of (1) we just need to make sure that $p < n$. See part (2).

For proving (2), let $x = e^{-|m|+1} \in I_m^+$ and $j = \min\{p,n\} - 1$. Then using the first part of (1) and property $(A3)$, we have

$$\begin{aligned} |f_a^{j+1}(x) - \xi_{j+1}^+(a)| &= |f_a^j(f_a(x)) - f_a^j(-1)| \\ &= (f_a^j)'(y)|f_a(x)+1|, \; y \in (-1, f_a(e^{-|m|+1})) \\ &\geq \frac{K_2}{B_1} D_j^+(a) \frac{|x|^s}{s}. \end{aligned}$$

Now by using binding condition and taking into account that a satisfies (EG_{n-1}), from the last inequality it follows that

$$\frac{K_2}{B_1 s} \lambda^j e^{-(|m|+2)s} \leq e^{-\beta(j+1)},$$

and from the above inequality it can be work out that

$$j \leq \frac{|m|s}{\beta + \log\lambda} + \frac{2s - \log(\frac{K_2}{B_1 s}) - \beta}{\beta + \log\lambda}.$$

Therefore if $|m|$ is large enough, we may conclude that

$$j \leq \frac{|m|(s+1)}{\beta + \log\lambda} - 1. \tag{4.2.1}$$

Since $|m| \leq [\alpha n] - 1$, from (4.2.1) we have

$$\begin{aligned} j &\leq \frac{([\alpha n]-1)(s+1)}{\beta + \log\lambda} - 1 \leq \frac{(\alpha n - 1)(s+1)}{\beta + \log\lambda} - 1 \\ &\leq \frac{(\alpha n)(s+1)}{\beta + \log\lambda} - 1 < n - 1, \end{aligned}$$

where the last inequality holds since $\beta \geq s\alpha$ and $\alpha < \log\lambda$. Hence $j = p-1$ and from (4.2.1) the result follows. Let us now prove (3). Clearly, by the binding condition

$$|f_a^p([-1, f_a(e^{-|m|+1})])| \geq e^{-\beta(p+1)}. \tag{4.2.2}$$

Thus by the mean value theorem, for some $z \in (-1, f_a(e^{-|m|+1}))$ and for some $y \in (0, e^{-|m|+1})$, we have

$$|f_a^p\big([-1, f_a(e^{-|m|+1})]\big)| = (f_a^p)'(z) f_a'(y) e^{-|m|+1}. \tag{4.2.3}$$

From (4.2.2) and (4.2.3), we obtain

$$(f_a^p)'(z) \geq \frac{e^{-\beta(p+1)+|m|-1}}{f_a'(y)}.$$

Now using the above inequality, property $(A3)$ and part (1), for any $x \in I_m^+$, we get

$$\begin{aligned}
(f_a^{p+1})'(x) &= (f_a^p)'(f_a(x)) f_a'(x) \\
&\geq \frac{1}{B_1} D_p^+(a) f_a'(x), \text{ since } f_a(x) \in [-1, f_a(e^{-|m|+1})] \\
&\geq \frac{1}{B_1^2} (f_a^p)'(z) f_a'(x), \text{ since } z \in [-1, f_a(e^{-|m|+1})] \\
&\geq \frac{1}{B_1^2} e^{-\beta(p+1)+|m|-1} \cdot \frac{f_a'(x)}{f_a'(y)} \\
&\geq \frac{1}{B_1^2} e^{-\beta(p+1)+|m|-1} \cdot \frac{K_2 |x|^{s-1}}{K_1 |y|^{s-1}}.
\end{aligned}$$

Since $|x| \geq e^{-|m|-2}$, $|y| \leq e^{-|m|+1}$ and from part (2) we have $p < \frac{s+1}{\beta+\log\lambda}|m|$. Hence the result concluded from the above inequality, providing Δ is sufficiently large so that $\frac{K_2}{K_1 B_1^2} e^{-(3s+\beta-2)} \geq e^{-\frac{\beta}{\beta+\log\lambda}|m|}$. □

Now we are intended to find similar bounds, as in the above lemma, when $p(a,m)$ is constant in small parameter intervals. In this regard, for a parameter interval ω such that either $\xi_n^+(\omega)$ or $\xi_n^-(\omega)$ is contained in some I_m^+, with $|m| \geq \Delta$. Then we define

$$p(\omega, m) = \min_{a \in \omega} p(a, m).$$

Note that by the above definition $p(\omega,m) \leq p(a,m)$ and

$$|f_a^{j-1}\big([-1, f_a(e^{-|m|+1})]\big)| \leq e^{-\beta j},$$

for all $1 \le j \le p(\omega,m)$ and for every $a \in \omega$. Furthermore, $p(\omega,m) = p(\omega,-m)$ and $p(\omega,m) \le p(a,m)$, therefore for every $a \in \omega$ items (1) and (2) of Lemma 4.2.2 follow directly. But it requires some more work in order to prove part (3) and this is what we are going to establish in the remaining section. We take a parameter interval $\omega \subset R_{n-1}$ with n sufficiently large and under this hypothesis the next two results are consequence of exponential growth of the lengths of $\xi_k^{\pm}(\omega)$, $k \ge 1$.

Proposition 4.2.3. *Let $\omega \subset R_{n-1}$ be a parameter interval, then for every $a,b \in \omega$*

$$|a-b| \le 4A\lambda^{-n}.$$

Proof. By using Proposition 4.1.2 and mean value theorem, for some $d \in \omega$, we have

$$\begin{aligned} 2 \ge |\xi_n^+(\omega)| &= (\xi_n^+)'(d)|\omega| \ge (\xi_n^+)'(d)|a-b| \\ &\ge \frac{1}{A} D_{n-1}^+(d)|a-b| \ge \frac{1}{A}\lambda^{n-1}|a-b|, \end{aligned}$$

where the last inequality holds since $d \in R_{n-1}$. And from the above inequality, we get

$$|a-b| \le 2A\lambda^{-(n-1)} \le 4A\lambda^{-n}.$$

□

Lemma 4.2.4. *Let $\omega \subset R_{n-1}$ be a parameter interval and either $\xi_n^+(\omega)$ or $\xi_n^-(\omega)$ is contained in I_m^+ with $\Delta \le |m| \le [\alpha n] - 1$, then for every $a,b \in \omega$ and every $1 \le j \le p(\omega,m)$,*

$$\left| |\xi_j^{\pm}(a)|^{s-1} - |\xi_j^{\pm}(b)|^{s-1} \right| \le e^{-\beta j}.$$

Proof. We need to prove the result just in the case of ξ_j^+, the other one can be prove in the same way. If $a = b$ then it is trivial. So let us assume $a \ne b$. From the inequality (4.1.4) in the proof of Proposition 4.1.2, we have

$$\frac{|(\xi_{j+1}^{+})'(a)|}{D_j^{+}(a)} \leq A_2\left(1+\sum_{k=1}^{j}\frac{1}{D_{k-1}^{+}(a)}\right),$$

and since $\omega \subset R_{n-1}$ and $j \leq p(\omega, m) \leq n-1$, we get

$$\frac{|(\xi_{j+1}^{+})'(a)|}{D_j^{+}(a)} \leq A_2\left(1+\sum_{k=1}^{j}\frac{1}{\lambda^{k-1}}\right) \leq A_2\left(1+\sum_{k=1}^{\infty}\frac{1}{\lambda^{k-1}}\right) \leq A_3,$$

for some $A_3 > 0$. Now if $1 < s \leq 2$, since the modulus function $|\cdot|$ is differentiable everywhere except 0. Therefore, using above inequality and mean value theorem, we get

$$\begin{aligned}\left||\xi_j^{+}(a)|^{s-1} - |\xi_j^{+}(b)|^{s-1}\right| &\leq \left||\xi_j^{+}(a)| - |\xi_j^{+}(b)|\right| \\ &= \left|\frac{\xi_j^{+}(d)}{|\xi_j^{+}(d)|}(\xi_j^{+})'(d)\right||a-b|, \quad d \in (a,b) \\ &\leq \frac{|(\xi_j^{+})'(d)|}{D_{j-1}^{+}(d)} D_{j-1}^{+}(d)|a-b| \\ &\leq A_3 D_{j-1}^{+}(d)|a-b|. \end{aligned} \tag{4.2.4}$$

And if $s > 2$, again using the mean value theorem, we get

$$\begin{aligned}\left||\xi_j^{+}(a)|^{s-1} - |\xi_j^{+}(b)|^{s-1}\right| &\leq (s-1)|\xi_j^{+}(d)|^{s-2}|(\xi_j^{+})'(d)||a-b|, \quad d \in (a,b) \\ &\leq (s-1)\frac{|(\xi_j^{+})'(d)|}{D_{j-1}^{+}(d)} D_{j-1}^{+}(d)|a-b| \\ &\leq A_s D_{j-1}^{+}(d)|a-b|, \end{aligned} \tag{4.2.5}$$

where $A_s = (s-1)A_3$. By Lemma 4.2.2 and the mean value theorem, for $y \in (-1, f_d(e^{-|m|+1}))$, we have

$$\begin{aligned}\left|f_d^{j-1}\left([-1, f_d(e^{-|m|+1})]\right)\right| &= |(f_d^{j-1})'(y)||[-1, f_d(e^{-|m|+1})]| \\ &\geq \frac{1}{B_1} D_{j-1}^{+}(d)|[-1, f_d(e^{-|m|+1})]|. \end{aligned} \tag{4.2.6}$$

From the inequalities (4.2.4), (4.2.5) and (4.2.6), we obtain

$$\left||\xi_j^+(a)|^{s-1} - |\xi_j^+(b)|^{s-1}\right| \le A_s B_1 |a-b| \frac{|f_d^{j-1}\big([-1, f_d(e^{-|m|+1})]\big)|}{|[-1, f_d(e^{-|m|+1})]|}, \tag{4.2.7}$$

Using property $(A3)$, we have

$$\begin{aligned} |[-1, f_d(e^{-|m|+1})]| &= 1 + f_d(e^{-|m|+1}) \\ &\ge \frac{K_2 e^{(-|m|+1)(s-1)}}{s} \\ &\ge K_2 e^{-|m|s} \ge K_2 e^{-\alpha ns}. \end{aligned} \tag{4.2.8}$$

And from the binding condition, we have

$$|f_d^{j-1}\big([-1, f_d(e^{-|m|+1})]\big)| \le e^{-\beta j}. \tag{4.2.9}$$

Using (4.2.8), (4.2.9) and Proposition 4.2.3 in (4.2.7), we get

$$\left||\xi_j^+(a)|^{s-1} - |\xi_j^+(b)|^{s-1}\right| \le \frac{A_s B_1}{K_2} 4A\lambda^{-n} e^{-\beta j} e^{\alpha sn}. \tag{4.2.10}$$

By the choice of α, $e^{\alpha s} < \lambda$ and for sufficiently large n, $4A\frac{A_s B_1}{K_2}(\frac{e^{\alpha s}}{\lambda})^n \le 1$. Hence the result directly follows from (4.2.10). □

Lemma 4.2.5. *Let $\omega \subset R_{n-1}$ be a parameter interval and either $\xi_n^+(\omega)$ or $\xi_n^-(\omega)$ is contained in I_m^+ with $\Delta \le |m| \le [\alpha n] - 1$, then there exists a positive constant $B_2 = B_2(\alpha, \beta)$ such that for every $a, b \in \omega$ and $x, y \in I_m^+$,*

$$\frac{(f_a^j)'(f_a(x))}{(f_b^j)'(f_b(y))} \le B_2 \quad \forall j = 1, \dots, p(\omega, m).$$

Proof. We may assume that $\xi_n^+(\omega) \subset I_m^+$. Since $x, y \in I_m^+$, $f_a(x), f_a(y) \in [-1, f_a(e^{-|m|+1})]$. Thus by using Lemma 4.2.2, we have

$$\frac{(f_a^j)'(f_a(x))}{(f_b^j)'(f_b(y))} \cdot \frac{D_j^+(a)}{D_j^+(b)} \cdot \frac{D_j^+(b)}{D_j(a)} \le B_1^2 \cdot \frac{D_j^+(a)}{D_j^+(b)}.$$

Now if $a = b$ then there is nothing to prove. So let us assume that $a \neq b$. Using chain rule, we have

$$\frac{D_j^+(a)}{D_j^+(b)} = \frac{\prod_{i=1}^j f_a'(\xi_i^+(a))}{\prod_{i=1}^j f_b'(\xi_i^+(b))},$$

which implies

$$\begin{aligned}\frac{D_j^+(a)}{D_j^+(b)} &= \prod_{i=1}^{j}\Big(1+\frac{f_a'(\xi_i^+(a))-f_b'(\xi_i^+(b))}{f_b'(\xi_i^+(b))}\Big)\\ &\leq \exp\Big(\sum_{i=1}^{j}\Big|\frac{f_a'(\xi_i^+(a))-f_b'(\xi_i^+(b))}{f_b'(\xi_i^+(b))}\Big|\Big). \qquad (4.2.11)\end{aligned}$$

Therefore to conclude the result we need to prove that

$$\sum_{i=1}^{j}\frac{|f_a'(\xi_i^+(a))-f_b'(\xi_i^+(b))|}{f_b'(\xi_i^+(b))}$$

is uniformly bounded. By using mean value theorem, property $(A3)$ and Lemma 4.2.4, we get

$$\begin{aligned}f_a'(\xi_i^+(a))-f_b'(\xi_i^+(b)) &\leq K_1|(\xi_i^+(a))|^{s-1} - K2|(\xi_i^+(b))|^{s-1}\\ &\leq K'\big||(\xi_i^+(a))|^{s-1} - |(\xi_i^+(b))|^{s-1}\big|, \quad \text{fore some large } K'\\ &\leq K'e^{-\beta i}. \qquad (4.2.12)\end{aligned}$$

Thus by using basic assumption and Lemma 4.2.4, we obtain

$$\begin{aligned}f_b'(\xi_i^+(b)) &\geq f_a'(\xi_i^+(a)) - K'e^{-\beta i}\\ &\geq K_1|\xi_i^+(a)|^{s-1} - K'e^{-\beta i}\\ &\geq K_1e^{-\alpha(s-1)i} - K'e^{-\beta i}\\ &\geq K_1e^{-\alpha(s-1)i}\big(1-\frac{K'}{K_1}e^{(\alpha(s-1)-\beta)i}\big)\\ &\geq K^*e^{-\alpha(s-1)i}, \qquad (4.2.13)\end{aligned}$$

where $K^* = K_1(1 - \frac{K'}{K_1} e^{\alpha(s-1)-\beta})$. Finally using iequalities (4.2.12) and (4.2.13), it follows that

$$\sum_{i=1}^{j} \frac{|f'_a(\xi_i^+(a)) - f'_b(\xi_i^+(b))|}{f'_b(\xi_i^+(b))} \le \frac{K'}{K^*} \sum_{i=1}^{\infty} e^{(\alpha(s-1)-\beta)i}$$

$$< \infty, \quad \text{since } \beta \ge s\alpha \, .$$

Hence the result follows. □

Finally we have the following lemma.

Lemma 4.2.6. *Let $\omega \subset R_{n-1}$ be a parameter interval and let either $\xi_n^+(\omega)$ or $\xi_n^-(\omega)$ is contained in I_m^+ with $\Delta \le |m| \le [\alpha n] - 1$. Set $p = p(\omega, m)$, then we have the following:*

(1) There exists a constant $B_1(\alpha,\beta)$ such that for every $k = 1, \ldots, p$:

$$(a) \quad \frac{1}{B_1} \le \frac{(f_a^k)'(y)}{D_k^+(a)} \le B_1 \quad if\ y \in [-1, f_a(e^{-|m|+1})],$$

$$(b) \quad \frac{1}{B_1} \le \frac{(f_a^k)'(y)}{D_k^-(a)} \le B_1 \quad if\ y \in [f_a(-e^{-|m|+1}), 1];$$

(2) $p < \frac{s+1}{\beta + \log\lambda}|m|$;

(3) Let $\kappa_2 = \beta \frac{s+3}{\beta+\log\lambda}$ and $x \in I_m^+$. Then for every $a \in \omega$ and $x \in I_m^+$ we have

$$(f_a^{p+1})'(x) \ge e^{(1-\kappa_2)|m|}.$$

Proof. We just need to prove (3). We may choose $a_* \in \omega$ such that $p(\omega,m) = p(a_*,m)$, then from Lemma 4.2.5, we have

$$\frac{(f_{a_*}^p)'(f_{a_*}(x))}{(f_a^p)'(f_a(x))} \le B_2.$$

Now from the above inequality, using property $(A3)$, we get

$$\begin{aligned}\frac{|(f_{a_*}^{p+1})'(x)|}{|(f_a^{p+1})'(x)|} &= \frac{f'_{a_*}(x)}{f'_a(x)}\frac{(f_{a_*}^p)'(f_{a_*}(x))}{(f_a^p)'(f_a(x))}\\ &\le \frac{K_1|x|^{s-1}}{K_2|x|^{s-1}}\frac{(f_{a_*}^p)'(f_{a_*}(x))}{(f_a^p)'(f_a(x))} \le \frac{K_1}{K_2}B_2.\end{aligned}$$

Using part (3) of Lemma 4.2.2 in the above inequality, we obtain

$$\begin{aligned}|(f_a^{p+1})'(x)| &\ge \frac{K_2}{K_1B_3}|(f_{a_*}^{p+1})'(x)|\\ &\ge \frac{K_2}{K_1B_3}exp\big((1-\beta\frac{s+2}{s+\log\lambda})|m|\big)\\ &\ge exp\big((1-\beta\frac{s+3}{s+\log\lambda})|m|\big),\end{aligned}$$

where the last inequality holds provided Δ is sufficiently large. □

4.3 Basic Construction

Now we define precisely the sets $(R_n)_{n\in\mathbb{N}}$ and for $a\in R_n$ the sequences $(\gamma_i)_{i\in\mathbb{N}}$ and $(p)_{i\in\mathbb{N}}$ as referred before. First we subdivide each I_m, $m\ge\Delta$ into m^2 intervals of equal length by introducing, for $1\le k\le m^2$, the following subintervals

$$I_{m,k} = [e^{-m} - k\frac{|I_m|}{m^2}, e^{-m} - (k-1)\frac{|I_m|}{m^2}),$$

and

$$I_{\Delta-1,k} = [e^{-\Delta}, e^{-\Delta} + k\frac{|I_{\Delta-1}|}{(\Delta-1)^2}),\ k\ge 1.$$

We extend the above definitions for $m\le-(\Delta-1)$ by setting $I_{m,k} = -I_{|m|,k}$. Therefore for $|m|\ge\Delta$ we have a partition of I_m into intervals of equal length, i.e., $I_m = I_{m,m^2}\cup\cdots\cup I_{m,1}$, and each $I_{m,k}$ has two adjacent intervals: $I_{m,k-1}$ and $I_{m,k+1}$ for $I_{m,k}$ with $1<k<m^2$, $I_{m-1,(m-1)^2}$ and $I_{m,2}$ for $I_{m,1}$, $I_{m+1,1}$ and I_{m,m^2-1} for I_{m,m^2}. We set $I^+_{m,k} = I_{m_1,k_1}\cup I_{m,k}\cup I_{m_2,k_2}$, where I_{m_1,k_1} and I_{m_2,k_2} are the adjacent intervals to $I_{m,k}$. Note that $I_{m,k}\subset I_m$, $I^+_{m,k}\subset I^+_m$ and $|I^+_{m,k}|\le\frac{3|I_m|}{m^2}$ if $k\ne 1$

and $|I^+_{m,k}| \le \frac{5|I_m|}{m^2}$ if $k = 1$, provided Δ is large enough. It is useful also to consider the sets $I^+_{\Delta-1,1} = (0,1]$ and $I^+_{1-\Delta,1} = [-1,0)$.

Related to the above splitting of U_Δ, we will define inductively partitioning $\mathcal{P}_n$ of the parameter intervals in order to have bounded distortion of $\xi_n^\pm$ and $D^\pm_{n-1}$ on $\omega \in \mathcal{P}_{n-1}$. Then we define

$$R_n = \bigcup\{\omega : \omega \in \mathcal{P}_n\}.$$

Now we start our induction by taking the parameter interval $[0, a_0]$ and the integer N_1 provided by Proposition 4.1.5. For $i = 1, \cdots, N_1 - 1$, we set $R_i = [0, a_0]$ and $\mathcal{P}_i = \{[0, a_0]\}$. Now assume by induction on $n \ge N_1$ that the following assertions are true for every $\omega \in \mathcal{P}_{n-1}$:

1. There is a sequence of parameter intervals $[0, a_0] = \omega_1 \supset \cdots \supset \omega_{n-1} = \omega$ such that $\omega_k \in \mathcal{P}_k$ for $k = 1, \cdots, n-1$.

2. There is a set $\mathcal{R}_{n-1}(\omega) = \{\gamma_0, \cdots, \gamma_\nu\}$, with $\gamma_0 = 1$, which is the set of the return times of ω up to $n-1$ and for $k < n-1$, $\mathcal{R}_k(\omega_k) = \mathcal{R}_k(\omega) \cap \{1, \cdots, k\}$. Note that when $\mathcal{R}_{n-1}(\omega) = \{1\}$, ω has no return.

3. For any return $\gamma_i \in \mathcal{R}_{n-1}(\omega)$, $i = 0, \ldots, \nu$, it is associated the intervals $I^+_{m_i,k_i}$ and $I^+_{-m_i,k_i}$, $|m_i| \ge \Delta$, such that $\xi^+_{\gamma_i}(\omega_{\gamma_i}) \subset I^+_{m_i,k_i}$ and $\xi^-_{\gamma_i}(\omega_{\gamma_i}) \subset I^+_{-m_i,k_i}$. We call $I^+_{m_i,k_i}$ and $I^+_{-m_i,k_i}$ as host intervals for ω. We put $p = p(\omega_{\gamma_i}, m_i)$, the pound periods associated to the returns γ_i and, for sake of notation, we set $p_0 = -1$. The periods $\{\gamma_i + p + 1, \ldots, \gamma_{i+1} - 1\}(i < \nu)$ and $\{\gamma_\nu + p_\nu + 1, \ldots, n-1\}$ (if $n > \gamma_\nu + p_\nu$) are called the free periods after the returns γ_i. During the free times $j = 1, \ldots, q_i$,

$$\xi^\pm_{\gamma_i+p+j}(\omega) \cap U_\Delta = \varnothing$$

and then by Lemma 4.1.1, the assertions (4.0.1) and (4.0.2) are satisfied for every $a \in \omega$.

4. For $k = 1, \cdots, n-1$, ω_k satisfies (BA_k) and (EG_k). Therefore for each return $\gamma_i \in \mathcal{R}_{n-1}(\omega)$, ω_{γ_i} satisfies (BA_{γ_i}) and (EG_{γ_i-1}) and then Lemma 4.2.6 guarantees that $p_i < \frac{s+1}{\beta+\log\lambda}|m_i|$, i.e., the bound period is finite. On the other hand, since $\omega \subset \omega_{\gamma_i}$, again using Lemma

4.2.6, for every $a \in \omega$ we have

$$(f_a^{p_i+1})'(\xi_{\gamma_i}^{\pm}(a)) \geq exp\big((1-\beta\frac{s+3}{s+\log\lambda})|m_i^+|\big) \geq 1,$$

which is assertion (4.0.3). Again by Lemma 4.2.6, for every $k = 1,\ldots,p_i$

$$(f_a^k)'(\xi_{\gamma_i+1}^{\pm}(a)) \geq \frac{1}{B_1}\cdot D_k^{\pm}(a).$$

Since a satisfies (EG_{γ_i-1}) and $k \leq p_i \leq \frac{s+1}{\beta+\log\lambda}\cdot|m_i^+| < \frac{s+1}{\beta+\log\lambda}\cdot\alpha\gamma_i < \gamma_i$, we have

$$(f_a^k)'(\xi_{\gamma_i+1}^{\pm}(a)) \geq \frac{1}{B_1}\lambda^k.$$

Therefore assertion (4.0.4) is also satisfied.

Notice that all the above properties are trivially verified for $n \leq N_1$ by taking $\mathcal{R}_{n-1}(\omega) = \{\gamma_0\}$, i.e., there is no return till $N_1 - 1$. Now we move towards the induction step. First we consider a supplementary partitioning $\mathcal{Q}_n$ containing portion of $\omega \in \mathcal{P}_{n-1}$ which satisfy (BA_n). Taking $\omega \in \mathcal{P}_{n-1}$, there can be following possible situations:

(a) If $\mathcal{R}_{n-1}(\omega) \neq \{1\}$ and $n \leq \gamma_{\nu-1} + p_{\nu-1}$, i.e., n belongs to the bound period associated to previous return then we put $\omega \in \mathcal{Q}_n$ and set $\mathcal{R}_n(\omega) = \mathcal{R}_{n-1}(\omega)$.

(b) If either $\mathcal{R}_{n-1}(\omega) = \{1\}$ or $n \leq \gamma_{\nu-1} + p_{\nu-1}$ and $\xi_n^{\pm}(\omega) \cap U_\Delta \subset I_{\Delta,1} \cup I_{-\Delta,1}$, we again put $\omega \in \mathcal{Q}_n$ and set $\mathcal{R}_n(\omega) = \mathcal{R}_{n-1}(\omega)$. We call n a free time for ω.

(c) If we are not in the above situations, then ω must have a returning situation at time n. In this case we can have two possibilities:

   (i) $\xi_n^+(\omega)$ do not cover completely some interval $I_{m,k}$. Clearly same holds for $\xi_n^-(\omega)$. Since $n \geq N_1$ we have that ω satisfies conditions (i) and (ii) of Proposition 4.1.2, so as mentioned before, $\xi_n^{\pm}|_\omega$ is an isomorphism. Also as ω is an interval by the assumption of induction, therefore $\xi_n^{\pm}(\omega)$ are intervals and must contain in some $I_{m,k}^+$ and $I_{-m,k}^+$. We put $\omega \in \mathcal{Q}_n$ and set $\mathcal{R}_n(\omega) = \mathcal{R}_{n-1}(\omega) \cup \{n\}$. We call n as an inessential return time for ω and refer $I_{m,k}^+$ and $I_{-m,k}^+$ as host intervals of the return.

(ii) $\xi_n^+(\omega)$ contains at least one interval $I_{m,k}$ with $|m| \geq \Delta$. Then $\xi_n^-(\omega)$ covers $I_{-m,k}$. In this case we say that ω has an essential return situation at time n and consider the following sets

$$\begin{aligned}\omega'_{m,k} &= (\xi_n^+)^{-1}(I_{m,k}) \bigcap \omega = (\xi_n^-)^{-1}(I_{-m,k}) \bigcap \omega,\\ \omega^1 &= (\xi_n^+)^{-1}\big([0,1] \setminus U_\Delta\big) \bigcap \omega = (\xi_n^-)^{-1}\big([-1,0] \setminus U_\Delta\big) \bigcap \omega,\\ \omega^2 &= (\xi_n^+)^{-1}\big([-1,0] \setminus U_\Delta\big) \bigcap \omega = (\xi_n^-)^{-1}\big([0,1] \setminus U_\Delta\big) \bigcap \omega.\end{aligned}$$

Let $\mathcal{A}$ be the set of indices (m,k) such that $\omega_{m,k}$ is non-empty, we have

$$\omega \setminus (\xi_n^+)^{-1}(0) = \omega \setminus (\xi_n^-)^{-1}(0) = \bigcup_{(m,k)\in\mathcal{A}} \omega'_{m,k} \cup \omega^1 \cup \omega^2.$$

Again since, $\xi_n^\pm|_\omega$ is an isomorphism, so $\omega'_{m,k}$ is an interval. Moreover $\xi_n^+(\omega'_{m,k})$ and $\xi_n^-(\omega'_{-m,k})$covers the whole $I_{m,k}$ and $I_{-m,k}$, respectively, except for two extreme end intervals. We join $\omega'_{m,k}$ to its adjacent interval when $\xi_n^+(\omega'_{m,k})$ do not cover $I_{m,k}$ completely and get a new decomposition of $\omega \setminus (\xi_n^+)^{-1}(0)$ into intervals $\omega_{m,k}$ such that $I_{m,k} \subset \xi_n^+(\omega_{m,k}) \subset I^+_{m,k}$ and $I_{-m,k} \subset \xi_n^-(\omega_{m,k}) \subset I^+_{-m,k}$. Now we put $\omega_{m,k} \in \mathcal{Q}_n$ if and only if $m \leq [\alpha n] - 1$ and set $I^+_{m,k}$ and $I^+_{-m,k}$ as its host intervals. Note that the portion of ω excluded is an interval whose image under $\xi_n^\pm$ contained in $U_{[\alpha n]-1}$. If $m \geq \Delta$ we set $\mathcal{R}_n(\omega_{m,k}) = \mathcal{R}_{n-1}(\omega) \cup \{n\}$ and call n as an essential return for $\omega_{m,k}$. If $m = \Delta - 1$ then we set $\mathcal{R}_n(\omega_{m,k}) = \mathcal{R}_{n-1}(\omega)$, then $\omega_{m,k}$ is called an escape component and n an escaping situation for $a \in \omega_{m,k}$.

Now we can easily check that any descendant of an $\omega \in \mathcal{P}_{n-1}$ that belongs to $\mathcal{Q}_n$ satisfies (BA_n):

(a) If n is a bound time, i.e., $n = \gamma + j$, $1 \leq j \leq p \leq n-1$, where γ and p are returns and bound periods for ω. Then by using binding condition, for all $a \in \omega$, we obtain

$$|\xi^\pm_{\gamma+j}(a)| \geq |\xi^\pm_j(a)| - e^{-\beta j}.$$

Since every $a \in \omega$ satisfies (BA_j) by the induction hypothesis, therefore from Lemma 4.2.6 and above inequality, we have

$$\begin{aligned} |\xi_n^{\pm}(a)| = |\xi_{\gamma+j}^{\pm}(a)| &\geq e^{-\alpha j} - e^{-\beta j} \\ &= (1 - e^{(\alpha-\beta)j})e^{-\alpha j} \\ &\geq e^{-\alpha n}, \text{ for large } N_1. \end{aligned}$$

(b) If n is a free time then $|\xi_n^{\pm}(a)| \geq e^{-(\Delta+1)}$ for every $a \in \omega$ and therefore $|\xi_n^{\pm}(a)| \geq e^{-\alpha n}$, providing N_1 sufficiently large.

(c) If n is a returning situation for ω.

(i) n is an inessential return for ω, i.e., $\xi_n^{+}(\omega)$ and $\xi_n^{-}(\omega)$ do not cover some interval $I_{m,k}$. If ω does not satisfy (BA_n) then there exists a $y \neq 0$ which contained one of $\xi_n^{\pm}(\omega)$ with $|y| < e^{\alpha n}$. Let us assume that $y \in \xi_n^{+}(\omega)$, then the host interval $I_{m,k}^{+}$ of ω at time n must having $|m| \geq [\alpha n] - 1$. Thus $|\xi_n^{+}(\omega)| \leq |I_{m,k}^{+}| \leq \frac{5|I_m|}{(m)^2} < e^{-\alpha n}$, which is not possible since we will prove later in this section (Lemma 4.3.3) that

$$|\xi_n^{+}(\omega)| \geq e^{-\alpha n}.$$

(ii) n is an essential returning situation. Since $\xi_n^{+}(\omega_{m,k}) \subset I_{m,k}^{+}$ and $\xi_n^{-}(\omega_{m,k}) \subset I_{-m,k}^{+}$ for every descendant $\omega_{m,k}$ of ω with $|m| \leq [\alpha n] - 1$, which means that $\xi_n^{\pm}(\omega_{m,k}) \cap U_{[\alpha n]} = \varnothing$, i.e., $|\xi_n^{\pm}(\omega_{m,k})| \geq e^{-\alpha n}$.

Now we set

$$\mathcal{P}_n = \{\omega \in \mathcal{Q}_n : H_n(a) \geq (1-\alpha)n) \text{ for every } a \in \omega\ \}$$

and

$$R_n = \bigcup \{\omega : \omega \in \mathcal{P}_n\}.$$

Then R_n satisfies (BA_n), (EG_{n-1}) and (H_n) and thus satisfies (EG_n) as explained earlier in this chapter before section 4.1.

Each $a \in R_n$ belongs to only one $\omega_k \in \mathcal{P}_k$, for every $k = 1, \ldots, n$. We construct these intervals as follows. We set $[0, a_0] = \omega_1 = \ldots = \omega_{N_1 - 1}$ and by Proposition 4.1.5, $\xi^+_{N_1}(\omega) \supset U_\Delta$, so $\gamma_1(a) = N_1$ is an essential returning situation for ω_{N_1-1}. Therefore we subdivide $[0, a_0]$ into intervals $\mathcal{J}_{(m,k)}$ with $\mathcal{J}_{(m,k)} \in \mathcal{P}_{N_1}$ for $\Delta - 1 \le |m| \le [\alpha N_1] - 1$. Since $a \in \mathcal{P}_n$, there is some (m_1, k_1) such that $\omega_{\gamma_1(a)} = \mathcal{J}_{(m_1,k_1)}$. Put $\omega_k = \mathcal{J}_{(m_1,k_1)}$ for $k = \gamma_1(a) + 1, \ldots, \gamma_2(a) - 1$, where $\gamma_2(a)$ is the next essential returning situation for $\mathcal{J}_{(m_1,k_1)}$. Now we split $\mathcal{J}_{(m_1,k_1)}$ and get a new component $\mathcal{J}_{(m_1,k_1),(m_2,k_2)}$ of $\mathcal{P}_{\gamma_2(a)}$ and we set $\omega_{\gamma_2(a)} = \mathcal{J}_{(m_1,k_1),(m_2,k_2)}$. By continuing in the same way we obtain sequences $\gamma_1, \ldots, \gamma_\nu$ and $(m_1, k_1), \ldots, (m_\nu, k_\nu)$ $(\nu = \nu(a, n))$ such that

$$\begin{aligned} &\omega_{\gamma_i} = \mathcal{J}_{(m_1,k_1),\ldots,(m_i,k_i)}, \\ &\omega_{\gamma_i} \subset \omega_{\gamma_i - 1}, \\ &\omega_k = \omega_{\gamma_i} \text{ for } k = \gamma_i, \ldots, \gamma_{i+1} - 1, \end{aligned}$$

with

$$I_{m_i,k_i} \subset \xi^+_{\gamma_i}(\omega_{\gamma_i}) \subset I^+_{m_i,k_i}$$

and

$$I_{-m_i,k_i} \subset \xi^-_{\gamma_i}(\omega_{\gamma_i}) \subset I^+_{-m_i,k_i}.$$

Moreover since $\xi^\pm_{\gamma_i}|_{\omega_{\gamma_i - 1}}$ are homeomorphisms, every $\omega \in \mathcal{P}_n$ is equal to some $\mathcal{J}_{(m_1,k_1),\ldots,(m_i,k_i)}$ for some unique sequence $(m_1, k_1), \ldots, (m_i, k_i)$ with $|m_i| \ge \Delta - 1$.

The next lemmas of this sections are proved for the critical value -1 and one can prove in the case of critical value 1 in the same way. The following lemma reveals that the escape components return very big as compared with U_Δ.

Lemma 4.3.1. *If $\omega \in \mathcal{P}_\theta$ is an escape component, then in the next returning situation γ for ω we have*

$$|\xi_\gamma^+(\omega)| \geq e^{-\kappa_3 \Delta},$$

where $\kappa_3 = \beta \dfrac{s+5}{\beta + \log \lambda}$.

Proof. If $\xi_\gamma^+(\omega)$ is not completely contained in U_1, then the result follows immediately. Thus we may assume that $\xi_\gamma^+(\omega) \subseteq U_1$. Since ω is an escape component with escaping time θ, so $I_{m,1} \subseteq \xi_\theta^+(\omega)$ with $|m| = \Delta - 1$. Without loss of generality assume that $m > 0$. Let p be the bound period after the return θ and $q = \gamma - \theta - p - 1$ be the free period before the return γ. Since γ is the return after θ, therefore it is not in the binding period of the return θ, i.e., $\gamma - \theta > p$. Now we may have two possible situations:

First if $\xi_\theta^+(\omega) \subseteq I_m$. Since ω is an interval so let us assume $\omega = (a,b)$. Therefore by using Lemma 4.1.1, Lemma 4.2.6 and tne mean value theorem, we obtain

$$\begin{aligned}
|\xi_\gamma^+(\omega)| &= |(f_a^{\gamma-1}(-1), f_b^{\gamma-1}(-1))| = |(f_a^{\gamma-\theta}(f_a^{\theta-1}(-1)), f_b^{\gamma-\theta}(f_b^{\theta-1}(-1)))| \\
&\geq |(f_a^{\gamma-\theta}(f_a^{\theta-1}(-1)), f_a^{\gamma-\theta}(f_b^{\theta-1}(-1)))| \\
&= |f_a^{\gamma-\theta}(f_a^{\theta-1}(-1), f_b^{\theta-1}(-1))| \\
&= (f_a^{\gamma-\theta})'(f_c^{\theta-1}(-1))|f_a^{\theta-1}(-1) - f_b^{\theta-1}(-1)|, \text{ for some } c \in \omega. \\
&= (f_a^q)'(f_a^{p+1}(f_c^{\theta-1}(-1)))(f_a^{p+1})'(f_c^{\theta-1}(-1))|\xi_\theta^+(\omega)| \\
&\geq \frac{1}{e}\lambda^q e^{(1-\beta\frac{s+3}{\beta+\log\lambda})\Delta}|\xi_\theta^+(\omega)|, \text{ since } f_c^{\theta-1}(-1) \subset \xi_\theta^+(\omega) \subset I_m \subset I_m^+. \\
&\geq \frac{1}{e\Delta^2}\lambda^q e^{\frac{2\beta}{\beta+\log\lambda}\Delta} e^{(1-\beta\frac{s+5}{\beta+\log\lambda})\Delta} e^{-\Delta} \\
&\geq e^{-\beta\frac{s+5}{\beta+\log\lambda}\Delta}, \text{ for } \Delta \text{ large enough,}
\end{aligned}$$

where second last inequality holds since θ is an escape time time for ω, thus $|\xi_\theta^+(\omega)| \geq \frac{e^{-(\Delta-1)} - e^{-\Delta}}{(\Delta-1)^2} > \frac{e^{-\Delta}}{\Delta^2}$.

Secondly if $\xi_\theta^+(\omega) \supseteq I_m$, we have

$$\begin{aligned}|\xi_\gamma^+(\omega)| &= |(f_a^{\gamma-1}(-1), f_b^{\gamma-1}(-1))| = |(f_a^{\gamma-\theta}(f_a^{\theta-1}(-1)), f_b^{\gamma-\theta}(f_b^{\theta-1}(-1)))| \\ &\geq |(f_a^{\gamma-\theta}(f_a^{\theta-1}(-1)), f_a^{\gamma-\theta}(f_b^{\theta-1}(-1)))| \\ &= |f_a^{\gamma-\theta}(f_a^{\theta-1}(-1), f_b^{\theta-1}(-1))| \\ &\geq |f_a^{\gamma-\theta}(I_m)| = (f_a^{\gamma-\theta})'(x)|I_m| \text{ for some } x \in I_m.\end{aligned}$$

Hence the result follows from the above inequality in similar way as of the previous case. □

In the following lemma we obtain the estimates on the length of ξ_k^+ at a return k.

Lemma 4.3.2. *Let γ_1 be a return for $\omega \in \mathcal{P}_{n-1}$ with host interval $I_{m,k}$. Let $p = p(\omega, m)$ be the bound period for the return γ_1, then for sufficiently large Δ, we have the following*

(a) If $\gamma_2 \leq n$ is the next return after γ_1, then by setting $q = \gamma_2 - \gamma_1 - p - 1$, we have

(i) $|\xi_{\gamma_2}^+(\omega)| \geq \lambda^q e^{(1-\kappa_3)|m|} |\xi_{\gamma_1}^+(\omega)| \geq 2|\xi_{\gamma_1}^+(\omega)|$,

(ii) $|\xi_{\gamma_2}^+(\omega)| \geq \lambda^q e^{-\kappa_3|m|}$, if γ_1 is an essential return;

(b) If n is a free time and γ_2 is the last return up to n, then putting $q = n - p$, we have

(i) $|\xi_{\gamma_2}^+(\omega)| \geq c\lambda^q e^{(1-\kappa_3)|m|} |\xi_{\gamma_1}^+(\omega)| \geq 2|\xi_{\gamma_1}^+(\omega)|$,

(ii) $|\xi_{\gamma_2}^+(\omega)| \geq c\lambda^q e^{-\kappa_3|m|}$, if γ_1 is an essential return.

Proof. By writing

$$\frac{|\xi_{\gamma_2}^+(\omega)|}{|\xi_{\gamma_1}^+(\omega)|} = \frac{|\xi_{\gamma_2}^+(\omega)|}{|\xi_{\gamma_1+p+1}^+(\omega)|} \cdot \frac{|\xi_{\gamma_1+p+1}^+(\omega)|}{|\xi_{\gamma_1}^+(\omega)|},$$

it follows from Lemma 4.1.4 that for some $a, b \in \omega$,

$$\frac{|\xi_{\gamma_2}^+(\omega)|}{|\xi_{\gamma_1}^+(\omega)|} \geq \frac{1}{A^4} \cdot |(f_a^q)'(\xi_{\gamma_1+p+1}^+(a))| \cdot |(f_b^{p+1})'(\xi_{\gamma_1}^+(b))|.$$

Now using Lemma 4.2.6, from the above inequality, we get

$$\frac{|\xi_{\gamma_2}^+(\omega)|}{|\xi_{\gamma_1}^+(\omega)|} \geq \frac{1}{A^4} \cdot |(f_a^q)'(\xi_{\gamma_1+p+1}^+(a))| \cdot e^{(1-\beta\frac{s+3}{\beta+\log\lambda})|m|}. \tag{4.3.1}$$

(a) Since $f_a^q(\xi^+_{\gamma_1+p+1}(a)) = \xi^+_{\gamma_2}(a) \in U_\Delta$, therefore from the inequality (4.3.1), using Lemma 4.1.1, we obtain

$$\frac{|\xi^+_{\gamma_2}(\omega)|}{|\xi^+_{\gamma_1}(\omega)|} \geq \frac{1}{A^4} \cdot \lambda^q \cdot e^{(1-\beta\frac{s+3}{\beta+\log\lambda})|m|}. \tag{4.3.2}$$

Hence (i) simply follows from the inequality (4.3.2) for Δ sufficiently large. In fact (ii) also follows from (4.3.2) by taking into account that γ_1 is an essential return, i.e., $\xi^+_{\gamma_1}(\omega) \supset I_{m,k}$, thus $\xi^+_{\gamma_1}(\omega) \geq \frac{e^{-|m|}}{m^2}$.

(b) The proof is analogous to part (a) and the constant c appears since in this situation we can just use part (1) of Lemma 4.1.1, which assures that $|(f_s^q)'(\xi^+_{\gamma_1+p+1}(s))| \geq c\lambda^q$.

□

Next lemma guarantees that if n is a returning situation for ω then the length of $\xi^+_n(\omega)$ is large as compared with $|U_{[\alpha n]}|$.

Lemma 4.3.3. *If n is a returning situation for $\omega \in \mathcal{P}_{n-1}$, then*

$$|\xi^+_n(\omega)| \geq e^{-\alpha n}.$$

Proof. Since n is a returning situation for ω, so it is not in bound period of the previous return. Let $\gamma_0 \leq n-1$ be the smallest integer such that $\omega_{\gamma_0} = \omega$, i.e., γ_0 is either and escape situation or an essential return for ω.

Now if γ_0 is an escape time, then the result immediately follows by Lemma 4.3.1, provided N_1 is sufficiently large so that $e^{-\beta\frac{s+5}{\beta+\log\lambda}\Delta} \geq e^{-\alpha n}$.

And if γ_0 is a an essential return for ω. Let $I_{m,k} \subset \xi^+_{\gamma_0}(\omega) \subset I^+_{m,k}$ with $\Delta \leq |m| \leq [\alpha n] - 1$. We set $n = \gamma_\nu$ and $\{\gamma_i\}_{i=1}^{\nu}$ as the returns after γ_0. Then there can be two cases:

(i) If $\nu = 1$, i.e., n is the return next to γ_0, then using Lemma 4.3.2, we have

$$|\xi^+_n(\omega)| \geq e^{-\beta\frac{s+5}{\beta+\log\lambda}|m|}.$$

(ii) If $\nu > 1$, then we may write

$$|\xi_n^+(\omega)| = |\xi_{\gamma_1}^+(\omega)| \cdot \prod_{i=2}^{\nu} \frac{|\xi_{\gamma_i}^+(\omega)|}{|\xi_{\gamma_{i-1}}^+(\omega)|}. \tag{4.3.3}$$

From Lemma 4.3.2 we know that $\frac{|\xi_{\gamma_i}^+(\omega)|}{|\xi_{\gamma_{i-1}}^+(\omega)|} \geq 2$, for $i = 2, \ldots, \nu$, and $|\xi_{\gamma_1}^+(\omega)| \geq e^{-\beta \frac{s+5}{\beta+\log\lambda}|m|}$. Then from (4.3.3), we obtain

$$|\xi_n^+(\omega)| \geq e^{-\beta \frac{s+5}{\beta+\log\lambda}|m|} \cdot \prod_{i=1}^{\nu} 1.$$

Therefore form both the above cases, we get

$$\begin{aligned} |\xi_n^+(\omega)| &\geq e^{-\beta \frac{s+5}{\beta+\log\lambda}|m|} \\ &\geq e^{-\beta \frac{s+5}{\beta+\log\lambda}\alpha n}, \quad \text{since } |m| \leq [\alpha n] - 1 \\ &\geq e^{-\alpha n}, \end{aligned}$$

where the last inequality holds since $\beta \frac{s+5}{\beta+\log\lambda} < 1$. Hence the result follows. □

Chapter 5

Main Results

In this chapter we are going to present our main results about the statistical instability of a class of maps in the contracting Lorenz-like family $\{f_a\}_{a\geq 0}$. In this regard, we will prove that the Rovella maps are not statistically stable if we consider a set consists of Rovella parameters and some other parameters which we will call as super-stable parameters. We remind that the Rovella maps lie in the family $\{f_a\}_{a\geq 0}$ and admit unique *SRB* measures (physical measure), as explained in chapter 3. It was proved by Alves and Soufi [3] that the map

$$R \ni a \mapsto g_a$$

is continuous in the L^1-norm at every point in the set of Rovella parameters R, where g_a is the density of physical measure μ_a for the map f_a. Thus Rovella maps are strongly statistically stable if we confine ourself on the set R. This chapter is organized as follows.

In section 5.1 we will present a result, given as Lemma 5.1.2, which guarantees the existence of critically-stable maps in the family $\{f_a\}_{a\geq 0}$ and consequently those maps admit a physical measure which is supported on the super-attractor. Then it is a question of great interest to study the statistical stability of Rovella maps on a larger class of maps, in the contracting Lorenz-like family, consists of Rovella maps and critically-stable maps.

In section 5.2 we will introduce the notion of critical measure and prove that if a critical measure for a Rovella map exists then it is accumulated by physical measures of the critically-stable maps (see Theorem). Finally in section 5.3 we will focus ourselves in answering weather the Rovella maps are statistically stable on an extended class of maps mentioned in the previous paragraph. And at the end we will conclude that the Rovella maps are not statistically stable on that extended class of maps in the contracting Lorenz-like family which is given by Theorem 5.3.2.

5.1 The Extended Set of Parameters

Recall from chapter 4, we constructed inductively the set of Rovella parameters R for the family $\{f_a\}_{a\geq 0}$ such that the critical orbits of each map corresponding to set R have slow recurrence to the critical point and the derivatives grow exponentially along critical orbits. We started the inductive step with the interval $[0,a_0]$ provided by Proposition 4.1.5. There exist $\lambda_0 > 1$ and a natural number N_1 such that for every $1 \leq j \leq N_1 - 1$, $\xi_j^{\pm}([0,a_0]) \cap U_\Delta = \phi$ and $\xi_{N_1}^{\pm}([0,a_0]) \supset U_\Delta$, and $(f_a^j)'(\pm 1) \geq \lambda_0^j$, where $U_\Delta = (-e^{-\Delta}, e^{-\Delta})$ for a sufficiently large integer Δ. By setting $\mathcal{P}_1 = \mathcal{P}_2 = \ldots = \mathcal{P}_{N_1-1} = \{[0,a_0]\}$, we made the inductive step by assuming that $\mathcal{P}_{n-1}$ consists of parameter intervals such that each parameter a lies in some interval in $\mathcal{P}_{n-1}$ satisfies basic assumption (BA_{n-1}):

$$|\xi_j^{\pm}(a)| \geq e^{-\alpha j} \quad \text{for } j = 1, \ldots, n-1,$$

where $\alpha > 0$ is sufficiently small and $\xi_j^{\pm}(a) = f_a^{j-1}(\mp 1)$, and the exponential growth property (EG_{n-1}):

$$(f_a^j)'(\pm 1) \geq \lambda^j \quad \text{for } j = 1, \ldots, n-1,$$

where $1 < \lambda \leq \lambda_0$.

For every parameter interval ω we associated free periods, returns and bound periods following the returns. The returns correspond to times when ω visits a small neighbourhood of 0 which we denote as $(-\delta, \delta)$, where $\delta = e^{-\Delta}$, i.e., γ is said to be a return for ω if $\xi_\gamma^{\pm}(\omega) \cap (-\delta, \delta) \neq \varnothing$. The bound period after the return γ is the set of consecutive integers

$\{j : \gamma + 1 \leq j \leq p\}$ such that for every $a \in \omega$ and for some $\beta > 0$

$$|\xi^{\pm}_{\gamma+j}(a) - \xi^{-}_{j}(a)| \leq e^{-\beta j}, \quad \text{if } \xi^{\pm}_{\gamma}(\omega) \cap (-\delta, \delta) \subset [-1, 0),$$

and

$$|\xi^{\pm}_{\gamma+j}(a) - \xi^{+}_{j}(a)| \leq e^{-\beta j}, \quad \text{if } \xi^{\pm}_{\gamma}(\omega) \cap (-\delta, \delta) \subset (0, 1],$$

for $j = 1, \ldots, p$. The bound period after any return is finite, by Lemma (4.2.6). And a free period represented by the time starting after the bound period, i.e., from $\gamma + p + 1$, and ends till the next return.

Then the partitioning $\mathcal{P}_n$ is obtained as follows: for a parameter interval $\omega_{n-1} \in \mathcal{P}_{n-1}$, if n is in a free period or in a bound period after a return then we do not make any change in ω_{n-1} and keep as it is in $\mathcal{P}_n$. But if n is a return for ω_{n-1} then we decide weather ω_{n-1} should break up further into smaller intervals and needs some parameter exclusions. There are two type of returns.

(i) If $\xi^{\pm}_{n}(\omega_{n-1})$ do not cover some interval of the form $I_{m,k}$ with $|m| \geq \Delta$ then again we pass ω_{n-1} to $\mathcal{P}_n$ and call n as inessential return time for ω_{n-1}.

(ii) If $I_{m,k} \subset \xi^{\pm}_{n}(\omega_{n-1})$ for some $|m| \geq \Delta - 1$, then if necessary, first we exclude the parameters from ω_{n-1} which do not satisfy (BA_n) and the excluded part is also an interval. Then we make the partitioning of remaining parts of ω_{n-1} into subintervals ω^{m}_{n} and ω^{es}_{n} such that $I_{m,k} \subset \xi^{\pm}_{n}(\omega^{m}_{n}) \subset I^{+}_{m,k}$, for $|m| \geq \Delta$, and $I_{m,1} \subset \xi^{\pm}_{n}(\omega^{es}_{n}) \subset I^{+}_{m,1}$, for $|m| = \Delta - 1$. In this case we call n as an essential return for ω^{m}_{n} and an escape situation for every parameter $a \in \omega^{es}_{n}$, where ω^{es}_{n} is said to be an escape component for a. Then we keep the intervals ω^{m}_{n} such that each $a \in \omega^{m}_{n}$ satisfies (H_n), i.e.,

$$H_n(a) \geq (1 - \alpha)n,$$

where $H_n(a)$ denotes the sum of free periods up to time n for the parameter a.

Hence we obtain a partitioning $\mathcal{P}_n$ of parameter intervals such that each parameter a, inside $\mathcal{P}_n$, satisfies (BA_n) and (EG_n). We set

$$R_n = \bigcup\{\omega : \omega \in \mathcal{P}_n\},$$

and finally we get

$$R = \bigcap_{n=1}^{+\infty} R_n.$$

Note that the implication of (H_n) assures that any parameter interval in $\mathcal{P}_n$ spends most of the time in the free periods, up to time n. Then as a consequence of (H_n) and Lemma 4.1.1, for every parameter $a \in R$ we obtain an infinite sequence $\{\theta_k\}_{k\geq 1}$ of escape times and the corresponding sequence $\{\omega_k(a)\}_{k\geq 1}$ of escape components.

5.1.1 Hyperbolic Periodic Repellers

We recall from the properties of the family of maps $\{f_a\}_{a\geq 0}$ given in section 3.1, that each map f_a is differentiable at every point in $I \setminus \{0\}$ with $f_a''(x) < 0$ for $x \in [-1,0)$ and $f_a''(x) > 0$ for $x \in (0,1]$, ± 1 are critical values for $f_a$with $f_a(-1)$ close to -1 and $f_a(1)$ close to 1, therefore the graph of f_a holds two connected components $\big[f_a(-1),1\big)$ and $\big(-1, f_a(1)\big]$. This further suggests that the graph of the map f_a^2 consists of four connected components $\big[f_a^2(-1),1\big)$, $\big(-1, f_a(1)\big)$, $\big(f_a(-1),1\big)$ and $\big(-1, f_a^2(1)\big]$, which are respectively the images of the points lie in the intervals $\big[-1, O^-(a)\big)$, $\big(O^-(a),0\big)$, $\big(0, O^+(a)\big)$ and $\big(O^+(a),1\big]$ under the map f_a^2. Also f_a^2 has three discontinuities at $O^-(a)$, 0 and $O^+(a)$, where the points $O^-(a)$ and $O^+(a)$, introduced in section 4.1, are zeros of the map f_a located on the left and the right side of 0, respectively.

Thus the graph of f_a^2 intersects the identity map in two disjoint intervals $\big((O^-(a),0\big)$ and $\big(0, O^+(a)\big)$ such that the bottom of the graph of f_a^2 in the intervals $\big(O^-(a),0\big)$ and $\big(0, O^+(a)\big)$ is near -1 with its ceiling close to 1 which assures that the derivative of f_a^2 at that points of intersection with identity map is greater than 1. Thus the map f_a has a repelling periodic orbit of period 2. Moreover, since the map f_a has negative Schwarzian derivative, therefore that repelling periodic orbit is hyperbolic from Guckenheimer's theorem [17] which states that every compact invariant set for f_a which does not contain critical point and all of its periodic points are hyperbolic repelling is a hyperbolic set.

The arguments given above also advocate that the map f_a has more hyperbolic repelling periodic orbits of period p for $p > 2$.

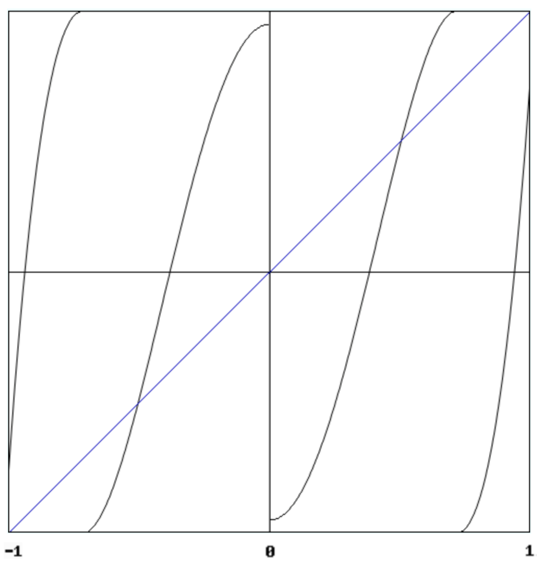

Fig. 5.1 Graph of f_a^2 in black and graph of identity map in blue

5.1.2 Critically-stable and Post-critically Finite Maps near Rovella Maps

In this section we present a result ensures the existence of parameters, outside the set R, admitting physical measures. First we precisely define some relevant terms.

Definition 5.1.1. A map f_a in the contracting Lorenz-like family is called

1. *critically-stable* if there is some $k \geq 1$ such that $\xi_k^+(a) = 0$ or $\xi_k^-(a) = 0$, and in such case we define $f_a(0) = -1$ or $f_a(0) = 1$, respectively; if both situations occur, we consider for definiteness $f_a(0) = -1$;

2. *post-critically finite* if there is some $k \geq 1$ such that $f_a^k(1)$ or $f_a^k(-1)$ has a repelling periodic orbit.

In case (1) we say that the orbit of 0 is a *super-attractor*. By extension, we call the parameters associated to critically-stable and post-critically finite maps as super-stable and post-critically finite parameters, respectively.

We also remind that there are constants $K_1, K_2 > 0$ and $s > 1$ such that for any parameter a associated with the contracting Lorenz-like family and for every $x \in I \setminus \{0\}$, we have

$$K_2|x|^{s-1} \leq f_a'(x) \leq K_1|x|^{s-1}. \tag{5.1.1}$$

The above property is given as $(A3)$ in section 3.1. Furthermore, observe that from the properties (A0)-(A3) it follows that, for a_0 sufficiently close to 0, $f_a'(x) \gg 1$ for each $a \in [0, a_0]$ and for every $x \in [-1, O^-(a)]$.

For the sake of notations we will denote by $[(a,b)]$ the open interval between a and b, not necessarily in order, and by $\ell(a,b)$ the length of interval (a,b). We also denote by $\Lambda_a \subset I$ the hyperbolic set consists of a repelling periodic orbit of period p for the map f_a, and we fix some point $y^-(a) \in \Lambda_a$ contained in the interval $\big(O^-(a), 0\big)$. Notice that as any map f_a in the contracting Lorenz-like family is smooth in the intervals $[-1,0)$ and $(0,1]$, thus we may find a neighbourhood $\mathcal{N}$ of the set Λ_a such that f_a is smooth in $\mathcal{N}$. Therefore the arguments of De Melo and Van Strien [13] can be adopted to show that the set Λ_a varies continuously with the parameter a.

The proof of next lemma is based on the idea that whenever a parameter faces the escape situation, the escape component containing that parameter returns big enough to a small neighbourhood of origin so that with a finite number of further iterations it crosses 0.

Lemma 5.1.2. *Consider a Rovella parameter $a \in R$ and let Λ_a be the hyperbolic set for f_a and $y(a) \in \Lambda_a$. Let θ_k be a large escape time for the parameter a with escaping component $\omega_{\theta_k}(a)$ and let τ_k be the next returning situation for $\omega_{\theta_k}(a)$. Then there are two parameters $a_s, a_p \in \omega_{\theta_k}(a)$ and two non-negative integers ρ_s and σ_p, with $\rho_s, \sigma_p < M$ for some large number M, such that*

(a) f_{a_s} has a super-attractor of period $\tau_k + \rho_s$;

(b) $\xi_i^+(a_p) \neq y(a_p)$ for $i < \sigma_k$ and $\xi_{\sigma_k}^+(a_p) = y(a_p)$ for $\sigma_k \leq \tau_k + \sigma_p$.

Proof. Since τ_k is a returning time for $\omega_{\theta_k}(a)$, thus $\xi^+_{\tau_k}(\omega_{\theta_k}(a)) \cap (-\delta,\delta) \neq \emptyset$, where $\delta = e^{-\Delta}$. Also from Lemma 4.3.1, we have

$$|\xi^+_{\tau_k}(\omega_{\theta_k}(a))| \geq e^{-\frac{\beta}{\beta+\log\lambda}\Delta\cdot(s+5)}.$$

Now the idea of the proof works as follows:

(a) If $0 \in \xi^+_{\tau_k}(\omega_{\theta_k}(a))$, then we conclude part (a) by taking $\rho_s = 0$. But if $0 \notin \xi^+_{\tau_k}(\omega_{\theta_k}(a))$, then we may take an interval $(b,d) \subset \omega_{\theta_k}(a)$ such that $|\xi^+_{\tau_k}(d) - \xi^+_{\tau_k}(b)| = \delta^{(s+5)} - \delta^{2(s+5)}$. Without loss of generality we can assume that the interval $\xi^+_{\tau_k}(\omega_{\theta_k}(a))$ takes place on the right side of the origin and $\xi^+_{\tau_k}(d) = \delta^{(s+5)}$ and $\xi^+_{\tau_k}(b) = \delta^{2(s+5)}$. Let $x^+ > 0$ be such that $x^+ \simeq 0$, by mean value theorem, for some $x_1 \in (x^+, \delta^{2(s+5)})$ and $x_2 \in (x^+, \delta^{(s+5)})$, we have

$$f_b(\delta^{2(s+5)}) - f_b(0^+) \simeq f_b'(x_1)\delta^{2(s+5)} \tag{5.1.2}$$

and

$$f_d(\delta^{(s+5)}) - f_d(0^+) \simeq f_d'(x_2)\delta^{(s+5)}, \tag{5.1.3}$$

where $f_b'(x_1)$ and $f_d'(x_2)$ are the slopes of chords joining the points $(x^+, f_b(x^+))$ and $(\delta^{2(s+5)}, f_b(\delta^{2(s+5)}))$, and the points $(x^+, f_d(x^+))$ and $(\delta^{(s+5)}, f_d(\delta^{(s+5)}))$, respectively. Since $b \sim d$ and the derivative of each map f_a, $a \in [0, a_0]$, is increasing in the interval (0,1], therefore $f_d'(x_2) \geq f_b'(x_1)$. Also the inequality (5.1.1) assures that $f_d'(x_2)$ is bounded away from 0. Then by taking into account that $f_b(0^+) = f_d(0^+) = -1$, one may assume, by using (5.1.2) and (5.1.3), that the distance of $\xi^+_{\tau_k+1+i}(b)$ from -1 will increase with a rate slower than $\left(f_b'(f_b(-1))\right)^i \cdot \delta^{2(s+5)}$ as for as the distance of $\xi^+_{\tau_k+1+i}(d)$ from -1 will be increasing with a rate faster than $\left(f_d'(f_d(O^-(d)))\right)^i \cdot \delta^{(s+5)}$, for $i \geq 1$ such that $\xi^+_{\tau_k+1+i}(d)$ remains in the interval $(-1, O^-(d))$. Set $y_i = f_d'\left(f_d^i(\delta^{(s+5)})\right)$ and $z_i = f_b'\left(f_b^i(\delta^{2(s+5)})\right)$, then for $i \geq 1$ with $\xi^+_{\tau_k+1+i}(d) \in (-1, O^-(d))$, we have

$$\ell\big((\xi^+_{\tau_k+1+i}(b),\xi^+_{\tau_k+1+i}(d))\big) \geq y_i\cdot\delta^{(s+5)} - z_i\cdot\delta^{2(s+5)} \tag{5.1.4}$$

$$= z_i\cdot\delta^{(s+5)}\big(\frac{y_i}{z_i} - \delta^{(s+5)}\big). \tag{5.1.5}$$

We may assume θ_k large enough so that $b\sim d$ and then $f'_b(x)\sim f'_d(x)$, for every $x\in I\setminus 0$. Since for every $x\in[-1,O^-(a)]$, $f'_a(x)\gg 1$ for any $a\in[0,a_0]$ with $a\leq b$, and δ is sufficiently small, therefore the inequality (5.1.4) indicate that the length of the interval $(\xi^+_{\tau_k+1}(b),\xi^+_{\tau_k+1}(d))$ start increasing continuously by the further iterations and eventually for some i_1,

$$\ell(\xi^+_{\tau_k+1+i_1}(b),\xi^+_{\tau_k+1+i_1}(d)) \simeq \ell(-1,O^-(d)).$$

On the other hand, since $f'_0(-1)\geq f'_a(-1)$ and $f'_{a_0}(O^-(a_0))\leq f'_a(O^-(a))$ for every $a\in[0,a_0]$, thus by keeping an eye on (5.1.5), one may notice that

$$i_1 \leq -\frac{(s+5)}{\log({c_1}/{c_2})}\log\delta,$$

where $c_1=f'_0(-1)$ and $c_2=f'_{a_0}(O^-(a_0))$, and $\log({c_1}/{c_2})>0$ since $c_2<c_1$. Therefore the interval $(\xi^+_{\tau_k+\rho_s}(b),\xi^+_{\tau_k+\rho_s}(d))$ will cross the origin for $\rho_s=i_1+2$ or $\rho_s=i_1+3$, with its left end still in a small neighbourhood of -1, and hence there exists $a_s\in\omega_{\theta_k}(a)$ such that $\xi^+_{\tau_k+\rho_s}(a_s)=0$.

(b) If $\xi^+_{\sigma'_k}(a_p)=y(a_p)$ for some $a_p\in\omega_{\theta_k}$ and $\sigma'_k\leq\tau_k$. Then we choose σ_k to be the least integer such that $\xi^+_{\sigma_k}(a_p)\in\Lambda_{a_p}$ concluded par (b) by taking some $\sigma_p\leq p$. Let us consider that $\xi^+_{\tau_k}(\omega_{\theta_k})\cap\Lambda_b=\varnothing$ for all $b\in\omega_{\theta_k}$. So we may may take an interval $(b',d')\in\omega_{\theta_k}$ such that $|\xi^+_{\tau_k}(d')-\xi^+_{\tau_k}(b')|=\frac{1}{2}(\delta^{(s+5)}-\delta^{2(s+5)})$. Again we can assume that $\xi^+_{\tau_k}(d')=\frac{\delta^{(s+5)}}{2}$ and $\xi^+_{\tau_k}(b')=\frac{\delta^{2(s+5)}}{2}$. Then by the similar arguments as in part (a), $\xi^+_{\tau_k+\sigma'}(d')$ will cross the origin, for some $\sigma'\leq\rho_s+i_2$, where i_2 is such that $\big(f'_d(\frac{\delta^{(s+5)}}{2})\big)^{i_2}\geq 2$. Clearly $i_2\leq M_1$ for some $M_1>0$, since δ is small and $f'_d(\frac{\delta^{(s+5)}}{2})\gg 1$. But $\xi^+_{\tau_k+\sigma'}(b')$ will be still in a small neighbourhood of -1.

Now as Λ_a moves continuously with a, so $[(y^-(b'),y^-(d'))]$ will be a small interval contained in the interval $(O^-(d'),0)$, therefore

$$[(y^-(b'),y^-(d'))] \subset \left(\xi^+_{\tau_k+\sigma'}(b'),\xi^+_{\tau_k+\sigma'}(d')\right). \tag{5.1.6}$$

Since $0 \notin \xi^+_{\tau_k+\sigma'-1}\left((b',d')\right)$, therefore $\xi^+_{\tau_k+\sigma'}$ is a diffeomorphism on the interval (b',d') and then $\xi^+_{\tau_k+\sigma'}(a)-y^-(a)$ is continuous on the (b',d'). From (5.1.6), $\xi^+_{\tau_k+\sigma'}(a)-y^-(a)$ changes sign on the interval (b',d'), thus by the intermediate value theorem there exists $a_p \in (b',d')$ such that $\xi^+_{\tau_k+\sigma'}(a_p)=y^-(a_p)$ and then $\xi^+_{\tau_k+\sigma_p}(a_p)=y(a_p)$ for some $\sigma_p \le \sigma'+p-1$. Hence this part concluded by taking $\sigma_k = \tau_k+\sigma_p$.

□

Remark 5.1.3. We may choose δ sufficiently small such that the 2-periodic repelling points of the map f_0 lie outside the interval $(-\delta,\delta)$. Since the absolute values of 2-periodic points for any map f_a, $a \in [0,a_0]$, is bigger than the absolute values of 2-periodic points of f_0, therefore 2-periodic points of f_a remain outside $(-\delta,\delta)$. Let us denote by Λ^δ_a the hyperbolic set of f_a consists of a repelling periodic orbit of period $p \ge 2$ such that $\Lambda^\delta_b \cap (-\delta,\delta)=\varnothing$ for every $b \in [0,a_0]$. Then it is to be noted that if we consider Λ^δ_a in Lemma 5.1.2 then $a_p \in R$: we can take θ_k large enough so that $e^{-\alpha\gamma_k} \le \frac{e^{-2(s+3)}}{2}$, then the parameter a_p satisfies (BA_{γ_k}). On the other hand, as the parameter a_p satisfies the condition (H_{γ_k}) so does every $a \in \omega_{\theta_k}$, and since $\Lambda^\delta_{a_p} \cap (-\delta,\delta)=\varnothing$ therefore after the time γ_k the orbit of a_p always stays outside the interval $(-\delta,\delta)$, i.e., the parameter a_p satisfies (BA_n) and (H_n) for all $n \ge 1$. Consequently, a_p never excluded in the construction of the set of Rovella parameters.

5.1.3 The Extended Set

Lemma 5.1.2 provides us some elements of the set $[0,a_0]$ which correspond to either critically-stable or post-critically finite maps. Let us denote by S the set of super-stable parameters in $[0,a_0]$. It is to be noted that if a map has a super-attractor then it can not have an *SRB* measure, so the set S is disjoint from the set of Rovella parameters R. Therefore by setting

$$\mathcal{E} = R \cup S,$$

we possess a larger set of parameters for the contracting Lorenz-like family such that the corresponding maps admit unique *SRB* measures.

5.2 Accumulation of Critical Sum by Physical Measures

This section is devoted to prove a result which states that if the critical measure for a Rovella map exists then there is a sequence of super-stable parameters such that the corresponding sequence of physical measures converges to the critical measure in the weak*-topology. First we give the following definition.

Definition 5.2.1. Let f_a be a map in the contracting Lorenz-like family and let c be one of its critical values. Let us denote $\mu_a^n(c) := \frac{1}{n}\sum_{k=0}^{n-1}\delta_{f_a^k(c)}$ the convex combination of delta Dirac measures on the first n terms of the critical orbit. If the limit

$$\lim_{n\to\infty}\mu_a^n(c)$$

exits in the weak*-topology then we call this limit as *critical measure* and denote by $\mu_a(c)$.

The following Proposition is analogous to [28, Lemma 4] and can be transformed straightforward into our context. Let us denote by $d_H(A,B)$ the Hausdorff distance between the sets A and B, which is defined as:

$$d_H(A,B) = \max\left\{\sup_{a\in A}\{\text{dist}(a,B)\}, \sup_{b\in B}\{\text{dist}(b,A)\}\right\},$$

where $\text{dist}(a,B) = \inf_{b\in B}\{\text{dist}(a,b)\}$.

Proposition 5.2.2. *Let* $\omega = (b,d) \subset [0,a_0]$ *be a parameter interval such that every* $a \in \omega$ *satisfies* (BA_n) *and* (EG_n). *Let* $\xi_n^+(\omega) \subset I_{m,k}^+$ *and* $\xi_n^-(\omega) \subset I_{-m,k}^+$ *for some* $|m| \geq \Delta$ *and* p *denotes the bound period corresponding to the return* n. *If* d *is sufficiently close to* 0, *then* *there exists a constant* $C > 0$ *such that for any* $a \in \omega$

$$d_H\big(\xi^{\pm}_{n+1+j}(\omega), f^j_a(\xi^{\pm}_{n+1}(\omega))\big) \leq C|f^j_a(\xi^{\pm}_{n+1}(\omega))|$$

for every $1 \leq j \leq p$.

Now we are in a position to state the main theorem of this section. We present this theorem for the critical value $c = -1$, similar result can be prove for the critical value $c = 1$.

Theorem 5.2.3. *For every* $a \in R$*, there exits a sequence of super-stable parameters* $\{a_k\}_{k=1}^{\infty}$ *such that if the critical measure* $\mu_a(c)$ *for the map* f_a *exits, then*

$$\mu_{a_k} \stackrel{weak*}{\longrightarrow} \mu_a(c),\ k \to \infty.$$

Proof. Since each $a \in R$ encounters infinitely many escape situations so we can consider the sequence $\{\theta_k\}_{k\geq 1}$ of escape times for a and $\{\omega_{\theta_k}(a)\}_{k\geq 1}$ be the corresponding escape components. Then by using Lemma 5.1.2 we obtain a sequence $\{a_k \in \omega_{\theta_k}(a)\}_{k\geq 1}$ of super-stable parameters such that f_{a_k} has a super-attractor of period ρ_k. Since ω_{θ_k} is the partitioning element contained in $\mathcal{P}_{\theta_k}$, therefore each $b \in \omega_{\theta_k}(a)$ satisfies (EG_{θ_k}), i.e.,

$$D^+_j(b) \geq \lambda^j \quad \text{for } j = 1, \ldots, \theta_k,$$

where $\lambda > 1$. On the other hand, since $0 \notin \xi^+_{\theta_k}(\omega_{\theta_k}(a))$ for any $k \geq 1$, thus by mean value theorem for any $k \geq 1$, we have

$$|\xi^+_{\theta_k+1}(\omega_{\theta_k}(a))| = |(\xi^+_{\theta_k+1})'(b_k)||\omega_{\theta_k}(a)|,$$

for some $b_k \in \omega_{\theta_k}(a)$. Then using Proposition 4.1.2 in the above equation, we obtain

$$\begin{aligned}
|\omega_{\theta_k}(a)| &= \frac{1}{|(\xi^+_{\theta_k+1})'(b_k)|}|\xi^+_{\theta_k+1}(\omega_{\theta_k}(a))| \\
&= \frac{D^+_{\theta_k}(b_k)}{|(\xi^+_{\theta_k+1})'(b_k)|}|\xi^+_{\theta_k+1}(\omega_{\theta_k}(a))|\frac{1}{D^+_{\theta_k}(b_k)} \\
&\leq 2A\lambda^{-\theta_k}.
\end{aligned}$$

Therefore the length of ω_{θ_k} decreasing and consequently $\rho_k \uparrow \infty$, $k \to \infty$. We need to show that for any real valued continuous function φ on the interval I

$$\lim_{n\to\infty} \int \varphi \, d\mu_{a_k} = \int \varphi \, d\mu_a(c).$$

Since the Lipschitz continuos functions are dense in the space of continuous functions on I, therefore it is enough to prove the above assertion for Lipschitz continuous test functions. Let ℓ_c be the Lipschitz constant for the function φ, then for any a_k, we have

$$\Big|\int \varphi \, d\mu_{a_k} - \int \varphi \, d\mu_a(c)\Big| \le \Big|\int \varphi \, d\mu_{a_k} - \frac{1}{\rho_k}\sum_{i=1}^{\rho_k} \varphi(\xi_i^+(a_k))\Big| + \Big|\frac{1}{\rho_k}\sum_{i=1}^{\rho_k}\big(\varphi(\xi_i^+(a_k)) - \varphi(\xi_i^+(a))\big)\Big|$$
$$+ \Big|\frac{1}{\rho_k}\sum_{i=1}^{\rho_k} \varphi(\xi_i^+(a)) - \int \varphi \, d\mu_a(c)\Big|. \tag{5.2.1}$$

Since μ_{a_k} is the physical measure supported on the super-attractor of f_{a_k}, thus the first term of the inequality (5.2.1) is 0. And since $\rho_k \uparrow \infty$, $k \to \infty$, by the definition of $\mu_a(c)$ for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that

$$\Big|\frac{1}{\rho_k}\sum_{i=1}^{\rho_k} \varphi(\xi_i^+(a)) - \int \varphi \, d\mu_a(c)\Big| < \varepsilon$$

for all $k \ge n_0$. Therefore to conclude this theorem we just need to show that $\Big|\sum_{i=1}^{\rho_k}\big(\varphi(\xi_i^+(a_k)) - \varphi(\xi_i^+(a))\big)\Big|$ is bounded by some constant independent of a_k.

Now by using the Lipschitz continuity of φ, we get

$$\Big|\frac{1}{\rho_k}\sum_{i=1}^{\rho_k}\big(\varphi(\xi_i^+(a_k)) - \varphi(\xi_i^+(a))\big)\Big| \le \frac{\ell_c}{\rho_k}\sum_{i=1}^{\rho_k} |\xi_i^+(a_k) - \xi_i^+(a)|$$

therefore we are going to show that the sum $\sum_{i=1}^{\rho_k} |\xi_i^+(a_k) - \xi_i^+(a)| =: \mathcal{S}$ is bounded by some constant independent of a_k. Let us denote

$$D_i = |\xi_i^+(a_k) - \xi_i^+(a)|,$$

and let τ_k be the last return before ρ_k for the interval ω_{θ_k} as in Lemma 5.1.2. We set $\gamma_\chi = \tau_k$ and consider the sequences $\{\gamma_j\}_{j=1}^{\chi}$ and $\{p_j\}_{j=1}^{\chi}$, respectively, as returns and the bound periods for the interval ω_{θ_k} up to the time τ_k. Just for the notations we set $\gamma_0 = p_0 = 0$. Then we can split the sum $\mathcal{S}$ as

$$\mathcal{S} = \sum_{j=0}^{\chi-1} (\mathcal{S}_j^1 + \mathcal{S}_j^2) + \mathcal{S}^3,$$

such that

$$\mathcal{S}_j^1 = \sum_{l=\gamma_j}^{\gamma_j+p_j} D_l, \qquad \mathcal{S}_j^2 = \sum_{l=\gamma_j+p_j}^{\gamma_{j+1}-1} D_l,$$

and

$$\mathcal{S}^3 = \sum_{l=\gamma_\chi}^{\rho_k} D_l.$$

Then observe that $\mathcal{S}_0^1$ is empty sum so it is equal to 0 and $\mathcal{S}_0^2$ is the sum until the first return. Again from Lemma 5.1.2, $\rho_k \le M$ for some $M \ge 0$, thus $\mathcal{S}^3$ is finite sum and therefore it is bounded. Now since τ_k is the next return to θ_k for the interval ω_{θ_k}, so every $b \in \omega_{\theta_k}$ satisfies (EG_{τ_k-1}) and therefore, by using Proposition 4.1.2 and mean value theorem, for some $b \in [(a_k, a)]$, for any $1 \le j \le \chi - 1$ and for every $1 \le n < \gamma_{j+1} - \gamma_j - p_j$, we have

$$\begin{aligned}
|\xi_{\gamma_{j+1}}^+(a_k) - \xi_{\gamma_{j+1}}^+(a)| &= |(\xi_n^+)'(b)||\xi_{\gamma_{j+1}-n}^+(a_k) - \xi_{\gamma_{j+1}-n}^+(a)| \\
&= D_n^+(b)\frac{|(\xi_n^+)'(b)|}{D_n^+(b)}|\xi_{\gamma_{j+1}-n}^+(a_k) - \xi_{\gamma_{j+1}-n}^+(a)| \\
&\ge \frac{1}{A}\lambda^n|\xi_{\gamma_{j+1}-n}^+(a_k) - \xi_{\gamma_{j+1}-n}^+(a)|.
\end{aligned}$$

Again since every $b \in \omega_{\theta_k}$ satisfies (EG_{τ_k-1}) and (BA_{τ_k-1}), thus by using above inequality and Lemma 4.3.2, we obtain

$$\sum_{j=0}^{\chi-1} \mathcal{S}_j^2 = \sum_{j=0}^{\chi-1} \sum_{l=\gamma_j+p_j}^{\gamma_{j+1}-1} D_l \le \sum_{j=0}^{\chi-1} \sum_{l=\gamma_j+p_j}^{\gamma_{j+1}-1} A\lambda^{-(\gamma_{j+1}-l)} D_{\gamma_{j+1}}$$
$$\le A_1 \sum_{j=1}^{\chi} D_{\gamma_j} \le A_1 \sum_{j=1}^{\chi} 2^{j-\chi} D_{\gamma_\chi} < \infty.$$

To conclude the result it remains to show that $\sum_{j=0}^{\chi-1} \mathcal{S}_j^1$ is bounded. Since $\omega_{\theta_k} \in \mathcal{P}_{\tau_k-1}$ and $\gamma_j < \gamma_\chi = \tau_k$ are returns, thus $\xi_{\gamma_j}^+[(a_k,a)] \subset I_{m_j,k_j}^+$, for some $|m_j| \ge \Delta$ and $k_j \le m_j^2$. Then using binding condition, mean value theorem, Lemma 4.2.6 and Proposition 5.2.2, for every $1 \le i \le p_j$ and for any $b \in [(a_k,a)]$, we obtain

$$D_{\gamma_j+i} \le C_1 \left| f_b^{i-1}\left(\xi_{\gamma_{j+1}}^+\left([(a_k,a)]\right)\right)\right| = C_1 \frac{\left|f_b^{i-1}\left(\xi_{\gamma_{j+1}}^+\left([(a_k,a)]\right)\right)\right|}{\left|f_b^{i-1}\left([-1,f_b(e^{-|m_j|+1})]\right)\right|} \left|f_b^{i-1}\left([-1,f_b(e^{-|m_j|+1})]\right)\right|$$
$$\le C_2 \frac{\left|\xi_{\gamma_{j+1}}^+\left([(a_k,a)]\right)\right|}{\left|[-1,f_b(e^{-|m_j|+1})]\right|} e^{-\beta i} \le C_2 \frac{\left|\xi_{\gamma_{j+1}}^+\left([(a_k,a)]\right)\right|}{\left|[f_b(e^{-|m_j|-2}),f_b(e^{-|m_j|+1})]\right|} e^{-\beta i}$$
$$\le C_2 \frac{f_b'(x)}{f_b'(y)} \frac{\left|\xi_{\gamma_j}^+\left([(a_k,a)]\right)\right|}{|I_{m_j}^+|} e^{-\beta i} \le C_2 \frac{\left|\xi_{\gamma_j}^+\left([(a_k,a)]\right)\right|}{|I_{m_j}|} e^{-\beta i}.$$

where $x \in \xi_{\gamma_j}^+\left([(a_k,a)]\right)$, $y \in I_{m_j}^+$, and the last inequality holds since $\xi_{\gamma_j}^+([(a_k,a)]) \subset I_{m_j,k_j}^+$ and the derivative increases when we move away from zero, thus $f_b'(x) \le f_b'(y)$. On the other hand since $|I_{m_j}| < 1$, thus

$$D_{\gamma_j} < \frac{\left|\xi_{\gamma_j}^+\left([(a_k,a)]\right)\right|}{|I_{m_j}|},$$

and therefore, we have

$$\mathcal{S}_j^1 \le C_2 \sum_{i=0}^{\infty} \frac{\left|\xi_{\gamma_j}^+\left([(a_k,a)]\right)\right|}{|I_{m_j}|} e^{-\beta i} \le C_3 \frac{\left|\xi_{\gamma_j}^+\left([(a_k,a)]\right)\right|}{|I_{m_j}|}$$
$$\le C_3 \frac{|I_{m_j,k_j}^+|}{|I_{m_j}|} \le C_3 \frac{5|I_{m_j}|}{m_j^2} \frac{1}{|I_{m_j}|} = C_3 \frac{5}{m_j^2}.$$

Hence by using Lemma 4.3.2, we obtain

$$\sum_{j=0}^{\chi-1} \mathcal{S}_j^1 \leq 5C_3 \sum_{m_j} \sum_{\substack{\text{returns} \\ \text{to } I_{m_j}}} m_j^{-2} \leq 5C_3 \sum_{m_j} \sum_{\substack{\text{last return} \\ \text{to } I_{m_j}}} m_j^{-2}$$
$$\leq 5C_3 \sum_{m \geq \Delta} m^{-2} \leq 5C_3 \sum_{m=1}^{\infty} m^{-2} < \infty.$$

□

5.3 Statistical Instability of the Rovella Maps

This is worth to start this section by recalling the notion of physical measure for a map f defined on I. A measure μ on I is called a physical measure for the mapping f if for any observable, i.e., continuous real valued function ϕ on I, the time average converges to the space average for a positive Lebesgue measure subset of I. More formally, An f-invariant measure μ is called a physical measure for f if the basin of μ, i.e., the set of points

$$\{x \in I : \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \int \varphi d\mu, \text{ for any continuous map } \varphi : I \to \mathbb{R}\}$$

has positive Lebesgue measure. One of the vital example of a physical measure is the ergodic absolutely continuous (with respect to Lebesgue) invariant probability measure which we called as an *SRB* measure. On the other extreme if a map owns an attracting periodic orbit then it admits a physical measure supported on that periodic orbit.

It is an important and interesting problem to study the statistical stability for a family of maps admitting unique physical measures. Recall that a map f in a family of maps $\mathcal{G}$, defined on I admitting unique physical measures, is statistically stable if the mapping

$$\mathcal{G} \ni g \longmapsto \mu_g$$

is continuous at f in the *weak** topology, where μ_g is the physical measure corresponding to the map g. The strong statistical stability refer as continuous variation of the densities of physical measures, if they exist, in the L^1-norm

It was Metzger [21] who proved that each Rovella map admits an *SRB* measure. Although, to establish the uniqueness of the *SRB* measures he considered a smaller class of maps. Recently, Alves and Soufi [3] showed that each Rovella map admits a unique *SRB* measure and then they proved that Rovella maps are strongly statistically stable if we restrict ourselves on the set of Rovella parameters R.

In section 5.1.3 we discovered an extended set of parameter $\mathcal{E}$ for the contracting Lorenz-like family $\{f_a\}_{a\geq 0}$ consists of Rovella parameters and super-stable parameters. Therefore the map f_a associated with any parameter $a \in \mathcal{E}$ admits a unique physical measures μ_a which is either an *SRB* measure or a measure supported on the super-attractor. Then it opens up the quest of statistical stability of the Rovella map on this extended class of maps which we are going to tackle in this chapter.

5.3.1 Accumulation of Rovella Maps by Post-critically Finite Rovella Maps

In this section we present a result which is essentially a corollary of Lemma 5.1.2 and it states that each Rovella parameter is accumulated by post-critically finite Rovella parameters. Recall that $\bar{A}$ denotes the closure of a set A.

Proposition 5.3.1. *Let $\Lambda = \Lambda_a^\delta$ be a hyperbolic set for f_a, $a \in R$, $y = y(a)$ be any point in Λ_a^δ and let Λ_b and $y(b)$ be the continuation of Λ and y. Then*

$$a \in \overline{\{b \in R \mid f_b^{\mathcal{N}}(-1) = y(b) \text{ for some integer } \mathcal{N} = \mathcal{N}(b)\}}.$$

Proof. Since each $a \in R$ encounters infinitely many escape situations so we can consider the sequence $\{\theta_k\}_{k\geq 1}$ of escape times for a and $\{\omega_{\theta_k}(a)\}_{k\geq 1}$ be the corresponding escape components. Then by using Lemma 5.1.2 and the Remark 5.1.3 we obtain a sequence $\{a_k \in \omega_{\theta_k}(a)\}_{k\geq 1}$ of parameters, contained in the set R, such that $\{\xi_j^+(a_k)\}_{k\geq 1}$ is pre-periodic to $y(a_k)$. Since ω_{θ_k} is the partitioning element contained in $\mathcal{P}_{\theta_k}$, therefore each $b \in \omega_{\theta_k}(a)$ satisfies (EG_{θ_k}), i.e.,

$$D_j^+(b) \geq \lambda^j \quad \text{for } j = 1, \dots, \theta_k,$$

where $\lambda > 1$. On the other hand, since $0^+ \notin \xi^+_{\theta_k}(\omega_{\theta_k}(a))$ for any $k \geq 1$, thus by mean value theorem for any $k \geq 1$, we have

$$|\xi^+_{\theta_k+1}(\omega_{\theta_k+1}(a))| = |(\xi^+_{\theta_k+1})'(b_k)||\omega_{\theta_k}(a)|,$$

for some $b_k \in \omega_{\theta_k}(a)$. Then using Proposition 4.1.2 in the above equation, we obtain

$$\begin{aligned}
|\omega_{\theta_k}(a)| &= \frac{1}{|(\xi^+_{\theta_k})'(b_k)|}|\xi^+_{\theta_k+1}(\omega_{\theta_k}(a))| \\
&= \frac{D^+_{\theta_k}(b_k)}{|(\xi^+_{\theta_k+1})'(b_k)|}|\xi^+_{\theta_k+1}(\omega_{\theta_k}(a))|\frac{1}{D^+_{\theta_k}(b_k)} \\
&\leq 2A\lambda^{-\theta_k}. \qquad (5.3.1)
\end{aligned}$$

Now for every $\varepsilon > 0$ there is a positive integer k_1 such that $\lambda^{-\theta_{k_1}} \leq \varepsilon$ and since $a \in \omega_{\theta_k}(a) \subset \omega_{\theta_{k_1}}(a)$ for every $k \geq k_1$, therefore by using the inequality (5.3.1), we get

$$|a_k - a| \leq |\omega_{\theta_k}(a)| \leq |\omega_{\theta_{k_1}}(a)| \leq \varepsilon \quad \text{for every } k \geq k_1.$$

From the above inequality we conclude that $a_k \to a$, $k \to \infty$ and hence a is accumulated by a sequence lie in the set

$$\{b \in R \mid f_b^{\mathcal{N}}(-1) = y(b) \text{ for some integer } \mathcal{N} = \mathcal{N}(b)\}.$$

□

5.3.2 Statistical Instability

Here we conclude this chapter by presenting our main result about the statistical instability of Rovella maps in the set $\mathcal{E}$. Proposition 5.3.1 is the crucial step towards the proof of that result and then the proof is accomplished by following the approach of Thunberg [28] for the Benedicks-Carleson quadratic maps.

The idea of proof is based on the following strategy: for any Rovella parameter a, first we make use of Proposition 5.3.1 to obtain a sequence of post-critically finite parameters lie in the set of Rovella parameters and converging to a. In the next step, using Lemma 5.1.2, for each post-critically finite Rovella parameter b in that sequence we find a sequence of super-stable parameters converging to b and the sequence of physical measures associated to critically-stable maps converges to the measure supported on a repelling periodic orbit of f_b in the weak*-topology. Finally hyperbolicity of the repelling periodic orbit enables us to extract a sequence of super-stable parameters converging to a and the corresponding sequence of physical measures converges, in the weak*-topology, to the measure supported on repelling periodic orbit for f_a, which is obviously not an *SRB* measure for f_a. Hence the Rovella maps are statistically unstable in the set $\mathcal{E}$.

Now all is set to present the main theorem of our work.

Theorem 5.3.2. *The map $\mathcal{E} \ni a \mapsto \mu_a$ is not continuous in the weak*-topology at any point in the set of Rovella parameters R.*

Proof. Let $a \in R$ and $\Lambda_a^\delta = \{x_1(a), \dots, x_p(a)\}$ be the hyperbolic repeller for f_a. Then from Proposition 5.3.1 we obtain a sequence $\{a_n\}_{n=1}^\infty \subset R$ such that $a_n \to a$, $n \to \infty$, and the critical orbit of f_{a_n} is pre-periodic to some point in $\Lambda_{a_n}^\delta$, for every $n \geq 1$. For arbitrary fixed n let $L = L(n)$ be the smallest natural number such that $f_{a_n}^L(-1) \in \Lambda_{a_n}$, and let $f_{a_n}^L(-1) = x_1(a_n)$.

Now for sufficiently small $r > 0$, using Lemma 5.1.2 we can obtain a sequence of parameter intervals $\{\Omega_{n,j} : \Omega_{n,j} \subset \omega_{n,j}(a_n)\}_{j=1}^\infty$, where $\omega_{n,j}(a_n)$ is the escape component of the parameter a_n, and a strictly increasing sequence of positive integers $\{m_j\}_{j=1}^\infty$, $m_1 = L$, such that

(i) $a_n \in \Omega_{n,j}$ for all $j \geq 1$, $\Omega_{n,j+1} \subset \Omega_{n,j}$, and $|\Omega_{n,j}| \to 0$ as $j \to \infty$;

(ii) $\xi_{m_i}^+(\Omega_{n,j}) \subset \bigcup_{k=1}^p (x_k(a_n) - r, x_k(a_n) + r)$ for $i = 1, \dots, j$;

(iii) $\xi_{m_j}^+(\Omega_{n,j}) = (x_{i_j}(a_n) - r, x_{i_j}(a_n) + r)$ for some $i_j \in \{1, \dots, p\}$;

(iv) There exists a natural number $\rho = \rho(r)$ such that for every j there is a positive integer $\rho_j \leq \rho$ such that $-1 \in \xi_{m_j+\rho_j}^+(\Omega_{n,j})$;

(v) $m_j - L + 1 = \ell_j p$ for some integer ℓ_j.

As a consequence of (i) and (iv), we obtain a sequence $\{a_{n,j} : a_{n,j} \in \Omega_{n,j}\}_{j=1}^\infty$ such that $a_{n,j} \to a_n$ as $j \to \infty$ and $f_{a_{n,j}}$ has a supper-attractor of length $m_j + \rho_j$ for every $j \geq 1$. From

(ii) and (iii), we have

$$\#\Big\{i \le m_j + \rho_j : f^i_{a_{n,j}}(-1) \notin \bigcup_{k=1}^{p}(x_k(a_n) - r, x_k(a_n) + r)\Big\} = (m_j + \rho_j) - (m_j - (L-1))$$
$$= \rho_j - 1 + L \le \rho + L.$$

Now we will show that $\mu_{a_{n,j}} \overset{weak*}{\longrightarrow} \frac{1}{p}\sum_{i=1}^{p}\delta_{x_i}(a_n) := \mu^{sing}_{a_n}$, $j \to \infty$. For that let us take a continuous map $\varphi : I \to \mathbb{R}$ and fix a sufficiently small $\varepsilon > 0$. Since φ is continuous on the closed interval I, thus it is bounded, i.e., there is a constant C>0, such that

$$\sup_{x\in I}\varphi(x) \le C,$$

and therefore for the physical measure $\mu_{a_{n,j}}$ of $f_{a_{n,j}}$, we have

$$\begin{aligned}\int \varphi\, d\mu_{a_{n,j}} &= \frac{1}{m_j+\rho_j}\sum_{i=1}^{m_j+\rho_j}\varphi\big(f^i_{a_{n,j}}(-1)\big)\\ &\le \frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\varphi\big(f^i_{a_{n,j}}(-1)\big) + \frac{\rho_j - 1 + L}{m_j+\rho_j}\sup_{x\in I}\varphi(x)\\ &\le \frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\varphi\big(f^i_{a_{n,j}}(-1)\big) + \frac{\rho + L}{m_j+1}C. \qquad (5.3.2)\end{aligned}$$

Now we are going to work out the first term of the above inequality. Again the continuity of φ on the closed interval I implies that it is uniformly continuous on I and therefore we can choose a small $r > 0$ such that

$$|\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2} \text{ whenever } |x - y| < r. \qquad (5.3.3)$$

On the other hand, since $f^L_{a_n}(-1) = x_1(a_n)$, thus by using (ii) and (iii), we have

$$f^i_{a_{n.j}}(-1) \in (x_i(a_n) - r, x_i(a_n) + r) \quad \text{for all } L \le i \le m_j\ ,$$

that is

$$|f^i_{a_{n.j}}(-1) - f^{i-L}_{a_n}(x_1(a_n))| < r \quad \text{for all } L \le i \le m_j\ . \qquad (5.3.4)$$

Then by taking into account the inequalities (5.3.3) and (5.3.4), we obtain

$$\begin{aligned}\frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\varphi\big(f^i_{a_{n,j}}(-1)\big) &= \frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\varphi(f^i_{a_{n.j}}(-1))\\ &= \frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\big(\varphi(f^{i-L}_{a_n}(x_1(a_n))+\varphi(f^i_{a_{n.j}}(-1))-\varphi(f^{i-L}_{a_n}(x_1(a_n)))\big)\\ &\le \frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\big(\varphi(f^{i-L}_{a_n}(x_1(a_n))+|\varphi(f^i_{a_{n.j}}(-1))-\varphi(f^{i-L}_{a_n}(x_1(a_n)))|\big)\\ &\le \frac{1}{m_j+\rho_j}\sum_{i=L}^{m_j}\big(\varphi(f^{i-L}_{a_n}(x_1(a_n)))+\frac{\varepsilon}{2}\big). \qquad (5.3.5)\end{aligned}$$

From (v) we can write $m_j - L + 1 = \ell_j p$ for some positive integer ℓ_j, thus from the above inequality, we get

$$\begin{aligned}\sum_{i=L}^{m_j}\big(\varphi(f^{i-L}_{a_n}(x_1(a_n)))+\frac{\varepsilon}{2}\big) &= \sum_{i=1}^{m_j-L+1}\big(\varphi(f^{i-1}_{a_n}(x_1(a_n)))+\frac{\varepsilon}{2}\big)\\ &= (\ell_j p)\Big(\frac{1}{\ell_j p}\sum_{i=1}^{\ell_j p}\big(\varphi(f^{i-1}_{a_n}(x_1(a_n)))+\frac{\varepsilon}{2}\big)\Big)\\ &= (m_j-L+1)\Big(\frac{1}{p}\sum_{i=1}^{p}\big(\varphi(f^{i-1}_{a_n}(x_1(a_n)))+\frac{\varepsilon}{2}\big)\Big)\\ &= (m_j-L+1)\Big(\frac{1}{p}\sum_{i=1}^{p}\big(\varphi((x_i(a_n)))+\frac{\varepsilon}{2}\big)\Big)\\ &= (m_j-L+1)\Big(\int\varphi\, d\mu^{sing}_{a_n}+\frac{\varepsilon}{2}\Big). \qquad (5.3.6)\end{aligned}$$

Using the inequalities (5.3.5) and (5.3.6) in the inequality (5.3.2), we have

$$\int\varphi\, d\mu_{a_{n,j}} \le \frac{m_j-L+1}{m_j+\rho_j}\Big(\int\varphi\, d\mu^{sing}_{a_n}+\frac{\varepsilon}{2}\Big)+\frac{\rho+L}{m_j+1}C.$$

Clearly the second term of the above inequality goes to zero as $m_j \to \infty$, therefore for sufficiently large m_j, we get

$$\int\varphi\, d\mu_{a_{n,j}} \le \int\varphi\, d\mu^{sing}_{a_n}+\varepsilon.$$

Following the similar arguments as above, we can get

$$\int \varphi \, d\mu_{a_{n,j}} \geq \int \varphi \, d\mu_{a_n}^{sing} - \varepsilon.$$

Therefore $\mu_{a_{n,j}} \overset{weak*}{\longrightarrow} \frac{1}{p}\sum_{i=1}^{p} \delta_{x_i}(a_n)$, $j \to \infty$. From the fact that Λ_a^δ moves continuously with a, we obtain a sequence $\{a_{n,j_n}\}_{n\geq 1}$ such that $f_{a_{n,j_n}}$ has a super-attractor with $a_{n,j_n} \to a$ as $n \to \infty$, and

$$\mu_{a_{n,j}} \overset{weak*}{\longrightarrow} \frac{1}{p}\sum_{i=1}^{p} \delta_{x_i(a)}.$$

Since $x_i(a) \in \Lambda_a^\delta$, $1 \leq i \leq p$, where Λ_a^δ, is a hyperbolic repeller, thus $\frac{1}{p}\sum\limits_{i=1}^{p} \delta_{x_i(a)}$ is not an *SRB* measure for the map f_a. Hence the mapping

$$\mathcal{E} \ni a \mapsto \mu_a$$

is not continuous at any $a \in R$. □

Finally from the above theorem we conclude the statistical instability of Rovella maps in the class of mappings associated with the set of parameters $\mathcal{E}$.

Chapter 6

Final Comments

Recall from chapter 2, Rovella [23] considered a vector field X on $\mathbb{R}^3$ which is linear in a neighbourhood U of the origin $(0,0,0)$ containing the cube $\{(x,y,z) : |x| \le 1, |y| \le 1, |z| \le 1\}$. The derivative of X at $(0,0,0)$, which is the only singularity of X, admits three real eigenvalues λ_1, λ_2 and λ_3 which satisfy

$$0 < \lambda_1 < -\lambda_3 < -\lambda_2.$$

We denote by Σ, the roof $\{(x,y,z) : |x| \le 1, |y| \le 1, z = 1\}$ of the cube, which is a cross-section for the flow of X and it is foliated by the stable leaves parallel to the x-axis. We have P as the Poincaré first return map from $\Sigma \setminus \Gamma$ to Σ, where $\Gamma = \{(x,y,z) : x = 0, |y| \le 1, z = 1\}$, with the return time function $\tau : \Sigma \to \mathbb{R}$. By making the quotient space of $\Sigma \setminus \Gamma$ with the stable leaves, projecting the stable leaves $\{x = constant\} \cap \Sigma$ to the line $\{(x,y,z) : |x| \le 1, y = -1, z = 1\}$ through the map π, we get a one-dimensional map $f : I \setminus \{0\} \to I$, such that

$$f \circ \pi = \pi \circ P$$

and f has two critical values -1 and 1.

Rovella considered a one-parameter family of vector fields near X and the corresponding one-parameter family of one-dimensional maps which we named as contracting Lorenz-like family. He also discovered that there is a positive Lebesgue measure set of parameters R

such that the derivatives of the corresponding maps have exponential growths along the critical orbits and those orbits have slow recurrence to the critical points. Later on, Metzger [21] proved that each Rovella map admits a unique absolutely continuous (w.r.t. Lebesgue) invariant probability measure (SRB). We shall refer these measures as ACIP measures in the sequel. In the previous chapter we explore a set of parameters $\mathcal{E}$, for the contracting Lorenz-like family, consists of Rovella parameters and the super-stable parameters such that each map corresponding to set $\mathcal{E}$ has a physical measure which either an ACIP measure or measure supported on the attracting periodic orbit.

The physical measure μ_f for the contracting Lorenz-like map f on the interval I may be lifted to a physical measure μ_X for the flow X^t of the vector field X on the contracting Lorenz attractor Λ. To define the physical measures for the flows we may distinguish two cases: one corresponding to ACIP measures and the other one for the measures supported on the attracting periodic orbits.

6.1 Lifting of ACIP Measures

In this section we will define a lift μ_X for the ACIP measure μ_f of the one-dimensional map f on the interval I to the contracting Lorenz attractor Λ which is ultimately a physical measure for the flow X^t. Like Alves and Soufi [4], we may use the approach given in [6]. We shall first pass through a physical measure for the the Poincaré map P on the cross-section Σ.

6.1.1 Physical Measure for the Poincaré Map

Let μ_f be the ACIP measure for the interval map f in the contracting Lorenz-like family. We may lift the measure μ_f to a measure μ_P on Σ. For any bounded function $\varphi : \Sigma \to \mathbb{R}$, let $\varphi_\pm : I \to \mathbb{R}$ be defined as

$$\varphi_+(x) = \sup_{x' \in \pi^{-1}(x)} \varphi(x') \quad \text{and} \quad \varphi_-(x) = \inf_{x' \in \pi^{-1}(x)} \varphi(x').$$

The following Lemma may be obtained in the similar way as [6, Lemma 6.1].

Lemma 6.1.1. *Given any continuous function $\varphi : \Sigma \to \mathbb{R}$, both the limits*

$$\lim_{n\to+\infty} \int (\varphi \circ P^n)_+ \, d\mu_f \quad \textit{and} \quad \lim_{n\to+\infty} \int (\varphi \circ P^n)_- \, d\mu_f$$

exist and they coincide.

Then we have the following corollary of the above Lemma similar as [6, corollary 6.2].

Corollary 6.1.2. *There exists a unique P-invariant probability measure μ_P on Σ such that*

$$\int \varphi \, d\mu_P = \lim_{n\to+\infty} \int (\varphi \circ P^n)_+ \, d\mu_f = \lim_{n\to+\infty} \int (\varphi \circ P^n)_- \, d\mu_f. \tag{6.1.1}$$

In fact μ_P is a physical measure for the Poncaré map P (c.f. [6]).

6.1.2 Physical Measure for the Flow

We may define an equivalence relation ~ on $\Sigma \times \mathbb{R}$ generated by $(x, \tau(x)) \sim (P(x), 0)$, that is $(x,u) \sim (x',u')$ if and only if there exits

$$(x,s) = (x_0,u_0),(x_1,u_1),\ldots,(x_k,u_k) = (x',u')$$

in $\Sigma \times \mathbb{R}$ such that, for every $1 \le i \le k$

$$\text{Either} \quad x_i = P(x_{i-1}) \quad \text{and} \quad u_i = u_{i-1} - \tau(x_{i-1});$$
$$\text{or} \quad x_{i-1} = P(x_i) \quad \text{and} \quad u_{i-1} = u_i - \tau(x_i),$$

where τ is the return time function on Σ defined in Section 2.2. We denote by $V = \Sigma \times \mathbb{R}/\sim$ the corresponding quotient space and by $\Pi : \Sigma \to V$ the canonical projection which induces on V a topology and Borel σ-algebra of measurable subsets of V.

The flow of X on the space V is given as

$$X^t(\Pi(x,u)) = \Pi(x,u+t),$$

for every $(x,u) \in \Sigma \times \mathbb{R}$ and $t \in \mathbb{R}$. We consider the set

$$D = \{(x,u) \in \Sigma \times \mathbb{R} : 0 \leq u < \tau(x), \text{if } \tau(x) \text{ is finite}\},$$

which is a fundamental domain for the equivalence relation ~ (c.f. [31]).

We need to make sure that the return time function τ is integrable with respect to the measure μ_P. In this regard we may use the following result by H. Cui and Y. Ding [12].

Theorem 6.1.3. *For every Rovella map f the density $\dfrac{d\mu_f}{dm}$ of the SRB measure μ_f with respect to the Lebesgue measure m belongs to some $L^p(m)$ with $p > 1$, where p depends only on the (side) orders of the critical point.*

It gives rise to an interesting problem to show that the density $\dfrac{d\mu_f}{dm}$ is uniformly bounded in $L^p(m)$ for some $1 < p < \infty$ as long as we have $\tau \in L^q(m)$ for all $q > 1$, then by using the Hölder inequality we may conclude that

$$\int \tau \, d\mu_P < +\infty,$$

since τ is measurable, bounded away from zero, $\tau \equiv +\infty$ on Γ and $\tau(z) \approx \log(d(z,\Gamma))$ with z close to Γ (c.f [30]). Therefore we may define a probability measure μ_X on V as

$$\int \psi \, d\mu_X = \frac{1}{\int \tau \, d\mu_P} \int \int_0^{\tau(x)} \psi(\Pi(x,t)) \, dt \, d\mu_P(x) \tag{6.1.2}$$

for every bounded measurable $\psi : V \to \mathbb{R}$. This measure is indeed a physical measure for the flow of the vector field X (c.f. [6]).

6.2 Lifting for the Measures supported on the Attracting Periodic Orbits

In this section we will consider a map f in the contracting Lorenz-like family which corresponds a super-stable periodic attractor and consequently a physical measure μ_f supported on the super-attractor. As in the case of ACIP, we can lift the measure μ_f to a physical measure

μ_X for the flow X^t on the contracting Lorenz attractor Λ. Again we shall first pass through the physical measure for Poncaré map P.

6.2.1 Physical Measure for the Poincaré Map

Let $\{z,\ldots,f^{k-1}(z)\}$ be the attracting periodic orbit for the map f. It follows that f^k has an attracting fixed point $z \in I$. This implies that the corresponding iterate of the Poincaré map P^k has an invariant stable leaf γ_z. As P^k restricted to the invariant stable leaf γ_z is a contraction on a disk, it necessarily has some fixed point $p \in \gamma_z \subset \Sigma$. It is not difficult to see that $\{p,\ldots,P^{k-1}(p)\}$ is an attracting periodic orbit for P. Hence

$$\tilde{\mu}_P = \frac{1}{k}\left(\delta_p + \cdots + \delta_{P^{k-1}(p)}\right) \tag{6.2.1}$$

is a physical measure for P. Then it can be seen easily that the lift μ_P of the measure

$$\mu_f = \frac{1}{k}\left(\delta_z + \cdots + \delta_{f^{k-1}(z)}\right)$$

defined in the similar way as in (6.1.1) coincides with the measure $\tilde{\mu}_P$.

6.2.2 Physical Measure for the Flow

Assume now that $\{p,\ldots,P^{k-1}(p)\}$ is an attracting periodic orbit for the Poincaré map P on Σ. It is straightforward to check that the orbit of p is an attracting periodic orbit for the flow of the vector field $X : U \to \mathbb{R}$. For each $j = 0,\ldots,k-1$, let τ_j be the time the flow of X takes to get from $P^j(p) \in \Sigma$ to $P^{j+1}(p) \in \Sigma$. Given any continuous $\varphi : U \to \mathbb{R}$, define

$$\int \varphi d\tilde{\mu}_X = \frac{1}{\tau_0 + \cdots + \tau_{k-1}} \sum_{j=0}^{k-1} \int_0^{\tau_j} \varphi(X(P^j(p),t))dt. \tag{6.2.2}$$

It is not difficult to see that $\tilde{\mu}_X$ coincide with the measure μ_X, on the contracting Lorenz attractor Λ, which is defined in the similar way as in (6.1.2) through the measure $\tilde{\mu}_P$. Hence $\tilde{\mu}_X$ is a physical measure for the flow of the vector field X.

6.3 Inverse Procedure

In the previous sections we have seen a procedure of defining a physical measure μ_X for the flow of the vector field X corresponding to a physical measure μ_f for the map f. In this section our aim is to describe an inverse procedure, i.e., we try to define a physical measure $\hat{\mu}_f$ for the map f corresponding to the physical measure μ_X for the flow of the vector field X such that $\hat{\mu}_f$ coincide with the measure μ_f.

6.3.1 Physical measure for the Poincaré map

Viana and Oliveira, in [31], introduced a technic to define a physical measure $\hat{\mu}_P$ for the Poincaré map P provided the physical measure μ_X for the flow of vector field X. We may define $\hat{\mu}_P$ as follows:

For every $\rho > 0$, we denote $\Sigma_\rho = \{x \in \Sigma : \tau(x) \geq \rho\}$. Given any $A \subset \Sigma_\rho$ and $\sigma \in (0, \rho]$, define $A_\sigma = \{X^t(x) : x \in A \text{ and } 0 \leq t < \sigma\}$. Then observe that the map $(x,t) \mapsto X^t(x)$ is a bijection from $A \times (0, \sigma]$ to A_σ. We have the following Lemma.

Lemma 6.3.1. *[6] Let A be a measurable subset of Σ_ρ for some $\rho > 0$. Then the function*

$$\sigma \mapsto \frac{\mu_X(A_\sigma)}{\sigma}$$

is constant in the interval $(0, \rho]$.

Given any measurable subset A of Σ_ρ, we define

$$\hat{\mu}_P(A) = \frac{\mu_X(A_\rho)}{\rho},$$

and given any measurable subset A of Σ

$$\hat{\mu}_P(A) = \sup_\rho \hat{\mu}_P(A \cap \Sigma_\rho).$$

Then $\hat{\mu}_P$ is a physical measure for the Poincaré map P [31]. It can be deduce through easy calculations that $\hat{\mu}_P = \mu_P$ on the cross-section Σ.

6.3.2 Physical measure for the One-dimensional Map

We want to define an inverse procedure for the lift defined by (6.1.1), i.e., assign to a P-invariant measure μ_P on Σ an f-invariant measure $\hat{\mu}_f$ on I whose lift coincides with μ_P. The natural candidate is the push-forward by π, the projection from the Poincaré section onto the interval,

$$\hat{\mu}_f = \pi_* \mu_P. \tag{6.3.1}$$

Let us now see that the lift $\hat{\mu}_P$ of $\hat{\mu}_f$ actually coincides with μ_P. Observe that for all continuous $\phi : \Sigma \to \mathbb{R}$ and all $n \in \mathbb{N}$ we have

$$(\phi \circ P^n)^- \circ \pi \leq \phi \circ P^n \leq (\phi \circ P^n)^+ \circ \pi.$$

It follows that

$$\int (\phi \circ P^n)^- \circ \pi d\mu_P \leq \int \phi \circ P^n d\mu_P \leq \int (\phi \circ P^n)^+ \circ \pi d\mu_P.$$

Using the fact that μ_P is P-invariant and basic properties of the push-forward, we deduce from the above inequality that

$$\int (\phi \circ P^n)^- d(\pi_* \mu_P) \leq \int \phi d\mu_P \leq \int (\phi \circ P^n)^+ d(\pi_* \mu_P).$$

Using (6.3.1) in the above inequality, we get

$$\int (\phi \circ P^n)^- d\hat{\mu}_f \leq \int \phi d\mu_P \leq \int (\phi \circ P^n)^+ d\hat{\mu}_f.$$

Taking limits in n in the above inequality and using (6.1.1), we conclude

$$\int \phi d\hat{\mu}_P = \int \phi d\mu_P,$$

which finally gives $\hat{\mu}_P = \mu_P$.

6.4 Final Remark: work in progress

It is an interesting problem to prove that the inverse procedure of defining a physical measure for the map f given a physical measure for the flow of X is continuous, which is our work in progress. Then as a corollary of Theorem B, we may conclude the statistical stability of the contracting Lorenz flow.

References

[1] Alves, J. F. (2004). Strong statistical stability of non-uniformly expanding maps. *Nonlinearity*, 17(4):1193.

[2] Alves, J. F., Bonatti, C., and Viana, M. (2000). SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Inventiones mathematicae*, 140(2):351–398.

[3] Alves, J. F. and Soufi, M. (2012). Statistical stability and limit laws for Rovella maps. *Nonlinearity*, 25(12):3527.

[4] Alves, J. F. and Soufi, M. (2014). Statistical stability of geometric lorenz attractors. *Fundamenta Mathematicae*, 224:219–231.

[5] Alves, J. F. and Viana, M. (2002). Statistical stability for robust classes of maps with non-uniform expansion. *Ergodic Theory and Dynamical Systems*, 22(01):1–32.

[6] Araujo, V., Pacifico, M., Pujals, E., and Viana, M. (2009). Singular-hyperbolic attractors are chaotic. *Transactions of the American Mathematical Society*, 361(5):2431–2485.

[7] Araújo, V. and Pacifico, M. J. (2010). Three-dimensional flows, volume 53 of ergebnisse der mathematik und ihrer grenzgebiete. 3. folge. a series of modern surveys in mathematics [results in mathematics and related areas. 3rd series. a series of modern surveys in mathematics].

[8] Benedicks, M. and Carleson, L. (1985). On iterations of $1-ax^2$ on $(-1,1)$. *Annals of Mathematics*, pages 1–25.

[9] Benedicks, M. and Carleson, L. (1991). The dynamics of the Hénon map. *Annals of Mathematics*, 133(1):73–169.

[10] Benedicks, M. and Young, L.-S. (1992). Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps. *Ergodic Theory and Dynamical Systems*, 12(1):13–37.

[11] Bowen, R. and Ruelle, D. (1975). The ergodic theory of Axiom A flows. In *The Theory of Chaotic Attractors*, pages 55–76. Springer.

[12] Cui, H. and Ding, Y. (2015). Invariant measures for interval maps with different one-sided critical orders. *Ergodic Theory and Dynamical Systems*, 35(3):835–853.

[13] De Melo, W. and Van Strien, S. (2012). *One-dimensional dynamics*, volume 25. Springer Science & Business Media.

[14] Freitas, J. M. (2005). Continuity of SRB measure and entropy for Benedicks–Carleson quadratic maps. *Nonlinearity*, 18(2):831.

[15] Freitas, J. M. (2006). *Statistical stability for chaotic dynamical systems*. PhD thesis, Ph. D. thesis, Universidade do Porto, 2006, http://www. fc. up. pt/pessoas/jmfreita/homeweb/publications. htm.

[16] Freitas, J. M. (2010). Exponential decay of hyperbolic times for Benedicks–Carleson quadratic maps. *Port. Math*, 67:525–540.

[17] Guckenheimer, J. (1979). Sensitive dependence to initial conditions for one dimensional maps. *Communications in Mathematical Physics*, 70(2):133–160.

[18] Guckenheimer, J. and Williams, R. F. (1979). Structural stability of Lorenz attractors. *Publications mathématiques de l'IHÉS*, 50:59–72.

[19] Keller, G. (1982). Stochastic stability in some chaotic dynamical systems. *Monatshefte für Mathematik*, 94(4):313–333.

[20] Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the atmospheric sciences*, 20(2):130–141.

[21] Metzger, R. J. (2000). Sinai-Ruelle-Bowen measures for contracting Lorenz maps and flows. In *Annales de l'IHP Analyse non linéaire*, volume 17, pages 247–276.

[22] Morales, C. A., Pacifico, M. J., and Pujals, E. R. (2004). Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers. *Annals of mathematics*, pages 375–432.

[23] Rovella, A. (1993). The dynamics of perturbations of the contracting Lorenz attractor. *Boletim da Sociedade Brasileira de Matemática-Bulletin/Brazilian Mathematical Society*, 24(2):233–259.

[24] Ruelle, D. (1976). A measure associated with Axiom A attractors. *American Journal of Mathematics*, pages 619–654.

[25] Sinai, Y. G. (1972). Gibbs measures in ergodic theory. *Russian Mathematical Surveys*, 27(4):21.

[26] Smale, S. (1998). Mathematical problems for the next century. *The Mathematical Intelligencer*, 20(2):7–15.

[27] Soufi, M. (2012). *Statistical stability for Rovella maps and Lorenz-like flows*. PhD thesis, Ph. D. thesis, Universidade do Porto, 2012.

[28] Thunberg, H. (2001). Unfolding of chaotic unimodal maps and the parameter dependence of natural measures. *Nonlinearity*, 14(2):323.

[29] Tucker, W. (1999). The Lorenz attractor exists. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 328(12):1197–1202.

[30] Viana, M. (1997). *Stochastic dynamics of deterministic systems*, volume 21. IMPA Rio de Janeiro.

[31] Viana, M. and Oliveira, K. (2016). *Foundations of ergodic theory*, volume 151. Cambridge University Press.