Social and economic games

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Ilustração da página anterior:

Amadeo de Souza-Cardoso
Os Galgos, 1911, óleo sobre tela, 100 x 73 cm

To my grandmothers,

with whom i started every cycle,

except the next one.

Acknowledgments

The task of acknowledging seems cold, in its core word, as i would like to dedicate warm thank you's, and it appears as subtly hard, since leaving someone out is a mere certainty. As such, i would like to restrict my attention to formal academic acknowledgments that have directly contributed to the work in hand.

I would like to start by thanking my mother Maria José Araújo and my father Renato Soeiro (the real one), for teaching me how to play games, how to interpret the world, how to think, how to be critique and how to doubt all that with a smile at hand. My brothers Francisco and José, and my sister Ana, for growing with me and sharing the empirical study on the world of interactions, from day one. I would like to thank all my family for providing all the tools along the way. I would also like to extend my gratitude to some friends who have discussed with me the work and its implications: all the people at Porto jazz school, in particular to Pedro Barreiros, and my partners in crime in combo G. I would like to thank all the people from *cascalhão, estouvados assumidos* and my *mathematical brides*, for always acompanying and supporting me throughout the fieldwork, and in particular Helena Peixoto for all the technical support. I would like to thank Nuno Moniz for the generous and insightful discussions, and in the international department, Miguel Semedo and Daniel Viana. I would like to thank Ana Barreira for being attentive. A special thank you to Luísa Matos, in particular for the patience of ilustrating boring subjects.

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Abstract

The work is divided in two parts. We start by studying a finite decision model where the utility function is an additive combination of a personal valuation component and a social interaction component. Individuals are characterized according to these two components (their valuation type and externality type), and also according to their crowding type (how they influence others). The social interaction component has two main properties, it is formed by dyadic interactions and based on whether individuals make the same or different decisions. We characterize pure and mixed Nash equilibria, namely through the study of type symmetries imposed by the social profile on the personal valuation space. In the second part we study duopolies where firms engage in a Bertrand competition and consumers choose strategically taking into account the consumption choice of other consumers. We propose an index that measures the *social propensity* of a market and allows a classification of markets according to the social interdependence of its consumer choices. Through the notion of subgame-perfect equilibrium, with first stage local pure price equilibrium for firms, we characterize local market equilibria and show that with social propensity duopolies have non-monopolistic outcomes. Furthermore, we characterize prices, demand and revealed personal preferences.

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Chapter 1

Introduction

The image chosen to feature on the cover page of this thesis is a work by portuguese painter Amadeo Souza Cardoso, called *Os Galgos*. In the 1911 painting, the sugestion of action about to unroll relates to two main ideas behind the present work. Interaction can be seen through the eyes of the symmetry effect it induces between the characteristics of actors and actions. As context changes, interaction induces a dynamic interdependence on the change of actions. The first part of this work develops the first idea and studies the dependence of actions on the symmetries that the social characteristics of individuals and their interactions impose on the space of personal profiles. This is done through the equilibrium notion on a decision game. The second part of the work builds on the second idea, finding local pure price solutions for a duopoly derived from the implicit changes that interactions provoke, and proposing a classification of markets according to an index of social interdependence of its consumer choices.

The decision game

A decision is in general a course of action resulting from a process which involves selecting among several possible alternatives. The collective co-existence and constant interaction of individuals necessarily creates a social frame in which decisions are made and a social context to which the decision leads. Regardless of whether these interactions are voluntary or not, they play a significant role in the global patterns of behavior that emerge from the individual decisions. Understanding what underlies a global behavior means understanding not only the interactions among decision-makers, the personal evaluations of alternatives and the interdependence between the two, but also having a grasp on the relation between the characteristics of the decision-makers and the characteristics of the global outcome. In fact, each decision composing this outcome, conveys information about the decision maker, as it reveals a choice, be it either a selection of a product to buy or a public service; be it an economic strategy, a political option or a social behavior; be it a life changing choice or a daily life decision, like choosing a bar to go to friday night. Thus, the study of the global behaviour both presupposes and enhances an understanding of what governs individual decisions. This is particularly relevant if it is assumed that individuals act rationally and the choice is a best response over the evaluation of the alternatives, in the sense of the existence of a Von Neumann-Morgerstern utility (1944).[1] At the core of a game theoretical approach to the problem is the modelling of interactions between decision-makers. Assuming decisions as a global mutual best response, one may use the concept of Nash equilibrium (1951) to retrieve information not only on a global interacting level, but also on an internal individual level, by analysing the interdependence of these social

[1]The issue of rationality is beyond the scope of this work, nevertheless, as we will be looking at the outcome and not at the decision process itself, and as we will be working in a complete information setting where the parameters are open to interpretation, underlying is in fact a very mild rationality assumption.

characteristics and the individual personal evaluation processes. The focus of the first chapter is the relation between the characteristics of individuals and the characteristics of the outcome, where the outcome is seen as the set of Nash equilibria of a finite non-cooperative game.

Positioning the approach

The study of the dependence of global behaviour (as an equilibrium) on the characteristics of individuals, and in particular the study of the dependence of individual decision rules on the strategies of others, has a long tradition. Namely, in the work of Schelling (1971, 1973, 1978) where, for example, different distributions of the level of tolerances of individuals lead to residential segregations with different properties; or in the work of Granovetter (1978), where small differences on the distribution of individual thresholds can lead to completely different collective behaviour; or in Mas-Collel (1984), Pascoa (1993) where an atomless distribution of types leads to symmetric equilibria; or other symmetry properties as in Wooders, Cartwright and Selten (2006) and Wooders and Cartwright (2014), which, as in this work, describe partitions of the set of players into groups that arise in equilibrium.

Our approach is to model the outcome of a decision process as the Nash equilibria of a finite (both in players and strategies) non-cooperative, simultaneous move game. The value of a given decision is measured through an utility function that is an additive combination of two components: (i) how much the individual personally values the decision, independently of the strategies of others; (ii) the externalities arising from social interactions with those individuals who make that same decision. This is, of course, a very broad class of utility functions included in many models in the literature. The crucial aspect is the choice of how to model the form of dependence on the strategies of others, i.e. how to model social interactions. Let us highlight three main features of our approach to this choice, and position our work in relation to the different approaches in the literature.

A first key feature we consider is dyadic interactions (see for example [8], [20]). Dyadic means that, for any given strategy, the influence/impact of an individual i on an individual j is independent of the decisions of others. A class of games that focuses only on this kind of interactions is for example the class of polymatrix games, see [23], [35].[2] Another option would be to introduce (also or instead) a dependence of this influence on the whole strategy profile. In general, excluding such a component usually means excluding some form of non-linear anonymous aggregate dependence on the strategies of others. In fact, many games can be captured by an appropriate dyadic component by using such an excluding assumption, as for example by making the appropriate restriction on singleton weighted congestion games, or on the games presented in [8], [10], [36]. The dyadic component is sometimes refered to as the local component of social interactions and the latter dependence as the global component.[3] Focusing on dyadic interactions seems like a suitable approach for the case we wish to study.

A second feature is assuming the influence from interactions to be presence-based. Presence, means social interactions have a dichotomic nature, in the sense of being restricted to whether individuals are using the same strategy or a different one, a type of Independence of Irrelevant Alternatives assumption [37]. In this work it can be better described and motivated in the following manner: given a strategy profile, if an individual i changes her decision, the change only affects those in her new decision, because she will start interacting with them, and those in her old decision, because she will no longer interact with them. Her influence on the rest of the individuals was that she was making a different decision, and that hasn't changed. This is also in the spirit of Independence of Irrelevant Choices as in [27], or no

[2] A first formal reference appears to be due to E. B. Yanovskaya in 1968.

[3] The use of the terms local and global in this context seem amenable to critique, since one could think of 'global dyadic components' or 'local aggregative components', hence we prefer the terms dyadic and aggregative.

spillovers as in [28]. This assumption is important for some of our results in the first chapter, which would not hold without it.

The third feature is that we will allow for social interactions to give rise to both positive and negative externalities, and then study their effects on equilibria and society formation. The question here could be whether to restrict the dependence to be in some sense 'positive' or complementary, leading to a conformity effect; or 'negative' leading to a congestion effect. On the strand of literature that treats conformity effects (those leading to a common (or symmetric) action which may overcome personal or intrinsic preferences) are for instance the works on behavioral conformity by Wooders, Cartwright and Selten [52], a theory of conformity by Bernheim [7], a model of herd behavior by Banarjee [5], the threshold models of collective action as in Granovetter [21], or even the equilibrium symmetry in supermodular games as in Cooper and John [15]. On the strand of literature focusing on congestion effects is for example the class of congestion games as first proposed by Rosenthal [38], later generalized by Milchtaich [30]; or the works of Quint and Shubik [36], Konishi, Le Breton and Weber [27], to name a few.

Social interactions, regardless of whether they exhibit a conformity or congestion effect, should depend not only on the number of individuals in each choice, but also on the characteristics of those individuals. This is a crucial aspect in the works of Wooders ([49, 50, 51]) and of Conley and Wooders ([12, 13, 14]). Wooders's earlier papers allow preferences to depend on the characteristics of agents (their types), while Conley and Wooders separate two sorts of characteristics: crowding characteristics, which determine the effects of a player on others, and tastes. In our model we will use a type profile that characterizes individuals, or distinguishes, according to three different aspects, or attributes. (Keep in mind though that for us type does not mean Bayesian type, as we will be working on a complete information setting and the type profile is something completely determined a priori.)

Following the work of Conley and Wooders, we will start with the use of a crowding space, which distinguishes individuals by their impact on the utility of others. The use of a crowding space has the advantage that allows the characterization of classes of strategies where the relevant information is the number of individuals with the same crowding type in each decision. Observe that there is no restriction here: depending on the choice of the crowding space individuals may be all distinguishable or totally anonymous. We then characterize individuals according to their utility function, i.e. taste type, but we will subdivide the taste type into two components, using the two additive components of the utility function. This allows the characterization of Nash equilibria according to the restrictions imposed on the relation between these two components. Furthermore, dividing the taste type in this way, separates the social part of the model, that captures the social interactions, from the 'personal' part given by the valuation component (sometimes called intrinsic preference, which we wittingly avoid). A key advantage of the separate analysis of the valuation component is that, besides comprising the intrinsic and personal perceived benefit of the decision, it captures exogeneous changes and/or characteristics associated to each decision. Namely, dependending on the decision in question, it may represent prices, taxes, product quality, road quality, marketing, political campaings, bribes, etc... In particular, for the second part of this thesis, the valuation component reveals how individual choices depend on prices.

The work on the first chapter starts as an extension of the two types dichotomic model by Soeiro et al. [43] to a wider finite setting where there may be any number of types of individuals facing a choice among any number of possible alternatives, and is primarily based on Soeiro et al. [42]. The former work finds its inspiration from Brida et al. [9], a socio-economic model that analyses how the choice of a service is influenced by the profile of users of that same service; and, on a different line, from Almeida et al. [2], where game theory and the field of social psychology are related through the theories of Planned Behavior or Reasoned Action, proposing the Bayesian-Nash equilibrium as one of the many possible mechanisms behind the transformation of human intentions into behaviors.

The duopoly game

A main driver of the study of price competition in oligopoly theory has been the determination of factors that, within a simple analytical framework, can sustain a pure price equilibrium with firms earning positive profits. In the context of a Bertrand competition, with price as the only strategic variable and uniform pricing (the same for all consumers), the search for asymmetric equilibria assumes particular relevance. In general, the departure from the paradoxical zero profit equilibrium involves either breaking the symmetry on the firms side or on the consumers side, by introducing some degree of heterogeneity. These are the general Hotelling vs Edgeworth approaches. An asymmetry on the firms side of the market is usually introduced through the Edgeworthian approach, by allowing different cost structures or capacity constraints, which often leads to indeterminateness of prices and non-existence of pure price equilibrium. This is dealt with some appropriate continuity assumptions and produces mixed strategy price solutions, which are in general hard to compute and often face critiques as to their interpretation. On the consumers side of the market (what we abusively called the Hotelling approach), the asymmetry usually relies on some heterogeneity, either from the usual vertical or product differentiation, or from other sources like search or switching costs, incomplete information or different price sensitivities. In some cases pure price solutions are known to exist, although also depending on appropriate assumptions which essencially rely on heterogeneity and some further continuity assumption (see for example [11] and references therein). Other general approaches to the problem would involve leaving the standard Bertrand framework and considering other strategic variables for firms, for example by allowing firms to also compete by choosing quantities as in Cournot, choosing/investing in quality, or others. Note that solutions based on the temporal dimension can in fact be seen as introducing timing as a strategic variable. [4]

From a game theoretic perspective though, there is an inherent strategic asymmetry in the original Bertrand framework: the set of players is composed of firms and consumers, and while firms play strategically and their best response depends on the whole strategy profile (prices and consumer choices), consumers best response depends only on prices, hence ignoring part of the game's strategic profile. Notwithstanding that in many markets this assumption may still be appropriate (like the classical mineral water example), in most markets today, consumption behavior shows increasing *social propensity*. In particular with the growth of internet, the emergence of social networks and the increase of data availability, the asymmetry of information between firms and consumers has reduced, and the very role that consumers play on each others' choices is today of greater importance. Moreover, the idea of consumption externalities, its relevance and economic implications are now better understood and extensively studied (see for example [25], [19] or [24]).

The decision game of the first chapter provides a road to close the strategic asymmetry gap and create duopolistic market solutions in pure price strategies. In a first stage firms simultaneously choose prices and in a second stage consumers play a decision game based on those prices. A market equilibrium will be a subgame perfect equilibrium of this two stage game.

[4] For general reviews we refer to the classical book references on industrial organization, for example [46, 48] or microeconomic analysis, e.g. [47].

Socially prone duopolies

A decisive main characteristic of a market is how demand changes, in particular how it reacts to price. The introduction of strategic interaction among consumers changes the price elasticity of demand and in general disrupts the zero profit paradox. Whether it is a more intricated form of social influence, status seeking or a simpler consumption externality, consumers might prefer or get stuck in a more expensive service, or at least have a smoother reaction to price undercutting strategies. The effect of a price deviation is mitigated or amplified by the effect consumers exert upon eachother, and the demand behavior no longer responds solely to the price difference. We propose a *social propensity index* to characterize markets and outcomes according to how changes are captured by the social component of a market. The interpretation is that it reveals how consumers may change their strategy in response to local changes in the overall consumer profile, which is reflected in the demand response to prices. We say that a duopoly is *socially prone* if it has a non-null social propensity index.

In the finite case, the drawback in having consumers act strategically is the coordination problem posed by the multiplicity of equilibria, that now extends beyond the region where firms charge the same price. Furthermore, for pure price solutions to exist with both firms earning positive profits, it is necessary to ensure a continuous demand response to price deviations in the neighbourhood of an equilibrium. Our approach to solve both these problems is to assume that, locally, consumers using pure strategies will continue to use pure strategies, and consumers using mixed strategies will continue to use mixed strategies. This will implicitly define a unique local continuous response to small price deviations, which works as a natural coordination device for firms. Naturally, there are discontinuous alternatives, which are credible since they are a Nash equilibrium of the consumers subgame, nevertheless they seem less plausible from an economic perspective. It's hard to envision a situation where firms believe that small price changes create a disruptive consumer behavior, especially when there is a credible smooth alternative. We prove the existence of local market equilibrium for these continuous deviations, with shared demand and positive profits. We characterize prices and show that equilibria reveal consumers personal preferences. The conditions are rather general and rely exclusively on the properties of the social profile of consumers through the social propensity index. Socially prone duopolies thus disrupt the Bertrand paradox and provide pure price solutions. These solutions do not rely on heterogeneity to exist nor to be asymmetrical.

Social propensity index and local influence network

In the duopoly case under consideration, the social profile is based on a social externality function whose properties will be inherited by demand, and reveal how consumers interact and the interdependence of their choices. The choice and characteristics of the social externality function are thus decisive to determine the type of duopoly and consequent results. Two main lines can be identified as crucial in this choice: the degree of social heterogeneity and its functional form. Nevertheless, to understand demand changes, the crucial aspect is not the externality function itself, but rather how changes in some consumer strategy affect the rest of consumers. As an example, think of a social network like facebook. There may be a large network of conections between users, which naturally provoke externalities, but the decision itself depends on how users look and interpret this conections, how they are influenced by them, which need not be by the whole of their conections. In our context this superstructure within the actual consumer network structure is what we call the *local influence network*. The nodes in the network are consumers and the edges represent the influence two consumers have on eachother, which is dependent on the context created by the consumers choice. The network is thus directed, weighted and state-dependent. Hence, consumers may have diferent influence on eachother, and that influence need not be symmetric nor have the same value throughout the network. Furthermore, it is state-dependent in the sense that the weight will depend on the consumers choice. A natural way to represent the network is through its weighted adjacency matrix. In order to use the standard notation and to provide a more intuitive representation, an weighted directed edge from i to j should represent the influence i has on j, which should be the value of the entry ij. The adjacency matrix of the influence network is defined as the transpose of the Jacobian matrix of *social differentiation*.[5] For a given consumers choice, social differentiation is the difference between the externalities consumers incur in each service, at that 'moment'. The influence network reveals changes in social differentiation provoked by a change in consumers strategy. This in turn provokes changes in the consumers utility differential. Note, however, that for consumers using pure strategies, this may not result in a strategy change if the Nash equilibrium condition is not strict. We call consumers using pure strategies, loyal consumers. When a change needs to be sufficiently high to result in a change of their best response, we say that loyal consumers have *lower sensitivity*. This means that for interior points the crucial aspect to capture local changes in demand is the non-loyal consumers influence network. The idea that loyal consumers may have lower sensibility and not always contribute to social propensity is rather natural, and intuitive to the very notion of brand loyalty. We observe though, that loyalty differs from installed base, since being loyal is a strategical behavior (those who opt for pure strategies) and not an exogeneously imposed choice, or a choice deriving from some switch cost or other stabilizing variable. Each network has a social propensity index which proposes to give information on how consumers react to changes and classify markets

[5] We hope this comment avoids more confusion than it creates. This is just a clarification, as it will only be used in the graphical representation.

accordingly. For markets with social propensity, pure price equilibrium exist and prices will be dependent on social propensity. A negative index will slow down demand response to price undercutting strategies, as for some consumers the incentive of turning to a cheaper service is overcome by the externality, similarly to a congestion, snob or Veblen effect. A positive index amplifies the demand response similarly to a conformity, herd or bandwagon effect, allowing firms to take advantage of the contextual presence of some consumers and in general leads to monopolistic settings or type-symmetric consumer response to prices. Interestingly, when consumer interactions are dyadic the externality becomes additively separable and social propensity locally constant. In this case equilibrium demand varies linearly with price, proportionally to social propensity, and the consumers profile determines the equilibria. So preferences are revealing, besides revealed.

Related literature

The literature on price competition is vast. We will refrain from a general review and focus on the main distinguishing features of our approach in relation to the literature. Rubinsteind and Osborne ([34], page 6) define the oligopoly problem as centered around the potential indeterminateness of price equilibria with a few number of competitors. We build up the duopoly on a finite set of consumers and are able to stabilize prices in pure strategies by allowing consumers to use mixed strategies, which, with a consumption externality, exist on a conected price region. The majority of the literature approaches the existence problem departing from a continuous set of consumers. This doesn't mean the approaches have many qualititively different results. Naturally, mixed strategies are linear combinations of pure strategies, and this poses a limitation to the kind of social externality we are considering. Nevertheless, our approach seems to hold for any C^1 externality function defined over a continuous space of strategies. An advantage of the approach is that pure price solutions based on consumer mixed strategies provide in general simpler analitical solutions and more natural interpretations than mixed strategies for firms. Consumption has in many contexts a smaller temporal frame than pricing strategies, and may be seen as a repeated choice on the temporal frame of pricing. Furthermore, one need not consider loyal consumers as installed bases of non-strategic consumers, but as consumers using pure strategies, hence being strategic consumers.

In our approach we allow for a general externality function, which can be either aggregated or based on a network, and we impose no specific fuctional form (nor require). We assume uniform pricing (the same price for all consumers), negative prices are not allowed and consumption is mandatory (in the duopoly consumers do not have a third option of not buying). We characterize local pure price solutions and study the effects of the heterogeneity of consumers and the effects of symmetries of the consumer profile. However, we do not need nor require heterogeneity or homogeneity of consumers. These are the main features that allow to position our work and its contribution.

An interesting survey including consumer demand under network effects and social influence can be found in [45]. On the literature and importance of network economics [24], [19], [25]. The existence of a pure price equilibrium for example in [11] or [16]. On price competition subject to aggregated consumption externalities, which do not depend on specific consumers but on an aggregated demand variable, we highlight the conection with the work of Grilo et al.(2001) [22] that "*combine the consumption externality model and the spatial models of product differentiation*". Nevertheless, the results are essentially derived under a specific functional form, firms have installed bases, the consumers set is continuous and heterogeneity based on spatial models. Hackner, Nyberg (1996) [26] study welfare aspects of negative reciprocal externalities, of which congestion is a special case. Acemoglu and Ozdaglar

(2007) [1] analyze price competition and efficiency of oligopoly equilibria where the allocation of network flows is subject to congestion costs captured by a route-specific nondecreasing convex latency function. We do not focus on negative or positive externalities, but allow both forms. On (positive) network externalities there is strand of literature building on Katz and Shapiro (1985), but which assumes Cournot competition, rather than price competition. On price competition with a consumer network Banarji and Dutta (2009) study the dependence of market segmentation on the underlying network structure, consumers have however the option of not buying. A recent ongoing work by Aoyagi (2013) [4] studies the dependence of the equilibrium on the underlying network structure, but in this case negative prices are allowed. An interesting work by Allen and Thisse (1992) in [3], although assuming non-strategic consumers is worth mention due to the study of pure price equilibria for an homogeneous product oligopoly market where consumers have different price sensitivities.

Our focus is on uniform pricing, however, there is an interesting growing literature on price discrimination which does not connect directly to the present work, but could be an interesting future possibility (see for example Fainmesser and Galeotti (2015) [17] and references therein).

Organization of the work

The work unfolds as follows: in the next chapter, first section, we set up the model and present a map characterizing the profile of individuals; in section 2.1.1 we present a conformity obstruction lemma which allows us to characterize the conditions in the individuals profile for a given strategy to be admissible or feasible as a Nash equilibrium; in section 2.2.2 we present the relation of our model to the concept of society introduced in [52]; in section 2.2.3 we define the Nash domain of a strategy (in terms of utility parameters) and characterize it completely; in section 2.2.5 we discuss conditions on social profile for the existence problem to be independent of the personal profile; and finally, in the end of the sections we prove the results.

In the third chapter we start by setting up the duopoly game in the first section, and in section 3.2 we present our main result characterizing market equilibria. In the following section 3.3 we study symmetry properties of the influence weights and the dependence of the underlying network, and in section 3.4 we study the case where consumer interaction is dyadic.

In the final chapter we present conclusions and directions for future work.

In the appendix we present a game and interface programmed for Netlogo.

Software

The free software R was used for computations, simulations and for the network figures in the second chapter, where routines where created for future applications. Netlogo was used for testing with the class of decision games up to 4 types and 4 possible actions, programming the game in appendix. The rest of the figures regarding tilings, Nash domains, monopolies and demand have been made using Adobe Illustrator (except for figure 2.5 which was computed using Matlab). The option of using Illustrator had the aim of better conveying ideas, hence improving images which attempt to ilustrate an idea rather than providing rigourous mathematical statements.

Notation

Throughout the work we will use in general: boldface for variables that convey information about the whole set of players, called generally profiles; caligraphic letters for spaces of such profiles and greek letters for specific parameters of a game. The symbol $\equiv$ is used for definitions.

Chapter 2

The decision game

The main idea behind the results in this chapter is that the difference in payoffs of similar individuals is bounded by the externality they provoke on eachother. Using the type map this allows us to consider a social profile and characterize the set of equilibria according to the symmetries imposed on the space of personal preferences. We study how positive externalities lead to strong type symmetries, while negative externalities allow the existence of equilibria that are not type-symmetric and have a stronger dependence and sensibility to the parameter space.

2.1 Game setup

The decision model we present is based on a finite non-cooperative game. We consider a finite set of individuals $\mathcal{I} \equiv \{1, \ldots, n_{\mathcal{I}}\}$, each having to choose independently an element from a finite set of alternatives $\mathcal{A} \equiv \{1, \ldots, n_{\mathcal{A}}\}$ (the common strategy set).[1] We describe the decisions of the individuals by a *strategy map* $s : \mathcal{I} \to \mathcal{A}$ associating to each individual $i \in \mathcal{I}$ her decision $s_i \equiv s(i) \in \mathcal{A}$ and defining a (pure) strategy profile $\mathbf{s} = (s_1, \ldots, s_{n_{\mathcal{I}}}) \in \mathcal{S} \equiv \mathcal{A}^{n_{\mathcal{I}}}$. The strategy profile $\mathbf{s}$ has a value for each individual $i \in \mathcal{I}$ determined

[1] Later on we show that it is possible to consider that individuals have different set of actions $\mathcal{A}^i$ and the results will hold.

by an *utility function* $u : \mathcal{I} \times \mathcal{S} \to \mathbb{R}$ and denoted $u(i; \mathbf{s})$. The game consists in each individual independently making a choice that maximizes her value.

Personal and social separability (PSS). *The utility function has the personal and social separability property if there are maps* $\omega : \mathcal{I} \times \mathcal{A} \to \mathbb{R}$ *and* $e : \mathcal{I} \times \mathcal{S} \to \mathbb{R}$ *such that*

$$u(i; \mathbf{s}) = \omega(i, s_i) + e(i; \mathbf{s}).$$

With the PSS property the utility becomes an additive combination of: (i) a *personal map* $\omega(i, a)$ which determines how much an individual $i \in \mathcal{I}$ personally values each alternative $a \in \mathcal{A}$, independently of the strategies of the others; and (ii) a *social externality map* $e(i; \mathbf{s})$ which determines determines the social impact of the strategy profile $\mathbf{s}$ on individual i, that is, the externalities arising from social interactions. In a first look it may look counterintuitive that the strategy of individual i is still part of both components. This is, however, the key point of driving the model into a game theoretical framework. If the strategy of individual i was not included in the social component, her best response would not depend on the strategy of other individuals, which would remove the interaction part of the model. The idea for the PSS property comes from the theories of planned behavior and reasoned action (see for example [2]), and a main advantage of the PSS property is allowing a separate analysis on the personal component and the social component of the decision. The variable also allows to explicitly acomodate variable transformations, for example, if each individual has its own action set $\mathcal{A}^i$, we can consider a common strategy set $\mathcal{A} = \bigcup \mathcal{A}^i$ and a personal map such that $\omega(i, a) = -\infty$ whenever $a \notin \mathcal{A}^i$. This leads to the same set of Nash equilibria since a will never be chosen by i. The separate study on the properties of the social externality map, means we are able to focus on the properties of the impact of others on individual's i decision.

A first natural property one may consider for the social component is dyadic interactions. Dyadic means that, for any given strategy, the influence/impact of an individual i on an individual j is independent of the decision of others, that is, social externalities are a result of pairwise interactions. Networks are a natural example of such interactions.

Dyadic interactions (DI). *The social externality map is based on Dyadic Interactions if for every individual $i \in \mathcal{I}$ it is additively separable in the strategy of the other individuals, i.e. for every strategy profile* $\mathbf{s}$ *it is given by*

$$e(i; \mathbf{s}) = \sum_{j \neq i} e(i, j; s_i, s_j).$$

A second natural property to consider is that social interactions are based on presence, in the sense of being restricted to whether individuals are using the same strategy or a different one. This restriction on social interactions is in line with some common assumptions in the game theoretic literature, as that of Independence of Irrelevant Choices in [27], or no spillovers in [28]. These are in general assumptions in the spirit of what's most commonly known as a type of Independence of Irrelevant Alternatives assumption (which has long been used, but sometimes differs depending on the context, see for example [37]). In our case, the assumption is in fact one of *dichotomic social influence*, as we stated in the introduction. That is, individuals are influenced by other individuals who make the same decision, and also by those who make a different decision, but just by the fact that they made a different decision, independently of what decision that is. With that in mind we call this a presence-based influence.

Presence-based influence (PBI). *The social externality map has the presence-based influence property if $e(i, j; s_i, s_j) = e(i, j; s_i, s'_j)$ whenever $s_j \neq s_i \neq s'_j$.*

A variable transformation as is done in [43] shows that with DI and PBI we can consider a map $\alpha : \mathcal{I} \times \mathcal{I} \times \mathcal{A} \to \mathbb{R}$ describing pairwise interactions determined by *social weight coordinates* α_a^{ij}, which may be interpreted as how much individual i is influenced by an individual j when they are both making decision a. (Note that in general this need not be a symmetrical map.) The component only depending on individual i is called *personal value coordinate* and denoted by $\omega_a^i \in \mathbb{R}$, and may be interpreted as how much an individual personally likes or dislikes to make a certain decision.[2] Let us denote the set of individuals who choose $a \in \mathcal{A}$ in a strategy profile $\mathbf{s}$ by $s^{-1}(a) \subset \mathcal{I}$. An utility function with properties PSS, DI and PBI can be written as

$$u(i; \mathbf{s}) = \omega_{s_i}^i + \sum_{j \in s^{-1}(s_i) \setminus \{i\}} \alpha_{s_i}^{ij}.$$

Let $\mathcal{U}$ be the space of such utility functions. For a given utility function $u \in \mathcal{U}$, we call the decision model with the above properties a *decision game* $\Gamma \equiv \Gamma(\mathcal{I}, \mathcal{A}, u)$. We will sometimes refer to decision games where there are only positive externalities as *social conformity games*; and to games where there are only negative externalities as *social congestion games*.

2.1.1 The profile of individuals

We will study different invariances that arise in a decision game where the utility function has the PSS, DI and PBI properties, and then characterize games from different invariance classes. These classes are related to how individuals may be distinguished in the game, be it either because they have different utility functions or because they have different impact on the utility function of others; or both. The interpretation is that an individual

[2] The idea is that we may assume $e(i, j; s_i, s_j) = 0$ whenever $s_j \neq s_i$ and obtain an isomorphic set of equilibria. Furthermore, we are in fact assuming that $\alpha_a^{ii} = \omega_a^i$. There is a slight abuse in using the same letter for the personal value coordinate and the personal map, but it makes things more clear, as they in fact represent the same thing, although with the variable transformation there might be a displacement.

is characterized in three main lines: how the individual *sees* the decisions, how she *sees* others and how others *see* her.

Crowding types

We start by analyzing the invariance derived from those characteristics of individuals that influence the utility of others. Following the work of Conley and Wooders ([12, 13, 14]), these are called the crowding type of the individuals. Let C be the set of possible crowding types and let $\mathbf{c} \equiv \mathbf{c}(\Gamma) = (c_1, \ldots, c_{n_{\mathcal{I}}}) \in \mathcal{C} \equiv C^{n_{\mathcal{I}}}$ denote the *crowding profile* of individuals in a decision game Γ. Two individuals $j_1, j_2 \in \mathcal{I}$ have the same crowding type $c_{j_1} = c_{j_2} = c \in C$ if for all $i \in \mathcal{I}$ and $a \in \mathcal{A}$ we have $\alpha_a^{ij_1} = \alpha_a^{ij_2} \equiv \alpha_a^{ic}$. We will use the standard notation $(s_i; \mathbf{s}_{-i})$ to represent strategy profile $\mathbf{s}$, but highlighting the component of individual i and the remaining strategy profile $\mathbf{s}_{-i}$. The utility function for an individual $i \in \mathcal{I}$ can be rewritten using the crowding space,

$$u^i(s_i; \mathbf{s}_{-i}, \mathbf{c}_{-i}) \equiv u(i; \mathbf{s}) = \omega_{s_i}^i + \sum_{j \in s^{-1}(s_i) \setminus \{i\}} \alpha_{s_i}^{ic_j}.$$

The use of a crowding space in the characterization of a game has the advantage that the utility of an individual i associated with a strategy profile $\mathbf{s}$ is invariant under permutations of strategies of other individuals with the same crowding type. Thus, the crowding space C induces a natural equivalence relation in the strategy space $\mathcal{S}$.

Externalities and valuations

The second step in our approach is to distinguish individuals according to the two additive components of the utility function, namely separating the part that measures the externality effects from the part that measures the individual's personal valuation of the alternatives $\mathcal{A}$. We will categorize individuals according to these two components so that we can then characterize a Nash equilibrium according to the restrictions it imposes on the relation between these two components. The two components are: (i) the column vector of personal personal values $\vec{\omega}_i \equiv \omega^i(\mathcal{A}) \in \mathbb{R}^{n_\mathcal{A}}$; and (ii) the matrix of social weights given to each crowding type, the *social externality matrix* $e_i \equiv e_i(\mathcal{A}, C) \in \mathbb{R}^{n_A \times n_C}$;

$$\vec{\omega}_i \equiv \begin{pmatrix} \omega_1^i \\ \vdots \\ \omega_{n_\mathcal{A}}^i \end{pmatrix}, \quad e_i \equiv \begin{pmatrix} \alpha_1^{i1} & \dots & \alpha_1^{in_C} \\ \vdots & \ddots & \vdots \\ \alpha_{n_\mathcal{A}}^{i1} & \dots & \alpha_{n_\mathcal{A}}^{in_C} \end{pmatrix}.$$

We observe that the impact of the personal value vectors $\vec{\omega}_i$ in this relation will not be given by the precise value of their coordinates, but rather by the relative preferences they induce, namely the difference between each pair of coordinates. That is, if a given decision d is a best response for an individual i, then if we changed her vector of personal values by the same amount in each coordinate, d would still be a best response. We will take this into account using a valuation space V with the following property: if two individuals $i, j \in \mathcal{I}$ have the same *valuation type* $v_i = v_j \equiv v \in V$, then their vectors of personal values are in the same relative valuation space. More precisely, the *relative valuation space spanned by* $\vec{\omega}_i$ is

$$W(\vec{\omega}_i) \equiv \{\vec{\omega}_i + k\vec{1} : k \in \mathbb{R}\}.$$

Hence, if the two individuals have the same valuation type v, then $W(\vec{\omega}_i) = W(\vec{\omega}_j)$. However, we do not ask the equivalence class to be maximal, i.e. there might be individuals with diferent valuation types $v_i \neq v_j$ such that the corresponding vectores of personal values $\vec{\omega}_i$ and $\vec{\omega}_j$ satisfy $W(\vec{\omega}_i) = W(\vec{\omega}_j)$. With a slight abuse of notation we will refer to the personal values vector of individuals with the same valuation type v as $\vec{\omega}_v$. The profile of personal value vectors of all individuals is denoted by $\boldsymbol{\omega} \equiv \boldsymbol{\omega}(\mathcal{I}; \mathcal{A}) \equiv (\vec{\omega}_1, \dots, \vec{\omega}_{n_\mathcal{I}}) \in (\mathbb{R}^{n_\mathcal{A}})^{n_\mathcal{I}}$. The valuation profile is denoted by $\mathbf{v} \equiv \mathbf{v}(\mathcal{I}) \equiv (v_1, \dots, v_{n_\mathcal{I}}) \in$

$\mathcal{V} \equiv V^{n_\mathcal{I}}$. Note that a profile of personal values might or not be compatible with a valuation profile. The set of all social externality matrices associated with the crowding profile $\mathbf{c}$ of a given game is described by the externality profile $\mathbf{e} \equiv \mathbf{e}(\mathcal{I}; \mathcal{A}, C) \equiv (e_1, \ldots, e_{n_\mathcal{I}}) \in \mathcal{E} \equiv E^{n_\mathcal{I}} \equiv (\mathbb{R}^{n_\mathcal{A} \times n_C})^{n_\mathcal{I}}$. We will use $\alpha_a^{e_i c_j}$ to refer to coordinates $\alpha_a^{i c_j}$ of an individual with externality type e_i.

Type map

The categorization of individuals can now be done according to their personal and social profile through a type map $t \equiv t_\Gamma : \mathcal{I} \to T$ which indicates the type of an individual in the type space $T = C \times E \times V$. The subscript on the type map (which we will omit) is there to reinforce that when we say type we do not mean bayesian type, rather the type map reveals symmetries of the utility profile, hence of a particular decision game Γ, and thus it is something known a priori. The type map defines a type profile for the game given by the triplet $\mathbf{t} = (\mathbf{c}, \mathbf{e}, \mathbf{v})$ in the space $\mathcal{T} = (C \times E \times V)^{n_\mathcal{I}}$, composed of: (i) a crowding profile $\mathbf{c}$ characterizing individuals according to their *crowding type*; (ii) an externality profile $\mathbf{e}$ characterizing individuals according to their *externality type*; and (iii) a valuation profile $\mathbf{v}$ characterizing individuals according to their *valuation type*. Note that the pair (e_i, v_i) is what is usually called an individual's taste type. An advantage of separating the taste into two components is that now the pairs (c_i, e_i) are responsible for the 'social' part of the model; they capture the social interactions in the model. We refer to this pair as the *social type* of an individual. The valuation type component v_i, that represents the way an individual values the possible choices, may be analysed separately.

The type profile of a decision game conveys information, or imposes restrictions, on the characteristics of its Nash equilibria. On the subsequent sections we will study the information one can retrieve about the structure of the utility profile of a decision game from studying the restrictions imposed by the type profile on the set of Nash equilibria.

A decision game has two main characteristics: (*i*) the type space, hence the number of types; and (*ii*) the number of decisions. We refer to different decision games by these two distinguishing features and denote the corresponding class of games by $\Gamma_{n_T,n_{\mathcal{A}}}$. Changing one dimension or the other has diferent impacts and produces different chalenges. Note for example that it is not possible to construct a game with only two actions without the presence-based influence property (PBI).

2.2 Pure Nash equilibria

A strategy profile $\mathbf{s}$ is a pure Nash equilibrium, if for every individual $i \in \mathcal{I}$, $u(i; s_i, \mathbf{s}_{-i}) \geq u(i; a, \mathbf{s}_{-i})$, for every $a \in \mathcal{A}$.

2.2.1 Feasibility

A first natural problem is whether individuals of the same type may use different strategies in a Nash equilibrium, hence, whether all Nash equilibria are type-symmetric. The first lemma is a result on the relation between a social type and its valuations in a Nash equilibrium. Let us start by defining for two individuals i and j the following measure of influence in a strategy profile $\mathbf{s}$, called their *influence relation*

$$R_{ij}(\mathbf{s}) \equiv \alpha^{ij}_{s_j} + \alpha^{ji}_{s_i}.$$

The influence relation reveals the bilateral externalities that two individuals would incur were they to change their decision. That is, if two individuals have a positive (resp. negative) influence relation in $\mathbf{s}$, then at least one of them would incur a positive (resp. negative) externality by changing (unilaterally) her strategy and joining the other in her decision. When $R_{ij}(\mathbf{s}) > 0$ we say that the individuals i and j have a *tendency to conform*, given by a *positive externality relation* in $\mathbf{s}$. Similarly, if $R_{ij}(\mathbf{s}) < 0$ we say that individuals i and j have a *negative externality relation* in $\mathbf{s}$.

Let $dist(\cdot, \cdot)$ be the distance given by the supnorm.

Lemma 1 (Conformity obstruction)**.** *Consider a decision game* Γ *and a Nash equilibrium* $\mathbf{s}$*. If* $i, j \in \mathcal{I}$ *and* $s_i \neq s_j$ *then*

$$dist(\vec{\omega}_{v_i}, \vec{\omega}_{v_j}) \geq R_{ij}(\mathbf{s})/2 - n_{\mathcal{I}} dist(e_i, e_j).$$

We call this an obstruction because, in the case of positive externalities, individuals need to be sufficiently different to make different decisions at a Nash equilibrium. Thus, their personal valuation of alternatives obstructs their tendency to conform to the decision of one another. When negative externalities are in place, this is not the case, since they do not have a tendency to conform. The conformity obstruction lemma leads to the following theorem for positive externalities.

Theorem 1 (Positive externality)**.** *Let* $i, j \in \mathcal{I}$ *be two individuals of the same social type* $(c_i, e_i) = (c_j, e_j)$*, and* $\mathbf{s} \in \mathcal{S}$ *a Nash equilibrium such that* $s_i \neq s_j$*. If* $R_{ij}(\mathbf{s}) > 0$*, then*

$$dist(W(\vec{\omega}_i), W(\vec{\omega}_j)) \geq R_{ij}(\mathbf{s})/2$$

The theorem reveals that, in a Nash equilibrium, individuals of the same social type with a tendency to conform need to have different valuations of the alternatives in order to make different decisions. We say that a strategy profile $\mathbf{s} \in \mathcal{S}$ is *admissible* with respect to a type profile $\mathbf{t} = (\mathbf{c}, \mathbf{e}, \mathbf{v})$ if the following property holds: if $i, j \in \mathcal{I}$ are two individuals of the same social type $(c_i, e_i) = (c_j, e_j)$ with $s_i \neq s_j$ and $R_{ij}(\mathbf{s}) > 0$, then they have different valuation types $v_i \neq v_j$. Equivalently, if $v_i = v_j$ and $R_{ij} > 0$ then $s_i = s_j$.

Corollary 1 (Nash equilibrium admissibility)**.** *A strategy* $\mathbf{s} \in \mathcal{S}$ *to be* $(\mathbf{c}, \mathbf{e}, \mathbf{v})$ *admissible is a necessary condition for* $\mathbf{s}$ *to be a Nash equilibrium.*

Note that we have not imposed any condition so far on individuals of different social types, and we will make that clear. Given a type profile $\mathbf{t} = (\mathbf{c}, \mathbf{e}, \mathbf{v})$, we say that a strategy profile $\mathbf{s} \in \mathcal{S}$ is $\mathbf{t}$ *feasible*, if $\mathbf{s}$ satisfies the following two properties: (i) $\mathbf{s}$ is $\mathbf{t}$ admissible; and (ii) if $i, j \in \mathcal{I}$ are two individuals with different social types $(c_i, e_i) \neq (c_j, e_j)$, then $v_i \neq v_j$. (Note that this does not mean i and j have different personal values, but rather that they are allowed to have different ones.)

Theorem 2 (Nash equilibrium feasibility). *Given a strategy profile* $\mathbf{s} \in \mathcal{S}$ *and a type profile* $\mathbf{t} \in \mathcal{T}$*, if* $\mathbf{s}$ *is* $\mathbf{t}$ *feasible then there is a profile of personal values* $\boldsymbol{\omega} \in \mathbb{R}^{n_{\mathcal{A}} \times n_{\mathcal{I}}}$ *compatible with the valuation profile* $\mathbf{v} \in \mathcal{V}$*, such that* $\mathbf{s}$ *is a Nash equilibrium.*

We note that given a type profile $\mathbf{t}$ and a strategy profile $\mathbf{s}$, to be $\mathbf{t}$ feasible is not a necessary condition for $\mathbf{s}$ to be a Nash equilibrium.

2.2.2 Social contexts and equilibrium partitions

The set of Nash equilibria of a decision game can be partitioned according to the information conveyed by the type map, i.e. according to the individuals profile. The set of characteristics of individuals can be distinguished between those on a more internal level, the taste of individuals, and those on a more external, or visible level, the crowding type. It is therefore natural to start partitioning equilibria according to the crowding profile of individuals. Given a strategy profile $\mathbf{s}$ and a crowding profile $\mathbf{c}$, we define *social context* as the pair $(\mathbf{s}, \mathbf{c})$. In studying social contexts that are based on a Nash equilibrium strategy $\mathbf{s}$, the characterization of the structure of the utility profile is naturally limited to studying subsets of individuals that are dintinguishable in that social context, and therefore provide different information. Using the crowding space, the utility function can be fully characterized by the following *(reduced) utility matrix* for each individual $i \in \mathcal{I}$,

$$U_i \equiv U(i; \mathcal{A}, C) \equiv \begin{pmatrix} \omega_1^i & \alpha_1^{i1} & \dots & \alpha_1^{in_C} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n_{\mathcal{A}}}^i & \alpha_{n_{\mathcal{A}}}^{i1} & \dots & \alpha_{n_{\mathcal{A}}}^{in_C} \end{pmatrix}.$$

The utility matrix defines the taste (or utility) type of an individual, and the utility profile $\mathbf{U} \equiv \mathbf{U}(\mathcal{I}; \mathcal{A}, C) \equiv (U_1, \dots, U_{n_{\mathcal{I}}}) \in (\mathbb{R}^{n_{\mathcal{A}} \times (1+n_C)})^{n_{\mathcal{I}}}$ determines a decision game. The set of Nash equilibria of a decision game will naturally depend on the utility profile. Nevertheless, different utility profiles may lead to the same Nash equilibria. Hence, we will study properties of utility matrices of decision games for which a given strategy class is a Nash equilibrium.

Consider a partition $\mathcal{P}(\mathbf{s},\mathbf{c})$ of the set of individuals $\mathcal{I}$ according to the social context $(\mathbf{s},\mathbf{c})$, meaning that every pair (d,c) creates a block $P(d,c)$ of the partition whose elements are all the individuals $i \in \mathcal{I}$ with the same crowding type $c_i = c$ and using the same strategy $s_i = d$. That is,

$$P(d,c) \equiv \{i \in \mathcal{I} : (s_i, c_i) = (d,c) \in \mathcal{A} \times C\},$$

$$\mathcal{P}(\mathbf{s},\mathbf{c}) \equiv \{P(d,c) : (d,c) \in \mathcal{A} \times C\}.$$

This kind of partitions is particularly interesting to relate to the notion of society defined in [52], and in fact inspired by it. A society is an element of a subpartition of a block $P(d,c)$ with an aditional property of convexity as defined properly below. Let us first denote convex hull by $con(\cdot)$ and without ambiguity let us use the same notation for the convex hull formed by the utilities of some individuals $J \subset \mathcal{I}$, thus

$$con(J) \equiv \left\{ \sum_{j \in J} \lambda_j U_j : \lambda_j \in \mathbb{R}_0^+ \text{ and } \sum_{j \in J} \lambda_j = 1 \right\}.$$

A set of individuals $S \in P(d,c)$ is called a *society* if it satisfies the following convexity property: if for $i \in \mathcal{I}$, $c_i = c$ and $U_i \in con(S)$, then $i \in S$ (see [52]). The society is maximal if there is no other society $S' \in P(d,c)$ such that $S \subset S'$. Given a decision game and a block $P(d,c)$ of a social context, let us denote by $SP(d,c) \equiv \{S_1, \ldots, S_k\}$ a partition of $P(d,c)$. Let now

$$\mathcal{SP}(\mathbf{s},\mathbf{c}) \equiv \bigcup_{i \in \mathcal{I}} SP(s_i, c_i).$$

The partition $\mathcal{SP}(\mathbf{s},\mathbf{c})$ is called a *societal partition* if its blocks $SP(d,c)$ are formed by societies, and it is called a *minimal societal partition* if it is formed by maximal societies.

Definition 1 (Global minimum societal partition)**.** *A societal partition is a* global minimum *if all its societies coincide with the $P(d,c)$ block, i.e. for all $S \in \mathcal{SP}(\mathbf{s},\mathbf{c})$, $S = P(d,c)$.*

We observe that while a partition $\mathcal{P}(\mathbf{s},\mathbf{c})$ is based on a combinatorial concept, societies are based on a topological one. A fundamental question is understanding the minimal societal partition of a Nash equilibrium, and in particular if that partition is a global minimum. We will show that, in the context of our work, when there are only positive externality relations between the $P(d,c)$ blocks for a given strategy, the societal partition is a global minimum. In particular, in a conformity game, the minimal societal partition of a Nash equilibrium is always a global minimum, and thus there are at most $n_{\mathcal{A}} n_C$ societies. That is not the case however for games with negative externalities. We will show that social congestion games may not have global minimum societal partitions of its Nash equilibria, and there may be up to $n_{\mathcal{I}}$ maximal societies. For a given block $P(d,c)$, let

$$U(d,c) \equiv \{U_i : i \in P(d,c)\}.$$

We say that two sets of individuals $I, J \in \mathcal{I}$ have a tendency to conform in strategy profile $\mathbf{s}$, if for all $i \in I$ and $j \in J$, $R_{ij}(\mathbf{s}) > 0$. Note that $R_{ij}(\mathbf{s}) = \alpha_{s_j}^{ic_j} + \alpha_{s_i}^{jc_i}$.

Theorem 3 (Positive externalities)**.** *Let $(\mathbf{s},\mathbf{c})$ be a social context and $\mathbf{s}$ a Nash equilibrium. If for two distinct decisions $d, d' \in \mathcal{A}$ and a crowding type $c \in C$, the blocks $P(d,c)$ and $P(d',c)$ have a tendency to conform in $\mathbf{s}$, then*

$$con(U(d,c)) \cap con(U(d',c)) = \emptyset.$$

Theorem 3 relates directly to the notion of societies, and in particular to the concept of global minimum societal partition.

Corollary 2 (Positive externalities)**.** *Let* $(\mathbf{s}, \mathbf{c})$ *be a social context and* $\mathbf{s}$ *a Nash equilibrium. For every* $c \in C$ *let* $P(d, c)$ *and* $P(d', c)$ *have a tendency to conform in* $\mathbf{s}$*, for every* $d, d' \in \mathcal{A}$*, with* $d \neq d'$*. There is a global minimum societal partition.*

In particular, for every Nash equilibrium of a social conformity game the minimal societal partition of a Nash equilibrium is a global minimum.

2.2.3 Strategy classes and Nash domains

The crowding space allows the characterization of classes of strategies where the relevant information is the number of individuals with the same crowding type in each decision. We thus define the *crowding-aggregate decision matrix* $\mathbf{L}(\mathbf{s}, \mathbf{c})$ whose coordinates, $l_a^c = l_a^c(\mathbf{s})$, indicate the number of individuals with crowding type $c \in C$ who choose alternative $a \in \mathcal{A}$ in strategy profile $\mathbf{s}$,

$$\mathbf{L}(\mathbf{s}, \mathbf{c}) \equiv \begin{pmatrix} l_1^1 & \dots & l_1^{n_C} \\ \vdots & \ddots & \vdots \\ l_{n_\mathcal{A}}^1 & \dots & l_{n_\mathcal{A}}^{n_C} \end{pmatrix}.$$

We denote by $\mathcal{L} \equiv \{\mathbf{L}(\mathbf{s}, \mathbf{c}) \in \mathbb{R}^{n_\mathcal{A} \times n_C} : \mathbf{s} \in \mathcal{S}, \mathbf{c} \in \mathcal{C}\}$ the set of all possible crowding-aggregate decision matrices in a given game. Given a matrix $\mathbf{L} \in \mathcal{L}$, there is always a subset of strategy profiles $S \in \mathcal{S}$ such that, for any $\mathbf{s}_1, \mathbf{s}_2 \in S$, we have $\mathbf{L}(\mathbf{s}_1, \mathbf{c}) = \mathbf{L}(\mathbf{s}_2, \mathbf{c}) = \mathbf{L}$. Thus, the set $\mathcal{L}$ characterizes the crowding equivalence relation in the strategy space $\mathcal{S}$ induced by the crowding profile $\mathbf{c}$, and we will refer to the *strategy class* $\mathbf{L} \in \mathcal{L}$ to mean the equivalence class $\{\mathbf{s} \in \mathcal{S} : \mathbf{L}(\mathbf{s}, \mathbf{c}) = \mathbf{L}\}$.

The *utility Nash Domain* $\mathcal{N}(\mathbf{s}, \mathbf{c})$ of a given social context $(\mathbf{s}, \mathbf{c})$ is defined as the set of all utility profiles $\mathbf{U}$ for which $\mathbf{s}$ is a Nash equilibrium under the crowding profile $\mathbf{c}$. For an individual $i \in \mathcal{I}$ the best response utility domain

$N_i(\mathbf{s}, \mathbf{c})$ of a social context $(\mathbf{s}, \mathbf{c})$ is the set of all utility matrices U_i such that s_i is a best response of individual i to $\mathbf{s}_{-i}$ under the crowding profile $\mathbf{c}$.

Remark 1 (Nash domain cone structure)**.** *Let* $(\mathbf{s}, \mathbf{c})$ *be a social context. We have,*

(i) $\mathcal{N}(\mathbf{s}, \mathbf{c}) = N_1(\mathbf{s}, \mathbf{c}) \times \cdots \times N_{n_{\mathcal{I}}}(\mathbf{s}, \mathbf{c})$*;*

(ii) if $U_1, U_2 \in N_i(\mathbf{s}, \mathbf{c})$ *then* $\lambda U_1 + \mu U_2 \in N_i(\mathbf{s}, \mathbf{c})$*, for all* $\lambda, \mu > 0$*;*

(iii) if $s_i = s_j$ *and* $c_i = c_j$ *then* $N_i(\mathbf{s}, \mathbf{c}) = N_j(\mathbf{s}, \mathbf{c})$*.*

We note that by condition (ii) Remark 1 the best response utility domains $N_i(\mathbf{s}, \mathbf{c})$ have a cone structure. Let $s(I)$ (the image by the strategy map s) be the subset of decisions chosen by individuals $I \subset \mathcal{I}$ in the associated strategy profile $\mathbf{s}$. Individuals with the same crowding type retrieve the same information from the aggregated structure of a strategy class $\mathbf{L}$, and if they are using the same strategy, they in fact share a best response utility domain (hence (iii)). Therefore, these domains can be described using the crowding-aggregate matrix, i.e. $N_i(\mathbf{s}, \mathbf{c}) = N(s_i, c_i; \mathbf{L}(\mathbf{s}, \mathbf{c}))$, and we can rewrite the utility Nash domain of a social context as follows

$$\mathcal{N}(\mathbf{s}, \mathbf{c}) = \times_{d \in s(\mathcal{I}), c \in C} N(d, c; \mathbf{L}(\mathbf{s}, \mathbf{c}))^{l_d^c}.$$

Given a crowding profile $\mathbf{c}$, a strategy profile $\mathbf{s}$ is a Nash equilibrium if, and only if, for every non-empty block $P(d, c)$ of the partition of the respective social context $(\mathbf{s}, \mathbf{c})$, we have

$$U(d, c) \subset N(d, c; \mathbf{L}).$$

We note that the domains $N(d, c; \mathbf{L})$ do not *preserve externalities* in the following sense: given two best response utility domains $N(d, c; \mathbf{L})$ and $N(d', c; \mathbf{L})$, there are some utilities in $N(d, c; \mathbf{L})$ which would provoke a 'positive externality relation with' some utilities in $N(d', c; \mathbf{L})$, and there are some utilities in $N(d, c; \mathbf{L})$ which would provoke a 'negative externality relation' with some utilities in $N(d, c'; \mathbf{L})$. That is, the influence relation is not preserved. Since it will be useful to study sets that preserve these relations, we will add an externality profile to the social context, extending it so that we can fiber the best response and utility Nash domains by the externality profile $\mathbf{e}$. Let $(\mathbf{s}, \mathbf{c}, \mathbf{e})$ be the *social context extension* to externality profile $\mathbf{e}$. For an individual $i \in \mathcal{I}$ the *best response valuation domain* $N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c}))$ of a social context extension $(\mathbf{s}, \mathbf{c}, \mathbf{e})$ is the set of all vectors $\vec{\omega}_i$ such that s_i is a best response to $\mathbf{s}_{-i}$, in the profile context $\mathbf{c}, \mathbf{e}$. We observe that if $\vec{\omega}_i \in N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c}))$, then $W(\vec{\omega}_i) \subset N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c}))$. Furthermore, the sets $N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c}))$ are convex, non-empty and preserve externalities. The *Nash valuation domain* of a social context extension $(\mathbf{s}, \mathbf{c}, \mathbf{e})$ is thus given by the cartesian product

$$\mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e}) = \times_{i \in \mathcal{I}} N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c})).$$

Theorem 4 (Positive externalities)**.** *Let $i, j \in \mathcal{I}$ be two individuals of the same social type $(c_i, e_i) = (c_j, e_j)$ and $\mathbf{s} \in \mathcal{S}$ a Nash equilibrium with $s_i \neq s_j$. If i and j have a tendency to conform in $\mathbf{s}$, then*

$$N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c})) \cap N(s_j, c_j, e_j; \mathbf{L}(\mathbf{s}, \mathbf{c})) = \emptyset.$$

Let $\mathcal{I}_t$ be set of individuals with type $t \in T$, and recall that individuals of the same type have the same valuation of alternatives. For a given type $t = (c, e, v) \in \mathcal{T}$, the type best response valuation domain is

$$N(t; \mathbf{L}(\mathbf{s}, \mathbf{c})) \equiv \bigcap_{i \in \mathcal{I}_t} N(s_i, c, e; \mathbf{L}(\mathbf{s}, \mathbf{c})).$$

In a strategy profile $\mathbf{s}$, individuals of type $t \in T$ are using best responses if $\vec{\omega}_v \in N(t; \mathbf{L}(\mathbf{s}, \mathbf{c}))$. If type t has a tendency to conform, Theorem 4 poses a problem for strategies for which $s(\mathcal{I}_t)$ is not a singleton. Recall that being admissible required different valuations for individuals of the same social type with a tendency to conform but making a different decision. It is now more clear that being admissible with respect to the type profile, is a necessary condition for a strategy profile to be a Nash equilibrium (Corollary 1).

Theorem 5 (Nash domain characterization)**.** *If* $\mathbf{s}$ *is* $\mathbf{t}$ *admissible then for every* $t \in T$*,* $N(t; \mathbf{L}(\mathbf{s}, \mathbf{c}))$ *is a (non-empty) convex set that is the closure of an open set and*

$$\mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e}) = \times_{t \in T} N(t; \mathbf{L}(\mathbf{s}, \mathbf{c})) \neq \emptyset.$$

Furthermore, if $\mathbf{s}$ *is* $\mathbf{t}$ *feasible then every* $\boldsymbol{\omega} \in \mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e}) \neq \emptyset$ *is compatible with* $\mathbf{v}$*.*

Theorem 2 follows from the above theorem. Let $\mathcal{I}_c$ be the set of individuals with a given crowding type $c \in C$. Theorem 5 provides an interesting conection to the number of societies in a minimal societal partition of a Nash equilibrium. Take for instance for all the individuals $i \in \mathcal{I}$, $\alpha_d^{ic} = -1$, for every $d \in \mathcal{A}$ and $c \in C$. Hence, all individuals have the same social type given by the externality matrix with all entries -1. Since for any given $c \in C$, $N(t; \mathbf{L}(\mathbf{s}, \mathbf{c}))$ contains an open set, it is possible to choose an utility profile so that we can order the utilities of all the individuals along a line in $N(t; \mathbf{L}(\mathbf{s}, \mathbf{c}))$ with the order that we prefer. Each order of the individuals along the line creates a number of societies that only needs to be compatible with the combinatorics imposed by the number of individuals of $\mathcal{I}_c$ that are in each block $P(d, c)$. Thus, taking

$$M_c = \min\{2(n_c - \bar{p}_c) + 1, n_c\},$$

where $n_c = \#\mathcal{I}_c$ and $\bar{p}_c$ is the cardinality of the largest set $P(d, c) \subset \mathcal{I}_c$, we obtain the following corollary.

Corollary 3 (Negative externalities). *Given a social context* $(\mathbf{s}, \mathbf{c})$*, for every* $c \in C$ *choose* q_c *such that* $\#s(\mathcal{I}_c) \leq q_c \leq M_c$*. There are utility profiles* $\mathbf{U} \in \mathcal{N}(\mathbf{s}, \mathbf{c})$ *such that the minimal societal partition has cardinality* $\sum_{c \in C} q_c$*.*

As such, for any given social context $(\mathbf{s}, \mathbf{c})$, the following minimal societal partitions can arise:

- (global minimum) there are utility profiles $\mathbf{U} \in \mathcal{N}(\mathbf{s}, \mathbf{c})$ such that the minimal societal partition is the global minimum societal partition;

- (no global minimum) if for some $c \in C$ there are decisions $d, d' \in \mathcal{A}$, with $d \neq d'$, $\#P(d, c) \geq 1$ and $\#P(d', c) > 1$, then there are utility profiles $\mathbf{U} \in \mathcal{N}(\mathbf{s}, \mathbf{c})$ such that there is not a global minimum societal partition;

- (maximality) if $\sum_{c \in C} q_c = n_{\mathcal{I}}$, then there are utility profiles $\mathbf{U} \in \mathcal{N}(\mathbf{s}, \mathbf{c})$ such that the cardinality of the minimal societal partition is $n_{\mathcal{I}}$, and thus it is maximal.

2.2.4 Conformity thresholds and decision tiling

For the explicit characterization of the Nash valuation domains of a social context extension, let us start by the analysis of the individual's best responses. We will then define thresholds for the valuation domains of those best responses in terms of the personal values. For this analysis it will be useful to rewrite the utility function using the strategy classes $\mathcal{L}$. Recall that in this section when we say Nash equilibrium we always mean pure Nash equilibrium. As it is natural when dealing with pure Nash equilibria, we will have to make comparisons between pairs of decisions, and this can be done comparing lines in the utility matrices, since each line d of those matrices is associated with the utility of the individual i when using strategy $s_i = d$. Hence, it will be useful to introduce a notation for the line vectors associated with each decision. When the choice of an individual $i \in \mathcal{I}$ is

$d \in \mathcal{A}$, the social influence that she is subject to, in a given strategy profile $\mathbf{s} \in \mathcal{S}$, may be summarized by two vectors: the *social preferences vector* $\vec{\alpha}_i(d) \in \mathbb{R}^{n_C}$, comprised of the social weights given by individual i to the aggregates of each crowding type in decision d; and the *crowding-aggregate vector* $\vec{l}(d) \in \mathbb{R}^{n_C}$ whose coordinates correspond to the line d of matrix $\mathbf{L}$, and thus indicate the number of individuals with crowding $c \in C$ who make decision d in a given strategy class $\mathbf{L}$,

$$\vec{\alpha}_i(d) \equiv (\alpha_d^{i1}, \ldots, \alpha_d^{in_C}), \quad \vec{l}(d) \equiv (l_d^1, \ldots, l_d^{n_C}).$$

The utility function can now be rewritten for strategy classes through the above vectors. For an individual $i \in \mathcal{I}$ it is given by

$$u_i(s_i, c_i; \mathbf{L}) \equiv u_i(s_i; \mathbf{s}_{-i}, \mathbf{c}_{-i}) = \omega_{s_i}^i + \vec{\alpha}_i(s_i) \cdot \vec{l}(s_i) - \alpha_{s_i}^{ic_i}$$

where $\cdot$ denotes the usual inner product. Note that determining the utility of an individual using a strategy class $\mathbf{L}$ instead of a specific strategy profile $\mathbf{s}$, forces the need to add some extra information. Namely, each individual needs to know her own crowding type due to the subtraction of coordinate $\alpha_{s_i}^{ic_i}$. This is a consequence of removing individual i from the aggregate $l_{s_i}^{c_i}$ and assigning social weight to $l_{s_i}^{c_i} - 1$ instead. However, this only means that individual i has no social weight on her own utility, rather she has an individual value for that decision, $\omega_{s_i}^i$ (which might nevertheless encompass a social interpretation of personal values). The aforementioned need for the knowledge of an individual's own crowding type, reveals how individuals may retrieve different information from the same aggregated structure of a strategy class.

Given a decision game Γ and a strategy profile $\mathbf{s}$, the *best response* of individual $i \in \mathcal{I}$ is

$$\mathrm{br}_i(\mathbf{s}_{-i}) \equiv \mathrm{br}(c_i, e_i, v_i; \mathbf{L}(\mathbf{s}, \mathbf{c})) = \arg\max_{d \in \mathcal{A}} \{\omega_d^{v_i} + \vec{\alpha}_{e_i}(d) \cdot \vec{l}(d) - \alpha_d^{e_i c_i}\}.$$

A strategy profile $\mathbf{s}$ is a *(pure) Nash equilibrium* if, for every $i \in \mathcal{I}$, $s_i = \mathrm{br}_i(\mathbf{s}_{-i})$. In a given social context extension, individuals with a same social type $(c_i, e_i) = (c_j, e_j) = (c, e)$ that make the same decision $s_i = s_j = d$ have the same individual best response valuation domain $N(d, c, e; \mathbf{L}) \equiv N(s_i, c_i, e_i; \mathbf{L}(\mathbf{s}, \mathbf{c}))$. Note that the best response utility Nash domains of social contexts are characterized by the best response valuation domains of social context extensions, since

$$N(d, c; \mathbf{L}) = \bigcup_e N(d, c, e; \mathbf{L}).$$

To characterize the best response valuation domains, we are going to define for a given strategy profile $\mathbf{s}$ conformity thresholds $T_{e_i}(s_i \to d; \mathbf{s}_{-i})$, that represent the surplus quantity that individual i has from social externalities, that could create an incentive for her to change from her current decision s_i to decision d. This threshold does not depend on the valuation type of the individual, but rather on the externality context $(\mathbf{s}, \mathbf{c}, \mathbf{e})$. In particular, as referred, it depends on the individual social type and the strategy class to which $\mathbf{s}$ belongs. Let us first define the *auxiliar externality type-threshold* between two decisions $d, d' \in \mathcal{A}$,

$$\bar{T}_e(d', d; \mathbf{L}) \equiv \vec{\alpha}_e(d) \cdot \vec{l}(d) - \vec{\alpha}_e(d') \cdot \vec{l}(d').$$

Given a strategy profile $\mathbf{s}$, the *conformity thresholds* are given for each individual $i \in \mathcal{I}$ with externality type e_i and for all decisions $d \in \mathcal{A} \setminus \{s_i\}$, by

$$T_{e_i}(s_i \to d; \mathbf{s}_{-i}) \equiv \bar{T}_{e_i}(s_i, d; \mathbf{L}(\mathbf{s}, \mathbf{c})) + \alpha_{s_i}^{e_i c_i},$$

which will be useful to rewrite using strategy classes,

$$T_{(c_i,e_i)}(s_i \to d; \mathbf{L}(\mathbf{s},\mathbf{c})) \equiv T_{e_i}(s_i \to d; \mathbf{s}_{-i}).$$

The notation reflects the idea of social incentive towards decision d from strategy s_i. Thus, this is the quantity by which $\omega_{s_i}^{v_i}$ (the value of decision s_i) has to overcome $\omega_d^{v_i}$ (the value of decision d), so that decision s_i is still 'preferable' for an individual with social type (c_i, e_i) in the externality context $(\mathbf{s}, \mathbf{c}, \mathbf{e})$. Observe that when we talk about incentives for player i to change her decision, we might be talking about desincentives, depending upon the sign of the conformity threshold $T_{(c_i,e_i)}(s_i \to d; \mathbf{L}(\mathbf{s},\mathbf{c}))$. Two opposite extreme cases appear when $\vec{\alpha}_{e_i}(s_i)$ has only positive coordinates and $\vec{\alpha}_{e_i}(d)$ has only negative coordinates, making the threshold negative, thus a desincentive to change; or when the opposite happens, making the threshold positive, thus an incentive to change. Concluding, incentives or desincentives are provoked by the relation between negative and positive coordinates in the social preference matrix.

Lemma 2 (Best response valuation domains characterization)**.** *The best response valuation domains $N(d, c, e; \mathbf{L})$ consist of all $\vec{\omega} \in \mathbb{R}^{n_\mathcal{A}}$ with the following properties:*

(i) $\omega_d \in \mathbb{R}$;

(ii) $\omega_{d'} \in \mathbb{R}$ *satisfying the following threshold inequality*

$$\omega_{d'} \leq \omega_d - T_{(c,e)}(d \to d'; \mathbf{L}) \tag{2.1}$$

for every decision $d' \in \mathcal{A} \setminus \{d\}$.

Hence, $\mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e})$ is non-empty and contains an open set in the space $(\mathbb{R}^{n_\mathcal{A}})^{n_\mathcal{I}}$.

Decision tilings.

The characterization of the Nash valuation domains for every strategy profile $\mathbf{s}$, provides the full characterization of the relation between valuations and strategies for a given social profile $(\mathbf{c}, \mathbf{e})$. That is, for any given valuation profile we know what are the possible Nash equilibrium strategies under that social profile. This characterization is summarized in the *decision tiling*

$$\mathcal{DT}(\mathbf{c}, \mathbf{e}) \equiv \bigcup_{\mathbf{s}} \mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e}).$$

By theorem 5 we can describe the decision tiling according to the type map and strategy classes as the union of the product of convex sets,

$$\mathcal{DT}(\mathbf{t}) = \bigcup_{\mathbf{L}} \times_{t \in T} N(t; \mathbf{L}(\mathbf{s}, \mathbf{c})).$$

In figures 2.1, 2.2, 2.3 and 2.4 we show examples of decision tilings for the class of decision games $\Gamma_{2,2}$. The interaction variables are summarized by $A_{tt'} \equiv \alpha_1^{tt'} + \alpha_2^{tt'}$ which characterize the domains. The axis are given by $x = \omega_1^{t_1} - \omega_2^{t_1}$ and $y = \omega_1^{t_2} - \omega_2^{t_2}$. The advantage of dimension 2 is the geometric representation allowing a visualization of the results. Note that for all $i, j \in \mathcal{I}$, $R_{ij} \in \{A_{11}, A_{12}, A_{22}, A_{21}\}$. The strategies are characterized by the pair (l_1, l_2) which indicate, respectively, the number of individuals of type 1 and type 2 in decision 1. There are n_t individuals of each type. This is based on [43] where the full characterization of these decisions tilings is done.

2.2.5 Reciprocal relations and cycles

A natural follow-up question is whether the decision tiling covers the whole valuation space $\mathbb{R}^{n_{\mathcal{A}}}$. This amounts to the question of existence of a pure Nash equilibrium and can be formulated more precisely in the following

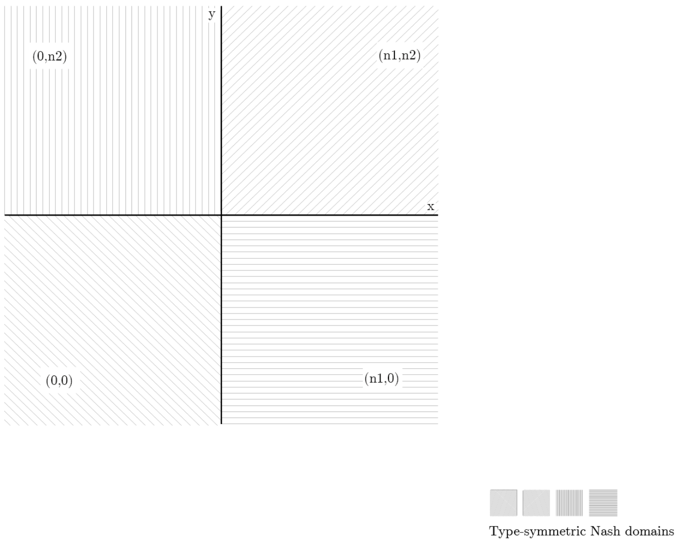

Figure 2.1: The benchmark decision tiling for the class of $\Gamma_{2,2}$ decision games. There are no interactions: $A_{11} = A_{22} = A_{12} = A_{21} = 0$. There are only type-symmetric strategies, except on the axis.

way: given a social profile $(\mathbf{c}, \mathbf{e})$, when is the problem of existence of a Nash equilibrium independent of the choice of the personal profile? Are there conditions that can be imposed on the social profile so that a pure Nash equilibrium always exist?

The question of existence is one of the major issues in game theory. In finite games, since a mixed equilibrium always exists, as proven by Nash [32], the question is whether a pure equilibrium exists. A major class of games that has drawn considerable atention in the literature for always possessing a pure Nash equilibrium is the class of potential games, introduced by Rosenthal [38], and later classified and generalized by Monderer and Shapley [31]. We will look at the problem of asking whether it is possible to impose symmetries in the relation of influences between individuals and guarantee the existence of a pure equilibrium, independent of their personal valuations. Note that if

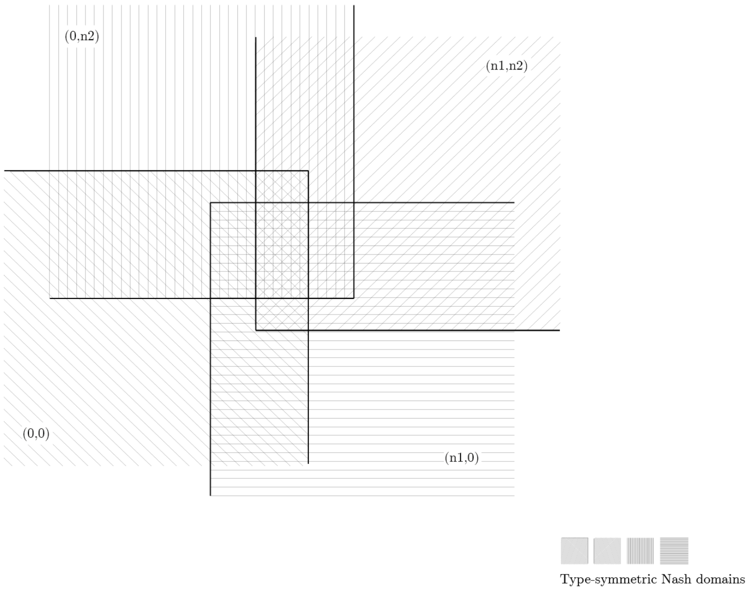

Figure 2.2: The decision tiling for a $\Gamma_{2,2}$ decision game when $A_{11} > 0$, $A_{22} > 0$, $A_{12} > 0$ and $A_{21} > 0$. All equilibria are type-symmetric. There are multiple equilibria in the intersection areas.

it would be possible to choose their personal valuation, then the first sections completely solve the problem.

Definition 2 (Strong reciprocity)**.** *A social profile has* strong reciprocity *if* $\alpha_a^{ij} = \alpha_a^{ji}$ *for every individuals* $i, j \in \mathcal{I}$ *and for every action* $a \in \mathcal{A}$.

A decision game has strong reciprocity if its social profile has strong reciprocity. Note that this is a strong symmetry property imposed on every pairwise relation of individuals. The effect is that when an individual changes its decision she will provoke the same externalities as she will incur, which leads improvements on best replies to 'flow' on the same direction, and ultimately to the existence of a potential function.

Result 1. *A decision game with strong reciprocity is a potential game.*

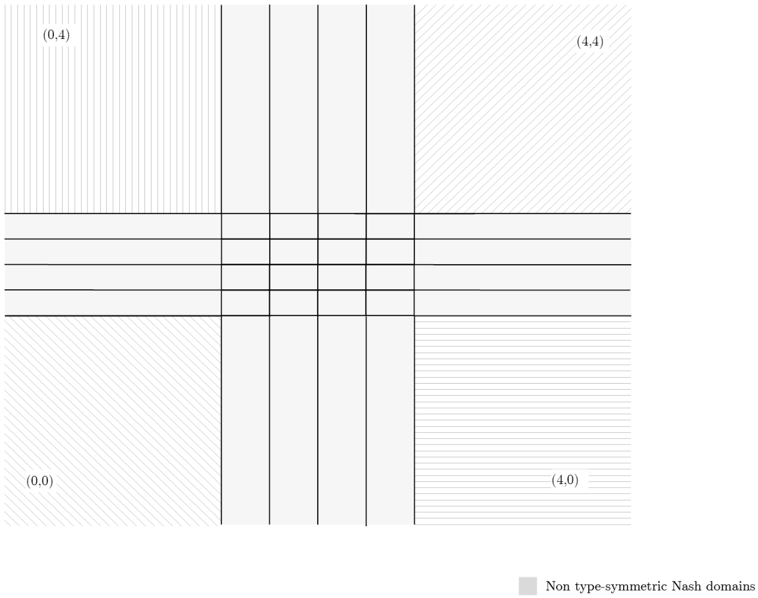

Figure 2.3: The area of the decision tiling for a $\Gamma_{2,2}$ decision game where non-type symmetric Nash domains will lie. In this case we considered 4 individuals in each type to show the non-type symmetric domains. Here there are only intra-type influences. $A_{22} < A_{11} < 0$, $A_{12} = A_{21} = 0$.

Result 2. *Every decision game with strong reciprocity has a pure Nash equilibrium.*

The two results are interconnected. Result 2 is a consequence of Result 1, as every potential game has a pure Nash equilibrium. The proof of existence is derived directly from a theorem by Le Breton and Weber in [8], and their proof, which is for a more general class of games is done by direct construction of a potential. The proof of result 1 in our case could be done directly from corollary 2.9 in [31] by Monderer and Shapley, without explicit construction of the potential, but we will omitt such a proof. Observe that a consequence of Result 2 is that for every social profile $(\mathbf{c}, \mathbf{e})$ with strong reciprocity the decision tiling covers the valuation space, i.e. $\mathcal{DT}(\mathbf{t}) = \mathbb{R}^{n_A}$. The strong reciprocity condition is a rather strong symmetry condition, which works as

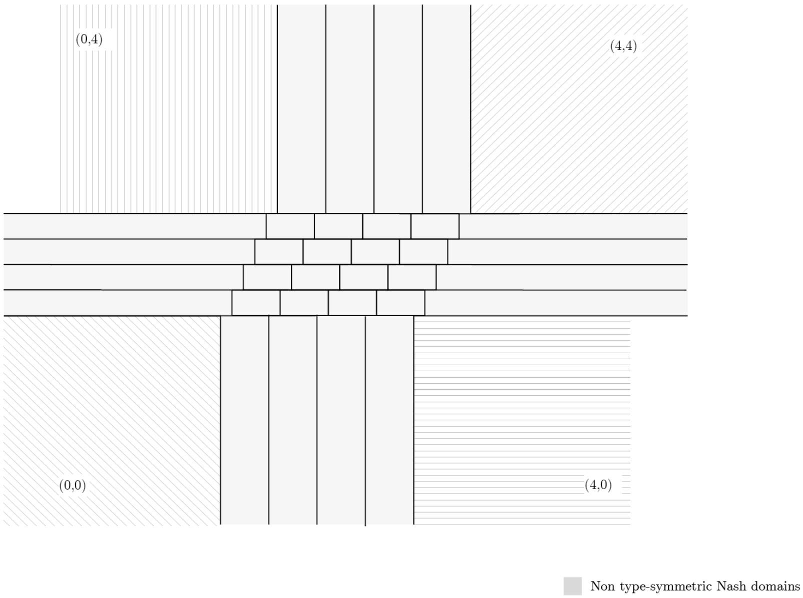

Figure 2.4: The area of the decision tiling for a $\Gamma_{2,2}$ decision game where non-type symmetric Nash domains will lie and inter-type interactions are *turned on*. In this case $A_{11} < 0$, $A_{22} < 0$, $A_{12} < 0$, $A_{21} = 0$.

a sufficiency condition for the existence, but is far from necessary, much in the same way being a potential game is a sufficient condition for existence, but not necessary. The next question we will focus is: under which conditions is it possible to relax the assumption, i.e. unbalance the relation $\alpha_a^{ij} \neq \alpha_a^{ji}$, and still guarantee a Nash equilibrium? Naturally, we need at least two types of individuals to do this, and then look at the intertype relations, since intratype relations are by definition symmetric. Hence, the following corollary holds for all games $\Gamma_{1,n_{\mathcal{A}}}$.

Corollary 4. *Every decision game with only one type of individuals has a pure nash equilibrium.*

The idea behind potential games guaranteing a Nash equilibrium, independently of the valuation profile, builds upon the fact that when an individual changes her decision she affects the utility of the remaining individuals the same exact same way she is affected by them. When that's not the case, an incentive for her to change could be a desincentive for other individuals to accept the change. Nevertheless, when only positive externalities are in place, the idea still holds, since the utility of every individual grows with each unilateral change and eventually individuals will stop changing, if not before, at least when they have made the same decision. This is the basic idea behind the proof of the next theorem.

Theorem 6. *Every social conformity game has a pure Nash equilibrium.*

Recall that social conformity games have only positive externalities, and as we have proven in previous section, there will be only type symmetric equilibria. Type-symmetry itself, however, does not play a relevant part, and is not a base for a sufficient condition. When we increase the number of types, the intertype externalities may be strong enough to prevent an equilibrium. When the number of types increase, the interactions between individuals of different types may be unbalanced (not strongly reciprocal) or with different signs, and this is where a new condition must work. Note that even if the interactions are balanced but with a different sign (hence introducing negative externalities) an equilibrium might not exist. The next example, although in a more informal language, gives a precise idea of the type of problems that can occur when the number of types grows, and we loosen the strong reciprocity condition. This also sets up the motivation for our next definition.

Example 1 (Cat and mouse externalities)**.** *Suppose there are two players (the cat and the mouse) and two possible actions (locations). Both players are personally indiferent between one action or the other. The mouse wants to choose a different strategy than the cat, and the cat wants to choose the same strategy as the mouse. No strategy can be an equilibrium, as one of them*

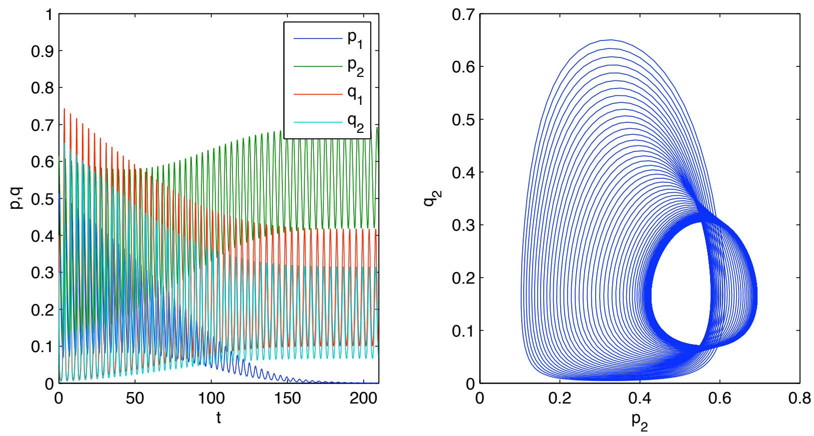

Figure 2.5: The time evolution of the probabilities of individuals under the replicator dynamics forming a stable cycle for a $\Gamma_{2,2}$ decision game. There are 2 individuals of each type and the parameters are $A_{11} = -0.1$, $A_{12} = 3$, $A_{21} = -10$, $A_{22} = 0$; $n_1 = n_2 = 2$, $x = 0.4$, $y = 0.6$. The initial probabilities are $p_1(0) = 0.5$, $p_2(0) = 0.6$ for type 1 and $q_1(0) = 0.4$ and $q_2(0) = 0.3$ for type 2.

always has an incentive to deviate. This is essencially general for decision games $\Gamma_{2,2}$ (with two types of individuals and two actions) where $A_{12} > 0$ and $A_{21} < 0$. No type-symmetric strategy will prevail, and non-symmetric strategies will require a balance between the remaining parameters. Similar examples can be found in a game leading to a limit cycle of the replicator dynamics in (Soeiro et al 2014 [43] which is shown in figure 2.5), and in an example where a celebrity and the public must choose locations in (Wooders 2006 [52]).

In such games with 'cat chases mouse' like externalities, if the utilities are based solely on externalities, there is no pure Nash equilibria. Note however, that using the results from the previous sections it is easy to find, for any of the players, a set of personal values for each decision such that an equilibrium exist. (Say the cat values action a_1 relative to a_2 high enough that is indiferent to the externality. Any allocation can be an equilibrium with the right set of personal values; those inside the Nash domains found

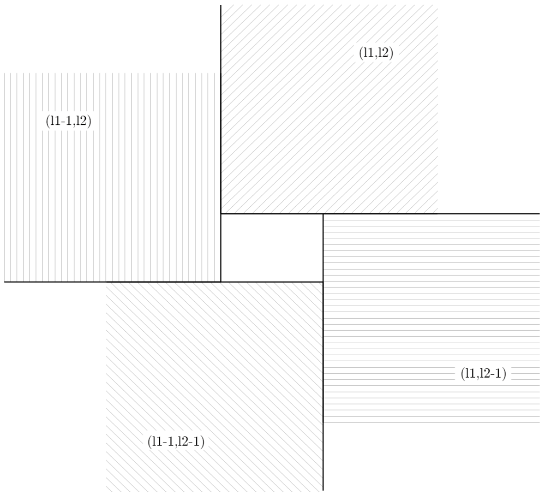

Figure 2.6: A relation between the non-type-symmetric Nash domains for the decision tiling for a $\Gamma_{2,2}$ decision game without weak reciprocity. In this case $A_{12} > 0$ and $A_{21} < 0$ which creates a space between the domains containing valuations for which there is no pure Nash equilibrium.

before.)

Definition 3 (Weak reciprocity)**.** *A social profile has* weak reciprocity *if for every individuals* $i, j \in \mathcal{I}$ *and actions* $a, a' \in \mathcal{A}$*,* $\mathrm{sgn}(\alpha_a^{ij}) = \mathrm{sgn}(\alpha_{a'}^{ji})$.

The condition of weak reciprocity implies that changes in the strategies of individuals will provoke externalities in the same direction, and thus break the type of 'cat chases mouse' externalities. This will prove to be crucial to decision games in $\Gamma_{2,2}$, such as that of the previous example.

Theorem 7. *A weakly reciprocal decision game with two types and two decisions has a pure Nash equilibrium.*

An example of a part of the decision tiling for the class of $\Gamma_{2,2}$ with and without weak reciprocity is shown in figures 2.6 and 2.7. The examples

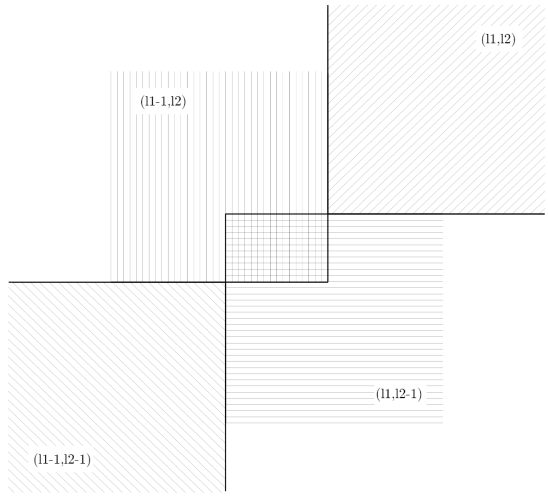

Figure 2.7: A relation between the non-type-symmetric Nash domains for the decision tiling of a $\Gamma_{2,2}$ decision game with weak reciprocity. In this case $A_{12} < 0$ and $A_{21} < 0$ which provokes the intersection of the domains creating multiplicity of equilibrium instead of an empty space.

ilustrate that externalities may create a hole on the decision tiling, i.e. ilustrate the cases when there will be a space between Nash domains with no equilibrium, and why weak reciprocity covers that space.

Corollary 5. *Social congestion games with two types and two actions have a pure Nash equilibrium.*

The fact that weak reciprocity does not hold when the number of types or decisions increases is because the type of 'cat chases mouse externalities' can be constructed on a second level of reasoning. Say there are three individuals and only negative externalities. The absolute value of externalities may induce that individual 1 is running away from individual 2, individual 3 is running away from individual 1, and individual 2 is running from 3. When there are only two possible actions, this creates a similar effect as if individual

1 was pursuing individual 3 (similar to following a positive externality). The same idea can be developped based on 3 decisions and 2 types. This is better explained in the next example.

Example 2 (The hermitt, the politician and the crowd.)**.** *Consider the following game construction. Suppose there are only negative externalities, so the game is weakly reciprocal. We will be working on the absolute values of those externalities. There are three types of players and two locations, the city and the countryside. The hermitt hates the crowd, and wants to be alone. The politician wants to build a society, so he hates the hermitt. The crowd is misinformed, so they hate the politician. The rest of externalities are negligible. All of them are indiferent between city and countryside. There is no pure Nash equilibrium, because no two types together can be a mutual best response. (Note that there are only type-symmetric best-responses, and even so there is no equilibrium.)*

Although the example shows why weak reciprocity does not hold as a sufficient condition in games with a higher number of types, it is in fact a necessary condition if the parameters of the game that are left free, include the values in the social profile. Just observe that in that case example 1 can always be constructed if weak reciprocity fails. For any number of players.

2.2.6 Proofs

The proofs are essencially divided into three groups. The first part is based on the idea of conformity obstruction, or tendency to conform, and uses a general version of lemma 1. The second part is based on the relation between best responses and conformity thresholds. The third part relates to the problem of existence and is based on improvement of best responses.

We start with an auxiliar result. Let us define for any two individuals $i, j \in \mathcal{I}$ and $d \in \mathcal{A}$, the following vector,

$$\vec{\varepsilon}_{ij}(d) \equiv \vec{\alpha}_{e_i}(d) - \vec{\alpha}_{e_j}(d).$$

Lemma 3. *Consider a decision game* Γ *and a Nash equilibrium* $\mathbf{s}$*. For every* $i, j \in \mathcal{I}$*, if* $s_i \neq s_j$ *then*

$$\omega_{s_i}^{v_i} - \omega_{s_i}^{v_j} + \omega_{s_j}^{v_j} - \omega_{s_j}^{v_i} \geq \alpha_{s_j}^{e_i c_j} + \alpha_{s_i}^{e_j c_i} + \vec{\varepsilon}_{ij}(s_j) \cdot \vec{l}(s_j) - \vec{\varepsilon}_{ij}(s_i) \cdot \vec{l}(s_i).$$

Proof. Consider a decision game Γ and let $\mathbf{s}$ be a Nash equilibrium of Γ. We have that

$$u_i(s_i; \mathbf{s}_{-i}) \geq u_i(s_j; \mathbf{s}_{-i})$$

and

$$u_j(s_j; \mathbf{s}_{-j}) \geq u_j(s_i; \mathbf{s}_{-j}).$$

Now observe that

$$u_i(s_j; \mathbf{s}_{-i}) = u_j(s_j; \mathbf{s}_{-j}) - \omega_{s_j}^{v_j} + \omega_{s_j}^{v_i} + \vec{\varepsilon}_{ij}(s_j) \cdot \vec{l}(s_j) + \alpha_{s_j}^{e_i c_j}$$

and similarly

$$u_j(s_i; \mathbf{s}_{-j}) = u_i(s_i; \mathbf{s}_{-i}) - \omega_{s_i}^{v_i} + \omega_{s_i}^{v_j} - \vec{\varepsilon}_{ij}(s_i) \cdot \vec{l}(s_i) + \alpha_{s_i}^{e_j c_i}.$$

which concludes the proof. □

Proofs based on confomity obstruction

Lemma 1

Proof. Lemma 3 imples that

$$\omega_{s_i}^{v_i} - \omega_{s_i}^{v_j} + \omega_{s_j}^{v_j} - \omega_{s_j}^{v_i} \geq \alpha_{s_j}^{e_i c_j} + \alpha_{s_i}^{e_j c_i} - 2n_{\mathcal{I}} dist(e_i, e_j)$$

and for d equal to s_i or s_j

$$2|\omega_d^{v_i} - \omega_d^{v_j}| \geq \alpha_{s_j}^{e_i c_j} + \alpha_{s_i}^{e_j c_i} - 2n_{\mathcal{I}} dist(e_i, e_j).$$

Thus,

$$dist(\vec{\omega}_{v_i}, \vec{\omega}_{v_j}) \geq R_{ij}(\mathbf{s})/2 - n_{\mathcal{I}} dist(e_i, e_j).$$

□

Theorem 1

Proof. Theorem 1 can be restated as follows: let $\mathbf{s}$ be a Nash equilibrium and $i, j \in \mathcal{I}$ be two individuals of the same social type $(c_i, e_i) = (c_j, e_j)$, such that $s_i \neq s_j$ and $R_{ij}(\mathbf{s}) > 0$. For all $\hat{\omega}_i \in W(\vec{\omega}_j)$ and $\hat{\omega}_j \in W(\vec{\omega}_i)$ and for all $0 < \varepsilon < R_{ij}(\mathbf{s})/2$, the open balls, in the l_∞ norm, centered at $\hat{\omega}_i$ and $\hat{\omega}_j$, with radius ε and $R_{ij}(\mathbf{s})/2 - \varepsilon$ do not intersect,

$$B_\varepsilon(\hat{\omega}_i) \cap B_{R_{ij}(\mathbf{s})/2-\varepsilon}(\hat{\omega}_j) = \emptyset.$$

This follows directly from Lemma 1. □

Theorems 3 and 4

Proof. (of Theorem 3) Let $(\mathbf{s}, \mathbf{c})$ be a social context and $\mathbf{s}$ be a Nash equilibrium. Observe that for an individual $i \in P(d,c)$, if her utility is replaced by any utility in $con(U(d,c))$, it follows from the cone structure of the Nash domain (see Remark 1) that $s_i = d$ is still a best response. Now note that if for two distinct decisions $d, d' \in \mathcal{A}$ and crowding type $c \in C$, individuals in $P(d,c)$ and $P(d',c)$ have a tendency to conform in $\mathbf{s}$, then for all $U_i \in con(U(d,c))$ and $U_j \in con(U(d',c))$, the individuals i and j would also have a tendency to conform. Thus, by Lemma 1, their utilities must differ at least in decision d and d'. □

Theorem 4 follows from Theorem 3.

Proofs based on conformity thresholds

Lemma 2

Proof. The strategy profile $\mathbf{s}$ is a Nash equilibrium if, and only if,

$$u_i(s_i, \mathbf{s}_{-i}) \geq u_i(d, \mathbf{s}_{-i})$$

for every $d \in D$ and $i \in \mathcal{I}$. Let $t_i = (c_i, e_i, v_i) = (c, e, v)$, the utility function can be rewritten explicitly as

$$u(i; \mathbf{s}) = \omega_{s_i}^v + \alpha_{s_i}^{ec}(l_{s_i}^c - 1) + \sum_{c' \neq c}^{n_C} \alpha_{s_i}^{ec'} l_{s_i}^{c'}(\mathbf{s}).$$

Letting $t = t_i$ and $l_d^t = l_d^t(\mathbf{s})$, we get

$$\omega_{s_i}^v - \alpha_{s_i}^{ec} + \sum_{c'=1}^{n_C} \alpha_{s_i}^{ec'} l_{s_i}^{c'} \geq \omega_d^v + \sum_{c'=1}^{n_C} \alpha_d^{ec'} l_d^{c'} \ .$$

Rearranging the terms, the previous inequality is equivalent to

$$\omega_{s_i}^v \geq \omega_d^v + \alpha_{s_i}^{ec} + \sum_{c'=1}^{n_C} \left(\alpha_d^{ec'} l_d^{c'} - \alpha_{s_i}^{ec'} l_{s_i}^{c'} \right) \ .$$

Hence, s_i is a best response for i if for every decision d

$$\omega_d^v \leq \omega_{s_i}^v - T_{(c,e)}(s_i \to d; \mathbf{L}).$$

□

Theorems 2 and 5

Start with the following construction. Let $\mathbf{t}$ be a type profile and $\mathbf{s}$ be a $\mathbf{t}$ admissible strategy profile, and denote its strategy class by $\mathbf{L} \equiv \mathbf{L}(\mathbf{s}, \mathbf{c})$. For every $t \in T$, let $\mathcal{S}_t \equiv \{s_i \in \mathcal{A} : i \in \mathcal{I}_t\}$ and let us use the superscript t on the parameters to mean the corresponding coordinate of t, for example, if $t = (c, e, v)$, then α_d^{tt} means α_d^{ec}. Since $\mathbf{s}$ is $\mathbf{t}$ admissible, for each type $t \in T$, there is at most one decision $d \in \mathcal{S}_t$ such that $\alpha_d^{tt} > 0$ (if there were two, they would violate the admissibility condition on the valuation map). Let us start by defining, for every type $t \in \mathcal{T}$,

$$d_t^* \equiv \arg\max_{d \in \mathcal{S}_t} \{\alpha_d^{tt}\}.$$

Let $i^* \in \mathcal{I}_t$ be an individual such that $s_{i^*} = d_t^*$, and let

$$\epsilon_t(\mathbf{s}) \equiv \begin{cases} 0 & \text{if } \alpha_{d_t^*}^{tt} \geq 0; \\ \\ -\dfrac{\alpha_{d_t^*}^{tt}}{2} & \text{if } \alpha_{d_t^*}^{tt} < 0. \end{cases}$$

Let $\Omega \equiv \times_{t \in T} \Omega_t$, where for a given type $t \in T$, Ω_t are the open sets of all $\boldsymbol{\omega}_t$ with the following properties:

(i) $\omega_{d_t^*}^t \in \mathbb{R}$;

(ii) if $\mathcal{A} \setminus \mathcal{S}_t \neq \emptyset$ then, for every $d \in \mathcal{A} \setminus \mathcal{S}_t$,

$$\omega_d^t \leq \min_{s_i \in \mathcal{S}_t} \{\omega_{s_i}^t - T_{(c,e)}(s_i \to d; \mathbf{L})\}; \tag{2.2}$$

(iii) if $\mathcal{S}_t \setminus \{d_t^*\} \neq \emptyset$, then, for every $s_i \in \mathcal{S}_t \setminus \{d_t^*\}$

$$\omega_{d_t^*}^t + T_{(c,e)}(s_i \to d_t^*; \mathbf{L}) + \epsilon_t(\mathbf{s}) \leq \omega_{s_i}^t \leq \omega_{d_t^*}^t - T_{(c,e)}(d_t^* \to s_i; \mathbf{L}) - \epsilon_t(\mathbf{s}). \tag{2.3}$$

Proof. (of Theorem 5) The proof is constructed over one type $t = (c, e, v) \in T$ by showing that $\emptyset \neq \Omega_t \in N(t; \mathbf{L})$, which holds for all $t \in T$, and thus $\Omega \in \mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e})$. As we will be refering always to the same type and to the same strategy, let us, for simplicity of notation, omit the subscript and the strategy class, hence, denote

$$T(d \to d') \equiv T_{(c,e)}(d \to d'; \mathbf{L})$$

Let's start by showing that $\Omega_t \neq \emptyset$. Observe that it is enough to show that equation (2.3) in the defintion of Ω_t refers to a non-degenerated interval, which translates into

$$-T(d_t^* \to s_i) - T(s_i \to d_t^*) \geq 2\epsilon_t(\mathbf{s}).$$

Hence, as

$$-T(d_t^* \to s_i) - T(s_i \to d_t^*) = -\alpha_{s_i}^{tt} - \alpha_{d_t^*}^{tt},$$

we get

$$-\alpha_{s_i}^{tt} - \alpha_{d_t^*}^{tt} \geq 0 \text{ when } \alpha_{d_t^*}^{tt} \geq 0,$$

and

$$-\alpha_{s_i}^{tt} - \alpha_{d_t^*}^{tt} \geq -\alpha_{d_t^*}^{tt} \text{ when } \alpha_{d_t^*}^{tt} < 0.$$

Now recall that being $\mathbf{t}$ admissible implies that for individuals i and j of the same type using different strategies $R_{ij}(\mathbf{s}) = \alpha_{s_i}^{tt} + \alpha_{s_j}^{tt} \leq 0$. As $\mathbf{s}$ is $\mathbf{t}$ admissible, there is for each type $t \in \mathcal{T}$ at most one decision $d \in \mathcal{S}_t$ such that $\alpha_d^{tt} > 0$, and that decision is by definition d_t^*, hence, $\alpha_{s_i}^{tt} \leq 0$.

To see that $\Omega \subset \mathcal{N}(\mathbf{s}, \mathbf{c}, \mathbf{e})$ we will show that the two equations setforth in the definition of the sets Ω_t are sufficient to guarantee that inequalities (2.1) in lemma 2 are satisfied for every individual and every decision. It is straightforward to see from equation (2.2) that no individual wants to change to decisions $d \notin \mathcal{S}_t$. Let's now check that equation (2.3) implies that individuals do not want to change between decisions within $\mathcal{S}_t$. Individuals choosing d_t^* do not want to change to other decisions in $\mathcal{S}_t$, since $\epsilon_t(\mathbf{s}) \geq 0$ and

$$\omega_{d_t^*}^t \geq \omega_{s_i}^t + T(d_t^* \to s_i) + \epsilon_t(\mathbf{s}).$$

An individual $i \neq i^* \in \mathcal{I}_t$ doesn't want to change to d_t^*, since $\epsilon_t(\mathbf{s}) \geq 0$ and

$$\omega_{s_i}^t \geq \omega_{d_t^*}^t + T(s_i \to d_t^*) + \epsilon_t(\mathbf{s}).$$

Finally, to see that an individual $i \neq j \in \mathcal{I}_t$ does not want to change to any decision $s_j \neq d_t^*$,

$$\omega_{s_i}^t - \omega_{s_j}^t \geq T(s_i \to d_t^*) + T(d_t^* \to s_j) + 2\epsilon_t(\mathbf{s}),$$

but

$$T(s_i \to d_t^*) + T(d_t^* \to s_j) = T(s_i \to s_j) + \alpha_{d_t^*}^{tt},$$

hence,

$$\omega_{s_i}^t \geq \omega_{s_j}^t + T(s_i \to s_j).$$

□

Theorem 2 follows from 5.

Proofs based on type improvement paths (existence of NE)

The following will be used in the proofs:

1) We call *t-rearangement* to a strategy profile $\mathbf{s}^*$ obtained from strategy $\mathbf{s}$ by changing all individuals of type $t \in T$ to their best response, while maintaining the remaining $\mathcal{I} \setminus \mathcal{I}_t$ players fixed. Note that by corollary 4 this is always possible.

2) Given a game with PBI and a strategy profile $\mathbf{s}$, add a new player i of type t_i, and place her in her best response a to strategy profile $\mathbf{s}$: i) for all other players only the payoff associated to action a has changed; ii) no individual of the same type that is also choosing a has an incentive to change (since they are of the same type, same payoffs).

Theorem 6

Proof. The proof follows by induction. By corollary 4 $\Gamma_{1,n_{\mathcal{A}}}$ always has a pure Nash equilibrium. We will now add one more player of a different type. Recall that there are only positive externalities. Place the new individual i_1, of the second type, in her best response, say it's action a. After a t_1-*rearangement* individuals of type t_1 can only change their decision towards decision a. But this increases the payoff of i_1, so she is still in her best response and we've reached an equilibrium. Add now a second player. If she also chooses a, the reasoning goes as before. If she chooses a', then, after a t_1-*rearangement* the only question is whether individual i_1 wants to move. If she does, it must be towards action a' for this is the only payoff that has increased. So let her change and we've reached an equilibrium, as before. The reasoning continues ad infinitum. Note that this reasoning did not depend on the type of the players added, nor on the number of decisions. □

Theorem 7

Proof. The proof follows by induction as before. By corollary 4 $\Gamma_{1,n_{\mathcal{A}}}$ always has a pure Nash equilibrium. We will now add a player of type t_2. Place the new individual i, of the second type, in her best response, say it's action a_1. After a t_1-*rearangement* individuals of type t_1 can only change their decision in two ways: *i*) out of decision a_1 if $\alpha_{a_1}^{12} < 0$; or *ii*) towards decision a_1 if $\alpha_{a_1}^{12} > 0$. In both cases the payoff of i increased, since the game is weakly reciprocal and $\mathrm{sgn}(\alpha_{a_1}^{21}) = \mathrm{sgn}(\alpha_{a_2}^{21}) = \mathrm{sgn}(\alpha_{a_1}^{12})$. Hence, we've reached an equilibrium. Add now a second player of type t_2. Two cases: (i) if she also chooses a_1, after a t_1-rearrangement, the reasoning goes as before; (ii) if she chooses a_2, after a t_1-*rearangement* the only question is whether individual i wants to move to decision a_2. If she does, we are in case (i). □

2.3 Mixed Nash equilibria

A mixed strategy is a probability distribution over the space of pure strategies $\mathcal{A}$. The space of mixed strategies is thus the simplex $\Delta^{n_{\mathcal{A}}}$. We represent the mixed strategy of an individual i by a vector $\vec{\sigma}_i = (\sigma_1^i, \dots, \sigma_{n_{\mathcal{A}}}^i)$ where $\vec{\sigma}_i(a) \equiv \sigma_a^i$ is the probability that $\vec{\sigma}_i$ assigns to $a \in \mathcal{A}$, and $\sum_{a\in\mathcal{A}} \sigma_a^i = 1$. The support of a mixed strategy, $supp(\vec{\sigma}_i)$, is the subset of pure strategies to which $\vec{\sigma}_i$ assigns positive probability. A mixed-strategy profile of the game is an element of $\Delta \equiv (\Delta^{n_{\mathcal{A}}})^{n_{\mathcal{I}}}$ with its coordinates being the mixed strategies of every individual $i \in \mathcal{I}$, denoted $\boldsymbol{\sigma} = (\vec{\sigma}_1, \dots, \vec{\sigma}_{n_{\mathcal{I}}})$.

The payoff to individual i of the mixed-strategy profile $\boldsymbol{\sigma}$ is the expected value with respect to $\boldsymbol{\sigma}$ of the pure strategy payoffs, which, with the standard slight abuse of notation, is denoted $u(i;\boldsymbol{\sigma}) \equiv \sum\limits_{\mathbf{s}\in S} \prod_{j=1}^{n_{\mathcal{I}}} \sigma_{s_j}^j u(i;\mathbf{s})$. In a decision game with the PSS property, the personal part will remain separable, and only the social component will have terms depending on the remaining strategies. Therefore, the payoffs for the mixed strategy profile can be written using general social externality functions $e^i : \mathcal{A} \times [0,1]^{(n-1)} \to \mathbb{R}$ as

$$u(i;\boldsymbol{\sigma}) = \sum_{a\in\mathcal{A}} \sigma_a^i \left(\omega_a^i + e_a^i(\boldsymbol{\sigma}_{-i})\right)$$

Consider a non-degenerate mixed-strategy Nash equilibrium $\boldsymbol{\sigma}^* \in \Delta$. Regarding the relation between the payoffs associated to each underlying pure strategy two general properties are useful:

$$u(i;\boldsymbol{\sigma}^*) = \omega_a^i + e_a^i(\boldsymbol{\sigma}_{-i}^*), \ \forall a \in supp(\vec{\sigma}_i^*) \tag{2.4}$$

$$u(i;\boldsymbol{\sigma}^*) \geq \omega_a^i + e_a^i(\boldsymbol{\sigma}_{-i}^*), \ \forall a \notin supp(\vec{\sigma}_i^*). \tag{2.5}$$

Remark 2 (Personal and social balance)**.** *Consider a non-degenerate mixed-strategy Nash equilibrium* $\boldsymbol{\sigma}^* \in \Delta$*. The following holds for every* $i \in \mathcal{I}$ *and* $a_1, a_2 \in \mathcal{A}$*,* *(i) if* $a_1, a_2 \in supp(\vec{\sigma}_i)$*,* $\Delta\omega^i(a_1, a_2) = \Delta e^i(a_2, a_1; \boldsymbol{\sigma}_{-i})$*;*

(ii) if $a_1 \in supp(\vec{\sigma}_i)$ *and* $a_2 \notin supp(\vec{\sigma}_i)$*,* $\Delta\omega^i(a_1, a_2) \geq \Delta e^i(a_2, a_1; \boldsymbol{\sigma}_{-i})$*.*

The remark is useful when looking at individuals with the same type, as differences in their personal and social balance will be given by differences in their strategy through the externality function.

2.3.1 Type symmetries under DI and PBI

Proposition 1. *In a decision game with* PSS*,* DI *and* PBI *properties, the expected value of* $\vec{\sigma}_i$ *with respect to* $\boldsymbol{\sigma}_{-i}$ *is given by*

$$u(i; \boldsymbol{\sigma}) = \sum_{a \in \mathcal{A}} \sigma_a^i \left(\omega_a^i + \sum_{j \neq i} \alpha_a^{ij} \sigma_a^j \right).$$

Lemma 4. *Let* $\boldsymbol{\sigma} \in \Delta$ *be a mixed-strategy Nash equilibrium and* $i, j \in \mathcal{I}$ *two individuals of the same social type* $(c_i, e_i) = (c_j, e_j) \equiv (c, e)$*. For any action* $a \in supp(\vec{\sigma}_i) \cap supp(\vec{\sigma}_j)$ *we have*

$$u(i; \boldsymbol{\sigma}) - u(j; \boldsymbol{\sigma}) = \omega_a^i - \omega_a^j - \alpha_a^{ec} \left(\sigma_a^i - \sigma_a^j \right)$$

The above lemma relates to that of conformity obstruction. Note that for two individuals of the same type (i.e. individuals that besides the same social type, have the same valuation type) we have $\omega_a^i = \omega_a^j$. Therefore, the diference in their payoffs in a Nash equilibrium is bounded by the externality they provoke on eachother, if their supports intersect. Consider the following partition of the set of pure strategies according to the influence individuals of type t have on eachother,

$$\mathcal{A}_t^- \equiv \{a \in \mathcal{A} : \alpha_a^{tt} < 0\}; \;\; \mathcal{A}_t^0 \equiv \{a \in \mathcal{A} : \alpha_a^{tt} = 0\}; \;\; \mathcal{A}_t^+ \equiv \{a \in \mathcal{A} : \alpha_a^{tt} > 0\}.$$

The set $\mathcal{A}_t^-$ is formed by the subset of pure strategies where individuals of type t do not like to be together. The subsets $\mathcal{A}_t^0$ and $\mathcal{A}_t^+$ are formed, respectively, by those pure strategies where individuals of type t are indifferent to eachother, and where they like to be together. Let us now partition the support of a mixed strategy $\vec{\sigma}_i$ of individual $i \in \mathcal{I}$ according to these subsets. Define $supp(\vec{\sigma}_i)^- \equiv supp(\vec{\sigma}_i) \cap \mathcal{A}_t^-$, and analogously, define $supp(\vec{\sigma}_i)^0$ and $supp(\vec{\sigma}_i)^+$. Putting together the result of lemma 4 and the above partitions leads to the following counterintuitive observations.

Remark 3 (Type utility ordering). *Let $\boldsymbol{\sigma} \in \Delta$ be a mixed-strategy Nash equilibrium and $i \in \mathcal{I}_t$. The following comparisons hold for all $j \in \mathcal{I}_t$,*

(i) if $a \in supp(\vec{\sigma}_i)^+$, then $\sigma_a^i < \sigma_a^j \Rightarrow u(i;\boldsymbol{\sigma}) > u(j;\boldsymbol{\sigma})$;

(ii) if $a \in supp(\vec{\sigma}_i)^-$, then $\sigma_a^i > \sigma_a^j \Rightarrow u(i;\boldsymbol{\sigma}) > u(j;\boldsymbol{\sigma})$;

(iii) if $a \in supp(\vec{\sigma}_i)^0$, then $u(i;\boldsymbol{\sigma}) \leq u(j;\boldsymbol{\sigma})$;

(iv) if $a \in supp(\vec{\sigma}_i)^0$ and $supp^0(\vec{\sigma}_j) \neq \emptyset$ then $u(i;\boldsymbol{\sigma}) = u(j;\boldsymbol{\sigma})$;

(v) if $a \in supp(\vec{\sigma}_i) \cap supp(\vec{\sigma}_j) \setminus \mathcal{A}_t^0$, then $u(i;\boldsymbol{\sigma}) = u(j;\boldsymbol{\sigma}) \Leftrightarrow \sigma_a^i = \sigma_a^j$.

Lemma 5 (Type-asymmetry obstructions). *Let $\boldsymbol{\sigma} \in \Delta$ be a non-degenerated mixed-strategy profile that is a Nash equilibrium. Consider two individuals of the same type $i, j \in \mathcal{I}_t$ such that $\vec{\sigma}_i \neq \vec{\sigma}_j$ and $u(i;\boldsymbol{\sigma}) > u(j;\boldsymbol{\sigma})$. The following holds*

(i) $supp(\vec{\sigma}_i)^+ \subseteq supp(\vec{\sigma}_j)^+$;

(ii) $supp(\vec{\sigma}_i)^0 = \emptyset$;

(iii) $supp(\vec{\sigma}_i)^- \neq \emptyset$.

Furthermore, if $supp(\vec{\sigma}_i) = supp(\vec{\sigma}_j) \equiv \underline{a}$, then

$$\sum_{a \in \underline{a}} \frac{1}{\alpha_a^{tt}} = 0. \tag{2.6}$$

The difference in individual strategies is limited by the choice of supports. In particular, when for some game the condition given by equation 2.6 never holds, then individuals choosing the same support will have to play the same strategy. The next definition and results are based on this idea.

Definition 4 (Mixed Type-Symmetry condition MTS)**.** *A support* $\underline{a} \subseteq \mathcal{A}$ *of a mixed strategy satisfies the mixed type-symmetry condition* (MTS) *for a given type* t *if*

$$\sum_{a^* \in \underline{a}} \prod_{a \neq a^*} \alpha_a^{tt} \neq 0.$$

Note that the MTS is in fact broken in equation 2.6 of lemma 5. As an example of why this condition can be useful, think of a game with only two possible actions, which leads to only three possibilities of distinct supports (including singletons). Individuals of the same type can either play a pure strategy or the same non-degenerate mixed strategy.

Remark 4. *Consider a decision game with only two possible actions* $\mathcal{A} = \{a_1, a_2\}$*. If for some type* t *we have* $\alpha_1^{tt} + \alpha_2^{tt} \neq 0$*, then all individuals of type* t *either play a pure strategy or the same strictly mixed strategy.*

Let us now go back to the general case and define a notation for the difference in payoffs for any two decisions chosen by an individual i, without considering the influence of another individual j_1,

$$\Delta u^i(a_1, a_2; \boldsymbol{\sigma}_{-\{j_1\}}) = \Delta\omega^i(a_1, a_2) + \sum_{j \neq i, j_1} \alpha_{a_1}^{ij} \sigma_{a_1}^j - \sum_{j \neq i, j_1} \alpha_{a_2}^{ij} \sigma_{a_2}^j.$$

Note that for two individuals of the same taste type, i.e. $(v_i, e_i) = (v_j, e_j) = (v, e)$, this difference is the same[3] $\Delta u^i(a_1, a_2; \boldsymbol{\sigma}_{-\{j\}}) = \Delta u^j(a_1, a_2; \boldsymbol{\sigma}_{-\{i\}})$. Using lemma 5 we will prove that for individuals of the same type, that is when we add the restriction of the same crowding for these individuals, the MTS condition will in fact force them to play the same, unique, strategy.

[3] This can be seen by writing $\Delta u^i(a_1, a_2; \boldsymbol{\sigma}_{-\{j\}}) = \Delta\omega^v(a_1, a_2) + \Delta e(a_1, a_2; \boldsymbol{\sigma}_{-\{i,j\}})$.

Let us first define for two individuals $i, j \in \mathcal{I}_t$, the quantity

$$h^t(a_1, a_2; \boldsymbol{\sigma}_{-\{i,j\}}) \equiv \Delta u^i(a_1, a_2; \boldsymbol{\sigma}_{-\{j\}}).$$

Theorem 8 (Type-symmetry)**.** *In a Nash equilibrium, if two individuals of the same type $i, j \in \mathcal{I}_t$ choose the same support $\underline{a} \subset \mathcal{A}$, and the support satisfies the* MTS *condition for type t, then $\vec{\sigma}_i = \vec{\sigma}_j$. Furthermore, the support $\underline{a}$ uniquely determines their (equilibrium) strategy in terms of the strategies of the other individuals $\boldsymbol{\sigma}_{-\{i,j\}}$. In the case of non-singleton supports it is given by*

$$\sigma_{a^*} = \frac{1 + \sum_a h^t(a, a^*; \boldsymbol{\sigma}_{-\{i,j\}})/\alpha_a^{tt}}{1 + \sum_{a \neq a^*} \alpha_{a^*}^{tt}/\alpha_a^{tt}}$$

for all $a^* \in \underline{a}$.

Note that the strategy characterized above is an intersection of best responses, not exactly the unique best response for the individuals. That's why in the theorem we mention it as an *equilibrium strategy* and not the best response to $\boldsymbol{\sigma}_{-\{i,j\}}$. Observe also, that by the MTS condition, the denominator is not zero and the strategy is well defined. Naturally it must satisfy being a probability. Another interesting remark is that the theorem does not hold for individuals of the same taste type but with different crowding types.

Similarly to what we've done in the case of pure strategies, let us define the *mixed type-aggregate decision matrix* $\mathbf{M}(\boldsymbol{\sigma})$ with coordinates, $M_a^t \equiv M_a^t(\boldsymbol{\sigma}) \equiv \sum_{j \in \mathcal{I}_t} \sigma_a^j$,

$$\mathbf{M}(\boldsymbol{\sigma}) \equiv \begin{pmatrix} M_1^1 & \dots & M_1^{n_T} \\ \vdots & \ddots & \vdots \\ M_{n_{\mathcal{A}}}^1 & \dots & M_{n_{\mathcal{A}}}^{n_T} \end{pmatrix}.$$

Let us denote the number of individuals of type t using a mixed strategy by

$$m_t(\boldsymbol{\sigma}) \equiv \{i \in \mathcal{I}_t : 0 < \sigma_a^i < 1,\ a \in \mathcal{A}\}.$$

The *mixed conformity threshold* excluding type t is defined as

$$T_t(a_1, a_2; \mathbf{M}_{-t}) \equiv \sum_{t' \neq t} \alpha_{a_2}^{tt'} M_{a_2}^{t'} - \sum_{t' \neq t} \alpha_{a_1}^{tt'} M_{a_1}^{t'} + \alpha_{a_2}^{tt}(m_t - 1).$$

Recall that

$$T_t(a_1, a_2; \mathbf{L}) = \sum_{t'} \alpha_{a_2}^{tt'} l_{a_2}^{t'} - \sum_{t'} \alpha_{a_1}^{tt'} l_{a_1}^{t'}.$$

Using Remark 4 we have the following corollary for $\Gamma_{n_T,2}$ games.

Corollary 6 (Dichotomous games)**.** *Consider the class $\Gamma_{n_T,2}$ of decision games with only two possible actions $\mathcal{A} = \{a_1, a_2\}$. If for some type t we have $\alpha_1^{tt} + \alpha_2^{tt} \neq 0$, then in a Nash equilibrium, the strategy of type t must be composed by a subset of individuals in pure strategies and a subset of individuals playing with the following mixed strategy,*

$$\sigma_{a_1} = \frac{T_t(a_1, a_2; \mathbf{L}) + T_t(a_1, a_2; \mathbf{M}_{-t}) - \Delta\omega_t(a_1, a_2)}{(\alpha_{a_2}^{tt} + \alpha_{a_1}^{tt})(m_t - 1)}$$

and $\sigma_{a_2} = 1 - \sigma_{a_1}$.

Note that the subsets mentioned in the above corollary might be empty. In particular, by theorem 4, if $\alpha_1^{tt} + \alpha_2^{tt} > 0$ then $m_t \in \{0, n_t\}$, that is, all strategies are type-symmetric in equilibrium. The next corollary is just a simplification of the above, but will be useful later. Let us denote for the class of homogeneous dichotomous case $\Gamma_{1,2}$ (only one type of individuals and two actions) the decision threshold

$$T(l_1) \equiv -(\alpha_1 + \alpha_2) l_1 + \alpha_2 (n - 1).$$

Corollary 7 (Homogeneous dichotomous games)**.** *Consider the class $\Gamma_{1,2}$ of decision games with only one type of individuals and two possible actions $\mathcal{A} = \{a_1, a_2\}$. If $\alpha_1 + \alpha_2 \neq 0$, then in a Nash equilibrium, there is a subset of individuals in pure strategies and a subset of individuals playing with the* *following mixed strategy,*

$$\sigma_{a_1} = \frac{T(l_1) - \Delta\omega(a_1, a_2)}{(\alpha_2 + \alpha_1)(m-1)}$$

with $\sigma_{a_2} = 1 - \sigma_{a_1}$.

As in the non-homogeneous case, by theorem 4, if $\alpha_1 + \alpha_2 > 0$ then $m \in \{0, n\}$, where n is the number of individuals.

In the general case, if individuals use different probabilities, then the intersection of their support must be contained in some 'null externality set'. This is concretized in the next remark.

Remark 5. *Let* $\underline{A}^* \subseteq supp(\vec{\sigma}_i) \cap supp(\vec{\sigma}_j)$ *be the maximal subset for which the* MTS *condition holds. If* $u^i = u^j$*, then for all* $a \in \underline{A}^*$ *players must use the same probability, that is* $\sigma^i_a = \sigma^j_a$.

2.3.2 Proofs

Proposition 1

Just observe that because of the DI property the payoff of each pure strategy where i and j interact is the same whatever are the strategies of other individuals. By linearity of the expected value, the product can be separated into a sum of all the strategies, which equate to 1 since they are a probability distribution.

Lemma 4

Proof. Observe that in a Nash equilibrium $\boldsymbol{\sigma}$ for all actions $a, a' \in supp(\vec{\sigma}_i)$, the payoffs equate $u(i; a, \boldsymbol{\sigma}_{-i}) = u(i; a', \boldsymbol{\sigma}_{-i})$ (see Equation 2.4). Now, for two individuals $i, j \in \mathcal{I}$ of the same social type $(c_i, e_i) = (c_j, e_j) \equiv (c, e)$, this means that, for i,

$$\omega_a^i + \sum_{j' \neq i,j} \alpha_a^{ej'} \sigma_a^{j'} + \alpha_a^{ec} \sigma_a^j = \omega_{a'}^i + \sum_{j' \neq i,j} \alpha_{a'}^{ej'} \sigma_{a'}^{j'} + \alpha_{a'}^{ec} \sigma_{a'}^j$$

and for j

$$\omega_a^j + \sum_{j' \neq i,j} \alpha_a^{ej'} \sigma_a^{j'} + \alpha_a^{ec} \sigma_a^i = \omega_{a'}^j + \sum_{j' \neq i,j} \alpha_{a'}^{ej'} \sigma_{a'}^{j'} + \alpha_{a'}^{ec} \sigma_{a'}^i$$

□

Lemma 5

Let us start by fixing a notation for the value of a given pure strategy a for individuals of type t not taking into account individuals i and j,

$$V_a^t(\boldsymbol{\sigma}_{-\{i,j\}}) \equiv \omega_a^t + \sum_{j' \neq i,j} \alpha_a^{tj'} \sigma_a^{j'}.$$

Now observe that if $\boldsymbol{\sigma}$ is a Nash equilibrium, for all $a \in supp(\vec{\sigma}_i)$ the utility of individual i of type t is $u(i;\boldsymbol{\sigma}) = V_a^t(\boldsymbol{\sigma}_{-\{i,j\}}) + \alpha_a^{tt} \sigma_a^j$.

Proof. Consider a Nash equilibrium $\boldsymbol{\sigma}$ where $\vec{\sigma}_i \neq \vec{\sigma}_j$ and $u(i;\boldsymbol{\sigma}) > u(j;\boldsymbol{\sigma})$.

Proof of (i). Suppose there is $a \in \mathcal{A}$ such that $a \in supp(\vec{\sigma}_i)^+$ and $a \notin supp(\vec{\sigma}_j)^+$. Let's observe that player j would have an incentive to change to the pure strategy a, which concludes the proof. As $\boldsymbol{\sigma}$ is a Nash equilibrium and $a \notin supp(\vec{\sigma}_j)^+$ we have $u(i;\boldsymbol{\sigma}) = V_a^t(\boldsymbol{\sigma}_{-\{i,j\}}) + \alpha_a^{tt} 0$. Since individuals i and j are of the same type, if j changes to a, then $u_j(a, \boldsymbol{\sigma}_{-j}) = V_a^t(\boldsymbol{\sigma}_{-\{i,j\}}) + \alpha_a^{tt} \sigma_a^i$ and $\alpha_a^{tt} > 0$. Hence, $u_j(a, \boldsymbol{\sigma}_{-j}) > u(i;\boldsymbol{\sigma}) > u(j;\boldsymbol{\sigma})$.

Proof of (ii). Suppose $a \in supp(\vec{\sigma}_i)^0$. Then, by the same reasoning $u_j(a, \boldsymbol{\sigma}_{-j}) = u(i;\boldsymbol{\sigma}) > u(j;\boldsymbol{\sigma})$.

Proof of (iii). Suppose $supp(\vec{\sigma}_i)^- = \emptyset$. From (i) and (ii) we have $supp(\vec{\sigma}_i) \subseteq supp(\vec{\sigma}_j)$. Furtermore $supp(\vec{\sigma}_i) = supp(\vec{\sigma}_i)^+$. By Remark 3,

$\sigma_a^i < \sigma_a^j$ for all $a \in supp(\vec{\sigma}_i)$. This is impossible since $\sum_i \sigma_a^i = \sum_j \sigma_a^j = 1$.

Let us now prove the last part of the theorem. First, note that as each individual earns the same in every pure strategy contained in the support of its mixed strategy, the following must hold for every pair of decisions $a, a' \in supp(\vec{\sigma}_i)$,

$$\alpha_{a'}^{tt}\sigma_{a'}^j - \alpha_a^{tt}\sigma_a^j = V_a^t(\boldsymbol{\sigma}_{-\{i,j\}}) - V_{a'}^t(\boldsymbol{\sigma}_{-\{i,j\}}).$$

Furthermore, this must hold both for i and j, since V does not depend on either. Suppose now that $supp(\vec{\sigma}_i) = supp(\vec{\sigma}_j)$. This originates a unique system of equations for the strategy of both individuals. If the strategies of two individuals of the same type are differente, i.e. $\vec{\sigma}_i \neq \vec{\sigma}_j$, this implies that the system associated to the above equations, together with the constraint $\sum_i \sigma_a^i = 1$, does not have a unique solution. Let us label the elements of the support $supp(\vec{\sigma}_i) = \{1, \ldots, n_\sigma\}$. Let us use the variable reduction $\sigma_{n_\sigma} = 1 - \sum_a \sigma_a$. The system can be reduced, and its coeficients can be represented by the following $(n_\sigma - 1) \times (n_\sigma - 1)$ square matrix

$$B \equiv \begin{pmatrix} \alpha_1^{tt} & -\alpha_2^{tt} & 0 & \cdots & \cdots & 0 \\ \alpha_1^{tt} & 0 & -\alpha_3^{tt} & 0 & \cdots & 0 \\ \vdots & & & & & \\ \vdots & & & & \ddots & 0 \\ \alpha_1^{tt} & 0 & \cdots & \cdots & 0 & -\alpha_{n_\sigma-1}^{tt} \\ \alpha_1^{tt} + \alpha_{n_\sigma}^{tt} & \alpha_{n_\sigma}^{tt} & \cdots & \cdots & \cdots & \alpha_{n_\sigma}^{tt} \end{pmatrix}$$

The first term of the determinant of the above matrix, calculated in terms of the last row, is

$$(-1)^{n_\sigma-1+1}(\alpha_1^{tt} + \alpha_{n_\sigma}^{tt})(-1)^{n_\sigma-2} \prod_{a \neq \{n_\sigma,1\}} \alpha_a^{tt},$$

and the remaining terms are (changing column 1 to 'place' $a^* - 1$)

$$\sum_{a^* \in \{2,\ldots,n_\sigma - 1\}} (-1)^{n_\sigma - 1 + a^*} \alpha^{tt}_{n_\sigma} (-1)^{1 + a^* - 1} \alpha^{tt}_1 (-1)^{n_\sigma - 3} \prod_{a \neq \{n_\sigma, a^*, 1\}} \alpha^{tt}_a .$$

Rearranging the last expression, (noting all exponents of the -1 terms even out) we get

$$\sum_{a^* \in \{2,\ldots,n_\sigma - 1\}} \prod_{a \neq a^*} \alpha^{tt}_a ,$$

hence,

$$det(B) = (\alpha^{tt}_1 + \alpha^{tt}_{n_\sigma}) \prod_{a \neq \{n_\sigma, 1\}} \alpha^{tt}_a + \sum_{a^* \in \{2,\ldots,n_\sigma - 1\}} \prod_{a \neq a^*} \alpha^{tt}_a ,$$

therefore

$$det(B) = \prod_{a \neq n_\sigma} \alpha^{tt}_a + \prod_{a \neq 1} \alpha^{tt}_a + \sum_{a^* \in \{2,\ldots,n_\sigma - 1\}} \prod_{a \neq a^*} \alpha^{tt}_a ,$$

and finally

$$det(B) = \sum_{a^* \in supp(\vec{\sigma}_i)} \prod_{a \neq a^*} \alpha^{tt}_a .$$

To conclude observe that for two players of the same type to play different strategies with the same support we wanted $\det(B) = 0$. Noting that $supp(\vec{\sigma}_i)^0 = \emptyset$

$$\det(B) = 0 \Rightarrow \sum_{a \in supp(\vec{\sigma}_i)} \frac{1}{\alpha^{tt}_a} = 0.$$

□

Theorem 8

Let us start by the following auxiliar result

Claim 1. *Let A be a matrix with zero in all entries except in the diagonal and columns j_1 and j_2. We have that*

$$\det(A) = (a_{j_1 j_1} a_{j_2 j_2} - a_{j_2 j_1} a_{j_1 j_2}) \prod_{i \neq \{j_1, j_2\}} a_{ii}$$

Proof. Start by observing that a diagonal matrix with one non zero column has as its determinant the product of the diagonal entries. Now, for the case with two columns non-zero j_1 and j_2, we will observe that the only minors with non-zero determinant are of the form mentioned before. Let's say we develop the determinant by column j_1. There are only two minors that do not have a column with zeros, and thus non-zero determinant. These are the ones with coefficient $a_{j_1 j_1}$ and $a_{j_2 j_1}$. The first minor is in the aforementioned case. The second minor, with a change of the (old) column j_2 to the place of the (old) column j_1 (which was eliminated), becames also the aforementioned case. □

Proof. (of Theorem 8) The proof follows from the proof of Lemma 5 by finding the solution of the system

$$\alpha_{a'}^{tt} \sigma_{a'}^{j} - \alpha_{a}^{tt} \sigma_{a}^{j} = V_a^t(\boldsymbol{\sigma}_{-\{i,j\}}) - V_{a'}^t(\boldsymbol{\sigma}_{-\{i,j\}})$$

using Cramer rule.[4] Note that it is sufficient to write the system in terms of

[4]This is probably not the simplest nor the more elegant approach.

decision 1. Let

$$B^{a^*} \equiv \begin{pmatrix} \alpha_1^{tt} & -\alpha_2^{tt} & 0 & \cdots & 0 & V_2^t - V_1^t & 0 & \cdots & 0 \\ \alpha_1^{tt} & 0 & -\alpha_3^{tt} & 0 & & V_3^t - V_1^t & & \cdots & 0 \\ \alpha_1^{tt} & 0 & & \ddots & 0 & \vdots & \vdots & \cdots & 0 \\ \vdots & & & & -\alpha_{a^*-1}^{tt} & \vdots & & \cdots & 0 \\ \vdots & & & & 0 & \vdots & 0 & \cdots & 0 \\ \vdots & 0 & & \cdots & & \vdots & -\alpha_{a^*+1}^{tt} & \cdots & 0 \\ \vdots & 0 & & \cdots & \vdots & \vdots & 0 & \ddots & 0 \\ \alpha_1^{tt} & 0 & \cdots & \cdots & 0 & \vdots & 0 & 0 & -\alpha_{n_\sigma-1}^{tt} \\ \alpha_1^{tt} + \alpha_{n_\sigma}^{tt} & \alpha_{n_\sigma}^{tt} & \cdots & \cdots & \alpha_{n_\sigma}^{tt} & V_{n_\sigma}^t - V_1^t + \alpha_{n_\sigma}^{tt} & \alpha_{n_\sigma}^{tt} & \cdots & \alpha_{n_\sigma}^{tt} \end{pmatrix}$$

For a subset $A^* \subset \mathcal{A}$ let us define $P(A^*) \equiv \prod_{a \in \mathcal{A} \setminus A^*} \alpha_a^{tt}$. We will omitt the superscripts relative to type t since the proof is done over type t only. Developping the determinant by the last row, there are two terms not in α_{n_σ}. These are (as before changing column 1 to the pl'place' $a^* - 1$ in the corresponding non-diagonal minor)

$$\begin{aligned} &(-1)^{n-1+1}(\alpha_1 + \alpha_{n_\sigma})(V_{a^*} - V_1)P(a^*, 1, n_\sigma)(-1)^{n-3} \\ +&(-1)^{n-1+a^*}(V_{n_\sigma} - V_1 + \alpha_{n_\sigma})(-1)^{a^*-2}P(a^*, n_\sigma)(-1)^{n-3}. \end{aligned}$$

This leads to

$$-(\alpha_1 + \alpha_{n_\sigma})(V_{a^*} - V_1)P(a^*, 1, n_\sigma) + (V_{n_\sigma} - V_1 + \alpha_{n_\sigma})P(a^*, n_\sigma),$$

and finally

$$P(a^*) - (V_{a^*} - V_1)P(a^*, n_\sigma) - (V_{a^*} - V_1)P(a^*, 1) + (V_{n_\sigma} - V_1)P(a^*, n_\sigma).$$

Now the other terms are (adjusting the matrix columns and using Claim 1)

$$\sum_{a\neq 1,a^*,n_\sigma} (-1)^{n-1+a}\alpha_{n_\sigma}(-1)^{a-2}\left(\alpha_1(V_{a^*}-V_1)-\alpha_1(V_a-V_1)\right)P(1,a,n_\sigma,a^*)(-1)^{n-4}$$

so we get

$$\sum_{a\neq 1,a^*,n_\sigma} \left((V_a-V_1)-(V_{a^*}-V_1)\right)P(a,a^*)$$

rearanging

$$\sum_{a\neq 1,a^*,n_\sigma}(V_a-V_1)P(a,a^*)-(V_{a^*}-V_1)\sum_{a\neq 1,a^*,n_\sigma}P(a,a^*).$$

Putting all terms together

$$\det(B^{a^*}) = P(a^*)+\sum_{a\neq a^*}(V_a-V_1)P(a,a^*)-(V_{a^*}-V_1)\sum_{a\neq a^*}P(a,a^*)$$

and finally

$$\det(B^{a^*}) = P(a^*)+\sum_{a}(V_a-V_{a^*})P(a,a^*).$$

The solution, by Cramer rule, is

$$\frac{\det(B^{a^*})}{\det(B)} = \frac{P(a^*)+\sum_a(V_a-V_{a^*})P(a,a^*)}{\sum_a\prod_{a'\neq a}\alpha^{tt}_{a'}}$$

when there are no zeros,

$$\frac{\det(B^{a^*})}{\det(B)} = \frac{1+\sum_a(V_a-V_{a^*})(\alpha^{tt}_a)^{-1}}{\alpha^{tt}_{a^*}\sum_a(\alpha^{tt}_a)^{-1}}$$

Note that $V_a - V_{a^*} = \Delta\omega_t(a,a^*)+\Delta e_t(a,a^*;\boldsymbol{\sigma}_{-\{i,j\}})$. Thus $V_a - V_{a^*} = h^t(a,a^*;\boldsymbol{\sigma}_{-\{i,j\}})$. □

Computations for corollaries of Theorem 8

Theorem 8 yields

$$\sigma_{a_1} = \frac{1 + h^t(a_2, a_1; \boldsymbol{\sigma}_{-\{i,j\}})/\alpha^{tt}_{a_2}}{1 + \alpha^{tt}_{a_1}/\alpha^{tt}_{a_2}}$$

With two types, $h^t(a_2, a_1; \boldsymbol{\sigma}_{-\{i,j\}})$ is given by

$$-\Delta\omega_t(a_1, a_2) - \sum_{j \neq i,j} \alpha^{ij}_{a_1}\sigma^j_{a_1} + \sum_{j \neq i,j} \alpha^{ij}_{a_2}\sigma^j_{a_2}$$

which by type

$$-\Delta\omega_t(a_1, a_2) - \sum_{t' \neq t} \alpha^{tt'}_{a_1}(l^{t'}_{a_1} + M^{t'}_{a_1}) + \sum_{t' \neq t} \alpha^{tt'}_{a_2}(l^{t'}_{a_2} + M^{t'}_{a_2})$$

$$+\alpha^{tt}_{a_2} l^t_{a_2} + \alpha^{tt}_{a_2}(m_t - 2)(1 - \sigma_{a_1}) - \alpha^{tt}_{a_1} l^t_{a_1} - \alpha^{tt}_{a_1}(m_t - 2)\sigma_{a_1}$$

therefore

$$-\Delta\omega_t(a_1, a_2) - \sum_{t' \neq t} \alpha^{tt'}_{a_1}(l^{t'}_{a_1} + M^{t'}_{a_1}) + \sum_{t' \neq t} \alpha^{tt'j}_{a_2}(l^{t'}_{a_2} + M^{t'}_{a_2})$$

$$+\alpha^{tt}_{a_2} l^t_{a_2} - \alpha^{tt}_{a_1} l^t_{a_1} + \alpha^{tt}_{a_2}(m_t - 2) - (\alpha^{tt}_{a_2} + \alpha^{tt}_{a_1})(m_t - 2)\sigma_{a_1}$$

so we get

$$\sigma_{a_1} = \frac{-\Delta\omega_t(a_1, a_2) + \sum_{t'} \alpha^{tt'}_{a_2} l^{t'}_{a_2} - \sum_{t'} \alpha^{tt'}_{a_1} l^{t'}_{a_1} + \sum_{t' \neq t} \alpha^{tt'}_{a_2} M^{t'}_{a_2} - \sum_{t' \neq t} \alpha^{tt'}_{a_1} M^{t'}_{a_1} + \alpha^{tt}_{a_2}(m_t - 1)}{(\alpha^{tt}_{a_2} + \alpha^{tt}_{a_1})(m_t - 1)}$$

which can be written using the thresholds

$$\sigma_{a_1} = \frac{T(a_1, a_2; \mathbf{L}) + T_{-t}(a_1, a_2; \mathbf{M}) - \Delta\omega_t(a_1, a_2)}{(\alpha^{tt}_{a_2} + \alpha^{tt}_{a_1})(m_t - 1)}.$$

Chapter 3

Socially prone duopolies

The main idea in this chapter is that small price changes are captured by consumers using non-degenerate mixed strategies (called non-loyal consumers). If pure strategy consumers (loyal consumers) are held constant, a change in price implicitly determines a unique continuous deviation for non-loyal consumers and this works as a coordination device for firms. Furthermore, as it is continuous it stabilizes firms in pure price equilibria. We define an influence network among consumers based on partial derivatives, and an index relating to its structural properties, which determines prices and personal preferences based on the existence of these solutions. When interactions are dyadic, the index is locally constant, demand depends linearly on prices and personal valuations determine the equilibrium demand and prices.

3.1 The duopoly setup

We consider the duopoly as being a two stage game. In the first stage, the *firms subgame*, two firms independently and simultaneously set a price for the service they provide: p_1 for the service provided by firm 1, p_2 for the service provided by firm 2, defining the price profile $\mathbf{p} \equiv (p_1, p_2)$. We assume firms have no costs in providing the service (neither variable nor fixed costs).[1] In the second stage, the *consumers subgame*, a finite set of consumers (individuals) $\mathcal{I} \equiv \{1, \dots, n\}$ observe the prices set in the first stage and each consumer $i \in \mathcal{I}$ independently decides the probability σ_1^i and σ_2^i of using each one of the two services $S \equiv \{s_1, s_2\}$, provided, respectively, by each one of the two firms. The choice is mandatory i.e. $\sigma_1^i + \sigma_2^i = 1$ (there is no reservation price set exogeneously), which means that given a pair of prices from the first stage, the choice of a consumer is a probability distribution over the set of services S provided by firms, i.e. over the space of pure strategies. Consumers are thus assumed to use standard mixed strategies in the space $\mathcal{S} \equiv S^n$. For simplicity of notation we will identify the distribution by a single parameter $(\sigma^i, 1 - \sigma^i) \equiv (\sigma_1^i, \sigma_2^i)$ and as such the space of mixed strategies can be identified with $[0, 1]^n$. The consumers choice is summarized in the profile of consumer (mixed) strategies denoted by $\boldsymbol{\sigma} \equiv (\sigma^1, \dots, \sigma^n) \in [0, 1]^n$. An outcome of the game is a pair $(\mathbf{p}, \boldsymbol{\sigma}) \in (\mathbb{R}_0^+)^2 \times [0, 1]^n$ formed by a pair of prices $\mathbf{p}$ and a consumers choice $\boldsymbol{\sigma}$. The characterization of outcomes that can arise in a market equilibrium will be done according to the notion of subgame perfect equilibrium, hence by characterizing the Nash equilibria of both stages.

3.1.1 Firms

The demand for each firm stems from the profiles of consumer choices that maximize their utility, and it is therefore contained in the set of Nash equilibrium of the consumers subgame. The *equilibrium choices* $\mathsf{EC}(\mathbf{p})$ of the consumers subgame is the set of consumers choices $\boldsymbol{\sigma}$ that are a Nash equilibrium of the consumers subgame for a given pair of prices $\mathbf{p}$ from the first stage. We say that an outcome $(\mathbf{p}, \boldsymbol{\sigma})$ is credible if $\boldsymbol{\sigma} \in \mathsf{EC}(\mathbf{p})$. In characterizing demand it is useful to use the partition of the set of individuals for a given consumers choice $\boldsymbol{\sigma}$ according to whether they use a pure

[1] Introducing a fixed cost structure does not change the results as it leads to an isomorphic set of equilibria through a change of parameters.

strategy or a nondegenerate mixed strategy. Let us call *loyal* consumers to those consumers who choose firm 1 or 2 with probability one, and *non-loyal* consumers to those using a nondegenerate mixed strategy. The partition is given by $\mathcal{L}_1(\boldsymbol{\sigma}) \cup \mathcal{L}_2(\boldsymbol{\sigma}) \cup \mathcal{M}(\boldsymbol{\sigma})$, where $\mathcal{L}_1(\boldsymbol{\sigma}) \equiv \{i \in \mathcal{I} : \sigma_1^i = 1\}$ and $\mathcal{L}_2(\boldsymbol{\sigma}) \equiv \{i \in \mathcal{I} : \sigma_2^i = 1\}$ are the sets of consumers respectively loyal to each firm and $\mathcal{M}(\boldsymbol{\sigma}) \equiv \{i \in \mathcal{I} : 0 < \sigma_1^i < 1\}$ is the subset of non-loyal consumers (those that play with non-integer probabilities). The cardinalities are respectively denoted by $l_1(\boldsymbol{\sigma}) \equiv \#\mathcal{L}_1(\boldsymbol{\sigma})$, $l_2(\boldsymbol{\sigma}) \equiv \#\mathcal{L}_2(\boldsymbol{\sigma})$, and $m(\boldsymbol{\sigma}) \equiv \#\mathcal{M}(\boldsymbol{\sigma})$. Note that $l_1(\boldsymbol{\sigma}) + l_2(\boldsymbol{\sigma}) + m(\boldsymbol{\sigma}) = n$. We call (l_1, l_2) the *loyalty characterization* of the outcome $(\mathbf{p}, \boldsymbol{\sigma})$, omitting the dependence when it is clear what outcome we are refereing to. The demand for each firm is, respectively, given by

$$D_1(\boldsymbol{\sigma}) \equiv l_1(\boldsymbol{\sigma}) + \sum_{i \in \mathcal{M}(\boldsymbol{\sigma})} \sigma^i, \quad D_2(\boldsymbol{\sigma}) \equiv l_2(\boldsymbol{\sigma}) + \sum_{i \in \mathcal{M}(\boldsymbol{\sigma})} (1 - \sigma^i).$$

Note that, as the choice is mandatory it always leads to full market coverage $D_1(\boldsymbol{\sigma}) + D_2(\boldsymbol{\sigma}) = n$. The *profit function* $\Pi : (\mathbb{R}_0^+)^2 \times [0,1]^n \to \mathbb{R}^2$ determines the profit of each firm, respectively, given by

$$\Pi_1(p_1, \boldsymbol{\sigma}) = p_1 D_1(\boldsymbol{\sigma}), \quad \Pi_2(p_2, \boldsymbol{\sigma}) = p_2 D_2(\boldsymbol{\sigma}).$$

Local deviation beliefs. For the characterization of price equilibria it is necessary to understand the dependence of consumer behavior on prices. In particular, to figure out if a given price is a best response, each firm needs to know how consumers would react to a price change. In this regard, we consider that firms have *local deviation beliefs*: given an outcome $(\mathbf{p}^*, \boldsymbol{\sigma}^*)$ and a neighbourhood $P_1(p_1^*) \times P_2(p_2^*) \subset (\mathbb{R}_0^+)^2$ of the outcome prices $\mathbf{p}^* = (p_1^*, p_2^*)$, local deviation beliefs are maps $\phi_1 : P_1 \to \mathsf{EC}(P_1, p_2^*)$ and $\phi_2 : P_2 \to \mathsf{EC}(p_1^*, P_2)$ that represent how firms believe consumers will respond to small price deviations. That is, firm 1 believes that a deviation from charging price p_1^* to charging price $p \in P_1$ will lead consumers to respond with a change from the given consumer choice $\boldsymbol{\sigma}^*$ to a consumer choice $\phi_1(p) \in \mathsf{EC}(p, p_2^*)$, producing demand $D_1(\phi_1(p))$. Analogously for ϕ_2, the deviation belief of firm 2. By definition $\phi_1(p_1^*) = \phi_2(p_2^*) = \boldsymbol{\sigma}^*$.[2] As deviation beliefs are a way for firms to evaluate if a deviation is profitable, and we will define market equilibrium through the notion of subgame perfect equilibrium, it is natural that we have restricted beliefs to credible outcomes. We say that a local deviation belief ϕ *preserves loyalty* if we have $\mathcal{L}_1(\phi) = \mathcal{L}_1(\boldsymbol{\sigma}^*)$ and $\mathcal{L}_2(\phi) = \mathcal{L}_2(\boldsymbol{\sigma}^*)$; and we say that firms have a *common local belief* if $\phi_1(p_1^* + \varepsilon) = \phi_2(p_2^* - \varepsilon)$. The profit expected from a small price deviation, taking into account local deviation beliefs, is $\Pi_1^*(p_1, \phi_1) = p_1 D_1(\phi_1(p_1))$ for firm 1, and $\Pi_2^*(p_2, \phi_2) = p_2 D_2(\phi_2(p_2))$ for firm 2.

3.1.2 Consumers

Given an outcome $(\mathbf{p}, \boldsymbol{\sigma})$ the payoff of a consumer is built on the utility derived from the use of each service which depends on three components: (i) the price of each service; (ii) the personal benefit derived from the use of each service; and (iii) the externality arising from the social influence exerted at each service by the choice of the other consumers. We assume that the utility has the PSS property and that the personal and social components are commensurable with money. Therefore, we can characterize the payoff a consumer i derives from the use of a service s through: (i) the personal component $\omega(i; s) = -p_s + b_s^i$, which is additively separable in price and personal benefit $b^i : S \to \mathbb{R}$; and (ii) the social component measured by a social externality function $e^i : S \times [0,1]^{(n-1)} \to \mathbb{R}$. With this, the use of each service respectively induces the following payoffs

$$u_1^i(p_1; \boldsymbol{\sigma}_{-i}) = -p_1 + b_1^i + e_1^i\left(\boldsymbol{\sigma}_{-i}\right), \qquad u_2^i(p_2; \boldsymbol{\sigma}_{-i}) = -p_2 + b_2^i + e_2^i\left(\boldsymbol{\sigma}_{-i}\right).$$

[2] This is not entirely a new concept, just a reinterpretation of mixed strategy in the context of multistage games (see for example [34] p103). In our case we only want the local part of deviation beliefs.

The utility function $u : \mathcal{I} \times (\mathbb{R}_0^+)^2 \times [0,1]^n \to \mathbb{R}$ is given by

$$u^i(\mathbf{p}, \boldsymbol{\sigma}) = \sigma_1^i u_1^i(p_1; \boldsymbol{\sigma}_{-i}) + \sigma_2^i u_2^i(p_2; \boldsymbol{\sigma}_{-i}).$$

Product differentiation and types. The form of duopoly in consideration and consequent results are naturally heavily dependent on the choice of the consumers utility function, since the relation between the personal and social parameters, and prices, will determine the Nash equilibria of the consumers subgame, and ultimately market equilibria. Nevertheless, whether a consumers choice is a Nash equilibrium or not is invariant to changes of parameters that do not affect the utility differentials $\Delta u^i = u_1^i - u_2^i$. Consequently, the characterization of equilibria can be done up to isomorphism through the differentials induced by Δu, namely, the price differential

$\Delta p \equiv p_1 - p_2$, (*price difference*).

and the differentials of personal benefit and social externalities, which characterize product differentiation and are given by

$\Delta b^i \equiv b_1^i - b_2^i$ (*standard product differentiation*);

$\Delta e^i(\boldsymbol{\sigma}_{-i}) \equiv e_1^i(\boldsymbol{\sigma}_{-i}) - e_2^i(\boldsymbol{\sigma}_{-i})$ (*social product differentiation*).

Observe that while standard product differentiation is 'intrinsic' to a consumer, social product differentiation has a contextual nature, in the sense of representing how consumers differentiate the product taking into account its momentaneous consumption profile.

Recall the type profile characterization of individuals in the beggining of chapter 1. A consumers type is defined by three characteristics: (i) her crowding type c^i; (ii) her social externality function e^i; and (iii) her personal benefit function b^i. This is given by a type[3] map $t : \mathcal{I} \to C \times E \times B$ and defines the type profile of consumers given by the triplet $\mathbf{t} = (\mathbf{c}, \mathbf{e}, \mathbf{b})$ in the

[3]The type of an individual is something known a priori, not a bayesian type.

space $\mathcal{T} = (C \times E \times B)^n$, where $\mathbf{c} \equiv (c^1, \ldots, c^n)$ is the *crowding profile* of individuals; $\mathbf{e} \equiv (e^1, \ldots, e^n)$ is the *externality profile*; and $\mathbf{b} \equiv (b^1, \ldots, b^n)$ is the *valuation profile*. Note that the pair (e^i, b^i) is what is usually called an individual's taste type, and the pair $(\mathbf{e}, \mathbf{b})$ determines the consumers *product differentiation profile*.

Local influence network. An advantage of separating the taste into two components is that the pairs (c^i, e^i) are responsible for the 'social' part of the model, and we refer to this pair as the *social type* of individuals. The personal benefit each consumer, or type, derives from her choice may be analyzed separately from social interactions. The profile pair $(\mathbf{c}, \mathbf{e})$ captures the social interactions in the model and is called the *social profile*. Based on the social profile we can build a local network of influences that reveals how small changes in the consumers strategy change social differentiation, and thus payoffs. Note however that for loyal consumers this may not result in a strategy change. When the best-reply contains a pure strategy and the Nash equilibrium condition is strict, a change needs to be sufficiently high to result in a change of their best response. When the best-reply is constant in the neighborhood of the outcome $(\mathbf{p}, \boldsymbol{\sigma})$, i.e. $\mathrm{br}_i(\mathbf{p}, \boldsymbol{\sigma}_{-i} + \varepsilon) = \mathrm{br}_i(\mathbf{p}, \boldsymbol{\sigma}_{-i})$ for some $\varepsilon > 0$, we say that loyal consumers have *lower sensitivity*. The crucial aspect to capture local changes in demand is the non-loyal consumers strategy. Let us define the network: the nodes are non-loyal consumers and the edges represent the influence two consumers have on eachother, which is captured by the partial derivatives $(\partial \Delta e^i / \partial \sigma^j)(\boldsymbol{\sigma})$ (which are well defined for interior points of non-loyal consumer strategies). The network is thus defined by the non-loyal Jacobian matrix $J_{\Delta e}(\boldsymbol{\sigma}; \mathcal{M}) \equiv \big[(\partial \Delta e^i / \partial \sigma^j)(\boldsymbol{\sigma}), i, j \in \mathcal{M}\big]$. Note that the network is directed, weighted and state-dependent. Hence, consumers may have diferent influence on eachother, and that influence need not be symmetric nor have the same value throughout the network. Furthermore, it is state-dependent in the sense that the weight will depend on the consumers choice.

3.1.3 Social propensity and communities

For a matrix M let $M^{(i)}$ denote the matrix obtained by replacing column i with 1 (from Cramer's rule).

Definition 5 (Social propensity index)**.** *The social propensity index* κ *of a consumers choice* $\boldsymbol{\sigma}$ *with* $\det\left[J_{\Delta e}(\boldsymbol{\sigma};\mathcal{M})\right] \neq 0$*, is defined as*

$$\kappa(\boldsymbol{\sigma}) \equiv \frac{\sum_i \det\left[J^{(i)}_{\Delta e}(\boldsymbol{\sigma};\mathcal{M})\right]}{\det\left[J_{\Delta e}(\boldsymbol{\sigma};\mathcal{M})\right]}.$$

The local influence network and the associated index of social propensity can be compactely represented and studied using invariances in the type profile.

Definition 6 (Community)**.** *Given a strategy profile* $\boldsymbol{\sigma}$*, a community is a subset of individuals* $Q \subseteq \mathcal{I}$ *of a same social type* (c, e) *that choose the same strategy* $q \in [0, 1]$*. That is,* $Q \equiv \{i \in \mathcal{I} : (c^i, e^i) = (c, e) \wedge \sigma^i = q\}$.

A strategy profile induces a partition of the set of individuals into communities $\mathcal{Q}(\boldsymbol{\sigma}, \mathbf{c}, \mathbf{e}) \equiv \{Q_1, \ldots, Q_{n_Q}\}$. In particular there are loyal and non-loyal communities. The advantage of communities is that strategy profiles that induce the same community partitions produce the same social externalities. Individuals within communities are not socially distinguishable so they instill the same externality to others, and they incur the same externality from others.[4] Thus, this induces an equivalence relation on the space of strategy profiles and the network needs only to be built on top of communities.

Given a comunity partition $\mathcal{Q}(\boldsymbol{\sigma}, \mathbf{c}, \mathbf{e})$, let $\mathbf{m} \equiv (m_1, \ldots, m_{n_Q})$ denote the number of individuals in each community and let $\mathbf{q} \equiv (q_1, \ldots, q_{n_Q})$ be the respective probabilities associated to each community. The *communities*

[4] The idea for the designation communities comes from the definition of societies in [52].

externality profile is given by $f : \mathcal{Q}\times(\{0,\ldots,n\}\times[0,1])^{n_Q} \to \mathbb{R}$, and for every individual $i \in Q$, $\Delta e^i(\boldsymbol{\sigma}_{-i}) = f_Q(\mathbf{m},\mathbf{q})$ which characterizes social product differentiation and the influence provoked by other communities. Note that this is just a reduction of the space needed to characterize social interactions, and not an imposition of any form. Consider a strategy profile $\boldsymbol{\sigma}$ and let Q_i and Q_j be two communities. The influence between the communities is $A_{ij}(\mathbf{m},\mathbf{q}) \equiv (\partial f_{Q_i}/\partial q_j)(\mathbf{m},\mathbf{q})$. Note that for $i \in Q_i$ and $j \in Q_j$ we have $A_{ij}(\mathbf{m},\mathbf{q}) = (\partial\Delta e^i/\partial\sigma^j)(\boldsymbol{\sigma})$. Let us define the *community influence matrix*

$$A(\mathbf{m},\mathbf{q}) \equiv \begin{pmatrix} A_{11}(\mathbf{m},\mathbf{q})\frac{m_1-1}{m_1} & A_{12}(\mathbf{m},\mathbf{q}) & \cdots & A_{1n_Q}(\mathbf{m},\mathbf{q}) \\ A_{21}(\mathbf{m},\mathbf{q}) & A_{22}(\mathbf{m},\mathbf{q})\frac{m_2-1}{m_2} & \cdots & A_{2n_Q}(\mathbf{m},\mathbf{q}) \\ \vdots & & \ddots & \vdots \\ A_{2n_Q}(\mathbf{m},\mathbf{q}) & \cdots & \cdots & A_{n_Qn_Q}(\mathbf{m},\mathbf{q})\frac{m_{n_Q}-1}{m_{n_Q}} \end{pmatrix}.$$

In figure 3.1 is a depiction of the relation between two communities in a community influence network.

Proposition 2 (Community social propensity index)**.** *Let* $\det[A(\mathbf{m},\mathbf{q})] \neq 0$. *The social propensity index* κ *of a consumers choice* $\boldsymbol{\sigma}$ *is determined by communities*

$$\kappa(\boldsymbol{\sigma}) = \kappa(\mathbf{m},\mathbf{q}) \equiv \frac{\sum_i \det\left[A^{(i)}(\mathbf{m},\mathbf{q})\right]}{\det[A(\mathbf{m},\mathbf{q})]}.$$

The proof is left for the end of the next section. Social propensity is a measure of how changes are captured by the social component of a market. The interpretation is that it reveals how consumers may change their strategy in response to local changes in the overall consumer profile. When $\kappa > 0$ changes in demand are amplified by social differentiation (similar to a conformity, herd or bandwagon effect). When $\kappa < 0$ changes in demand are mitigated by social differentiation (similar to a congestion, snob or Veblen effect). Hence, the response of demand to prices is amplified or mitigated by social differentiation, acording to the social propensity of non-loyal con-

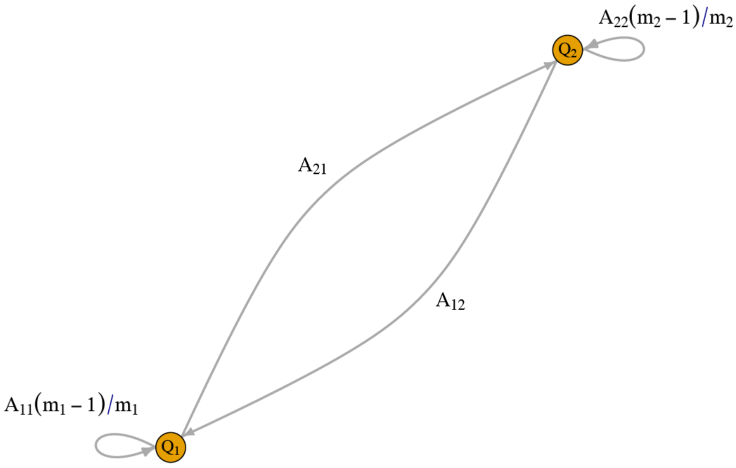

Figure 3.1: Depiction of the influence relation between Two communities Q_1 and Q_2.

sumers. We are interested in duopolies with some social propensity.

Socially prone duopolies. *We say that a duopoly is socially prone if there is a non-empty set of consumer choices* $\mathsf{Sp} \subset [0,1]^n$ *with the following properties: for every* $\boldsymbol{\sigma} \in \mathsf{Sp}$

(i) Loyal communities have lower sensitivity;

(ii) Non-loyal communities are socially prone: $\kappa(\mathbf{m}, \mathbf{q}) \neq 0$.

Socially prone outcomes are non-monopolistic. Property (i) means that in a neighborhood of the outcome the best-reply of a loyal individual is constant, i.e. for $i \in \mathcal{L}(\boldsymbol{\sigma})$, $\mathrm{br}_i(\mathbf{p} + \varepsilon, \boldsymbol{\sigma}_{-i} + \delta) = \mathrm{br}_i(\mathbf{p}, \boldsymbol{\sigma}_{-i})$. Furthermore, by the results of the first chapter, there is an open set of personal preferences $\mathbf{b}$ such that (i) holds. The set of consumer choices such that (ii) holds is dense in $[0,1]^n$. So Sp is a well behaved set, and in general $\mathsf{Sp} \neq \emptyset$.

There are some natural restricions that non-zero social propensity imposes on the network of non-loyal consumers. Namely, the strong components of the network cannot be singletons, i.e. there are no sinks or sources. The idea that loyal consumers may have lower sensibility and not contribute to social propensity is rather natural, and intuitive to the very notion of brand loyalty. Note however that loyalty differs from installed base, since being loyal is a strategical behavior (those who opt for pure strategies) and not an exogeneously imposed choice, or a choice deriving from some switch cost or other stabilizing variable. In figure 3.2 is an example of a small local influence network with two strong components and negative social propensity. A red (green) conection represents a negative (positive) influence weight. Thickness indicates relative strength of influence weights. The color of vertices indicates the consumer strategy in a grey scale, where black means $\sigma^i = 1$, and white means $\sigma^i = 0$.

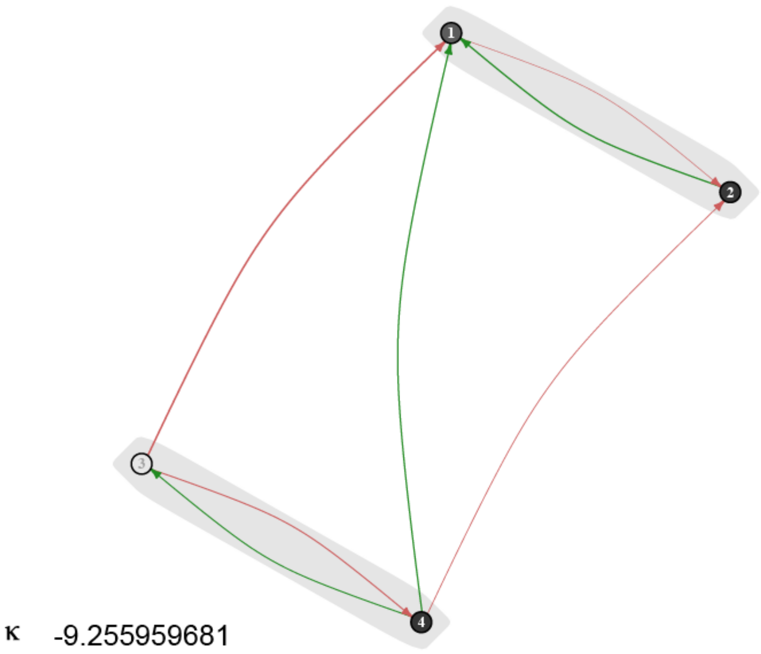

Figure 3.2: A small influence network with two strong components, positive and negative influences, and negative social propensity. A red (green) conection represents a negative (positive) influence weight. Thickness indicates relative strength of influence weights. The color of vertices indicates the consumer strategy in a grey scale, where black means $\sigma^i = 1$, and white means $\sigma^i = 0$.

3.2 Local Market Equilibria

An outcome $(\mathbf{p}^*, \boldsymbol{\sigma}^*)$ with associated local deviation beliefs ϕ_1, ϕ_2 forms a *local market equilibrium* if it is a subgame-perfect equilibrium for an open set containg $\mathbf{p}^*$, that is, if $\boldsymbol{\sigma}^* \in \mathsf{EC}(\mathbf{p}^*)$ and $\mathbf{p}^*$ is a local Nash equilibrium for the firms subgame taking into account their deviation beliefs. More formally, the prices $\mathbf{p}^*$ are a *local pure price equilibrium* for firms if there is a neighbourhood $P_1 \times P_2$ of prices, in which, for $j = 1, 2$, and for all $p_j \in P_j(p_j^*)$, we have $\Pi_j(p_j, \phi_j) \leq \Pi_j(p_j^*, \boldsymbol{\sigma}^*)$. Although we are using standard definitions, let us define local market equilibrium formally to emphasize the notion that it is local in prices and subgame-perfect.

Definition 7 (Local market equilibrium)**.** *An outcome* $(\mathbf{p}, \boldsymbol{\sigma})$ *with local deviation beliefs* ϕ_1, ϕ_2 *is a* local market equilibrium *if (i)* $\boldsymbol{\sigma}$ *is a Nash equilibrium of the consumers subgame, i.e.* $\boldsymbol{\sigma} \in \mathsf{EC}(\mathbf{p})$*; and (ii)* $\mathbf{p}$ *is a local pure price equilibrium for the firms subgame.*

Recall that both demand and beliefs must come from strategies contained in the Nash equilibria of the consumers subgame, hence as price is the unique strategic variable for firms, the characterization of local market equilibrium is essentially dependent on the local structure of $\mathsf{EC}(\mathbf{p})$. Namely, through the characterization of admissable local deviation beliefs, which rely on the existence or non-existence of multiple equilibria, the relation between loyal and non-loyal consumers and the price regions where they hold.

Main result

Suppose there is no product differentiation, meaning that $\Delta b^i = 0$ and $\Delta e^i = 0$ for all $i \in \mathcal{I}$. The game becomes essencially the original Bertrand framework. The paradox arises since $\mathsf{EC}(\mathbf{p})$ is a singleton except when $\Delta p = 0$, which induces the following unique demand beliefs: $D_1(\phi_1^*) = n$

if $\Delta p < 0$; $D_1(\phi_1^*) = 0$ if $\Delta p > 0$. [5] This means the only credible non-monopolistic outcomes have associated discontinuous beliefs, which leads to the paradox. Since $\mathcal{I}$ is finite, introducing standard product differentiation $\Delta b^i \neq 0$ for some subset of consumers $I \subseteq \mathcal{I}$, but no social differentiation $\Delta e^i = 0$, for all $i \in \mathcal{I}$, may lead to a shift to a monopoly equilibrium, but the behavior of consumers is identical. The set $\mathsf{EC}(\mathbf{p})$ still a singleton except for a finite set of prices where $\Delta p = \Delta b^i$ for at least some consumer $i \in \mathcal{I}$. For all other prices, consumers best response is unique and consumers will use pure strategies, which will again lead to beliefs that are either discontinuous or constant in a neighbourhood of the outcomes candidate for equilibria, thus creating an incentive for firms to deviate. The introduction of a social component will give rise to conected price regions where, not only are there multiple Nash equilibria for the consumers subgame, but these include non-degenerate mixed strategy equilibria with larger domains. This means there are price regions where the loyalty characterization is constant and small price changes may be captured by non-loyal consumers, leading to smooth changes in demand. Nevertheless, although $\mathsf{EC}(\mathbf{p})$ is no longer a singleton almost everywhere, if beliefs are contained in the pure strategy choices of consumers, a market equilibrium where both firms have positive profits will still fail to exist.

Lemma 6. *If firms beliefs are that consumers use only pure strategies, or use pure strategies for almost every price, i.e.* $\mathcal{M}(\phi) = \emptyset$ *a.s., then in a market equilibrium at least one firm has zero profit.*

Naturally, an equilibrium may not exist, but for any outcome, a deviation, if admissable, is profitable for firms. When we allow consumers to use mixed strategies, the effect of a price deviation may be captured by smooth changes in the non-loyal consumers probability through the externality function. The

[5] There are multiple equilibria only when $\Delta p = 0$, and in fact as any choice is a Nash equilibrium of the consumers subgame $\mathsf{EC}(p, p) = [0, 1]^n$.

properties of the externality function will be inherited by demand and allow the existence of continuous deviation beliefs that stabilize prices and create market equilibria where both firms earn positive profits. The drawback with these new equilibria for consumers is the coordination problem posed by the multiplicity of Nash equilibria of the consumers subgame. In particular, firms will face a coordination problem, since local deviation beliefs are in general not unique.

Lemma 7 (Demand responsiveness)**.** *Consider a socially prone duopoly and a credible outcome* $(\mathbf{p}^*, \boldsymbol{\sigma}^*)$ *with a socially prone consumer choice* $\boldsymbol{\sigma}^* \in \mathsf{Sp}$. *There is a unique continuous local deviation belief* ϕ^*. *Furthermore, this belief preserves loyalty, is common for both firms, and in equilibrium*

$$\frac{\partial D_s}{\partial p_s}(\phi^*(\mathbf{p})) = \kappa(\boldsymbol{\sigma}^*),\ \ s \in S.$$

Although discontinuous beliefs are credible alternatives, since they are contained in the set of Nash equilibria of the consumers subgame, they are hard to justify from an economic perspective. It's hard to envision a situation where firms will believe that small price deviations provoke a disruptive behavior in consumers, when there is a credible smooth alternative. The second part of lemma 7 shows why outcomes with positive social propensity will in general not allow for equilibria with positive profits, while negative social propensity will create the effect of slowing the demand response to price changes, opening the possibility of a shared market equilibrium. When social propensity tends to infinity it leads towards jumps, meaning consumers will be highly sensitive to changes, which may prevent firms from finding an equilibrium. On the other hand if social propensity gets close to zero, consumers will have low sensitivity to changes, which leads to the opposite effect, also preventing firms from finding non-monopolistic equilibria.

Theorem 9 (Local Market Equilibrium)**.** *Consider a socially prone duopoly and let* $\boldsymbol{\sigma} \in \mathsf{Sp}$*. The outcome* $(\mathbf{p}, \boldsymbol{\sigma})$ *is a local market equilibrium with continuous deviation beliefs and positive profits for both firms if, and only if,* $\kappa(\boldsymbol{\sigma}) < 0$*, prices are given by*

$$p_1 = -\frac{l_1 + \mathbf{m} \cdot \mathbf{q}}{\kappa(\mathbf{m}, \mathbf{q})}$$

$$p_2 = -\frac{l_2 + \mathbf{m} \cdot (\mathbf{1} - \mathbf{q})}{\kappa(\mathbf{m}, \mathbf{q})}$$

and personal preferences for non-loyal communities are

$$\Delta b^Q = \frac{l_2 - l_1 + \mathbf{m} \cdot (\mathbf{1} - 2\mathbf{q})}{\kappa(\mathbf{m}, \mathbf{q})} - f_Q(\mathbf{m}, \mathbf{q}).$$

Furthermore, there is a unique continuous local deviation belief, which is common for both firms and preserves communities.

Note that if $\boldsymbol{\sigma} \in \mathsf{Sp}$ then $0 < D_1(\boldsymbol{\sigma}), D_2(\boldsymbol{\sigma}) < n$. The theorem, which is the main result of this chapter, not only proves the existence of local market equilibria with shared demand and positive profits, but also characterizes completely its prices and reveals consumer personal preferences. The conditions are rather general and rely exclusively on the properties of the social profile through the social propensity index. Socially prone duopolies thus disrupt the Bertrand paradox and provide pure price solutions These solutions do not rely on heterogeneity to exist or to be asymmetrical. Note that this work focuses on local equilibria, so in order to obtain a global solution, one would need to discuss firms beliefs farther away from the local behavior of consumers. In these cases, firms need not coordinate on the same belief.

In figures 3.3, 3.4 and 3.5 are three ilustrative examples of local influence networks and the respective values of a local market equilibrium. In figure 3.3 an influence network with 40 consumers and respective social propensity and equilibrium values. In figure 3.4 is the reduced community network of figure

3.3. Note that the values are exactly the same. The size of the vertices represent the size of communities. A red (green) conection represents a negative (positive) influence weight. Thickness indicates relative strength of influence weights. The color of vertices indicates the consumer strategy in a grey scale, where black means $\sigma^i = 1$, and white means $\sigma^i = 0$. In figure 3.5 is an example of an network comprised only of positive interactios, but that, nonetheless, produces a shared equilibrium due to a high negative social propensity, which drives price close to zero.

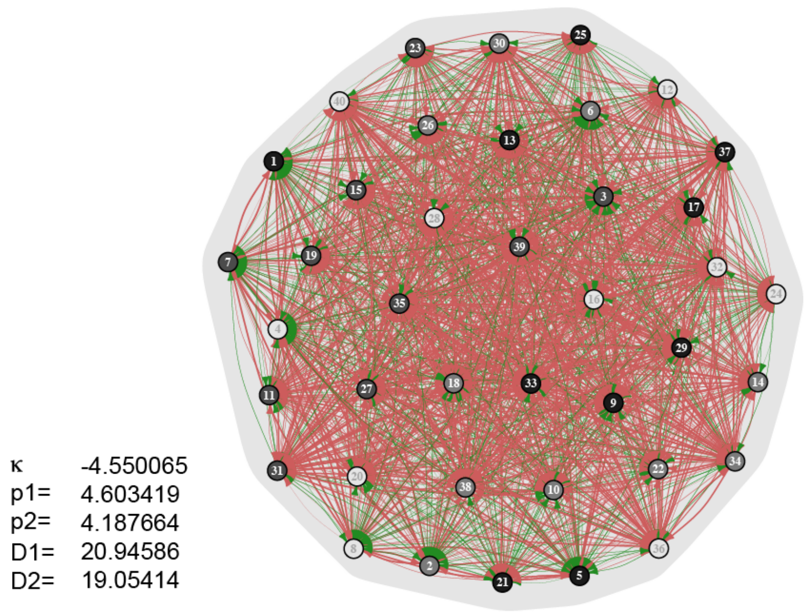

Figure 3.3: A influence network with 40 individuals and the respective social propensity and local market equilibrium. It's nearly impossible to uncover relations in such a condensed set of relations, nevertheless this can be dealt with using communities, as can be seen in figure 3.4. A red (green) conection represents a negative (positive) influence weight. Thickness indicates relative strength of influence weights. The color of vertices indicates the consumer strategy in a grey scale, where black means $\sigma^i = 1$, and white means $\sigma^i = 0$.

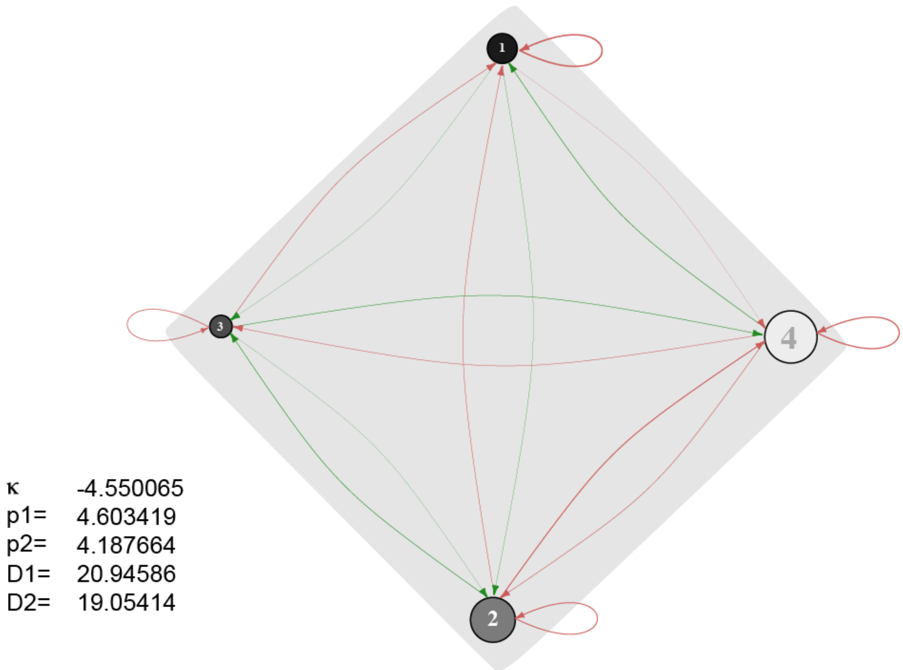

Figure 3.4: The community network for the individuals influence network in figure 3.3. The size of the vertices indicates the communities size, which in this case are respectively $8, 12, 6, 14$. The color of the vertices indicates the community strategy in a grey scale, where black means $\sigma^i = 1$, and white means $\sigma^i = 0$. Again, a red (green) conection represents a negative (positive) influence weight. Thickness indicates relative strength of influence weights.

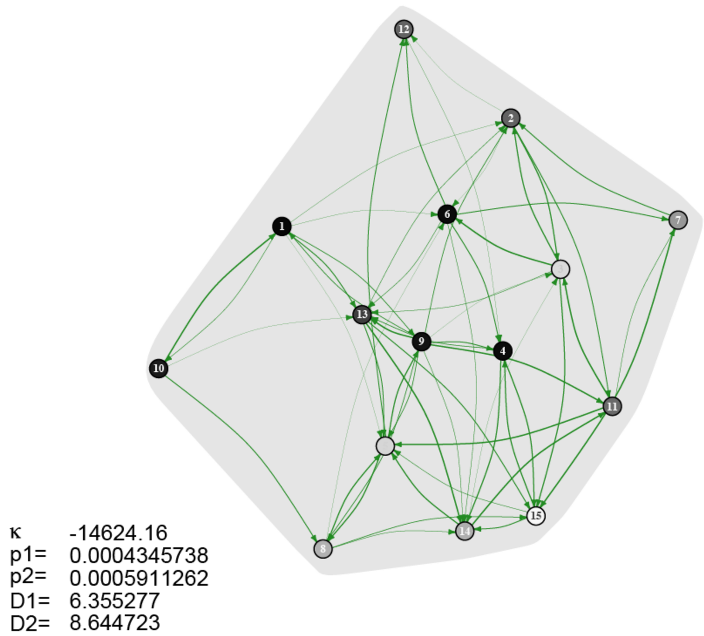

Figure 3.5: A influence network where there are only positive interactions but still a high negative social propensity.

Proofs

Lemma 6

Proof. Suppose by reductio ad absurdum there is an equilibrium where both firms charge positive profits. As demand is almost surely based on pure strategies and the game is finite, there is $\varepsilon > 0$, as small as desired, such that demand for $p_1^* \pm \varepsilon$ is either constant or a jump. In both cases there is an incentive to deviate. □

Lemma 7

Proof. Recall that $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}) = \Delta b^i - \Delta p + \Delta e^i(\boldsymbol{\sigma}_{-i})$. Since $\boldsymbol{\sigma}$ is a Nash equilibrium for consumers and $\boldsymbol{\sigma} \in \mathsf{Sp}$ (loyals have lower sensitive), we must have the following,

(i) $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}) > 0$ for $i \in \mathcal{L}_1(\boldsymbol{\sigma})$

(ii) $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}) < 0$ for $i \in \mathcal{L}_2(\boldsymbol{\sigma})$

(iii) $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}) = 0$ for $i \in \mathcal{M}(\boldsymbol{\sigma})$

Hence for $i \in \mathcal{L}(\boldsymbol{\sigma})$ the best response is constant in a neighbourhood $V_p(\boldsymbol{\sigma})$. Let $m = \#\mathcal{M}(\boldsymbol{\sigma})$ (number of mixed players), with $m > 1$. Index the players in the set $\mathcal{M}(\boldsymbol{\sigma}) \equiv \{1, \ldots, m\}$. Consider the function

$$\begin{aligned} \Delta U : (\mathbb{R}_0^+)^2 \times (0,1)^m &\rightarrow \mathbb{R}^m \\ (\mathbf{p}, \boldsymbol{\sigma}) &\mapsto \Delta u^1(\mathbf{p}, \boldsymbol{\sigma}), \ldots, \Delta u^m(\mathbf{p}, \boldsymbol{\sigma}) \end{aligned}$$

Let $(\mathbf{p}^*, \boldsymbol{\sigma}^*)$ be an outcome in the conditions of theorem. By (iii), and the fact that for loyals the best response is locally constant, any outcome in the neighbourhood of $(\mathbf{p}^*, \boldsymbol{\sigma}^*)$ such that $\Delta U(\mathbf{p}, \boldsymbol{\sigma}) = \mathbf{0}$ is a Nash equilibrium for consumers. Note now that ΔU is C^1 and recall that

$$J_{\Delta e}(\boldsymbol{\sigma}) \equiv \left[(\partial \Delta e^i / \partial \sigma^j)(\boldsymbol{\sigma}), i, j \in \mathcal{M}(\boldsymbol{\sigma}) \right],$$

Observe that the jacobian determinant for ΔU is

$$\det \left[J_\sigma \Delta U(\mathbf{p}^*, \boldsymbol{\sigma}^*) \right] = \det \left[J_{\Delta e}(\boldsymbol{\sigma}^*) \right] \neq 0 \text{ (by assumption).}$$

Therefore, using the Implicit Function Theorem, there exists an open set $V_p \times V_\sigma \subset (\mathbb{R}_0^+)^2 \times (0,1)^m$ and a unique C^1 function $\phi : V_p \rightarrow V_\sigma$ such that

$\Delta U(\mathbf{p}, \phi(\mathbf{p})) = 0$. Furthermore, in that neighbourhood

$$\sum_i \frac{\partial \Delta U}{\partial \phi^i(\mathbf{p})}(\mathbf{p}, \phi(\mathbf{p}))\frac{\partial \phi^i}{\partial p_1}(\mathbf{p}) = \frac{\partial \Delta U}{\partial p_1}(\mathbf{p}, \phi(\mathbf{p}))$$

Recall now that $\Delta u^i(\mathbf{p}, \boldsymbol{\sigma}) = \Delta b^i - \Delta p + \Delta e^i(\boldsymbol{\sigma}_{-i})$. Hence in the neighbourhood, $\Delta e^i(\boldsymbol{\sigma}_{-i}) = \Delta p - \Delta b^i$, and

$$J_{\Delta e}(\phi(\mathbf{p}))\frac{\partial \phi}{\partial p_1}(\mathbf{p}) = \mathbf{1}_m$$

Using Cramer rule

$$\frac{\partial \phi^i}{\partial p_1}(\mathbf{p}) = \frac{\det\left[J^{(i)}_{\Delta e}(\phi(\mathbf{p}))\right]}{\det\left[J_{\Delta e}(\phi(\mathbf{p}))\right]}$$

where $J^{(i)}_{\Delta e}(\phi(\mathbf{p}))$ is obtained by replacing column i with 1 in $J_{\Delta e}(\boldsymbol{\sigma}^*)$. As such in that neighbourhood

$$\frac{\partial D_1}{\partial p_1}(\phi(\mathbf{p})) = \frac{\sum_i \det\left[J^{(i)}_{\Delta e}(\phi(\mathbf{p}))\right]}{\det\left[J_{\Delta e}(\phi(\mathbf{p}))\right]}$$

□

Theorem 9

Proof. The isoprofit functions for outcomes $(\mathbf{p}^*, \boldsymbol{\sigma}^*)$, where $p_1, p_2 \neq 0$ are given, respectively, by

$$h_1(p_1; \mathbf{p}^*, \boldsymbol{\sigma}^*) = \frac{p_1^* D_1(\boldsymbol{\sigma}^*)}{p_1}, \quad h_2(p_2; \mathbf{p}^*, \boldsymbol{\sigma}^*) = \frac{p_2^*(n - D_1(\boldsymbol{\sigma}^*))}{p_2}.$$

which we abreviate to $h_1(p_1)$ and $h_2(p_2)$ when there is no ambiguity to which outcome we are refreing to. The isoprofit for firm 1 results from solving the following equality $(p_1^* + \delta)h_1(p_1^* + \delta) = p_1^* D_1(\boldsymbol{\sigma}^*)$ (and analogously for firm

2). The derivatives are

$$h_1'(p_1) = -\frac{p_1^* D_1(\boldsymbol{\sigma}^*)}{p_1^2}, \quad h_2'(p_2) = -\frac{p_2^*(n - D_1(\boldsymbol{\sigma}^*))}{p_2^2} \tag{3.1}$$

For an outcome to be a market equilibrium, it must satisfy the following conditions:

(i) $\frac{\partial D_1(\boldsymbol{\sigma}^*)}{\partial p_1} = h_1'(p_1^*)$;

(ii) $\frac{\partial D_2(\boldsymbol{\sigma}^*)}{\partial p_2} = h_2'(p_2^*)$;

(iii) $h_1'(p_1^*) = h_2'(p_2^*)$;

(iv) $D_1(\phi_1(p_1, p_2^*)) \leq h_1(p_1)$;

(v) $D_2(\phi_2(p_1^*, p_2)) \leq h_2(p_2)$.

Furthermore, since $D_1 + D_2 = n$, we have that

$$\frac{\partial D_2(\boldsymbol{\sigma}^*)}{\partial p_2} = -\frac{\partial D_1(\boldsymbol{\sigma}^*)}{\partial p_2}.$$

From (iii), (iv) and 3.1 we get

$$p_1^* = -\frac{D_1(\boldsymbol{\sigma}^*)}{\frac{\partial D_1(\boldsymbol{\sigma}^*)}{\partial p_1}}; \quad p_2^* = \frac{n - D_1(\boldsymbol{\sigma}^*)}{\frac{\partial D_1(\boldsymbol{\sigma}^*)}{\partial p_2}}.$$

Now observe that by lemma 7 this characterizes prices. To conclude the proof observe that $\Delta b^i = \Delta p - \Delta e^i(\boldsymbol{\sigma}_{-i})$. □

Proposition 2 (page 78)

Proof. Note that by lemma 7, the social propensity index is the unique solution to a system. □

3.3 Underlying network structure and homogeneities

The influence network allows the characterization of symmetries, and thus heterogeneity, based on the social type (c_i, e_i) of individuals. There are two main lines in analyzing the impact of social heterogeneity of a consumers social profile: (i) the underlying structure of the influence network, meaning the unweighted digraph (directed graph); and (ii) the distribution of weights in the network. In this section we analyze how social propensity will rely on the structural properties of the underlying network when there are strong symmetries (or homogeneities) on the influence weights. This will allow social propensity to be seen as a functional. For any given underlying network, with the appropriate choice of weights, the index can attain any value, either positive or negative. The idea works vice-versa, in terms of weights and underlying network.

Let $G \equiv G(\mathbf{c}, \mathbf{e})$ be the underlying unweighted network of consumer interactions. Let $\det[G]$ be the determinant of the correspondent adjacency matrix and $\det[G_{ij}]$ be determinant of its ij minor. Let $\det[G^{(i)}]$ be, as before, obtained from $\det[G]$ by replacing entries in column i by 1.

When the crowding space C is a singleton, there is *crowding anonymity*. This means individuals do not distinguish between one another and they are influenced by every individual equally. Nevertheless, different individuals might be influenced differently. In terms of the influence network it implies that for every individual $i \in \mathcal{I}$

$$\partial \Delta e^i / \partial \sigma^{j_1} = \partial \Delta e^i / \partial \sigma^{j_2}, \quad \forall j_1, j_2 \in \mathcal{I}.$$

Let $\theta^i(\boldsymbol{\sigma}) \equiv \partial \Delta e^i / \partial \sigma^j$ be the weight for individual i associated to the change of other individual j (the crowding homogeneous weight). Note that the influence matrix has an horizontal symmetry, as all individuals influence i by the same amount θ^i.

Proposition 3 (Crowding anonymity)**.** *Given a social profile with crowding anonimity, the social propensity index is given by*

$$\kappa_{ca}(\boldsymbol{\sigma}) = \frac{\sum_i \sum_j (-1)^{i+j} \det\left[G_{ij}\right] / \theta^i(\boldsymbol{\sigma})}{\det\left[G\right]}.$$

Example 3 (Aggregated externality)**.** *The simplest form of externality with crowding anonimity is probably when social influence is exerted through the aggregate choice of other consumers, thus, when the externality is derived from a function which looks only at the aggregate behavior. In that case, social differentiation can be measured through an aggregated externality function* $g^i : S \times [0, n-1] \to \mathbb{R}$*. The social differentiation in this case becomes* $\Delta e^i(\boldsymbol{\sigma}_{-i}) = g^i\left(D_{-i}\right)$ *where* $D_{-i} \equiv \sum_{j \neq i} \sigma^j$*.*

The same reasoning as the one presented in propostion 3 can be done with a vertical symmetry on the influence network matrix. In fact just replace i by j in the above expression. This is an interesting homogeneity property, since it means that every individual, in that strategy context, is being influenced in the same 'way', that is

$$\partial \Delta e^{i_1} / \partial \sigma^j = \partial \Delta e^{i_2} / \partial \sigma^j, \quad \forall i_1, i_2, j \in \mathcal{I}.$$

Nevertheless, this does not have a counterpart on the externality types of individuals, as it does with an horizontal symmetry and the crowding type. This is essencially due to the fact that the same externality type need not induce type symmetric strategies. The crowding type, however, is defined 'outside' the utility function. We define this as a *social differentiation homogeneity.* Let $\theta_j(\boldsymbol{\sigma}) \equiv \partial \Delta e^i / \partial \sigma^j$.

Proposition 4 (Social differentiation homogeneity)**.** *Given a social profile with social differentiation homogeneity, the social propensity index is*

$$\kappa_{eh}(\boldsymbol{\sigma}) = \frac{\sum_i \det\left[G^{(i)}\right]/\theta_i(\boldsymbol{\sigma})}{\det\left[G\right]}.$$

Although from the point of view of firms both horizontal and vertical symmetries appear as very similar, from a consumers best response point of view, they are quite distinct. In fact the following result does not hold for crowding anonimity.

Lemma 8 (Complete network with social differentiation homogeneity)**.** *When the network is complete and there is homogeneity of social differentiation, the social propensity index is*

$$\kappa_{eh}(\boldsymbol{\sigma}) = \frac{\sum_i (\theta_i)^{-1}}{m-1}.$$

Let's go back to the aggregated example, and assume all individuals have the same externality type, hence, they all look at the aggregate through the same aggregated externality function $g : S \times [0, n-1] \to \mathbb{R}$. Social differentiation becomes

$$\Delta e^i(\boldsymbol{\sigma}_{-i}) = g\left(D_{-i}\right).$$

Note that whilst the function is common, each consumer may be applying it in different points according to the aggregate they face, which, in this finite case, need not be the same. The next theorem shows how social propensity still depends on the individual i.

Theorem 10 (Aggregated externality)**.** *In a socially prone duopoly with socially homogeneous consumers and a C^1 aggregated externality function g, the social propensity index for $\boldsymbol{\sigma} \in \mathsf{Sp}$ is*

$$\kappa_g(\boldsymbol{\sigma}) = \frac{1}{m-1} \sum_i (1/g'(D_{-i})).$$

This does not exclude that when there is total symmetry (both horizontal and vertical) the social propensity index will rely essencially on the properties of the underlying network.

Proposition 5 (Total weight symmetry)**.** *When the influence matrix has all entries with the same weight $\theta(\boldsymbol{\sigma})$, then κ relies essencially on the properties of the underlying network, and is given by*

$$\kappa_{ts}(\boldsymbol{\sigma}) = -\frac{\sum_i \sum_j (-1)^{i+j} \det [G_{ij}]}{\det [G]\, \theta(\boldsymbol{\sigma})}.$$

Example 4. *In the complete network case, total weight symmetry leads to*

$$\kappa_{ts}(\boldsymbol{\sigma}) = \frac{m}{(m-1)\theta(\boldsymbol{\sigma})}.$$

Observe however that in the case of a total symmetry, although individuals all have the same social type, and are thus socially homogeneous, this does not mean they derive the same personal benefit from the use of each service, i.e. there may be i, j with $\Delta b^i \neq \Delta b^j$.

Proofs

Proposition 3

Proof. Observe that every non-zero entry of every column has the same value, θ^i, except for the columns on the numerator, whose entries have been replaced by 1.

$$\kappa_{ca}(\boldsymbol{\sigma}) = \frac{\sum_i \sum_j (-1)^{i+j} \prod_{i \neq j} \theta^i(\boldsymbol{\sigma}) \det [G_{ij}]}{\prod_i \theta^i(\boldsymbol{\sigma}) \det [G]},$$

which can be rewritten

$$\kappa_{ca}(\boldsymbol{\sigma}) = \frac{\sum_i \sum_j (-1)^{i+j} \prod_i \theta^i(\boldsymbol{\sigma})/\theta^j(\boldsymbol{\sigma}) \det [G_{ij}]}{\prod_i \theta^i(\boldsymbol{\sigma}) \det [G]},$$

and

$$\kappa_{ca}(\boldsymbol{\sigma}) = \frac{\sum_i \sum_j (-1)^{i+j} \det [G_{ij}] / \theta^j(\boldsymbol{\sigma})}{\det [G]}.$$

□

Proposition 5

Proof. Observe that every non-zero entry has the same value, $\theta \equiv \theta(\boldsymbol{\sigma})$, except for the columns on the numerator whose entries have been replaced by 1.

$$\kappa_{ts}(\boldsymbol{\sigma}) = \frac{\sum_i \theta^{m-1} \det \left[G^{(i)}\right]}{\theta^m \det [G]},$$

which can be rewritten

$$\kappa_{ts}(\boldsymbol{\sigma}) = \frac{\sum_i \sum_j (-1)^{i+j} \det [G_{ij}]}{\theta \det [G]}.$$

□

Lemma 8

Proof. For a complete graph G_m with m vertices, we have that $det(G) = (-1)^{m-1}(m-1)$. Observe now that $G^{(i)}$ just adds 1 in entry ii, and the minor G_{ii} is a replica of the complete graph with one less vertex. Thus,

$$det(G^{(i)}) = det(G_m) + det(G_{m-1}) = (-1)^{m-1}(m-1) + (-1)^{m-2}(m-2),$$

which equates to

$$det(G^{(i)}) = (-1)^{m-1}(m-1-m-1+1) = (-1)^{m-1}.$$

The proof of the lemma follows from direct aplication of the above result in the following expression

$$\kappa_{eh}(\boldsymbol{\sigma}) = \frac{\sum_i \det\left[G^{(i)}\right]/\theta_i}{\det\left[G\right]}.$$

which is

$$\kappa_{eh}(\boldsymbol{\sigma}) = \frac{\sum_i (\theta_i)^{-1}}{m-1}.$$

□

Theorem 10

The proof of the theorem follows straightforward from Lemma 8 by constructing the influence matrix and showing it has social differentiation homogeneity, which follows directly from social homogeneity. Nevertheless we leave an alternative proof based on the implicit function theorem.[6]

[6] The reason we leave the proof is because it was the starting proof of this work, later generalized for higher dimensions and heterogeneity in the main theorem. With the use of the determinant of a complete graph it becomes trivial.

Proof. Consider $\Delta u : (0, +\infty)^2 \times (0, n-1) \to \mathbb{R}$, which is given by

$$\Delta u(\mathbf{p}, D_{-i}) = \Delta b - \Delta p + g(D_{-i}),$$

(Observe that Δu is C^1 and defined on an open set which excludes prices equal to zero and imposes that at least one player different from i is playing in non-integer probability.)
Note that $\frac{\partial \Delta u}{\partial D_{-i}} = g'(D_{-i})$. If there is a point $(\mathbf{p}^*, d^*) \in (0, +\infty)^2 \times (0, n-1)$ such that $\Delta u(\mathbf{p}^*, d^*) = 0$ and $g'(d^*) \neq 0$, then (by the IFT) there are a ball $B(\mathbf{p}^*, \delta) \in (0, +\infty)^2$ and an interval $J = (d^* - \varepsilon, d^* + \varepsilon)$ such that $\Delta u^{-1}(0) \cap (B \times J)$ is the graphic of a function $y_i : B \to J$ of class C^1. For all $\mathbf{p} \in B$ we have

$$\frac{\partial y_i}{\partial p_s}(\mathbf{p}) = -\frac{\partial \Delta u}{\partial p_s}(\mathbf{p}, y_i(\mathbf{p})) / \frac{\partial \Delta u}{\partial D_{-i}}(\mathbf{p}, y_i(\mathbf{p})), \quad s = 1, 2$$

as $\dfrac{\partial \Delta u}{\partial p_s} = \mp 1$, thus,

$$\frac{\partial y_i}{\partial p_s}(\mathbf{p}) = \pm 1 / g'(y_i(\mathbf{p})), \quad s = 1, 2$$

hence $D_{-i} = y_i(\mathbf{p})$ is defined implicitly by $\Delta u(\mathbf{p}, D_{-i}) = 0$ and for every $\mathbf{p} \in B$ there is a unique $D_{-i} = y_i(\mathbf{p}) \in J$ such that $\Delta u(\mathbf{p}, D_{-i}) = 0$.

For an equilibrium with m players using nondegenerate mixed strategies, provided above is, for every $i \in \mathcal{M}(\boldsymbol{\sigma})$, the solution to the system

$$\sum_{j \neq i} \sigma^j = y_i(\mathbf{p}) - l_1$$

with $0 < \sigma^i, \sigma^j < 1$ and $0 < y_i(\mathbf{p}) - y_j(\mathbf{p}) < 1$ hence the equilibrium demand is

$$D_1(\boldsymbol{\sigma}) = \frac{\sum_i y_i(\mathbf{p}) - m l_1}{m - 1}$$

and

$$\frac{\partial D_1(\mathbf{p})}{\partial p_1} = \frac{1}{m-1} \sum_i (1/g'(D_{-i})).$$

□

3.4 Dyadic interactions

In this section we study the case of a social externality function based on dyadic interactions, i.e. we assume the utility function has the DI property. By proposition 1 the utility function $u : \mathcal{I} \times (\mathbb{R}_0^+)^2 \times [0,1]^n \to \mathbb{R}$ is given by the weighted combination of the following pure strategy payoffs

$$u_1^i(p_1; \boldsymbol{\sigma}_{-i}) = -p_1 + b_1^i + \sum_j \alpha_1^{ij} \sigma^j, \quad u_2^i(p_2; \boldsymbol{\sigma}_{-i}) = -p_2 + b_2^i + \sum_j \alpha_2^{ij} (1 - \sigma^j).$$

The externality function is thus additively separable, which means the entries in the influence matrix, that is the partial derivatives, depend only on every pair of players. Social differentiation becomes

$$\Delta e^i\,(\boldsymbol{\sigma}_{-i}) = \sum_j (\alpha_2^{ij} + \alpha_1^{ij})\sigma^j - \sum_j \alpha_2^{ij}$$

The influence network is locally constant and given by the following coordinates

$$(\partial \Delta e^i / \partial \sigma^j)(\boldsymbol{\sigma}) = \alpha_1^{ij} + \alpha_2^{ij}.$$

In the case of dyadic interactions, it is particular useful to study the symmetries imposed by the type profile.

Proposition 6. *In a duopoly with dyadic interactions, if the* MTS *condition holds for all types, each type has at most one non-loyal community.*

Proof. Recall that by corollary 6, if the mixed type-symmetric condition (MTS) holds for some type $t \in T$, then all individuals of type t must be using the same strategy at a Nash equilibrium. □

The community influence matrix does not depend on the probability chosen in the strategy, namely, for types t, t', we can define $A_{tt'} \equiv \alpha_1^{tt'} + \alpha_2^{tt'}$. The MTS in this case is $A_{tt} \neq 0$, and the type influence is given by the matrix

$$A(\mathbf{m}) \equiv \begin{pmatrix} A_{11}\frac{m_1-1}{m_1} & A_{12} & \cdots & A_{1n_T} \\ A_{21} & A_{22}\frac{m_2-1}{m_2} & \cdots & A_{2n_Q} \\ \vdots & & \ddots & \vdots \\ A_{2n_T} & \cdots & \cdots & A_{n_Q n_T}\frac{m_{n_T}-1}{m_{n_T}} \end{pmatrix}.$$

In particular this means the socially propensity index is locally constant

$$\kappa_\alpha(\mathbf{m}) \equiv \frac{\sum_i \det\left[A^{(i)}(\mathbf{m})\right]}{\det\left[A(\mathbf{m})\right]},$$

and equilibrium demand varies linearly with price,

$$D_1^* = \kappa_\alpha^* p_1^*.$$

Being locally constante, the influence network induces an invertible property on the set of local market equilibria.

Theorem 11 (Revealing preferences). *In a socially prone duopoly with dyadic interactions the personal profile* $\mathbf{b}$ *and non-loyal characterization* $\mathbf{m}$ *fully determine the local market equilibrium.*

We note that by theorem 9 $\det[A(\mathbf{m})] \neq 0$ is a necessary condition for the existence of a local market equilibrium with both firms earning positive profits. The equilibrium strategy for the consumers subgame is, in this case, given by the *type solution* to a linear system. Furthermore, by theorem 11, when $\det(A(\mathbf{m})) \neq 0$ the system has a unique solution. When the personal profile is such that the solution is in fact an interior point of a probability distribution, that is all coordinates lie in the interval $(0, 1)$, the equilibrium will be a duopoly equilibrium with positive profits as given by socially prone outcomes. As such, for every strategy class the loyal characterization (l_1, l_2) and the type non-loyal characterization $\mathbf{m}$ completely determine the local solution to the duopoly problem given by theorem 9. Therefore, given a profile of product differentiation $(\mathbf{e}, \mathbf{b})$ if firms have some previous information, or some ideia on what loyalty characterization to expect, they know exactly what are the local market equilibria. This is a rather reasonable assumption for any market, especially if there areprevious consumption moments for the services firms provide. Furthermore, this is the most natural interpretation for a 'loyal' consumer.

However, from a game theoretic point of view, when the choice of a consumer includes no previous information and must be made simultaneously with all other consumers, a coordination problem on the loyal characterization remains. Having multiplicity of equilibria and no coordination device, in the moment of choosing a strategy, consumers are left with the question of deciding which is the best strategy given that they don't know what the others will do. The following result provides a possible alternative solution to that of previous information. Let $\mathsf{AL}(\mathbf{b})$ be the set of admissable loyal characterizations under personal profile $\mathbf{b}$.

Proposition 7 (Focal equilibrium)**.** *In a socially prone duopoly with dyadic interactions, the following loyal characterization is a focal point for consumers,*

(i) if $A_{tt} < 0$ *then* $(l_1^t, l_2^t) = (0, 0)$*;*

(ii) if $A_{tt} > 0$ *then*

$$(l_1^t, l_2^t) = \begin{cases} (n, 0) & \text{if} \quad \Delta b^t - \Delta p > 0 \\ (0, n) & \text{if} \quad \Delta b^t - \Delta p < 0 \\ (0, 0) & \text{if} \quad \Delta b^t - \Delta p = 0 \end{cases} .$$

Furthermore, this equilibrium is fair for consumers and leads to a unique focal market equilibrium.

The notion of *focal point* is that of Schelling from [41], and is based on the idea that without communication and in the presence of multiple equilibria, players may coordinate in an equilibrium whose salience appears as a unique possible coordination device. The notion of *fair* equilibrium, is an equilibrium where individuals of the same type receive the same payoff. Establishing a focal equilibrium is always amenable to critiques, nevertheless it is yet another way of completing theorem 11.

Proofs

Theorem 11

Recall that given a matrix H we denote by $H^{(i)}$ the matrix obtained from matrix H by replacing column i by 1.

Claim 2. *Let H be a $n \times n$ matrix and $\mathbf{1}_n$ the $n \times n$ matrix with 1 in all entries. For any $r \in \mathbb{R}$,*

$$\det(H + r\mathbf{1}_n) = \det(H) + r \sum_i \det\left(H^{(i)}\right)$$

Proof. Let $\hat{h}_i$ be column i of matrix H. Note that

$$\det(H + r\mathbf{1}_n) = \det(\hat{h}_1 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1})$$

We start by separating the first column

$$\det(H + r\mathbf{1}_n) = \det(\hat{h}_1, \hat{h}_2 + r\hat{1} \ldots, \hat{h}_n + r\hat{1}) + \det(r\hat{1}, \hat{h}_2 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1})$$

Now observe that the second term $\det(r\hat{1}, \hat{h}_2 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1})$ leads to

$$\det(r\hat{1}, \hat{h}_2, \hat{h}_3 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1}) + \det(r\hat{1}, r\hat{1}, \hat{h}_3 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1})$$

and $\det(r\hat{1}, r\hat{1}, \hat{h}_3 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1}) = 0$. Hence

$$\det(r\hat{1}, \hat{h}_2 + r\hat{1}, \ldots, \hat{h}_n + r\hat{1}) = \det(r\hat{1}, \hat{h}_2, \hat{h}_3, \ldots, \hat{h}_n) = r \det\left(H^{(1)}\right)$$

The reasoning continues for every column. □

Proof. (of theorem 11) In equilibrium, by theorem 9 the following holds for non-loyal consumers $\Delta b^i = -\Delta D(\boldsymbol{\sigma})/\kappa(\boldsymbol{\sigma}) - \Delta e^i(\boldsymbol{\sigma}_{-i})$. Note that $\Delta D(\boldsymbol{\sigma}) = D_2 - D_1 = n - 2D_1$. Furthermore, with dyadic interactions, social propensity is locally constant, i.e. $\kappa(\boldsymbol{\sigma}) = \kappa_\alpha(\mathbf{m})$, furthermore, $\dfrac{\partial D_1}{\partial \sigma^j} = 1$ and $\dfrac{\partial \Delta e^i}{\partial \sigma^j} = A_{ij}$. We want to show that the following has a unique solution

$$\frac{\partial \Delta b^i}{\partial \sigma^j} = -2/\kappa_\alpha - A_{ij}$$

Hence we want to show that $\det(J_{\Delta \mathbf{b}}(\boldsymbol{\sigma})) \neq 0$ (the jacobian for $\Delta \mathbf{b}$). Now note that

$$\det(J_{\Delta \mathbf{b}}(\boldsymbol{\sigma})) = \det\left(-A(\mathbf{m}) - \frac{2}{\kappa_\alpha}\mathbf{1}_m\right)$$

By claim 2

$$\det(J_{\Delta \mathbf{b}}(\boldsymbol{\sigma})) = -\det(A(\mathbf{m})) - \frac{2}{\kappa_\alpha}\sum_i \det\left(A^{(i)}(\mathbf{m})\right)$$

but using the definition of $\kappa_\alpha(\mathbf{m})$, we get (supposing $\sum_i \det\left(A^{(i)}(\mathbf{m})\right) \neq 0$)

$$\det(J_{\Delta \mathbf{b}}(\boldsymbol{\sigma})) = -3\det(A(\mathbf{m})),$$

and $\det(A(\mathbf{m})) \neq 0$. □

Proposition 7

Proof. Note that consumers of the same type are indistinguishable, so whatever is the reasoning behind trying to antecipate the behavior of other individuals, the conclusion must be the same. Hence, they would all end up in a type symmetric strategy, which is in fact the one where they all play a mixed strategy, since, by corollary 6, if the mixed type-symmetric condition (MTS) holds all individuals of type t must be using the same strategy at a Nash equilibrium, which is precise. In the case of positive externalities, as there are three type-symmetric strategies, they can in fact choose the one they all prefer. □

3.4.1 The case with DI and homogeneous consumers

In this subsection we take the example of homogeneous consumers. The results are basicly aplications of all previous results and thus proofs are omitted. The purpose is to give cleaner explicit analytical expressions for prices and demand. Let the social externality function be formed by DI and consider socially homogeneous consumers, i.e. all individuals $i, j \in \mathcal{I}$ have the same personal profile $\Delta b^i = \Delta b^j = \Delta b$, and the same social profile $\alpha_1^{ij} = \alpha_1$ and $\alpha_2^{ij} = \alpha_2$. The influence network is complete and determined by a single weight parameter $\alpha_1 + \alpha_2$. Let us suppose the MTS condition holds, that is $\alpha_1 + \alpha_2 \neq 0$. The mixed strategy equilibria, by corollary 7, are given by some loyalty characterization (l_1, l_2) and a unique probability used by non-loyal consumers. Let $m = n - l_1 - l_2$ be the number of non-loyal consumers, and q_m the probability used by non-loyal consumers. Demand is given by $D_1(\boldsymbol{\sigma}^*) = l_1 + mq_m$. The social propensity index is thus completely determined locally by $\alpha_1 + \alpha_2$ and m, and given by

$$\kappa_\alpha(m) = \frac{m}{(m-1)(\alpha_1 + \alpha_2)}.$$

Remark 6 (Linear aggregate)**.** *Consider a linear aggregated externality function $g_1(d) = \alpha_1 d$ and $g_2(d) = \alpha_2(n-d)$ where $d \in [0, n-1]$. Then, the consumers game amounts to this same game with homogeneous consumers and dyadic interactions (which can be seen by theorem 10).*

When $m > 0$ let us define the consumers behavior *loyalty indices*, respectively the *market loyalty index* L, and the (marginal) *firms loyalty indices* L_1, L_2,

$$L(\boldsymbol{\sigma}) \equiv \frac{l_1 + l_2}{m}; \quad L_1(\boldsymbol{\sigma}) \equiv \frac{l_1}{m}, \quad L_2(\boldsymbol{\sigma}) \equiv \frac{l_2}{m}.$$

When $m = 0$ define $L = n$ and analogously for L_1, L_2. Note that the loyalty indices are in the range $0 \leq L \leq n$, n being full market loyalty. When $L = 0$, there are only non-loyal players. Naturally, a shared market equilibrium only exists if $L < n$.

Recall the decision threshold $T(l_1) \equiv -(\alpha_1 + \alpha_2)l_1 + \alpha_2(n-1)$.

Lemma 9 (Congestion effects)**.** *Let* $\alpha_1 + \alpha_2 < 0$. *The admissible demand for each* (l_1, l_2) *characterization is given by*

$$D_1^*(\mathbf{p}) = l_1 + m\frac{\Delta b - \Delta p - T(l_1)}{-(\alpha_2 + \alpha_1)(m-1)}$$

Proof. Follows directly from corollary 7. □

Theorem 12 (Congestion equilibrium prices)**.** *Let* $\alpha_1 + \alpha_2 < 0$. *The market equilibrium prices for each loyalty characterization are given by*

$$p_1^* = -\frac{1}{3}(\alpha_1 + \alpha_2)\left(n - L^* - L_1^* - 1\right) + \frac{1}{3}\left(\Delta b - \alpha_2(n-1)\right);$$

$$p_2^* = \frac{1}{3}(\alpha_1 + \alpha_2)\left(n - L^* - L_2^* - 1\right) - \frac{1}{3}\left(\Delta b + \alpha_1(n-1)\right).$$

Note that the price difference is

$$\Delta p^* = \frac{1}{3}\left((\alpha_1 + \alpha_2)\Delta L + 2\Delta b + (n-1)\Delta\alpha\right)$$

When the market grows, the main responsibles for price asymmetries are social product differentiation and changes in the market loyalty indices, which are also proportional to the social weights. In particular, it would be natural that a market grows and the loyalty indices are kept constant. The role of

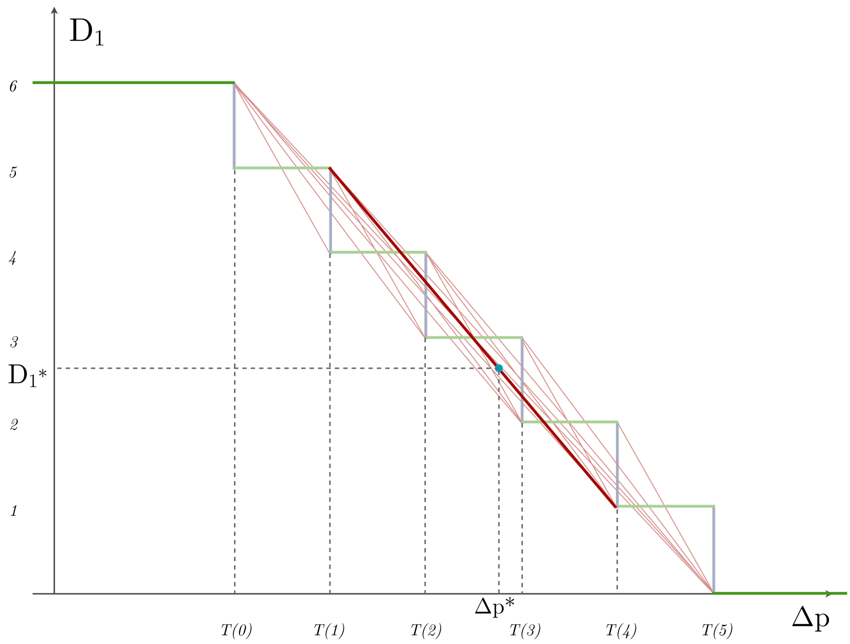

Figure 3.6: Equilbrium demand in a simplistic case of 6 consumers. Represented are the thresholds for pure strategies, and the mixed strategies in red. Highlighted is a particular local pure price equilibrium.

standard product differentiation in creating price asymmetries is overcome. Note that under congestion efects, heterogeneity is not necessary to resolve the classical Bertrand paradox, nor to achieve asymmetric pure price equilibria. In figure 3.6 is depicted equilibrium demand for the case of congestion effects.

Lemma 10 (Conformity effects)**.** *Let $\alpha_1 + \alpha_2 \geq 0$. The admissible demand has only one discontinuity which is the equilibrium point, and it is continuous out of equilibrium. It is of the form*

$$D_1^*(\mathbf{p}) = \begin{cases} n & \text{if} \quad p_1 < p_2 - c_b^* \\ nq_m & \text{if} \quad p_1 = p_2 - c_b^* \\ 0 & \text{if} \quad p_1 > p_2 - c_b^* \end{cases}$$

where $c_b^* \in [-\alpha_1(n-1), \alpha_2(n-1)]$.

We call c_b^* the consumer bias parameter, as it indicates how consumers will choose between two firms, when both monopolies are in their best response. Note that in the case where $\alpha_1 + \alpha_2 \geq 0$ the strategy of consumers is completely determined by the consumer bias parameter. When $c_b^* = 0$ we are in the Bertrand framework (consumers essentially ignore the multiple 'possibilities created by externalities).

Theorem 13 (Conformity equilibria)**.** *Let* $\alpha_1 + \alpha_2 \geq 0$. *The market equilibrium is*

(i) a monopoly for firm 1 with price $\mathbf{p} = (|c_b^*|, 0)$ *if* $c_b^* < 0$*;*

(ii) a monopoly for firm 2 with price $\mathbf{p} = (0, c_b^*)$ *if* $c_b^* > 0$*;*

(iii) a Bertrand zero profit equilibrium with $\mathbf{p} = (0, 0)$ *and demand* $D_1^* \in \{0, n/2, n\}$ *if* $c_b^* = 0$.

Note that in the positve case the price difference is

$$\Delta p^* = c_b^*.$$

In figure 3.7 is depicted equilibrium demand for the case of conformity effects.

3.4.2 Pure strategies and monopolies

When consumers have biased personal preferences $\Delta \mathbf{b}$, i.e. they are contained in pure type-symmetric Nash domains, firms have the possibility to have monopoly, or at least *type-monopolies*, meaning all individuals of the same type. Furthermore, the demand behavior results from the relative preferences $\Delta b^t - \Delta p$ induced by the pair of prices set by the firms in the first stage, and must be based on a strategy whose Nash domain contains the relative preferences. On one hand, there is a limited set of relative preferences

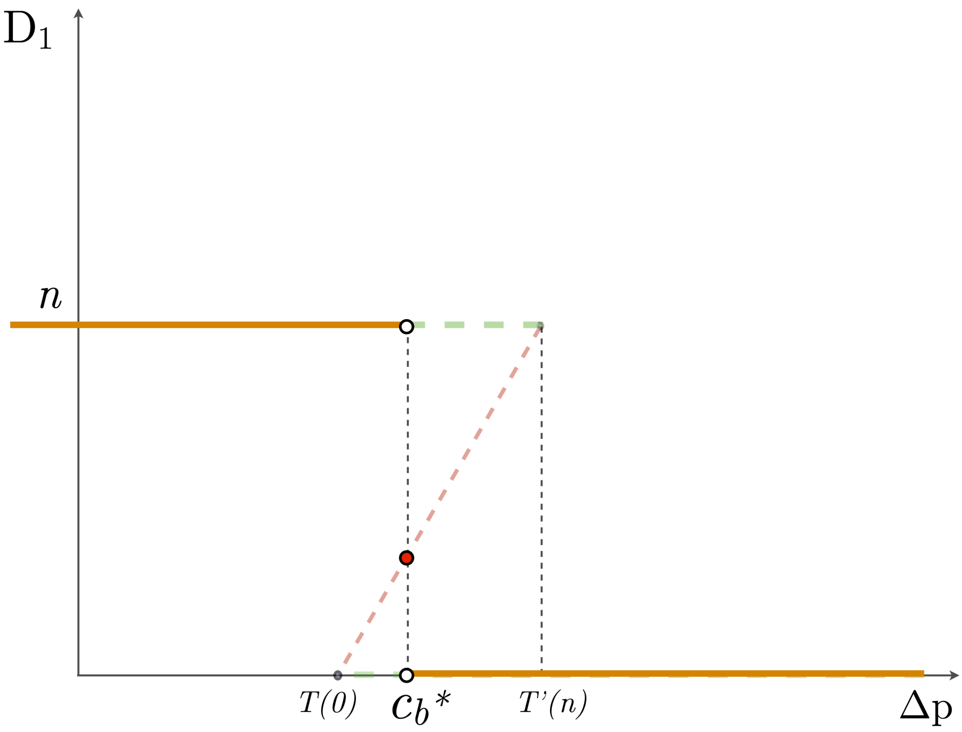

Figure 3.7: Equilbrium demand in the case of conformity effects. Shadowed are the multiple equilibria not chosen in the admissble demand. The consumer bias parameter c_b^* indicates the choice of consumers in the region where the three type-symmetric equilibrium exist.

that can be induced by a pair of prices, namely, as $\Delta b^t - \Delta p$, we have that for every pair of prices set by the firms, they will be in $\{\mathbf{\Delta b}+m\mathbf{1}_{n_T}\}$. On the other hand, these preferences need to be resistent to price deviations, and its neighbourhood must intersect the Nash domain of a different strategy, as the smallest price change must provoke a change in consumers behavior, or else it would be profitable to deviate. A pair of prices $\mathbf{p}^*$ is part of a subgame-perfect equilibria only if the relative preferences profile is in a singleton given by the intersection of $\Delta b^t - \Delta p$ with the boundary of the pure Nash domain. This is because in interior points demand is constant. We are thus able to identify the price candidates to a firm equilibrium geometrically.

In the benchmark case, when there is no consumer network, there are only pure strategies for individuals. The monopoly/competitive regions are characterized by

$$M_1 \equiv \left\{ \Delta \mathbf{b} \in (\mathbb{R}^+)^{n\mathcal{T}} : \forall t, t' \in \mathcal{T},\ \frac{n_t}{n} \leq \frac{\Delta b^{t'}}{\Delta b^t} \leq \frac{n}{n_{t'}} \right\}$$

$$M_2 \equiv \left\{ \Delta \mathbf{b} \in (\mathbb{R}^-)^{n\mathcal{T}} : \forall t, t' \in \mathcal{T},\ \frac{n_t}{n} \geq \frac{\Delta b^{t'}}{\Delta b^t} \geq \frac{n}{n_{t'}} \right\}$$

$$Z = \{\mathbf{0}\}$$

Note that, in the previous homogeneous case, if $A = 0$ then $M_1 \subset (\mathbb{R}^+)^{n_t}$, $M_2 \subset (\mathbb{R}^-)^{n_t}$ and $M_1 \cap M_2 = \{\mathbf{0}\}$.

The equilibium will be

(i) monopoly for firm 1 if $\Delta \mathbf{b} \in M_1 \subset \mathcal{N}(n, 0)$ with prices $\mathbf{p} = (\min \Delta b^t, 0)$;

(ii) monopoly for firm 2 if $\Delta \mathbf{b} \in M_2 \subset \mathcal{N}(0, n)$ with prices $\mathbf{p} = (0, \min |\Delta b^t|)$;

(iii) a equilibrium with zero profits if $\Delta \mathbf{b} = \mathbf{0}$, with $\mathbf{p} = \mathbf{0}$;

(iv) no equilibrium if $\Delta \mathbf{b} \notin M_1 \cup M_2$.

Before we set out for the geometric example, note that adding inter-type interactions, while leaving intra-type at zero ($A_{tt} = 0$)has the following effect on the monopoly regions.

$$M_1 \equiv \left\{ \Delta \mathbf{b} \in (\mathbb{R}^+)^{n\mathcal{T}} : \forall t, t' \in \mathcal{T},\ \frac{n_t}{n} \leq \frac{\Delta b^{t'} - \alpha_1^{t't}}{\Delta b^t - \alpha_1^{tt'}} \leq \frac{n}{n_{t'}} \right\}$$

$$M_2 \equiv \left\{ \Delta \mathbf{b} \in (\mathbb{R}^-)^{n\mathcal{T}} : \forall t, t' \in \mathcal{T},\ \frac{n_t}{n} \geq \frac{\Delta b^{t'} - \alpha_2^{t't}}{\Delta b^t - \alpha_2^{tt'}} \geq \frac{n}{n_{t'}} \right\}$$

$$Z = \times_{tt'} \left[\min\{-\alpha_1^{tt'}, \alpha_2^{tt'}\}, \max\{-\alpha_1^{tt'}, \alpha_2^{tt'}\} \right]$$

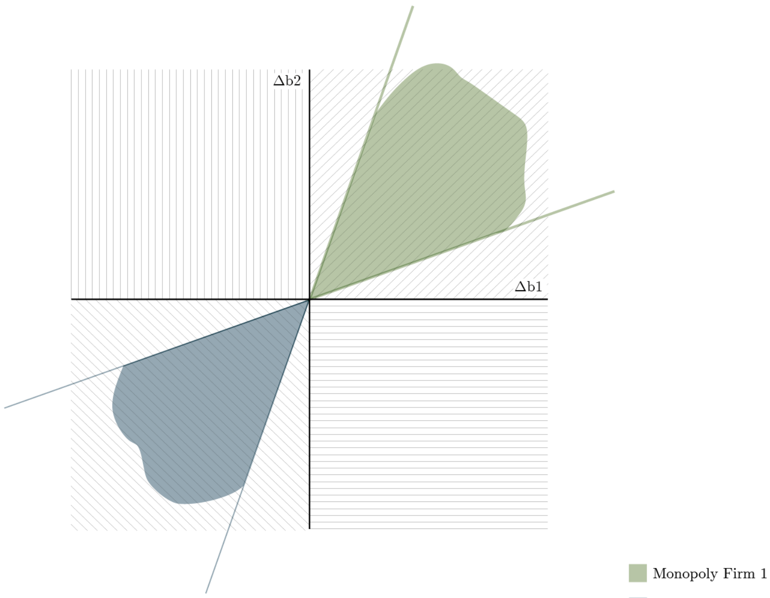

Figure 3.8: The benchmark case and monopoly areas. Note the Bertrand paradoxical zero profit equilibrium is at the origin. $A_{11} = A_{22} = A_{12} = A_{21} = 0$.

The case with two types of consumers ($\Gamma_{2,2}$)

Consider two types t_1, t_2 of consumers. The reduced influence network is determined by the parameters $A_{11}, A_{12}, A_{21}, A_{22}$. The social propensity index is

$$\kappa_\alpha(\mathbf{m}) = \frac{A_{11}(m_1-1)/m_1 + A_{22}(m_2-1)/m_2 - A_{12} - A_{21}}{A_{11}A_{22}(m_1-1)(m_2-1)/m_1m_2 - A_{12}A_{21}}.$$

In figures 3.13, 3.8, 3.12, 3.9, A.3, 3.11 and 3.14 are depicted the monopoly regions and their relation with changes in the parameters of the social profile.

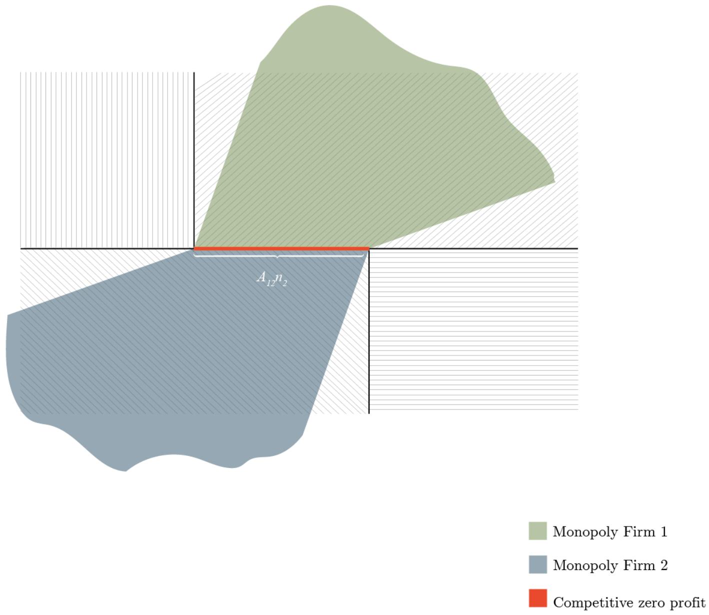

Figure 3.9: The effect of one positive intertype interactions. $A_{12} > 0$ and $A_{11} = A_{22} = A_{21} = 0$. The monopoly regions intersect and there is a region with zero profit equilibrium for both firms.

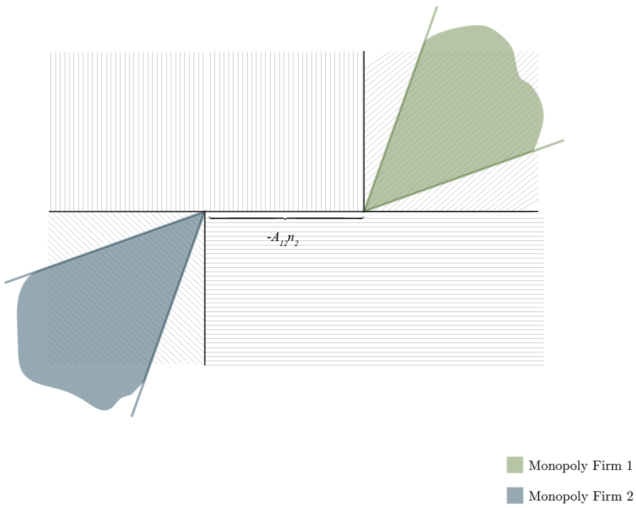

Figure 3.10: The effect of one negative intertype interactions. $A_{12} < 0$ and $A_{11} = A_{22} = A_{21} = 0$. The monopoly regions get separated and there is a region with no equilibrium.

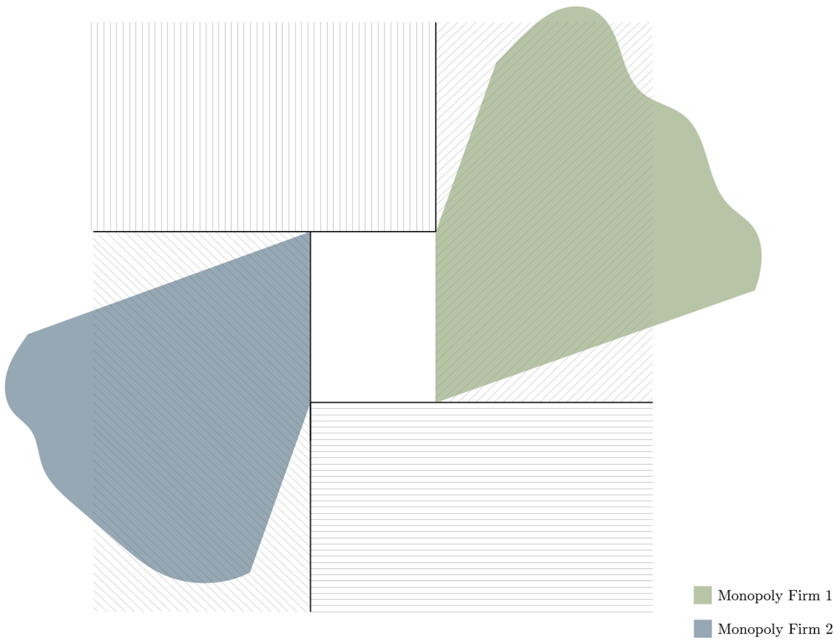

Figure 3.11: The effect of intertype interactions with different signs. $A_{12} > 0$, $A_{21} < 0$ and $A_{11} = A_{22} = 0$. There is a region with no pure equilibrium for consumers.

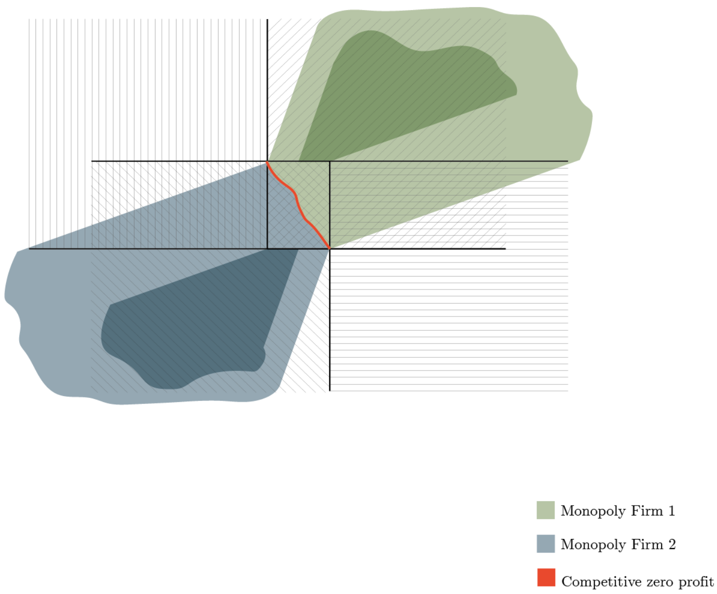

Figure 3.12: The effect of positive intertype or intratype interactions. Both produce the same effect. In the case $A_{12} > 0$, $A_{21} > 0$ and $A_{11} = A_{22} = 0$. There is a region where there is multiplicity equilibria, and the lighter colored areas of monopoly will only exist depending on the choice of consumers on the areas where multiple equilibria exist. Furthermore, in the middle square the consumer bias parameter will decide the position of the line corresponding to the competitive equilibria.

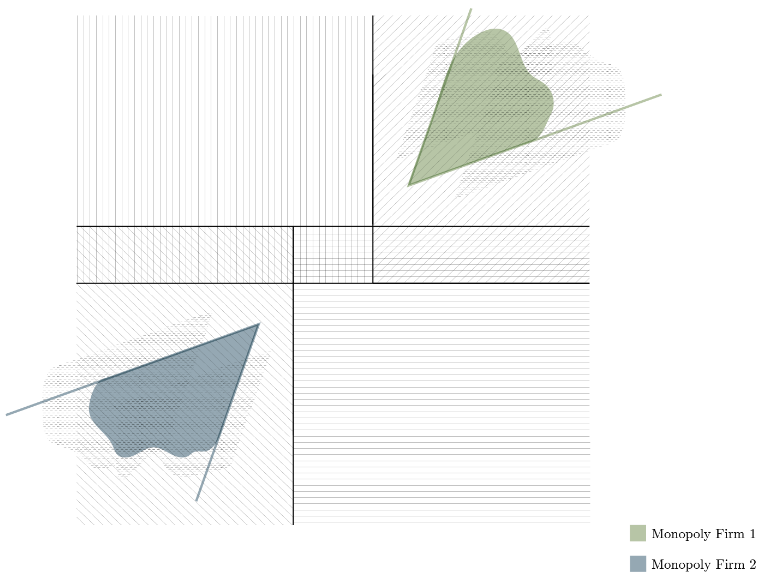

Figure 3.13: The effect of negative intratype interactions. The shaded regions around monopolies mean that the monopoly region will depend on the consumers choice on the area of intersection, that has multiple Nash equilibria. $A_{12} < 0$, $A_{21} < 0$ and $A_{11} = A_{22} = 0$.

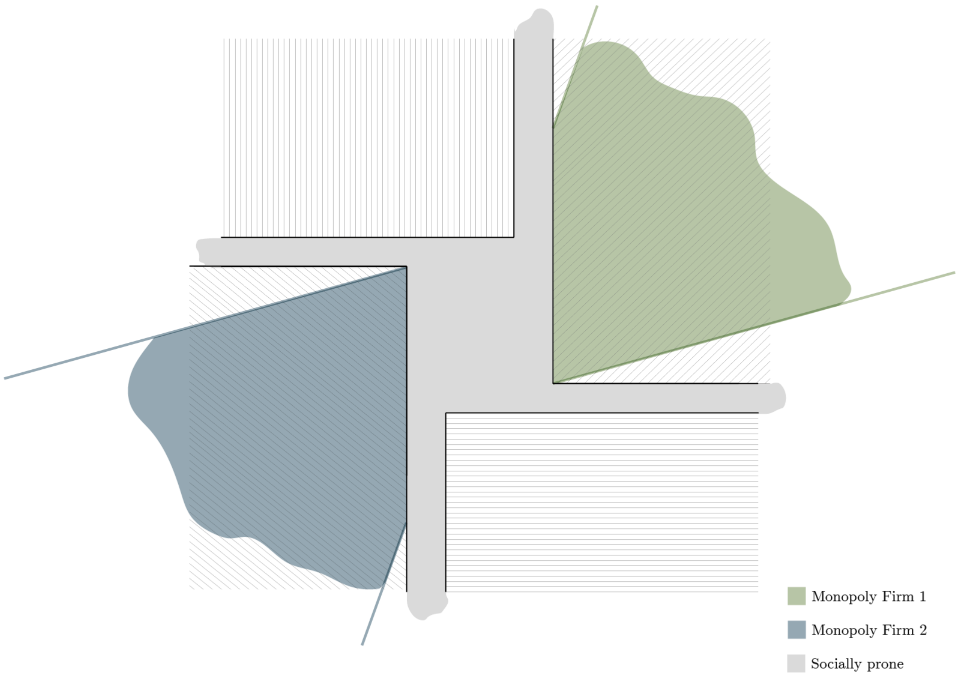

Figure 3.14: All parameters turned on. $A_{12} > 0$, $A_{21} < 0$ and $A_{11} < 0$, $A_{22} < 0$. The shaded areas are contain the socially prone outcomes.

Chapter 4

Conclusions and future work

We have set forth an idea of how to build up market equilibria with pure price strategies, and classify markets according to solutions based on an index of the interdependence of social interactions in consumption, called social propensity. This allows the departure from the paradoxical zero price equilibrium and the characterization of prices, demand and personal preferences using social propensity. Naturally, further study of the properties of the index, the dependence on particular parameters of a market, the study of what happens when the market changes, namely when it grows, or how it behaves for particular types of networks and canonical network examples, besides the complete network, should be develpped. Namely, for special configurations like small-world networks. In particular, an econometric analysis on real market data would be an interesting source of validation or refutation, and it could provide insights on whether to adjust the definition or create different indices. In this regard, the reduction of the influence network to communities appears as very useful. This was done based on *exact communities*, but it would naturally be more interesting to loosen the exactness and study the case where communities are comprised of similar individuals, up to some error, and see what this could produce in terms of the error in the index and computation of equilibria. The work on the notion of societies on the first chapter and the proof of the conformity obstruction lemma could set the pace for such an approach.

Another natural extension comes to mind: the extension of the duopoly situation to an oligopoly. On the one hand this is a natural extension and provides the basis for a more close connection to an econometric analysis; on the other, this would allow to relax the imposition that consumers use standard mixed strategies over the space of firms, and thus to relax the hypothesis of mandatory consumption. The option of not buying would naturally introduce some exogenous reservation price, and firms might, in some situations, be led to that boundary, and get stuck in shared market situations with pure consumer strategies, which we do not have in the mandatory consumption case. A natural departure point is the general class of decision games that we have already fully characterized and studied on the first chapter, where we allowed for any finite set of actions. The extension to an oligopoly situation would probably also require a new approach to creating a one valued index, since the natural extension of the one we presented would be an n-dimensional index.

We have focused this work on uniform pricing. This presents particular difficulties as to the existence of pure price equilibria. A different direction would be to allow firms to use price schemes, and with the knowledge of consumer interdependence, discriminate prices according to the influence each consumer exerts over other consumers. Furthermore, negative prices could also be allowed, as a form of subsidizing influential consumers, and change loyalty characterizations. Some of the results that we have on non-existence of equilibria for positive social propensity may be dealt with this approach.

The work has been based on the concept of Nash equilibrium. Naturally it would be interesting to think about other notions of equilibrium, and one such example is (social) welfare equilibrium. Our conjecture is that allowing a coordination device so that consumers maximize group welfare will lead them to fixate in pure strategies. In that sense, a natural follow-up would be to allow firms to use price distributions and try to characterize these price distribution in terms of the social profile. The change to a cooperative setting, at least to allow some coalitions to form for some groups of consumers, provides an approach with interesting interpretations for some particular contexts, and the study of bounds on the price of anarchy and price of stability would be interesting.

A generalization to consumer games with continuous space of strategies appears also as a possible road. The duopoly results seem to hold for C^1 social externality functions, as we have not used any special property of the finite case in their proofs. Beyond generalizations, there is yet a large body of work which is still to be done, and this includes, for example, marketing strategies, values of a market study, loyalty characterizations, static analysis or introducing uncertainty on the consumers or firms subgames.

In the first chapter, which was based on the characterization of the set of pure and mixed Nash equilibiria of a decision game with three essencial properties, a natural extension is the study of how breaking the properties assumed for the utility function impact the results. For the duopoly this has essencially been done. The question remains as to the Nash equilibria set forth in the first chapter. Dyadic interactions can be relaxed to study aggregative of forms of influence. Namely, the intertwining of the two and the study of the relative impact of close friendship dyadic relations and general aggregative influence seems fruitful. Breaking PBI will provoke a high impact on the existence of pure Nash equilibria, and an interesting approach is using it as a road to chaos. In fact, relaxing both assumptions has a particular high impact regarding the section based on reciprocal relations. Nevertheless, even maintaining the three essential properties, the study of somewhat reasonable symmetry properties to impose on the social profile to guarantee existence of pure Nash equilibria, independent of personal space, could be pursued in further detail, but we would recommend caution. We have proposed some approaches, namely based on reciprocal relations and potential games, and these aimed at finding a sufficiency condition. A condition based on the relation of absolute values should be further investigated. The existence of a necessary and sufficient condition which is relatively simple (at least simpler than what is supposed to ensure) and useful, does not seem plausible due to the examples mentioned (see also for example [29]). Nevertheless, there can be special cases of interest. The referendum game in appendix already provides a tool for an experimental study of games with and without these properties. Furthermore, it introduces the mixture when consumers care (possibly differently) about the general outcome of the game.

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Appendix A

The referendum game.

We have built on Netlogo a game with an action set $\mathcal{A} = \{Y, N, B, A\}$ to mean the votes yes, no, blank and abstention and that can have up to four types of individuals. This allows the study of the classes of $\Gamma_{4,4}$. We added the possibility of breaking all properties of the utility function by adding the following possibilities:

Quorum sensibility

$$f_t(s_i) \equiv \begin{cases} 0 & \text{if } s_i \neq A; \\ \\ \varepsilon_o(\omega_N^t - \omega_Y^t) e^{-\left(\frac{l_A^t / n_{\mathcal{I}} - 0.5}{\varsigma}\right)^2} & \text{if } s_i = A. \end{cases}$$

Tie sensibility

$$g_t(s_i) \equiv \begin{cases} 0 & \text{if } s_i \in \{Y, N\}; \\ \\ \varepsilon_o|\omega_N^t - \omega_Y^t| e^{-\left(\frac{l_Y^t - l_N^t}{\varsigma n_{\mathcal{I}}}\right)^2} & \text{if } s_i \in \{B, A\}. \end{cases}$$

The two interior parameters ε_o and ς reflect *sensibility* to outcomes and *strateginess*. The new utility function becomes

$$\bar{u}(i;\boldsymbol{\sigma}) \equiv u(i;\boldsymbol{\sigma}) + f_{t_i}(s_i) + g_{t_i}(s_i).$$

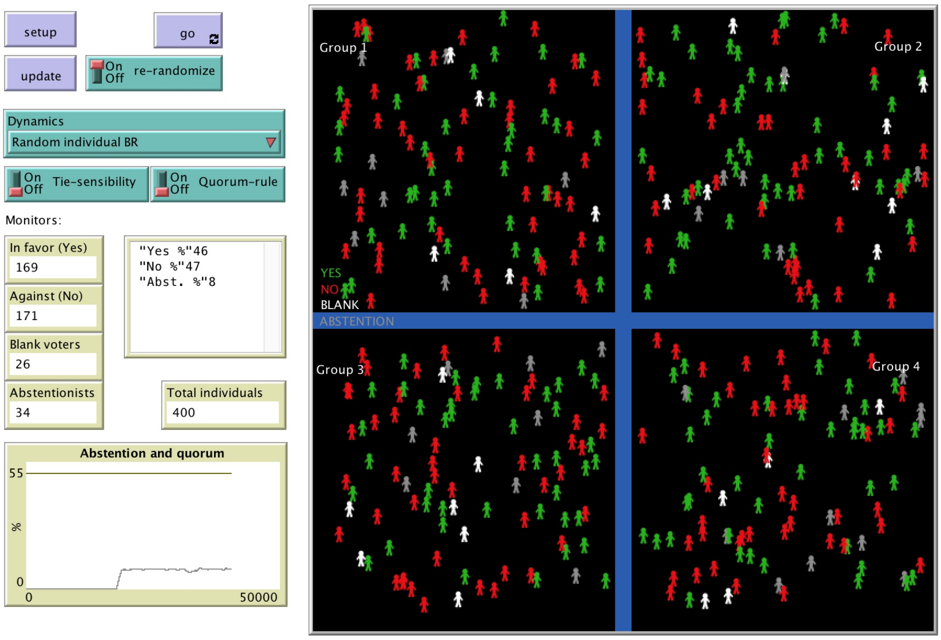

Figure A.1: The interface for the referendum game built in Netlogo, the part of main controls. The colors represent the players strategies: red means voting against, green in favor, white means voting blank, and grey abstention.

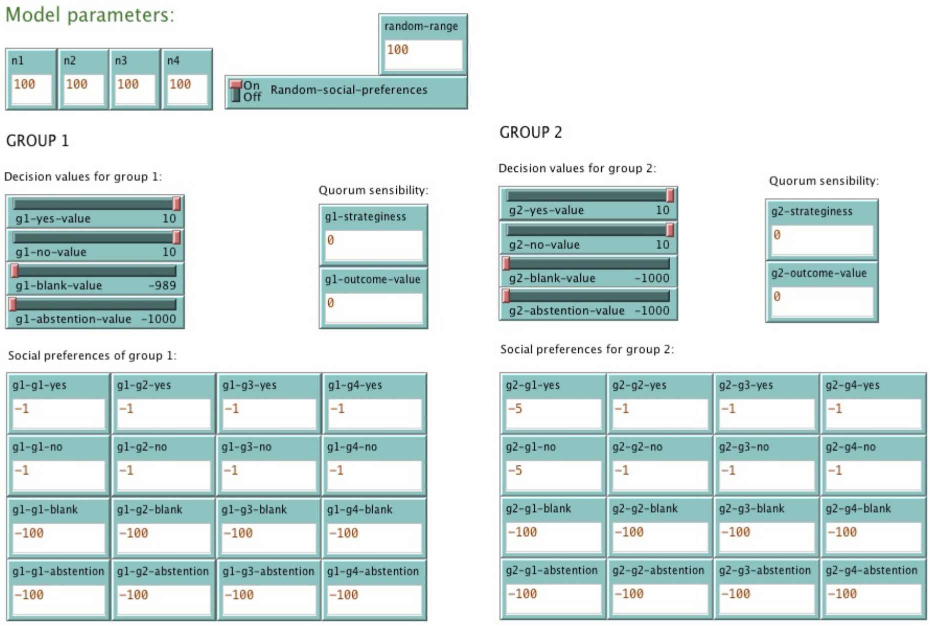

Figure A.2: The interface for the referendum game built in Netlogo, input part.

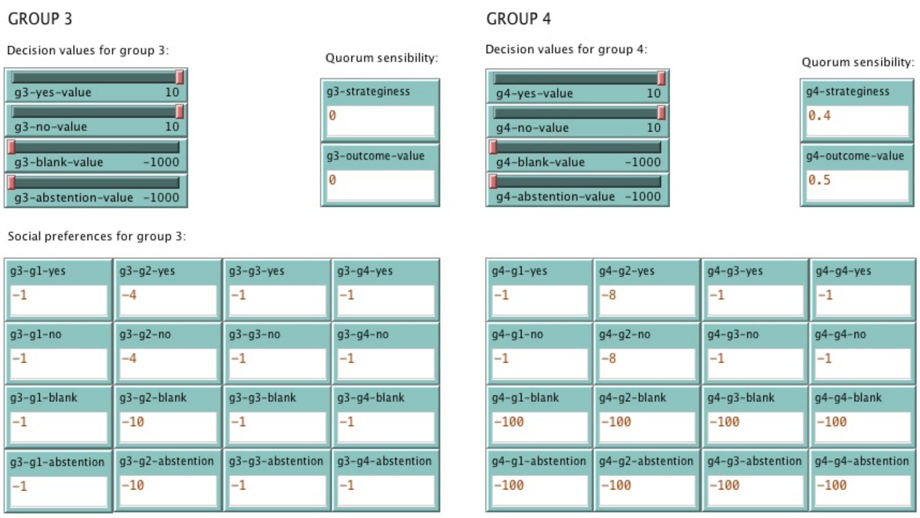

Figure A.3: The interface for the referendum game built in Netlogo, input part.