Automorphisms and derivations of $U_q(\mathfrak{sl}_4^+)$

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Abstract

We compute the automorphism group of the q-enveloping algebra $U_q(\mathfrak{sl}_4^+)$ of the nilpotent Lie algebra of strictly upper triangular matrices of size 4. The result obtained gives a positive answer to a conjecture of Andruskiewitsch and Dumas. We also compute the derivations of this algebra and then show that the Hochschild cohomology group of degree 1 of this algebra is a free (left) module of rank 3 (which is the rank of the Lie algebra $\mathfrak{sl}_4$) over the center of $U_q(\mathfrak{sl}_4^+)$.

Keywords: quantized enveloping algebra; automorphisms; derivations; Hochschild cohomology.

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Introduction

Let $\mathbb{K}$ be a field, $\mathcal{L}$ a Lie algebra over $\mathbb{K}$ and $U(\mathcal{L})$ its enveloping algebra. The group $\mathrm{Aut}_{\mathbb{K}}U(\mathcal{L})$ of $\mathbb{K}$-algebra automorphisms of $U(\mathcal{L})$ is still for the most part unknown (except in particular instances, e.g. $\dim \mathcal{L} \leq 2$). For example, if $\mathcal{L}$ is the two-dimensional abelian Lie algebra, then $U(\mathcal{L})$ is the polynomial algebra in two indeterminates x_1, x_2, whose group of automorphisms is generated by the *elementary* automorphisms of the form

$$x_i \mapsto \lambda x_i + f(x_j), \quad x_j \mapsto x_j \qquad (i \neq j)$$

with $\lambda \in \mathbb{K}^*$ and $f(x_j)$ a polynomial in the variable x_j ([16], [26]). In contrast with this simple description, the conjecture that the polynomial algebra in three variables over $\mathbb{K}$ has *wild* automorphisms (i.e. automorphisms not of the above type) has recently been settled (see [24]) assuming $\mathbb{K}$ has characteristic 0. Another example is the enveloping algebra of $\mathfrak{sl}_2$, which is known to have wild automorphisms by a result of Joseph [15].

Pertaining more to what is studied in this paper is the enveloping algebra of the three-dimensional Heisenberg Lie algebra, which is given by generators x, y and z, subject to the relations

$$[x, y] = z, \quad [z, x] = 0 = [z, y].$$

This algebra can also be seen as the enveloping algebra of the Lie algebra $\mathfrak{sl}_3^+$ of strictly upper triangular matrices of size 3. The infinite dimensional simple quotients of $U(\mathfrak{sl}_3^+)$ are isomorphic to the first Weyl algebra $\mathbb{A}_1(\mathbb{K})$, whose group of automorphisms was described by Dixmier in [10]. Yet, the full group of automorphisms of $U(\mathfrak{sl}_3^+)$ remains to be described, and Alev [1] proved the existence of wild automorphisms of this algebra.

Unlike the classical scenario, quantum algebras are believed to possess less symmetry (see [12, 1.1]) and the group of automorphisms of several algebras of this kind has been computed successfully. Making use of a general result relating automorphisms and derivations of $\mathbb{N}$-graded algebras,

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Alev and Chamarie [2] described the automorphism group of (the coordinate ring of) a quantum affine space, of the algebra of 2×2 quantum matrices and of the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$. Also, in [4] the authors found the automorphism groups of the quantum Weyl algebra, the Weyl-Hayashi algebra, the quantum Heisenberg algebra $U_q(\mathfrak{sl}_3^+)$ (see also [8]) and of other related algebras. Here the methods used included describing the set of normal elements of the algebras involved and using appropriate filtrations to carry out computations. In [22], Rigal used the invariance under automorphisms of the set of height 1 prime ideals of quantum Weyl algebras to describe their automorphism group. Related methods were employed by Gómez-Torrecillas and Kaoutit [11] regarding the coordinate ring of quantum symplectic space, and by Lenagan and the first author [19] regarding the algebra of non-square quantum matrices. In all of these cases, the automorphism group of the algebras involved does not differ from the natural torus which acts diagonally on the generators by more than a finite group and perhaps a copy of $\mathbb{Z}$.

In their paper [5], Andruskiewitsch and Dumas conjectured that, given a finite-dimensional complex simple Lie algebra $\mathfrak{g}$ with triangular decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$, then $\mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{g}^+)$, the group of $\mathbb{K}$-algebra automorphisms of the quantized enveloping algebra of the nilpotent Lie algebra $\mathfrak{g}^+$, is isomorphic to the semi-direct product of the torus $(\mathbb{K}^*)^n$ (n being the rank of $\mathfrak{g}$) with the group of order 1, 2 or 3 consisting of the diagram automorphisms of $\mathfrak{g}^+$, see [5, Prob. 1]. This conjecture holds for $\mathfrak{g}^+ = \mathfrak{sl}_3^+$ ([8], [4]) and recently the first author proved in [17] that it holds as well in the B_2 case, i.e., with $\mathfrak{g}^+ = \mathfrak{so}_5^+$.

In this paper we settle the conjecture of Andruskiewitsch and Dumas in the A_3 case, so that $\mathfrak{g}^+ = \mathfrak{sl}_4^+$ is the Lie algebra of strictly upper triangular matrices of size 4. We also compute the Lie algebra of derivations and the first Hochschild cohomology group of $U_q(\mathfrak{sl}_4^+)$, which is shown to be a free module of rank 3 over the center of $U_q(\mathfrak{sl}_4^+)$.

Let us briefly summarise what is done in the paper. There exist normal elements Δ_1, Δ_2 and Δ_3 such that the center of $U_q(\mathfrak{sl}_4^+)$ is the polynomial algebra in the variables $z_1 = \Delta_1\Delta_3$ and $z_2 = \Delta_2$. Given an automorphism ϕ of $U_q(\mathfrak{sl}_4^+)$, our strategy is to show that, up to the diagram automorphism and the diagonal action of the torus $(\mathbb{K}^*)^3$ on the Chevalley generators of $U_q(\mathfrak{sl}_4^+)$, ϕ fixes Δ_1, Δ_2 and Δ_3. Then, by using degree arguments, we conclude that ϕ is the identity.

The difficulty that arises is in showing that the central element Δ_2 is fixed. Hence we use the methods of [2] and [18] and determine the derivations of $U_q(\mathfrak{sl}_4^+)$. To do this, we first apply the deleting derivations algorithm of Cauchon [9] so that, after suitably localising, we can embed $U_q(\mathfrak{sl}_4^+)$ in a quantum torus $P(\Lambda)$. Extending a derivation D of $U_q(\mathfrak{sl}_4^+)$ to $P(\Lambda)$ we obtain, by a result of Osborn and Passman [21], a decomposition

$$D = \mathrm{ad}_x + \theta$$

with $x \in P(\Lambda)$ and θ a central derivation of $P(\Lambda)$. Using a sort of restoring derivations algorithm, we finish by deducing that $x \in U_q(\mathfrak{sl}_4^+)$ and that θ sends each Chevalley generator of $U_q(\mathfrak{sl}_4^+)$ to a multiple of itself by a central element of $U_q(\mathfrak{sl}_4^+)$.

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1 Basic aspects of $U_q(\mathfrak{sl}_4^+)$

Let $\mathbb{K}$ be a field of characteristic 0 and fix a parameter $q \in \mathbb{K}^*$ which we assume is not a root of unity. Consider, for $n \geq 2$, the Lie algebra $\mathfrak{sl}_n$ of $n \times n$ matrices of trace 0 and its maximal nilpotent subalgebra $\mathfrak{sl}_n^+$ consisting of the strictly upper triangular matrices of size n.

Throughout this paper $\mathbb{N}$ is the set of nonnegative integers. For $k \in \mathbb{N}$, the q-integer $[k]$ is defined by $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ and we use the notation $\widehat{q} = q - q^{-1}$.

1.1 q-Serre relations

The algebra $U_q(\mathfrak{sl}_4^+)$ is the q-deformation of the universal enveloping algebra of the nilpotent Lie algebra $\mathfrak{sl}_4^+$. It is the unital associative $\mathbb{K}$-algebra with generators e_1, e_2 and e_3, subject to the quantum Serre relations:

$$e_1e_3 - e_3e_1 = 0 \tag{1}$$

$$e_i^2e_j - (q+q^{-1})e_ie_je_i + e_je_i^2 = 0 \qquad \text{if } \ |i-j| = 1. \tag{2}$$

1.2 Weight space decomposition

Let $Q = \mathbb{Z}^3$ be the free abelian group with canonical basis α_1, α_2, α_3 and $Q^+ = \mathbb{N}^3$ be its submonoid. Since the quantum Serre relations are homogeneous in the given generators, there is a Q^+-grading on $U_q(\mathfrak{sl}_4^+)$ obtained by assigning to e_i degree α_i. We use the terminology *weight* instead of degree for this grading, and write $wt(u) = \beta$ if $u \in U_q(\mathfrak{sl}_4^+)$ has weight β.

1.3 PBW basis

Several authors have constructed PBW bases for quantized enveloping algebras (e.g. [27], [25], [23]). It will be convenient for us to use the following construction:

$$\begin{aligned} X_1 &= e_1, & X_2 &= e_1e_2 - q^{-1}e_2e_1, \\ X_4 &= e_2, & X_5 &= e_2e_3 - q^{-1}e_3e_2, \\ X_6 &= e_3, & X_3 &= e_1X_5 - q^{-1}X_5e_1. \end{aligned}$$

Then, the set of monomials $\left\{X_1^{b_1}\cdots X_6^{b_6} \mid b_i \in \mathbb{N}\right\}$ is a linear basis of $U_q(\mathfrak{sl}_4^+)$. Notice that all X_i are weight vectors.

1.4 Ring theoretical properties of $U_q(\mathfrak{sl}_4^+)$

Let R be a ring and let τ be an endomorphism of R. Recall that a (left) τ-derivation of R is an additive map $\delta : R \to R$ which satisfies the relation $\delta(ab) = \tau(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. Given R, τ and δ as above, we can form the skew polynomial ring $R[X;\tau,\delta]$. As a left R-module, $R[X;\tau,\delta]$ is free with basis $\{X^i \mid i \geq 0\}$ and the multiplication in $R[X;\tau,\delta]$ is determined by that of R and the rule:

$$Xa = \tau(a)X + \delta(a),$$

for $a \in R$. Naturally, if τ' is an endomorphism of $R[X;\tau,\delta]$ and δ' is a τ'-derivation of $R[X;\tau,\delta]$, this construction can be repeated to obtain an iterated skew polynomial ring $R[X;\tau,\delta][Y;\tau',\delta']$, and so on. It is well known that if R is a Noetherian domain and τ is an automorphism, then $R[X;\tau,\delta]$ is still a Noetherian domain. Also, if R is a $\mathbb{K}$-algebra and the maps τ and δ are $\mathbb{K}$-linear then the resulting skew polynomial ring is a $\mathbb{K}$-algebra in a natural way. The reader who is not familiar with skew polynomial rings can refer to [14] for more details and examples.

It was seen in [23] (see also [6, I.6.10] and references therein) that $U_q(\mathfrak{sl}_4^+)$ is an iterated skew polynomial ring. In terms of the PBW basis described above, we have

$$U_q(\mathfrak{sl}_4^+) = \mathbb{K}[X_1][X_2;\tau_2][X_3;\tau_3][X_4;\tau_4,\delta_4][X_5;\tau_5,\delta_5][X_6;\tau_6,\delta_6], \tag{3}$$

with τ_i a $\mathbb{K}$-algebra automorphism and δ_i a $\mathbb{K}$- linear (left) τ_i-derivation of the appropriate subalgebra. Thus $U_q(\mathfrak{sl}_4^+)$ is a Noetherian domain.

So that we can easily compute in $U_q(\mathfrak{sl}_4^+)$, and also because this information will be needed in Section 3.1, we specify these automorphisms and skew-derivations below by giving their values on the X_j ($\delta_i(X_j) = 0$ unless otherwise specified):

$$\tau_2(X_1) = q^{-1}X_1$$

$$\tau_3(X_1) = q^{-1}X_1, \qquad \tau_3(X_2) = q^{-1}X_2$$

$$\tau_4(X_1) = qX_1, \qquad \tau_4(X_2) = q^{-1}X_2, \qquad \tau_4(X_3) = X_3, \qquad \delta_4(X_1) = -qX_2$$

$$\tau_5(X_1) = qX_1, \qquad \tau_5(X_2) = X_2, \qquad \tau_5(X_3) = q^{-1}X_3,$$
$$\tau_5(X_4) = q^{-1}X_4, \qquad \delta_5(X_1) = -qX_3, \qquad \delta_5(X_2) = -\hat{q}X_3X_4$$

$$\tau_6(X_1) = X_1, \qquad \tau_6(X_2) = qX_2, \qquad \tau_6(X_3) = q^{-1}X_3$$
$$\tau_6(X_4) = qX_4, \qquad \tau_6(X_5) = q^{-1}X_5, \qquad \delta_6(X_2) = -qX_3, \qquad \delta_6(X_4) = -qX_5$$

Furthermore, for $4 \leq i \leq 6$, $\tau_i \circ \delta_i = q^{-2}\delta_i \circ \tau_i$, so the theory of deleting derivations of [9] applies to $U_q(\mathfrak{sl}_4^+)$. In particular, as shown in [23], all prime ideals of $U_q(\mathfrak{sl}_4^+)$ are completely prime.

1.5 Normal elements and the center

The elements $a, b \in U_q(\mathfrak{sl}_4^+)$ are said to *q-commute* if there is an integer λ such that $ab = q^\lambda ba$. If u q-commutes with the generators e_i of $U_q(\mathfrak{sl}_4^+)$ then we say that u is *q-central*. Clearly, q-central elements are normal and Caldero [8, Prop. 2.1] has shown the reciprocal of this statement, so that the normal elements of $U_q(\mathfrak{sl}_4^+)$ are just the q-central ones.

The following theorem was established (in the more general context of $U_q(\mathfrak{sl}_n^+)$) independently by Alev and Dumas [3] and by Caldero [7, 8].

Theorem 1.1. *There exist q-central weight elements $\Delta_i \in U_q(\mathfrak{sl}_4^+)$, $i = 1, 2, 3$, such that:*

(a) Δ_2 *is central and*

  (i) e_2 *commutes with* Δ_i*, for all* $i = 1, 2, 3$*;*

  (ii) $e_1\Delta_1 = q\Delta_1 e_1$, $e_1\Delta_3 = q^{-1}\Delta_3 e_1$*;*

  (iii) $e_3\Delta_1 = q^{-1}\Delta_1 e_3$, $e_3\Delta_3 = q\Delta_3 e_3$*;*

(b) *The subalgebra* $\mathbb{K}[\Delta_1, \Delta_2, \Delta_3]$ *generated by the* Δ_i *is a (commutative) polynomial algebra in* 3 *variables.*

(c) *The center* $Z(\mathfrak{sl}_4^+)$ *of* $U_q(\mathfrak{sl}_4^+)$ *is the polynomial algebra in the variables* $z_1 = \Delta_1\Delta_3$ *and* $z_2 = \Delta_2$.

The set of q-central elements of $U_q(\mathfrak{sl}_4^+)$ was also described by Caldero (see for example [8, Thé. 2.2]) in terms of the Δ_i and the longest element of the Weyl group of $\mathfrak{sl}_4$ (in the notation of [8], $\Delta_i = e_{s(\varpi_{4-i})}$). It follows from his analysis that every q-central element is an element of $\mathbb{K}[\Delta_1, \Delta_2, \Delta_3]$. So let $p = \sum_j c_j\Theta_j$ be q-central, with each $c_j \in \mathbb{K}^*$ and the Θ_j distinct monomials in the Δ_i. Take $\lambda \in \mathbb{Z}$ so that $e_1 p = q^\lambda p e_1$. By Theorem 1.1 (a), each Θ_j is q-central, so it must be that $e_1\Theta_j = q^\lambda\Theta_j e_1$ for all j, as $U_q(\mathfrak{sl}_4^+)$ is a domain and the Θ_j are distinct. Assume $\lambda \geq 0$ and write $\Theta_j = \Delta_1^\alpha\Delta_2^\beta\Delta_3^\gamma$. Then, once more by Theorem 1.1(a), $\lambda = \alpha - \gamma$ and so $\Theta_j = \Delta_1^\lambda u_j$ with $u_j = z_1^\gamma z_2^\beta$ central. Since j was arbitrary, we deduce that p is the product of Δ_1^λ and a central element. Had we assumed $\lambda \leq 0$, we would have obtained an analogous statement with Δ_1^λ replaced by $\Delta_3^{-\lambda}$. Conversely, it is clear that all elements of $\Delta_i^c Z(\mathfrak{sl}_4^+)$ are q-central, for $c \in \mathbb{N}$ and $i \in \{1, 3\}$, so we have established the following:

Lemma 1.2. *Let $u \in U_q(\mathfrak{sl}_4^+)$ be normal. Then there exists a central element z, a nonnegative integer c and $i \in \{1, 3\}$ such that $u = \Delta_i^c z$.*

In terms of the PBW basis we are using, the Δ_i are given by the formulae (see [7, Sec. 4] or [20, Sec. 4.1] but notice that we have ordered the PBW basis elements differently):

$$\Delta_1 = X_3, \tag{4}$$
$$\Delta_2 = X_2X_5 - qX_3X_4, \tag{5}$$
$$\Delta_3 = \hat{q}^{\,2}X_1X_4X_6 - q\hat{q}X_2X_6 - q\hat{q}X_1X_5 + q^2X_3. \tag{6}$$

2 The automorphism group of $U_q(\mathfrak{sl}_4^+)$

In this section we compute the group of algebra automorphisms of $U_q(\mathfrak{sl}_4^+)$ and confirm the conjecture of Andruskiewitsch and Dumas [5] for this case. Let $\mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ denote this group. We shall show that $\mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ is the semi-direct product of the 3-torus $(\mathbb{K}^*)^3$ and the group of order two generated by the diagram automorphism of $U_q(\mathfrak{sl}_4^+)$.

Let $\mathcal{H} = (\mathbb{K}^*)^3$. Each $\bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \mathcal{H}$ determines an algebra automorphism $\phi_{\bar{\lambda}}$ of $U_q(\mathfrak{sl}_4^+)$ with $\phi_{\bar{\lambda}}(e_i) = \lambda_i e_i$ for $i = 1, 2, 3$, with inverse $\phi_{\bar{\lambda}}^{-1} = \phi_{\bar{\lambda}^{-1}}$. Hence we think of $\mathcal{H}$ as a subgroup of $\mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ via this correspondence. There is also a diagram automorphism η of $U_q(\mathfrak{sl}_4^+)$ arising from the symmetry of the Dynkin diagram of type A, and defined on the generators by $\eta(e_i) = e_{4-i}$. Notice that η^2 is the identity morphism and that, up to nonzero scalars, η permutes Δ_1 and Δ_3, and fixes Δ_2. Finally, as is to be expected,

$$\eta \circ \phi_{(\lambda_1, \lambda_2, \lambda_3)} \circ \eta^{-1} = \phi_{(\lambda_3, \lambda_2, \lambda_1)}. \tag{7}$$

2.1 An $\mathbb{N}$-grading on $U_q(\mathfrak{sl}_4^+)$

In addition to the weight space decomposition of Section 1.2, $U_q(\mathfrak{sl}_4^+)$ has an $\mathbb{N}$-grading induced by the monoid homomorphism $a\alpha_1 + b\alpha_2 + c\alpha_3 \mapsto a + b + c$, from Q^+ to $\mathbb{N}$. Let

$$U_q(\mathfrak{sl}_4^+) = \bigoplus_{i \in \mathbb{N}} U_i \tag{8}$$

be the corresponding decomposition, with U_i the subspace of homogeneous elements of degree i. In particular, $U_0 = \mathbb{K}$ and U_1 is the 3-dimensional space spanned by the generators e_1, e_2, e_3. For $t \in \mathbb{N}$ set $U_{\geq t} = \bigoplus_{i \geq t} U_i$ and define $U_{\leq t}$ similarly.

We say that the nonzero element $u \in U_q(\mathfrak{sl}_4^+)$ has degree t, and write $\deg(u) = t$, if $u \in U_{\leq t} \setminus U_{\leq t-1}$ (using the convention that $U_{\leq -1} = \{0\}$). In such a case, if $u = \sum_{0 \leq i \leq t} u_i$ with $u_i \in U_i$ and $u_t \neq 0$, we set $\bar{u} = u_t$. By definition, $\bar{u} \neq 0$, $\overline{uv} = \bar{u}\bar{v}$ and $\deg(uv) = \deg(u) + \deg(v)$ for $u, v \neq 0$, as $U_q(\mathfrak{sl}_4^+)$ is a domain.

The hypotheses of [19, Prop. 3.2] can be slightly weakened to yield, with essentially the same proof, the following proposition.

Proposition 2.1. *Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}$-graded $\mathbb{K}$-algebra with $A_0 = \mathbb{K}$ which is generated as an algebra by $A_1 = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$. Assume that for each $i \in \{1, \ldots, n\}$ there exist $0 \neq a \in A$ and a scalar $q_{i,a} \neq 1$ such that $x_i a = q_{i,a} a x_i$. Then, given an algebra automorphism σ of A and a nonzero homogeneous element x of degree d, there exist $y_d \in A_d \setminus \{0\}$ and $y_{>d} \in A_{\geq d+1}$ so that $\sigma(x) = y_d + y_{>d}$.*

The algebra $U_q(\mathfrak{sl}_4^+)$, endowed with the grading just defined, satisfies the conditions of the above proposition. Indeed, the quantum Serre relations involving i and $i+1$ are equivalent to

$$e_i \left(e_i e_{i+1} - q^{-1} e_{i+1} e_i\right) = q \left(e_i e_{i+1} - q^{-1} e_{i+1} e_i\right) e_i \tag{9}$$

$$e_{i+1} \left(e_i e_{i+1} - q^{-1} e_{i+1} e_i\right) = q^{-1} \left(e_i e_{i+1} - q^{-1} e_{i+1} e_i\right) e_{i+1}. \tag{10}$$

Thus we have an analogue of [19, Cor. 3.3]:

Corollary 2.2. *Let $\sigma \in \mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ and $x \in U_d \setminus \{0\}$. Then $\sigma(x) = y_d + y_{>d}$, for some $y_d \in U_d \setminus \{0\}$ and $y_{>d} \in U_{\geq d+1}$.*

2.2 Invariance of the normal elements

Proposition 2.3. *Given $\sigma \in \mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$, there exist $\epsilon \in \{0, 1\}$ and nonzero scalars μ_1 and μ_3 such that $\eta^\epsilon \circ \sigma(\Delta_i) = \mu_i \Delta_i$ for $i = 1, 3$.*

Proof. Since Δ_1 is normal, so is $\sigma(\Delta_1)$. By Lemma 1.2 there exist $i \in \{1,3\}$, $c \in \mathbb{N}$ and a central element z such that $\sigma(\Delta_1) = \Delta_i^c z$. Furthermore, $c \geq 1$ as Δ_1 is not central. It follows from Corollary 2.2 that $c = 1$, as $\deg(\Delta_j) = 3$ for $j = 1, 3$. Thus,

$$\sigma(\Delta_1) = \Delta_i z. \tag{11}$$

If we repeat the argument above replacing Δ_1 by Δ_i and σ by its inverse, apply σ^{-1} to equation (11) and compute degrees, we find that z is a (nonzero) scalar. Similarly, $\sigma(\Delta_3)$ is a nonzero scalar multiple of Δ_j for some $j \in \{1,3\}$ with $j \neq i$. If $i = 1$ and $j = 3$, we take $\epsilon = 0$; if $i = 3$ and $j = 1$, we take $\epsilon = 1$. In either case, as η interchanges Δ_1 and Δ_3, $\eta^\epsilon \circ \sigma$ fixes Δ_1 and Δ_3 up to scalars. □

Remark. The normal element Δ_1 generates a completely prime ideal of $U_q(\mathfrak{sl}_4^+)$, hence so does $\sigma(\Delta_1)$. This observation also leads to the conclusion that $z \in \mathbb{K}^*$ in (11).

We have as a corollary of Proposition 2.3 that any algebra automorphism of $U_q(\mathfrak{sl}_4^+)$ acts on the central element $z_1 = \Delta_1\Delta_3$ as multiplication by a scalar. Since the center of $U_q(\mathfrak{sl}_4^+)$ is $\mathbb{K}[z_1, z_2]$ with $z_2 = \Delta_2$ and any $\sigma \in \mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ induces an automorphism of this polynomial algebra, it is not hard to see that $\sigma(\Delta_2) = \lambda\Delta_2 + p(z_1)$ with $\lambda \in \mathbb{K}^*$ and $p(z_1)$ a polynomial in z_1 with zero constant term (by Corollary 2.2). Unfortunately, this is not quite sufficient. In fact, if – as we claim – $\mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ is the semi-direct product of $\mathcal{H}$ and the order 2 group generated by η, it must be that $p(z_1) = 0$. Our next result, preceded by a preparatory lemma, provides this step.

Lemma 2.4. *For any $\sigma \in \mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$ there exist $\epsilon \in \{0,1\}$ and $\bar{\lambda} \in \mathcal{H}$ such that*

$$(\phi_{\bar{\lambda}} \circ \eta^\epsilon \circ \sigma - Id)(U_1) \subseteq U_{\geq 2}. \tag{12}$$

Proof. By Proposition 2.3, $\eta^\epsilon \circ \sigma(\Delta_1) = t\Delta_1$, for some $\epsilon \in \{0,1\}$ and $t \in \mathbb{K}^*$. Let $\psi = \eta^\epsilon \circ \sigma$. By Corollary 2.2, there exist $u_1 \in U_1 \setminus \{0\}$ and $u_{>1} \in U_{\geq 2}$ such that $\psi(e_1) = u_1 + u_{>1}$. If now we apply ψ to the relation $e_1\Delta_1 = q\Delta_1 e_1$ and equate the homogeneous terms of degree 4, we obtain $u_1\Delta_1 = q\Delta_1 u_1$. As u_1 is a linear combination of e_1, e_2 and e_3, Theorem 1.1(a) implies that $u_1 = \lambda_1 e_1$ for some $\lambda_1 \in \mathbb{K}^*$. Analogously, $\psi(e_i) = \lambda_i e_i + w_i$ for $\lambda_i \in \mathbb{K}^*$ and $w_i \in U_{\geq 2}$, $i = 2, 3$. Let $\bar{\lambda} = (\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1})$. Then $(\phi_{\bar{\lambda}} \circ \psi - Id)(U_1) \subseteq U_{\geq 2}$, since $\phi_{\bar{\lambda}}(U_{\geq 2}) \subseteq U_{\geq 2}$. □

Theorem 2.5. *Let σ be an algebra automorphism of $U_q(\mathfrak{sl}_4^+)$. Then there is a nonzero scalar $\mu_2 \in \mathbb{K}^*$ such that $\sigma(\Delta_2) = \mu_2\Delta_2$.*

Proof. Since the statement of the theorem is valid for the automorphisms η and $\phi_{\bar{\lambda}}$, $\bar{\lambda} \in \mathcal{H}$, we can assume by the previous lemma that $(\sigma - Id)(U_1) \subseteq U_{\geq 2}$. Thus, by [2, Lem. 1.4.2], there exist $\mathrm{d}_l \in \mathrm{D}(U_q(\mathfrak{sl}_4^+))$, $l \geq 0$, such that

$$\sigma(\Delta_2) = \sum_{l \geq 0} \mathrm{d}_l(\Delta_2), \tag{13}$$

where $\mathrm{D}(U_q(\mathfrak{sl}_4^+))$ is the $\mathbb{K}$-subalgebra of $\mathrm{End}_{\mathbb{K}}\left(U_q(\mathfrak{sl}_4^+)\right)$ generated by the $\mathbb{K}$-derivations of $U_q(\mathfrak{sl}_4^+)$. Furthermore, $\mathrm{d}_0(\Delta_2) = \Delta_2$ and $\mathrm{d}_l(\Delta_2)$ is the homogeneous component of $\sigma(\Delta_2)$ of degree $l + 4$, as Δ_2 is homogeneous of degree 4.

In Section 3 it will be shown (see Theorem 3.8) that $\delta(\Delta_2)$ is in the ideal of $U_q(\mathfrak{sl}_4^+)$ generated by Δ_2, for any derivation δ of $U_q(\mathfrak{sl}_4^+)$, and this will be done independently of Theorem 2.5. Therefore, $\mathrm{d}(\Delta_2) \in (\Delta_2)$ for all $\mathrm{d} \in \mathrm{D}(U_q(\mathfrak{sl}_4^+))$ and thus $\sigma(\Delta_2) \in (\Delta_2)$, by (13). This same reasoning applies to σ^{-1}, so that $(\sigma(\Delta_2)) = (\Delta_2)$. Since Δ_2 is central, it is then obvious that there exists a unit $\mu_2 \in U_q(\mathfrak{sl}_4^+)$ such that $\sigma(\Delta_2) = \mu_2\Delta_2$. However, the set of units of $U_q(\mathfrak{sl}_4^+)$ is precisely $\mathbb{K}^*$, so that $\mu_2 \in \mathbb{K}^*$, as desired. □

2.3 Determination of $\mathrm{Aut}_{\mathbb{K}} U_q(\mathfrak{sl}_4^+)$

We are now ready to compute the group of algebra automorphisms of $U_q(\mathfrak{sl}_4^+)$.

Proposition 2.6. *Let ψ be an algebra automorphism of $U_q(\mathfrak{sl}_4^+)$ with the property that $(\psi - Id)(U_1) \subseteq U_{\geq 2}$. Then ψ is the identity morphism.*

Proof. By the hypothesis on ψ, there exist $u_i \in U_{\geq(\deg(X_i)+1)}$ such that

$$\psi(X_i) = X_i + u_i$$

for all $1 \leq i \leq 6$. Also, by Proposition 2.3 and Theorem 2.5, we know that $\psi(\Delta_j) = \Delta_j$ for $j = 1, 2, 3$. In particular, $u_3 = 0$ as $\Delta_1 = X_3$. Define, for $1 \leq i \leq 6$, $d_i = \deg(\psi(X_i))$. It is enough to prove that $d_1 = d_4 = d_6 = 1$ as $X_1 = e_1$, $X_4 = e_2$ and $X_6 = e_3$ generate $U_q(\mathfrak{sl}_4^+)$ as an algebra. Let us assume, by way of contradiction, that this is not the case. Thus $d_1 + d_4 + d_6 > 3$.

Notice that by Corollary 2.2, $d_i \geq \deg(X_i)$ for all i. Looking at the expression (5) of Δ_2 in the PBW basis and using the fact that ψ fixes Δ_2, we can conclude that

$$d_2 + d_5 = d_3 + d_4 = 3 + d_4. \tag{14}$$

Also, since X_2 is a linear combination of X_1X_4 and X_4X_1, we have $2 \leq d_2 \leq d_1 + d_4$ and similarly $2 \leq d_5 \leq d_4 + d_6$. Therefore,

$$d_1 + d_4 + d_6 \geq \max\{d_2 + d_6, d_1 + d_5\} \qquad \text{and} \tag{15}$$
$$d_1 + d_4 + d_6 > 3 = d_3. \tag{16}$$

Since ψ fixes the degree 3 element Δ_3, the inequality in (15) cannot be strict, by (6). Hence either $d_1 + d_4 + d_6 = d_2 + d_6$ or $d_1 + d_4 + d_6 = d_1 + d_5$. These cases are symmetric and we can assume without loss of generality that $d_1 + d_4 + d_6 = d_2 + d_6$. Thus, using (14), $d_1 + d_4 = d_2 = 3 + d_4 - d_5$ and $d_1 + d_5 = 3$. Since $d_1 \geq 1$ and $d_5 \geq 2$, it must be $d_1 = 1$ and $d_5 = 2$. In other words, $u_1 = 0 = u_5$ and ψ fixes X_1 and X_5.

Now we apply ψ to the defining equation (5) of Δ_2 to obtain

$$u_2X_5 = qX_3u_4; \tag{17}$$

similarly, the relation $X_5X_4 = q^{-1}X_4X_5$ yields

$$X_5u_4 = q^{-1}u_4X_5 \tag{18}$$

after applying ψ; finally, ψ applied to equation (6) gives

$$\hat{q}\,(X_1X_4u_6 + X_1u_4X_6 + X_1u_4u_6) = q\,(X_2u_6 + u_2X_6 + u_2u_6)\,. \tag{19}$$

By (17), $u_2 = 0 \iff u_4 = 0$ and if this occurs then $\hat{q}X_1X_4u_6 = qX_2u_6$, on account of (19). If $u_6 \neq 0$ the latter implies $\hat{q}X_1X_4 = qX_2$, which is false as the X_i form a PBW basis. Thus $u_6 = 0$ and $d_1+d_4+d_6 = 3$, contradicting our assumption. Hence $u_4, u_2 \neq 0$. Likewise, if $u_6 = 0$ then (19) implies $\hat{q}X_1u_4 = qu_2$ and then by (17) followed by (18) we get $\hat{q}X_1X_5u_4 = qX_3u_4$, which is again a contradiction as $u_4 \neq 0$. Hence $d_2 = \deg(u_2) \geq 3$, $d_4 = \deg(u_4) \geq 2$ and $d_6 = \deg(u_6) \geq 2$.

To obtain the final contradiction, we just have to look at the degrees occurring in (19). Indeed, $\deg(X_1X_4u_6) = 2 + d_6 < 1 + d_4 + d_6 = \deg(X_1u_4u_6)$; similarly, $\deg(X_1u_4X_6) < \deg(X_1u_4u_6)$, $\deg(X_2u_6) < \deg(u_2u_6)$ and $\deg(u_2X_6) < \deg(u_2u_6)$. Therefore we must have $\deg(X_1u_4u_6) = \deg(u_2u_6)$ and, using the notation introduced in section 2.1,

$$\hat{q}\,X_1\bar{u}_4\bar{u}_6 = q\,\bar{u}_2\bar{u}_6, \tag{20}$$

so that $\hat{q}\,X_1\bar{u}_4 = q\,\bar{u}_2$. Multiplying this equation on the right by X_5, using relations $\bar{u}_2X_5 = qX_3\bar{u}_4$ and $\bar{u}_4X_5 = qX_5\bar{u}_4$, arising from (17) and (18), respectively, we obtain the equality $\hat{q}\,X_1X_5\bar{u}_4 = qX_3\bar{u}_4$, which leads to the contradiction $\hat{q}\,X_1X_5 = qX_3$. The contradiction was derived from the assumption that $d_1 + d_4 + d_6 > 3$. Consequently $d_1 = d_4 = d_6 = 1$ and ψ is the identity on $U_q(\mathfrak{sl}_4^+)$. □

At last, we prove our main result of this section, which gives a positive answer to the conjecture of Andruskiewitsch and Dumas [5] for $U_q(\mathfrak{sl}_4^+)$.

Theorem 2.7. $\mathrm{Aut}_{\mathbb{K}}U_q(\mathfrak{sl}_4^+)$ *is isomorphic to the semi-direct product of the* 3*-torus* $\mathcal{H}$ *and the group of order* 2 *generated by the diagram automorphism* η *of* $U_q(\mathfrak{sl}_4^+)$*.*

Proof. Let $\sigma \in \mathrm{Aut}_{\mathbb{K}}U_q(\mathfrak{sl}_4^+)$. By Lemma 2.4 and Proposition 2.6 there exist $\epsilon \in \{0,1\}$ and $\bar{\lambda} \in \mathcal{H}$ such that $\phi_{\bar{\lambda}} \circ \eta^{\epsilon} \circ \sigma$ is the identity on $U_q(\mathfrak{sl}_4^+)$. Thus,

$$\sigma = \eta^{\epsilon} \circ \phi_{\bar{\mu}}, \tag{21}$$

where $\bar{\mu} = \bar{\lambda}^{-1}$. Furthermore, the above expression is easily seen to be unique, so the theorem follows from (7). $\square$

3 Derivations of $U_q(\mathfrak{sl}_4^+)$

The aim of this section is to describe the Lie algebra of $\mathbb{K}$-derivations of $U_q(\mathfrak{sl}_4^+)$. In particular, we show that the Hochschild cohomology group of degree 1 of $U_q(\mathfrak{sl}_4^+)$ is a free module of rank 3 over the center of $U_q(\mathfrak{sl}_4^+)$. Our method consists of using previous results of Osborn and Passman, [21], on the the Hochschild cohomology group of degree 1 of a quantum torus, and then to use the theory of deleting derivations of Cauchon (see [9]) in order to transfer information on the derivations of a certain quantum torus (in which $U_q(\mathfrak{sl}_4^+)$ embeds) to the derivations of $U_q(\mathfrak{sl}_4^+)$ itself. This method was first used in [18] in order to describe the derivations of the algebra of quantum matrices and of some related algebras.

3.1 The deleting derivations algorithm in $U_q(\mathfrak{sl}_4^+)$

It follows from Section 1.4 that the theory of deleting derivations (see [9]) can be applied to the iterated Ore extension $R := U_q(\mathfrak{sl}_4^+) = \mathbb{K}[X_1]\dots[X_6;\tau_6,\delta_6]$. The corresponding deleting derivations algorithm constructs, for each $r \in \{6,5,4,3,2\}$, a family $(X_i^{(r)})_{i\in\{1,\dots,6\}}$ of elements of $\mathrm{Frac}(U_q(\mathfrak{sl}_4^+))$, defined as follows (see [9, Sec. 3.2]):

1. $X_1^{(6)} = X_1$, $X_2^{(6)} = X_2 - q\hat{q}^{-1}X_3X_6^{-1}$, $X_3^{(6)} = X_3$, $X_4^{(6)} = X_4 - q\hat{q}^{-1}X_5X_6^{-1}$, $X_5^{(6)} = X_5$ and $X_6^{(6)} = X_6$.

   In order to simplify the notations, we set $Y_i := X_i^{(6)}$ for all $i \in \{1,\dots,6\}$.

2. $X_1^{(5)} = Y_1 - q\hat{q}^{-1}Y_3Y_5^{-1}$, $X_2^{(5)} = Y_2 - qY_3Y_4Y_5^{-1}$, $X_3^{(5)} = Y_3$, $X_4^{(5)} = Y_4$, $X_5^{(5)} = Y_5$ and $X_6^{(5)} = Y_6$.

   In order to simplify the notations, we set $Z_i := X_i^{(5)}$ for all $i \in \{1,\dots,6\}$.

3. $X_1^{(4)} = Z_1 - q\hat{q}^{-1}Z_2Z_4^{-1}$, $X_2^{(4)} = Z_2$, $X_3^{(4)} = Z_3$, $X_4^{(4)} = Z_4$, $X_5^{(4)} = Z_5$ and $X_6^{(4)} = Z_6$.

   In order to simplify the notations, we set $T_i := X_i^{(4)}$ for all $i \in \{1,\dots,6\}$.

4. For all $r \in \{2,3\}$ and $i \in \{1,\dots,6\}$, $X_i^{(r)} = T_i$.

As in [9], for all $r \in \{6,5,4,3,2\}$, we denote by $R^{(r)}$ the subalgebra of $\mathrm{Frac}(R)$ generated by the elements $X_i^{(r)}$ for $i \in \{1,\dots,6\}$. Also, we denote by $\overline{R}$ the subalgebra of $\mathrm{Frac}(R)$ generated by the indeterminates obtained at the end of this algorithm, that is, $\overline{R} = R^{(2)}$ is the subalgebra of $\mathrm{Frac}(R)$ generated by the T_i, for each $i \in \{1,\dots,6\}$. Finally, by convention, we set $R^{(7)} := R$.

Recall from [9, Thé. 3.2.1] that, for all $r \in \{6,5,4,3,2\}$, $R^{(r)}$ can be presented as an iterated Ore extension over $\mathbb{K}$, with the generators $X_i^{(r)}$ adjoined in lexicographic order. Thus the ring $R^{(r)}$ is a Noetherian domain. Observe in particular that we have (with some abuse of notation):

$$R^{(6)} = \mathbb{K}[Y_1][Y_2;\tau_2][Y_3;\tau_3][Y_4;\tau_4,\delta_4][Y_5;\tau_5,\delta_5][Y_6;\tau_6], \tag{22}$$

$$R^{(5)} = \mathbb{K}[Z_1][Z_2;\tau_2][Z_3;\tau_3][Z_4;\tau_4,\delta_4][Z_5;\tau_5][Z_6;\tau_6], \tag{23}$$

$$\overline{R} = R^{(4)} = R^{(3)} = R^{(2)} = \mathbb{K}[T_1][T_2;\tau_2][T_3;\tau_3][T_4;\tau_4][T_5;\tau_5][T_6;\tau_6]. \tag{24}$$

Let $N \in \mathbb{N}^*$ and let $\Lambda = (\Lambda_{i,j})$ be a multiplicatively antisymmetric $N \times N$ matrix over $\mathbb{K}^*$; that is, $\Lambda_{i,i} = 1$ and $\Lambda_{j,i} = \Lambda_{i,j}^{-1}$ for all $i,j \in \{1,\dots,N\}$. We denote by $\mathbb{K}_\Lambda[T_1,\dots,T_N]$ the corresponding algebra of regular functions on the quantum affine space; that is, the $\mathbb{K}$-algebra generated by the N indeterminates $T_1,\dots,T_N$ subject to the relations $T_iT_j = \Lambda_{i,j}T_jT_i$ for all $i,j \in \{1,\dots,N\}$. Next, we denote by $P(\Lambda)$ the quantum torus associated to $\mathbb{K}_\Lambda[T_1,\dots,T_N]$, which is the localisation of $\mathbb{K}_\Lambda[T_1,\dots,T_N]$ with respect to the multiplicative system generated by the T_i. For $\gamma = (\gamma_1,\dots,\gamma_N) \in \mathbb{Z}^N$, set $T^\gamma := T_1^{\gamma_1}\dots T_N^{\gamma_N}$. Note that the monomials $(T^\gamma)_{\gamma\in\mathbb{Z}^N}$ form a PBW basis of $P(\Lambda)$.

It follows from [9, Prop. 3.2.1] that $\overline{R}$ is the algebra of regular functions on a quantum affine space over $\mathbb{K}$, given by indeterminates $T_1,\dots,T_6$. We denote by $P(\Lambda)$ the corresponding quantum torus. In the present case, the matrix that defines $\overline{R}$ is the following:

$$\Lambda = \begin{pmatrix} 1 & q & q & q^{-1} & q^{-1} & 1 \\ q^{-1} & 1 & q & q & 1 & q^{-1} \\ q^{-1} & q^{-1} & 1 & 1 & q & q \\ q & q^{-1} & 1 & 1 & q & q^{-1} \\ q & 1 & q^{-1} & q^{-1} & 1 & q \\ 1 & q & q^{-1} & q & q^{-1} & 1 \end{pmatrix}$$

For all $r \in \{6,5,4,3,2\}$, we denote by S_r the multiplicative system generated by the indeterminates T_i with $i \geq r$. Since $T_i = X_i^{(r)}$ for all $i \geq r$, S_r is a multiplicative system of regular elements of $R^{(r)}$. Moreover, the T_i with $i \geq r$ are normal in $R^{(r)}$. Hence S_r is an Ore set in $R^{(r)}$ and one can form the localisation:

$$A_r := R^{(r)}S_r^{-1}.$$

Clearly, the family $\left((X_1^{(r)})^{\gamma_1}(X_2^{(r)})^{\gamma_2}\dots(X_6^{(r)})^{\gamma_6}\right)$, with $\gamma_i \in \mathbb{N}$ if $i < r$ and $\gamma_i \in \mathbb{Z}$ otherwise, is a PBW basis of A_r. Further, recall from [9, Thé. 3.2.1] that $\Sigma_r := \{T_r^k \mid k \in \mathbb{N}\}$ is an Ore set in both $R^{(r)}$ and $R^{(r+1)}$, and that

$$R^{(r)}\Sigma_r^{-1} = R^{(r+1)}\Sigma_r^{-1}.$$

Hence we get the following result.

Lemma 3.1. *For all $r \in \{6,5,4,3,2\}$, we have $A_r = A_{r+1}\Sigma_r^{-1}$ with the convention that $A_7 := R = U_q(\mathfrak{sl}_4^+)$.*

Now, observe that T_1 is a normal element in A_2, so that one can form the Ore localisation $A_1 := A_2\Sigma_1^{-1}$, where Σ_1 is the multiplicative system generated by T_1. Naturally, A_1 is the quantum torus associated to $\overline{R}$. Hence we also denote A_1 by $P(\Lambda)$, and we deduce from Lemma 3.1 the following tower of algebras:

$$A_7 = R \quad\subset\quad A_6 = A_7\Sigma_6^{-1} \subset A_5 = A_6\Sigma_5^{-1} \subset A_4 = A_5\Sigma_4^{-1} \tag{25}$$

$$\subset\quad A_3 = A_4\Sigma_3^{-1} \subset A_2 = A_3\Sigma_2^{-1} \subset A_1 := P(\Lambda). \tag{26}$$

3.2 Action of the deleting derivations algorithm on the normal elements

Observe that the formulas expressing the Y_i in terms of the X_i can be rewritten in order to express the X_i in terms of the Y_i. In particular, one can easily check that:
$X_1 = Y_1$, $X_2 = Y_2 + q\hat{q}^{-1}Y_3Y_6^{-1}$, $X_3 = Y_3$, $X_4 = Y_4 + q\hat{q}^{-1}Y_5Y_6^{-1}$, $X_5 = Y_5$ and $X_6 = Y_6$.

In a similar manner, one can express the Y_i in terms of the Z_i, and the Z_i in terms of the T_i. More precisely, we have:
$Y_1 = Z_1 + q\hat{q}^{-1}Z_3Z_5^{-1}$, $Y_2 = Z_2 + qZ_3Z_4Z_5^{-1}$, $Y_3 = Z_3$, $Y_4 = Z_4$, $Y_5 = Z_5$ and $Y_6 = Z_6$
and
$Z_1 = T_1 + q\hat{q}^{-1}T_2T_4^{-1}$, $Z_2 = T_2$, $Z_3 = T_3$, $Z_4 = T_4$, $Z_5 = T_5$ and $Z_6 = T_6$.

Using these formulas, one can express the three normal elements Δ_1, Δ_2 and Δ_3 defined in Section 1.5 in terms of the Y_i, or in terms of the Z_i, or in terms of the T_i. Indeed, straightforward computations lead to the following results.

Lemma 3.2. *1.* $\Delta_1 = X_3 = Y_3 = Z_3 = T_3$.

2. $\Delta_2 = X_2X_5 - qX_3X_4 = Y_2Y_5 - qY_3Y_4 = Z_2Z_5 = T_2T_5$.

3.

$$\begin{aligned}\Delta_3 &= \hat{q}^{\,2}X_1X_4X_6 - q\hat{q}X_2X_6 - q\hat{q}X_1X_5 + q^2X_3\\ &= \hat{q}^{\,2}Y_1Y_4Y_6 - q\hat{q}Y_2Y_6\\ &= \hat{q}^{\,2}Z_1Z_4Z_6 - q\hat{q}Z_2Z_6\\ &= \hat{q}^{\,2}T_1T_4T_6\end{aligned}$$

3.3 Centers of the algebras A_i

First, recall that the center of $U_q(\mathfrak{sl}_4^+) = A_7$ has been computed by Alev and Dumas [3] and by Caldero [7, 8], who have shown that this is the polynomial algebra $\mathbb{K}[z_1, z_2]$, where $z_1 = \Delta_1\Delta_3$ and $z_2 = \Delta_2$.

On the other hand, the center of the quantum torus $A_1 = P(\Lambda)$ is easy to compute. Indeed, it is well known (see for instance [13]) that it is a Laurent polynomial ring over $\mathbb{K}$, and that it is generated by the monomials $T_1^{\gamma_1}T_2^{\gamma_2}\ldots T_6^{\gamma_6}$, with $\gamma_i \in \mathbb{Z}$, that are central. Easy computations show that such a monomial is central if and only if $\gamma_1 = \gamma_4 = \gamma_6 = \gamma_3$ and $\gamma_2 = \gamma_5$. Hence, we deduce from Lemma 3.2 that the center of $P(\Lambda)$ is the Laurent polynomial ring over $\mathbb{K}$ generated by z_1 and z_2, that is:

$$Z(P(\Lambda)) = Z(A_1) = \mathbb{K}[z_1^{\pm1}, z_2^{\pm1}].$$

It will be convenient to denote by $\mathcal{F}$ the set of all $\gamma \in \mathbb{Z}^6$ such that $T^\gamma \in Z(P(\Lambda))$, that is:

$$\mathcal{F} = \{\gamma \in \mathbb{Z}^6 \mid \gamma_1 = \gamma_4 = \gamma_6 = \gamma_3 \text{ and } \gamma_2 = \gamma_5\}. \tag{27}$$

In the sequel we will also need to know the center of A_4. Recall that A_4 is the localisation at the multiplicative system generated by T_4, T_5 and T_6 of $R^{(4)} = \overline{R}$, the algebra of regular functions on the quantum affine space. In particular, the monomials $(T_1^{\gamma_1}T_2^{\gamma_2}\ldots T_6^{\gamma_6})$, with $\gamma_i \in \mathbb{N}$ if $i \leq 3$ and $\gamma_i \in \mathbb{Z}$ otherwise, form a linear basis of A_4. The argument used above to compute the center of $P(\Lambda)$ also works for A_4, with the additional restrictions that $\gamma_i \geq 0$ for $i \leq 3$. So we have the following result.

Lemma 3.3. *1.* $Z(A_4) = Z(A_7) = \mathbb{K}[z_1, z_2]$.

2. $Z(A_1) = \mathbb{K}[z_1^{\pm1}, z_2^{\pm1}]$.

3.4 Derivations of $U_q(\mathfrak{sl}_4^+)$

Our aim in this section is to investigate the Lie algebra of $\mathbb{K}$-derivations of $U_q(\mathfrak{sl}_4^+)$, which we denote by $\mathrm{Der}(U_q(\mathfrak{sl}_4^+))$.

Let D be a derivation of $U_q(\mathfrak{sl}_4^+) = A_7$. It follows from Lemma 3.1 that D extends (uniquely) to a derivation of each of the algebras in the tower

$$A_7 \subseteq A_6 \subseteq \cdots \subseteq A_2 \subseteq A_1 = P(\Lambda).$$

In particular, D extends to a derivation of the quantum torus $P(\Lambda)$. So it follows from [21, Cor. 2.3] that D can be written as

$$D = \mathrm{ad}_x + \theta,$$

where $x \in P(\Lambda)$ and, in the terminology of [21], θ is a central derivation of $P(\Lambda)$, that is, $\theta(T_i) = \mu_i T_i$ with $\mu_i \in Z(P(\Lambda)) = \mathbb{K}[z_1^{\pm 1}, z_2^{\pm 1}]$.

Since the monomials $(T^\gamma)_{\gamma \in \mathbb{Z}^6}$ form a PBW basis of $P(\Lambda)$, one can write:

$$x = \sum_{\gamma \in \mathcal{E}} c_\gamma T^\gamma,$$

where $\mathcal{E}$ is a finite subset of $\mathbb{Z}^6$ and $c_\gamma \in \mathbb{K}$. Moreover, since $\mathrm{ad}_x = \mathrm{ad}_{x+z}$ for all $z \in Z(P(\Lambda))$, it can be assumed that no monomial T^γ, with $\gamma \in \mathcal{E}$, belongs to $Z(P(\Lambda))$, i.e., one can assume that $\mathcal{E} \cap \mathcal{F} = \emptyset$. Furthermore, by Lemmas 3.2 and 3.3 we can write, for each $i \in \{1, \dots, 6\}$, μ_i as follows:

$$\mu_i = \sum_{\gamma \in \mathcal{F}} \mu_{i,\gamma} T^\gamma,$$

where $\mu_{i,\gamma} \in \mathbb{K}$.

Lemma 3.4. *For all $i \in \{1, 2, 3, 4\}$, we have $x \in A_i$.*

Proof. We prove this lemma by induction on i. The case $i = 1$ is trivial. Hence we assume that $x \in A_{i-1}$ for some $2 \leq i \leq 4$.

It follows that

$$x = \sum_{\gamma \in \mathcal{E}} c_\gamma T^\gamma,$$

where $\mathcal{E}$ is a finite subset of $\{\gamma \in \mathbb{Z}^6 \mid \gamma_1 \geq 0, \dots, \gamma_{i-2} \geq 0\}$ with $\mathcal{E} \cap \mathcal{F} = \emptyset$. We need to prove that $\gamma_{i-1} \geq 0$.

Let $j \in \{1, \dots, 6\}$ with $j \neq i-1$. As we have previously observed, D extends uniquely to a derivation of A_i. Hence, since $T_j \in A_i$, we must have $D(T_j) \in A_i$, that is:

$$xT_j - T_j x + \mu_j T_j \in A_i. \tag{28}$$

We set

$$x_+ := \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} \geq 0} c_\gamma T^\gamma,$$

and

$$x_- := \sum_{\gamma \in \mathcal{E}, \gamma_{i-1} < 0} c_\gamma T^\gamma. \tag{29}$$

We shall prove that $x_- = 0$.

First, we deduce from (28) that

$$u := x_- T_j - T_j x_- + \mu_j T_j \in A_i.$$

Next, using the commutation relations between the T_k, we get

$$u = \sum_{\gamma\in\mathcal{E},\gamma_{i-1}<0} c'_{j,\gamma} c_\gamma T^{\gamma+\varepsilon_j} + \sum_{\gamma\in\mathcal{F}} \mu'_{j,\gamma} T^{\gamma+\varepsilon_j} \tag{30}$$

where ε_j denotes the j-th element of the canonical basis of $\mathbb{Z}^6$, $\mu'_{j,\gamma} = q^\bullet \mu_{j,\gamma}$ for some integer $\bullet$, and $c'_{j,\gamma} \in \mathbb{K}$ is defined by

$$x_- T_j - T_j x_- = \sum_{\gamma\in\mathcal{E},\gamma_{i-1}<0} c'_{j,\gamma} c_\gamma T^{\gamma+\varepsilon_j}.$$

Observe that since we assume that $\mathcal{E}\cap\mathcal{F} = \emptyset$, we have:

$$\text{for all } \gamma\in\mathcal{E} \text{ and all } \gamma'\in\mathcal{F},\ \gamma+\varepsilon_j \neq \gamma'+\varepsilon_j.$$

Hence, (30) gives the expression of u in the PBW basis of $P(\Lambda)$.

On the other hand, since u belongs to A_i, we get that:

$$u = \sum_{\gamma\in\mathcal{E}'} x_\gamma T^\gamma,$$

where $\mathcal{E}'$ is a finite subset of $\{\gamma\in\mathbb{Z}^6 \mid \gamma_1 \geq 0,\dots,\gamma_{i-1}\geq 0\}$. Comparing the two expressions of u in the PBW basis of $P(\Lambda)$ leads to $c'_{j,\gamma}c_\gamma = 0$ for all $\gamma\in\mathcal{E}$ such that $\gamma_{i-1}<0$, as $j\neq i-1$. Hence, we have

$$x_- T_j - T_j x_- = \sum_{\gamma\in\mathcal{E},\gamma_{i-1}<0} c'_{j,\gamma} c_\gamma T^{\gamma+\varepsilon_j} = 0,$$

for all $j\neq i-1$. In other words, x_- commutes with those T_j such that $j\neq i-1$.

Now, recall from Lemma 3.2 that $z_1 = \Delta_1\Delta_3 = \hat{q}^2 T_1T_4T_6T_3$ and $z_2 = \Delta_2 = T_2T_5$ are central in $P(\Lambda)$, so that x_- commutes with those T_j such that $j\neq i-1$, and with $T_1T_4T_6T_3$ and T_2T_5. Naturally this implies that x_- also commutes with T_{i-1}, so that $x_-\in Z(P(\Lambda))$. Thus one can write x_- as follows:

$$x_- = \sum_{\gamma\in\mathcal{F}} d_\gamma T^\gamma. \tag{31}$$

As $\mathcal{E}\cap\mathcal{F}=\emptyset$, it follows from (29) and (31) that $x_-=0$, so that $x = x_+\in A_i$, as desired. □

In particular, it follows from Lemma 3.4 that $x\in A_4$. Since the derivation D of $U_q(\mathfrak{sl}_4^+)$ extends to a derivation of A_4, we must have $D(T_i)\in A_4$ for all $i\in\{1,\dots,6\}$. Hence

$$D(T_i) = xT_i - T_ix + \mu_iT_i \in A_4.$$

Since $x\in A_4$, this implies that $\mu_iT_i\in A_4$ for all $i\in\{1,\dots,6\}$. On the other hand, recall that μ_i is central in $P(\Lambda)$ and can be written as:

$$\mu_i = \sum_{\gamma\in\mathcal{F}} \mu_{i,\gamma}T^\gamma,$$

where $\mathcal{F}$ is given by (27). Hence we get

$$\begin{aligned}\mu_iT_i &= \sum_{\gamma\in\mathcal{F}} \mu'_{i,\gamma}T^{\gamma+\varepsilon_i}\\ &= \sum_{\gamma=(\gamma_1,\gamma_2)\in\mathbb{Z}^2} \mu'_{i,\gamma} T_1^{\gamma_1+\delta_{1i}} T_2^{\gamma_2+\delta_{2i}} T_3^{\gamma_1+\delta_{3i}} T_4^{\gamma_1+\delta_{4i}} T_5^{\gamma_2+\delta_{5i}} T_6^{\gamma_1+\delta_{6i}} \in A_4,\end{aligned}$$

where $\mu'_{i,\gamma} = q^\bullet\mu_{i,\gamma}$ for some integer $\bullet$.

Assume now that $i \neq 2$. Then, since the monomials T^γ, with $\gamma \in \mathbb{N}^3 \times \mathbb{Z}^3$, form a PBW basis of A_4 , we get that $\mu'_{i,\gamma} = 0$ if either $\gamma_1 < 0$ or $\gamma_2 < 0$. Hence μ_i can be written as follows:

$$\mu_i = \sum_{\gamma=(\gamma_1,\gamma_2)\in\mathbb{N}^2} c_{i,\gamma} T_1^{\gamma_1} T_2^{\gamma_2} T_3^{\gamma_1} T_4^{\gamma_1} T_5^{\gamma_2} T_6^{\gamma_1}.$$

In other words, $\mu_i \in \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_4^+)$ since $z_1 = \Delta_1\Delta_3 = \hat{q}^2 T_1T_4T_6T_3$ and $z_2 = \Delta_2 = T_2T_5$ by Lemma 3.2.

Finally, assume that $i = 2$. One cannot yet prove that $\mu_2 \in U_q(\mathfrak{sl}_4^+) = A_7$. However, one can prove the following weaker result: $\mu_2 z_2 \in \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_4^+)$. Indeed, we already know that $\mu_2 T_2 \in A_4$. Hence, it follows from Lemma 3.2 that $\mu_2 z_2 = \mu_2 T_2 T_5 \in A_4$. Further, $\mu_2 z_2$ is central in $P(\Lambda) \supset A_4$, so that $\mu_2 z_2 \in Z(A_4) = \mathbb{K}[z_1, z_2]$, as desired.

To sum up, we have just proved the following result.

Corollary 3.5. *1. $\mu_2 z_2 \in Z(A_4) = \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_4^+)$.*

2. For all $i \neq 2$, $\mu_i \in \mathbb{K}[z_1, z_2] \subseteq U_q(\mathfrak{sl}_4^+)$.

We now have to deal with localisation at elements which are not normal. We do this in three steps.

First, recall from Lemma 3.1 that $A_4 = A_5\Sigma_4^{-1}$, where Σ_4 is the multiplicative system generated by $T_4 = Z_4$. Recall also that the monomials $Z_1^{\gamma_1} \dots Z_6^{\gamma_6}$, with $\gamma = (\gamma_1, \dots, \gamma_6) \in \mathbb{N}^4 \times \mathbb{Z}^2$, form a PBW basis of A_5. Of course, this implies that the monomials $Z_1^{\gamma_1} \dots Z_6^{\gamma_6}$, with $\gamma \in \mathbb{N}^3 \times \mathbb{Z}^3$, form a PBW basis of A_4. In order to simplify the notation we set, as usual,

$$Z^\gamma := Z_1^{\gamma_1} Z_2^{\gamma_2} \dots Z_6^{\gamma_6}$$

for all $\gamma \in \mathbb{N}^3 \times \mathbb{Z}^3$.

Corollary 3.6. *$\mu_2 Z_2 \in A_5$.*

Proof. We know that $\mu_2 z_2 \in Z(A_4) = Z(A_5)$, so that $\mu_2 z_2 \in A_5$. Now the result follows from the facts that $z_2 = Z_2Z_5$ (Lemma 3.2) and that Z_5 is invertible in A_5. □

We are now able to prove that $x \in A_5$.

Lemma 3.7. *1. $x \in A_5$.*

2. $\mu_2 = \mu_1 + \mu_4 \in Z(\mathfrak{sl}_4^+)$, where $Z(\mathfrak{sl}_4^+)$ still denotes the center of $U_q(\mathfrak{sl}_4^+)$.

3. $D(Z_i) = \mathrm{ad}_x(Z_i) + \mu_i Z_i$ for all $i \in \{1, \dots, 6\}$.

Proof. We proceed in three steps.

- *Step 1: We prove that $x \in A_5$.*

It follows from Lemma 3.4 that x belongs to A_4, so that x can be written as follows:

$$x = \sum_{\gamma\in\mathcal{E}} c_\gamma Z^\gamma,$$

where $\mathcal{E} \subseteq \mathbb{N}^3 \times \mathbb{Z}^3$.
We set

$$x_+ := \sum_{\gamma\in\mathcal{E},\gamma_4\geq 0} c_\gamma Z^\gamma,$$

and

$$x_- := \sum_{\gamma\in\mathcal{E},\gamma_4<0} c_\gamma Z^\gamma.$$

Assume that $x_- \neq 0$.

We denote by B the subalgebra of A_4 generated by the Z_j with $j \neq 4$, Z_5^{-1} and Z_6^{-1}. Since Z_4 q-commutes with Z_5 and Z_6 in A_4, it is easy to check that A_4 is a free left B-module with basis $(Z_4^a)_{a\in\mathbb{Z}}$, so that one can write:

$$x_- = \sum_{a=a_0}^{-1} b_a Z_4^a$$

with $a_0 < 0$, $b_a \in B$ and $b_{a_0} \neq 0$. (Observe that this makes sense since we are assuming that $x_- \neq 0$.)

As D extends to a derivation of A_5, we have $D(Z_1) \in A_5$. Recalling from Section 3.2 that $Z_1 = T_1 + q\hat{q}^{-1}T_2T_4^{-1}$, this leads to:

$$x_-Z_1 - Z_1x_- + \mu_1 Z_1 + q\hat{q}^{-1}(\mu_2 - \mu_1 - \mu_4)Z_2Z_4^{-1} \in A_5.$$

Since $\mu_1 \in U_q(\mathfrak{sl}_4^+) \subset A_5$ by Corollary 3.5 and $Z_1 \in A_5$, we get

$$x_-Z_1 - Z_1x_- + q\hat{q}^{-1}(\mu_2 - \mu_1 - \mu_4)Z_2Z_4^{-1} \in A_5. \quad (32)$$

Then, multiplying this expression by Z_4 (on the right) yields

$$(x_-Z_1 - Z_1x_-)Z_4 + q\hat{q}^{-1}(\mu_2 - \mu_1 - \mu_4)Z_2 \in A_5.$$

Since μ_1 and μ_4 belong to $U_q(\mathfrak{sl}_4^+) \subset A_5$ and $\mu_2 Z_2 \in A_5$ by Corollary 3.6, this leads to

$$u := (x_-Z_1 - Z_1x_-)Z_4 \in A_5,$$

that is:

$$u = \sum_{a=a_0}^{-1} b_a Z_4^a Z_1 Z_4 - \sum_{a=a_0}^{-1} Z_1 b_a Z_4^{a+1} \in A_5.$$

Now, an easy induction shows that

$$Z_4^{-k}Z_1 = q^{-k}Z_1Z_4^{-k} + q[k]Z_2Z_4^{-k-1}$$

for every positive integer k. Hence we have

$$u = \sum_{a=a_0}^{-1} (q^a b_a Z_1 - Z_1 b_a)\, Z_4^{a+1} + \sum_{a=a_0}^{-1} q[-a] b_a Z_2 Z_4^a \in A_5.$$

Since A_5 is a free left B-module with basis $(Z_4^a)_{a\in\mathbb{N}}$ and $u \in A_5$, one can write

$$u = \sum_{a=0}^{k} u_a Z_4^a$$

with $k \in \mathbb{N}$ and $u_a \in B$. Comparison of these two expressions of u in the basis of A_4 (viewed as a left B-module) shows that we must have $b_{a_0} = 0$, a contradiction. Hence, $x_- = 0$ and $x = x_+ \in A_5$, as desired.

- *Step 2: We prove that* $\mu_2 = \mu_1 + \mu_4$.

Since $x_- = 0$, we deduce from (32) that

$$(\mu_2 - \mu_1 - \mu_4)Z_2Z_4^{-1} \in A_5,$$

that is

$$(\mu_2 - \mu_1 - \mu_4)Z_2 \in A_5Z_4.$$

Mutliplying this by Z_5 on the right leads to

$$(\mu_2 - \mu_1 - \mu_4)z_2 \in A_5 Z_4,$$

since $z_2 = Z_2 Z_5$ by Lemma 3.2 and $Z_4 Z_5 = q^{-1} Z_5 Z_4$. We set $z := (\mu_2 - \mu_1 - \mu_4)z_2$ and $J := A_5 Z_4$, so that $z \in J$.

It follows from Corollary 3.5 that $\mu_1, \mu_4 \in \mathbb{K}[z_1, z_2]$ and $\mu_2 z_2 \in \mathbb{K}[z_1, z_2]$. Hence $z \in \mathbb{K}[z_1, z_2]$. We need to prove that $z = 0$. Let us write

$$z = \sum_{i,j \in \mathbb{N}} a_{i,j} z_1^i z_2^j,$$

with all $a_{i,j} \in \mathbb{K}$ equal to zero except for a finite number of them. Since $z_1 = q^{-1}\hat{q}^{\,2} Z_3 Z_1 Z_6 Z_4 - q\hat{q} Z_3 Z_2 Z_6$ (see Lemma 3.2), we get that $z_1 - (-q\hat{q}) Z_3 Z_2 Z_6 = q^{-1}\hat{q}^{\,2} Z_3 Z_1 Z_6 Z_4 \in J$. Then, using the fact that z_1 and $z_2 = Z_2 Z_5$ are central elements of A_5, and that Z_2, Z_3, Z_5 and Z_6 q-commute with each other, we easily verify that

$$z_1^i z_2^j - q^{\bullet} (-q\hat{q})^i Z_2^{i+j} Z_3^i Z_5^j Z_6^i \in J$$

for all $i, j \in \mathbb{N}$, where $\bullet$ denotes, as usual, an integer. Therefore, we obtain:

$$z - \sum_{i,j \in \mathbb{N}} q^{\bullet} (-q\hat{q})^i a_{i,j} Z_2^{i+j} Z_3^i Z_5^j Z_6^i \in J.$$

As we have already proved that $z \in J$, this forces

$$\sum_{i,j \in \mathbb{N}} q^{\bullet} (-q\hat{q})^i a_{i,j} Z_2^{i+j} Z_3^i Z_5^j Z_6^i \in J. \tag{33}$$

However, since Z_4 q-commutes with Z_5 and Z_6, every element of J can be written as

$$\sum_{\substack{\gamma \in \mathbb{N}^4 \times \mathbb{Z}^2 \\ \gamma_4 > 0}} c_\gamma Z_1^{\gamma_1} \dots Z_6^{\gamma_6} \tag{34}$$

in the PBW basis of A_5. Identifying the two expressions (33) and (34) leads to $a_{i,j} = 0$ for all i, j, so that $z = 0$. Thus we have proved that $(\mu_2 - \mu_1 - \mu_4)z_2 = 0$. Since $z_2 \neq 0$, we get $\mu_2 = \mu_1 + \mu_4$, as desired. Observe that, since μ_1 and μ_4 belong to $Z(\mathfrak{sl}_4^+)$ by Corollary 3.5, this implies that μ_2 also belongs to $Z(\mathfrak{sl}_4^+)$.

- *Step 3: We prove that $D(Z_i) = \mathrm{ad}_x(Z_i) + \mu_i Z_i$ for all $i \in \{1, \dots, 6\}$.*

If $i > 1$, this is trivial since $Z_i = T_i$ and we already know that $D(T_i) = \mathrm{ad}_x(T_i) + \mu_i T_i$.

Next, recall that $Z_1 = T_1 + q\hat{q}^{-1} T_2 T_4^{-1}$. Hence, we have

$$D(Z_1) = \mathrm{ad}_x(Z_1) + \mu_1 T_1 + q\hat{q}^{-1}(\mu_2 - \mu_4) T_2 T_4^{-1}.$$

Since $\mu_2 = \mu_1 + \mu_4$, this implies that

$$D(Z_1) = \mathrm{ad}_x(Z_1) + \mu_1 T_1 + q\hat{q}^{-1}\mu_1 T_2 T_4^{-1} = \mathrm{ad}_x(Z_1) + \mu_1 Z_1,$$

as desired.

□

We are now able to prove that $D(z_2)$ belongs to the ideal of $U_q(\mathfrak{sl}_4^+)$ generated by $z_2 = \Delta_2$. This result is crucial in order to compute the automorphism group of $U_q(\mathfrak{sl}_4^+)$ (see Theorem 2.5).

Theorem 3.8. *Let $D \in \mathrm{Der}(U_q(\mathfrak{sl}_4^+))$. Then there exists $z \in Z(\mathfrak{sl}_4^+)$ such that $D(z_2) = z z_2$.*

Proof. Let $D \in \mathrm{Der}(U_q(\mathfrak{sl}_4^+))$. Since $z_2 = \Delta_2 = Z_2 Z_5 \in A_5$ by Lemma 3.2, we deduce from Lemma 3.7 that $D(z_2) = \mathrm{ad}_x(z_2) + (\mu_2 + \mu_5) z_2$ with $x \in A_5$ and $\mu_2, \mu_5 \in Z(\mathfrak{sl}_4^+)$. Now the result easily follows from the centrality of z_2 in A_5. □

Having completed the proof of Theorem 2.5 and thus described the automorphism group of $U_q(\mathfrak{sl}_4^+)$, we proceed to obtain a complete description of $\mathrm{Der}(U_q(\mathfrak{sl}_4^+))$.

Using arguments similar to those in the proof of Lemma 3.7, one can prove the following two results.

Lemma 3.9. *1. $x \in A_6$.*

2. $\mu_3 = \mu_1 + \mu_5$.

3. $\mu_2 + \mu_5 = \mu_3 + \mu_4$.

4. $D(Y_i) = \mathrm{ad}_x(Y_i) + \mu_i Y_i$ for all $i \in \{1, \dots, 6\}$.

And also:

Lemma 3.10. *1. $x \in A_7 = U_q(\mathfrak{sl}_4^+)$.*

2. $\mu_3 = \mu_2 + \mu_6$.

3. $\mu_5 = \mu_4 + \mu_6$.

4. $D(X_i) = \mathrm{ad}_x(X_i) + \mu_i X_i$ for all $i \in \{1, \dots, 6\}$.

It is easy to check that we can define three derivations D_1, D_4 and D_6 of $U_q(\mathfrak{sl}_4^+)$ by setting:

$$\begin{array}{llll} D_1(X_1) = X_1 & D_1(X_2) = X_2 & D_1(X_3) = X_3 & \\ D_4(X_2) = X_2 & D_4(X_3) = X_3 & D_4(X_4) = X_4 & D_4(X_5) = X_5 \\ D_6(X_3) = X_3 & D_6(X_5) = X_5 & D_6(X_6) = X_6 & \end{array}$$

and $D_i(X_j) = 0$ otherwise.

Then it follows from Lemmas 3.7, 3.9 and 3.10 that any derivation D of $U_q(\mathfrak{sl}_4^+)$ can be written as follows:

$$D = \mathrm{ad}_x + \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6,$$

with $x \in U_q(\mathfrak{sl}_4^+)$ and $\mu_1, \mu_4, \mu_6 \in Z(\mathfrak{sl}_4^+)$.

Recall that the Hochschild cohomology group in degree 1 of $U_q(\mathfrak{sl}_4^+)$, denoted by $\mathrm{HH}^1(U_q(\mathfrak{sl}_4^+))$, is defined by:

$$\mathrm{HH}^1(U_q(\mathfrak{sl}_4^+)) := \mathrm{Der}(U_q(\mathfrak{sl}_4^+))/\mathrm{InnDer}(U_q(\mathfrak{sl}_4^+)),$$

where $\mathrm{InnDer}(U_q(\mathfrak{sl}_4^+)) := \{\mathrm{ad}_x \mid x \in U_q(\mathfrak{sl}_4^+)\}$ is the Lie algebra of inner derivations of $U_q(\mathfrak{sl}_4^+)$. It is well known that $\mathrm{HH}^1(U_q(\mathfrak{sl}_4^+))$ is a module over $\mathrm{HH}^0(U_q(\mathfrak{sl}_4^+)) := Z(\mathfrak{sl}_4^+)$. Our final result makes this latter structure precise.

Theorem 3.11. *1. Every derivation D of $U_q(\mathfrak{sl}_4^+)$ can be uniquely written as follows:*

$$D = \mathrm{ad}_x + \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6,$$

with $\mathrm{ad}_x \in \mathrm{InnDer}(U_q(\mathfrak{sl}_4^+))$ and $\mu_1, \mu_4, \mu_6 \in Z(\mathfrak{sl}_4^+)$.

2. $\mathrm{HH}^1(U_q(\mathfrak{sl}_4^+))$ is a free $Z(\mathfrak{sl}_4^+)$-module of rank 3 with basis $(\overline{D_1}, \overline{D_4}, \overline{D_6})$.

Proof. It just remains to prove that, if $x \in U_q(\mathfrak{sl}_4^+)$ and $\mu_1, \mu_4, \mu_6 \in Z(\mathfrak{sl}_4^+)$ with $\mathrm{ad}_x + \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6 = 0$, then $\mu_1 = \mu_4 = \mu_6 = 0$ and $\mathrm{ad}_x = 0$. Set $\theta := \mu_1 D_1 + \mu_4 D_4 + \mu_6 D_6$, so that $\mathrm{ad}_x + \theta = 0$. Since θ is a derivation of $U_q(\mathfrak{sl}_4^+)$, θ uniquely extends to a derivation $\tilde{\theta}$ of the quantum torus $P(\Lambda)$. Naturally, we still have $\mathrm{ad}_x + \tilde{\theta} = 0$. Futher, straightforward computations show that

$$\begin{array}{lll} \tilde{\theta}(T_1) = \mu_1 T_1 & \tilde{\theta}(T_2) = (\mu_1 + \mu_4) T_2 & \tilde{\theta}(T_3) = (\mu_1 + \mu_4 + \mu_6) T_3 \\ \tilde{\theta}(T_4) = \mu_4 T_4 & \tilde{\theta}(T_5) = (\mu_4 + \mu_6) T_5 & \tilde{\theta}(T_6) = \mu_6 T_6 \end{array}$$

Hence $\tilde{\theta}$ is a central derivation of $P(\Lambda)$, in the terminology of [21]. Thus we deduce from [21, Cor. 2.3] that $\mathrm{ad}_x = 0 = \theta$. Evaluating θ on X_1, X_4 and X_6 leads to $\mu_1 = \mu_4 = \mu_6 = 0$, as desired. □

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