

# Geometric characterizations of virtually free groups

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## ABSTRACT

Four geometric conditions on a geodesic metric space, which are stronger variants of classical conditions characterizing hyperbolicity (featuring  $\delta$ -thin polygons, the Gromov product or the mesh of triangles) are proved to be equivalent. They define the class of polygon hyperbolic geodesic metric spaces. In the particular case of the Cayley graph of a finitely generated group, it is shown that they characterize virtually free groups.

## 1 Introduction

Finitely generated virtually free groups constitute an important subclass of hyperbolic groups. A group  $G$  is *virtually free* if it has a free subgroup  $F$  of finite index. If  $G$  is finitely generated, we may assume the same for  $F$ .

Virtually free groups constitute probably the class of groups which admits the widest variety of characterizations. Following Diekert and Weiss [5, 4], we enumerate a few to indulge all tastes.

A finitely generated group  $G = \langle A \rangle$  is virtually free if and only if:

- $G$  is the fundamental group of a finite graph of finite groups [11] (see also [17, Theorem 7.3]);
- $G$  acts on a connected locally finite graph of finite treewidth, with finitely many orbits and finite node stabilizers [5];

- the Cayley graph  $\Gamma_A(G)$  has finite treewidth [12];
- $\Gamma_A(G)$  can be  $k$ -triangulated [14];
- there exists some  $\varepsilon \geq 0$  such that, for all coterminal geodesic  $\xi$  and path  $\xi'$  in  $\Gamma_A(G)$ ,  $\text{Im}(\xi) \subseteq D_\varepsilon(\text{Im}(\xi'))$  [1];
- there exist some finite generating set  $B$  of  $G$  and some  $k \geq 0$  such that every  $k$ -locally geodesic in  $\Gamma_B(G)$  is a geodesic [8];
- $G$  admits a finite presentation by some geodesic rewriting system [8];
- the language of all words on  $A \cup A^{-1}$  representing the identity is context-free (Muller-Schupp's Theorem [13, 6]);
- $G$  is the universal group of a finite pregroup [16];
- the monadic second-order theory of  $\Gamma_A(G)$  is decidable [12, 14];
- $G$  admits a Stallings section [18].

More details can be found in [1, 4].

The above list gives evidence of the interest which was devoted by different authors to the Cayley graph of a virtually free group. However, to our best knowledge, no explicit results were published on the classical geometric conditions used to define hyperbolicity [10], [7, Proposition 2.21].

Thus in Section 3 we establish equivalent geometric conditions for geodesic metric spaces which will allow us to characterize finitely generated virtually free groups when we consider their Cayley graphs in Section 4.

Our starting point were three of the most common characterizations of hyperbolicity, using respectively thin geodesic triangles (Rips condition), the Gromov product (the original definition by Gromov) and the mesh of a geodesic triangle. We succeed on replacing thin geodesic triangles by thin geodesic polygons, and the mesh of a geodesic triangle by the mesh of an arbitrary triangle. For the Gromov product, we consider inequalities with arbitrary long sequences of group elements.

## 2 Preliminaries

We present in this section well-known facts regarding hyperbolic spaces and hyperbolic groups. The reader is referred to [3, 7] for details.

If  $(X, d)$  is a metric space and  $\varepsilon \geq 0$ , we denote by  $D_\varepsilon(x)$  the closed ball of center  $x \in X$  and radius  $\varepsilon$ . If  $Y \subseteq X$  is nonempty, we write

$$D_\varepsilon(Y) = \bigcup_{y \in Y} D_\varepsilon(y).$$

A mapping  $\varphi : (X, d) \rightarrow (X', d')$  between metric spaces is called an *isometric embedding* if  $d'(x\varphi, y\varphi) = d(x, y)$  for all  $x, y \in X$ . A surjective isometric embedding is an *isometry*.

We consider the usual metric for  $\mathbb{R}$  and its subsets. A *path* in  $(X, d)$  is a continuous mapping  $\xi : [0, s] \rightarrow X$  ( $s \in \mathbb{R}_0^+$ ). If  $\xi' : [0, s'] \rightarrow X$  is another path and  $0\xi' = 0\xi$ ,  $s'\xi' = s\xi$ , we say that the paths  $\xi$  and  $\xi'$  are *coterminal*.

A metric space  $(X, d)$  is said to be *geodesic* if, for all  $x, y \in X$ , there exists some path  $\xi : [0, s] \rightarrow X$  which is an isometric embedding and such that  $0\xi = x$  and  $s\xi = y$ . We call  $\xi$  a *geodesic* of  $(X, d)$ .

A *quasi-isometric embedding* of metric spaces is a mapping  $\varphi : (X, d) \rightarrow (X', d')$  such that there exist constants  $\lambda \geq 1$  and  $K \geq 0$  satisfying

$$\frac{1}{\lambda}d(x, y) - K \leq d'(x\varphi, y\varphi) \leq \lambda d(x, y) + K$$

for all  $x, y \in X$ . We may call it a  $(\lambda, K)$ -quasi-isometric embedding if we want to stress the constants. If in addition

$$\forall x' \in X' \exists x \in X : d'(x', x\varphi) \leq K,$$

we say that  $\varphi$  is a *quasi-isometry*.

Two metric spaces  $(X, d)$  and  $(X', d')$  are said to be *quasi-isometric* if there exists a quasi-isometry  $\varphi : (X, d) \rightarrow (X', d')$ . Quasi-isometry turns out to be an equivalence relation on the class of metric spaces. A path  $\xi : [0, s] \rightarrow X$  which is a quasi-isometric embedding is called a *quasi-geodesic* of  $(X, d)$ .

Let  $(X, d)$  be a geodesic metric space. A *triangle* in  $(X, d)$  is a collection  $\Delta = [[\xi_0, \xi_1, \xi_2]]$  of three paths  $\xi_i : [0, s_i] \rightarrow X$  such that

$$s_i\xi_i = 0\xi_{i+1} \quad \text{for } i = 0, 1, 2.$$

Here and in many other instances of the paper, we consider the indices modulo 3 (or  $n+1$ ), so that  $\xi_3 = \xi_0$  in the formula above. If all the paths  $\xi_i$  are geodesics, we say that  $\Delta$  is a *geodesic triangle*.

More generally, we define a *geodesic polygon* to be a collection of  $n+1$  geodesics  $\Pi = [[\xi_0, \xi_1, \dots, \xi_n]]$  such that

$$s_i\xi_i = 0\xi_{i+1} \quad \text{for } i = 0, 1, \dots, n,$$

with  $n \geq 1$  arbitrary. We may call  $\Pi$  an  $(n+1)$ -gon. If  $n = 1$ , we have a *geodesic bigon*.

Given  $\delta \geq 0$ , a geodesic polygon  $[[\xi_0, \xi_1, \dots, \xi_n]]$  is said to be  $\delta$ -thin if

$$\text{Im}(\xi_n) \subseteq D_\delta(\text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1})).$$

We say that  $(X, d)$  is  $\delta$ -hyperbolic if every geodesic triangle in  $(X, d)$  is  $\delta$ -thin. We say that  $(X, d)$  is *hyperbolic* if  $(X, d)$  is  $\delta$ -hyperbolic for some  $\delta \geq 0$ .

There are several equivalent characterizations of hyperbolicity. The original one, introduced by Gromov in [10], uses the concept of Gromov product, which we now define. We note that it can be defined for every metric space  $(X, d)$ .

Given  $x, y, p \in X$ , we define

$$(x|y)_p = \frac{1}{2}(d(p, x) + d(p, y) - d(x, y)).$$

We say that  $(x|y)_p$  is the *Gromov product* of  $x$  and  $y$ , taking  $p$  as basepoint.

A third road to hyperbolicity uses the concept of *mesh*. Recall the notion of *diameter*. Given a nonempty  $Y \subseteq X$ , we write

$$\text{diam}(Y) = \sup\{d(y, y') \mid y, y' \in Y\}.$$

Given a triangle  $\Delta = [[\xi_0, \xi_1, \xi_2]]$  in a geodesic metric space  $(X, d)$ , we define

$$\text{mesh}(\Delta) = \inf\{\text{diam}(\{u_0, u_1, u_2\}) \mid u_i \in \text{Im}(\xi_i) \ (i = 0, 1, 2)\}.$$

The following equivalences are well known, see [7, Proposition 2.21] for a proof.

**Theorem 2.1** *The following conditions are equivalent for a geodesic metric space  $(X, d)$ :*

(i)  $(X, d)$  is hyperbolic;

(ii) there exists some  $\delta' \geq 0$  such that

$$(x_0|x_2)_p \geq \min\{(x_0|x_1)_p, (x_1|x_2)_p\} - \delta'$$

for all  $x_0, x_1, x_2, p \in X$ ;

(iii) there exists some  $\mu \geq 0$  such that

$$\text{mesh}(\Delta) \leq \mu$$

for every geodesic triangle  $\Delta$  in  $(X, d)$ .

Among the most important properties of hyperbolic spaces, stands the fact that they are closed under quasi-isometry.

Given  $Y, Z \subseteq X$  nonempty, the *Hausdorff distance* between  $Y$  and  $Z$  is defined by

$$\text{Haus}(Y, Z) = \max\{\sup_{y \in Y} d(y, Z), \sup_{z \in Z} d(z, Y)\}.$$

The following result of Gromov [10] (see also [7, Theorem 5.4.21]) is also important to us:

**Proposition 2.2** *Let  $\lambda \geq 1$  and  $K, \delta \geq 0$ . There exists a constant  $R(\delta, \lambda, K) \geq 0$  such that: if  $(X, d)$  is a  $\delta$ -hyperbolic space,  $\xi$  is a geodesic and  $\xi'$  a coterminal  $(\lambda, K)$ -quasi-geodesic in  $(X, d)$ , then*

$$\text{Haus}(\text{Im}(\xi), \text{Im}(\xi')) \leq R(\delta, \lambda, K).$$

We proceed now to define hyperbolic groups.

Given a subset  $A$  of a group  $G$ , we denote by  $\langle A \rangle$  the subgroup of  $G$  generated by  $A$ . We assume throughout the paper that generating sets are finite.

Given  $G = \langle A \rangle$ , we write  $\tilde{A} = A \cup A^{-1}$ . The *Cayley graph*  $\Gamma_A(G)$  has vertex set  $G$  and edges of the form  $g \xrightarrow{a} ga$  for all  $g \in G$  and  $a \in \tilde{A}$ . The *geodesic metric*  $d_A$  on  $G$  is defined by taking  $d_A(g, h)$  to be the length of the shortest path connecting  $g$  to  $h$  in  $\Gamma_A(G)$ .

Since  $\text{Im}(d_A) \subseteq \mathbb{N}$ , then  $(G, d_A)$  is not a geodesic metric space. However, we can remedy that by embedding  $(G, d_A)$  isometrically into the *geometric realization*  $\bar{\Gamma}_A(G)$  of  $\Gamma_A(G)$ , where vertices become points and edges become segments of length 1 in some (euclidean) space, intersections being determined by adjacency only. With the obvious metric,  $\bar{\Gamma}_A(G)$  is a geodesic metric space, and the geometric realization is unique up to isometry. We denote also by  $d_A$  the induced metric on  $\bar{\Gamma}_A(G)$ .

We say that the group  $G$  is *hyperbolic* if the geodesic metric space  $(\bar{\Gamma}_A(G), d_A)$  is hyperbolic. Since  $(\bar{\Gamma}_A(G), d_A)$  is quasi-isometric to  $(\bar{\Gamma}_B(G), d_B)$  for every alternative finite generating set  $B$  of  $G$ , the concept of hyperbolic group is independent from the finite generating set considered. However, the hyperbolicity constant  $\delta$  may vary with the generating set.

Let  $H$  be a subgroup of a hyperbolic group  $G = \langle A \rangle$  and let  $q \geq 0$ . We say that  $H$  is  *$q$ -quasi-convex* with respect to  $A$  if, for every geodesic  $\xi : [0, s] \rightarrow \bar{\Gamma}_A(G)$  with  $0\xi, s\xi \in H$ , we have

$$\text{Im}(\xi) \subseteq D_q(H).$$

We say that  $H$  is *quasi-convex* if it is  $q$ -quasi-convex for some  $q \geq 0$ . Like most other properties in the theory of hyperbolic groups, quasi-convex does not depend on the finite generating set considered.

A (finitely generated) subgroup of a hyperbolic group needs not be hyperbolic, but a quasi-convex subgroup of a hyperbolic group is always hyperbolic. The converse is not true in general. The reader is referred to [3, Section III.Γ.3] for details on quasi-convex subgroups.

### 3 Geodesic metric spaces

We prove in this section the equivalence of four geometric conditions on a geodesic metric space.

Given  $\delta \geq 0$ , we say that a geodesic metric space  $(X, d)$  is *polygon  $\delta$ -hyperbolic* if every geodesic polygon in  $(X, d)$  is  $\delta$ -thin. We say that  $(X, d)$  is *polygon hyperbolic* if it is polygon  $\delta$ -hyperbolic for some  $\delta \geq 0$ . We introduce also the notation

$$\text{mesh}(X, d) = \sup\{\text{mesh}(\Delta) \mid \Delta \text{ is a triangle in } (X, d)\}.$$

**Lemma 3.1** *Let  $\varphi : (X, d) \rightarrow (X', d')$  be a quasi-isometric embedding of geodesic metric spaces. Then:*

(i) *if  $(X', d')$  is polygon hyperbolic, so is  $(X, d)$ ;*

(ii) *if  $\text{mesh}(X', d')$  is finite, so is  $\text{mesh}(X, d)$ .*

**Proof.** (i) Assume that  $\varphi$  is a  $(\lambda, K)$ -quasi-isometry and  $(X', d')$  is polygon  $\delta'$ -hyperbolic. Let  $[[\xi_0, \dots, \xi_n]]$  be a geodesic polygon in  $(X, d)$ .

Each geodesic  $\xi_i$  induces a  $(\lambda, K)$ -quasi-geodesic  $\xi_i\varphi$  in  $X'$ . For  $i = 0, \dots, n$ , let  $\xi'_i$  be a geodesic in  $X'$  coterminal with  $\xi_i\varphi$ . Since  $(X', d')$  is  $\delta'$ -hyperbolic, it follows from Proposition 2.2 that

$$\text{Haus}(\text{Im}(\xi_i\varphi), \text{Im}(\xi'_i)) \leq R(\delta, \lambda, K) \quad (1)$$

for  $i = 0, \dots, n$ .

Write  $R = R(\delta, \lambda, K)$  and let  $y \in \text{Im}(\xi_n)$ . Then  $y\varphi \in \text{Im}(\xi_n\varphi)$  and by (1) we have  $d'(y\varphi, y') \leq R$  for some  $y' \in \text{Im}(\xi'_n)$ .

Now  $[[\xi'_0, \dots, \xi'_n]]$  is a geodesic polygon in  $X'$ . Since  $(X', d')$  is polygon  $\delta'$ -hyperbolic, there exist  $j \in \{0, \dots, n-1\}$  and  $z' \in \text{Im}(\xi'_j)$  such that  $d'(y', z') \leq \delta'$ . By (1), we have  $d'(z', z\varphi) \leq R$  for some  $z \in \text{Im}(\xi_j)$ , hence

$$\frac{1}{\lambda}d(y, z) - K \leq d'(y\varphi, z\varphi)$$

yields

$$d(y, z) \leq \lambda(d'(y\varphi, z\varphi) + K) \leq \lambda(d'(y\varphi, y') + d'(y', z') + d'(z', z\varphi) + K) \leq \lambda(2R + \delta' + K).$$

Since  $y \in \text{Im}(\xi_n)$  is arbitrary and  $z \in \text{Im}(\xi_j)$ , it follows that  $(X, d)$  is polygon  $\delta$ -hyperbolic for  $\delta = \lambda(2R + \delta' + K)$ .

(ii) Assume that  $\varphi$  is a  $(\lambda, K)$ -quasi-isometry and  $\text{mesh}(X', d') = \mu' < +\infty$ . Let  $[[\xi_0, \xi_1, \xi_2]]$  be a triangle in  $(X, d)$ . Then  $[[\xi_0\varphi, \xi_1\varphi, \xi_2\varphi]]$  is a triangle in  $(X', d')$ . Since  $\text{mesh}(X', d') = \mu'$ , there exist some  $u_i \in \text{dom}(\xi_i\varphi) = \text{dom}(\xi_i)$ , for  $i = 0, 1, 2$ , such that

$$\text{diam}(\{u_0\xi_0\varphi, u_1\xi_1\varphi, u_2\xi_2\varphi\}) \leq \mu'.$$

Since

$$\frac{1}{\lambda}d(u_i\xi_i, u_j\xi_j) - K \leq d'(u_i\xi_i\varphi, u_j\xi_j\varphi)$$

for all  $i, j \in \{0, 1, 2\}$ , we get

$$\text{diam}(\{u_0\xi_0, u_1\xi_1, u_2\xi_2\}) \leq \lambda(\mu' + K).$$

Therefore

$$\text{mesh}(X, d) \leq \lambda(\mu' + K) < +\infty.$$

□

**Theorem 3.2** *Let  $(X, d)$  be a geodesic metric space. Then the following conditions are equivalent:*

(i)  $(X, d)$  is polygon hyperbolic;

(ii) there exists some  $\delta' \geq 0$  such that

$$(x_0|x_n)_p \geq \min\{(x_0|x_1)_p, \dots, (x_{n-1}|x_n)_p\} - \delta'$$

holds for all  $n \geq 2$  and  $x_0, \dots, x_n, p \in X$ ;

(iii) there exists some  $\varepsilon \geq 0$  such that, for all coterminal geodesic  $\xi$  and path  $\xi'$  in  $(X, d)$ ,

$$\text{Im}(\xi) \subseteq D_\varepsilon(\text{Im}(\xi'));$$

(iv)  $\text{mesh}(X, d) < +\infty$ .

**Proof.** (i)  $\Rightarrow$  (ii). Assume that  $(X, d)$  is polygon  $\delta$ -hyperbolic. Let  $x_0, \dots, x_n, p \in X$  with  $n \geq 2$ . For  $i = 0, \dots, n$ , consider a geodesic  $\xi_i : [0, s_i] \rightarrow X$  such that  $0\xi_i = x_i$  and  $s_i\xi_i = x_{i+1}$ . Let  $\Pi = [[\xi_0, \dots, \xi_n]]$  and  $Z = \text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1})$ .

Since  $(X, d)$  is polygon  $\delta$ -hyperbolic, it is in particular  $\delta$ -hyperbolic. By [19, Lemmas 2.9, 2.31 and 2.32], we have

$$(x_i|x_{i+1})_p \leq d(p, \text{Im}(\xi_i)) \leq (x_i|x_{i+1})_p + 2\delta \quad (2)$$

for  $i = 0, \dots, n$ . Hence

$$\min\{(x_0|x_1)_p, \dots, (x_{n-1}|x_n)_p\} \leq \min\{d(p, \text{Im}(\xi_0)), \dots, d(p, \text{Im}(\xi_{n-1}))\} = d(p, Z). \quad (3)$$

Since  $\text{Im}(\xi_n)$  is compact, we have  $d(p, \text{Im}(\xi_n)) = d(p, q)$  for some  $q \in \text{Im}(\xi_n)$ . Applying our hypothesis to  $\Pi$  we get  $d(q, Z) \leq \delta$  and so

$$d(p, Z) \leq d(p, q) + d(q, Z) \leq d(p, \text{Im}(\xi_n)) + \delta \leq (x_n|x_0)_p + 3\delta \quad (4)$$

in view of (2). Now (3) and (4) together yield

$$(x_0|x_n)_p \geq d(p, Z) - 3\delta \geq \min\{(x_0|x_1)_p, \dots, (x_{n-1}|x_n)_p\} - 3\delta$$

and we are done.

(ii)  $\Rightarrow$  (i). Let  $[[\xi_0, \dots, \xi_n]]$  be a geodesic polygon in  $(X, d)$  with geodesics  $\xi_i : [0, s_i] \rightarrow X$  for  $i = 0, \dots, n$ . Write  $x_i = 0\xi_i = s_{i-1}\xi_{i-1}$  for  $i = 0, \dots, n$ . Let  $p \in \text{Im}(\xi_n)$  and let  $Z = \text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1})$ .

By [19, Lemmas 2.9, 2.30 and 2.32], the case  $n = 2$  in condition (ii) implies

$$(x_i|x_{i+1})_p \leq d(p, \text{Im}(\xi_i)) \leq (x_i|x_{i+1})_p + 2\delta' \quad (5)$$

for  $i = 0, \dots, n$ . Hence

$$d(p, Z) = \min\{d(p, \text{Im}(\xi_0)), \dots, d(p, \text{Im}(\xi_{n-1}))\} \leq \min\{(x_0|x_1)_p, \dots, (x_{n-1}|x_n)_p\} + 2\delta'.$$

By condition (ii), we get

$$d(p, Z) \leq (x_0|x_n)_p + 3\delta'.$$

Since  $p \in \text{Im}(\xi_n)$ , it follows from (5) that  $d(p, Z) \leq 3\delta'$ . Therefore  $(X, d)$  is polygon  $3\delta'$ -hyperbolic.

(i)  $\Rightarrow$  (iii). Assume that  $(X, d)$  is polygon  $\delta$ -hyperbolic. Let  $\xi : [0, s] \rightarrow X$  be a geodesic and  $\xi' : [0, s'] \rightarrow X$  be a coterminal path. Since  $\text{Im}(\xi')$  is compact, we can find finitely many  $x_0 = s\xi, x_1, \dots, x_{n-1}, x_n = 0\xi \in \text{Im}(\xi')$  such that  $d(x_i, x_{i-1}) \leq 1$  for  $i = 1, \dots, n$ . Now we build a geodesic polygon  $\Pi = [[\xi_0, \dots, \xi_n]]$ , where each  $\xi_i$  is a geodesic with endpoints  $x_i$  and  $x_{i+1}$ , and  $\xi_n = \xi$ .

Let  $p \in \text{Im}(\xi)$ . Since  $(X, d)$  is polygon  $\delta$ -hyperbolic, there exist some  $j \in \{0, \dots, n-1\}$  and  $q \in \text{Im}(\xi_j)$  such that  $d(p, q) \leq \delta$ . Hence

$$d(p, \text{Im}(\xi')) \leq d(p, x_j) \leq d(p, q) + d(q, x_j) \leq \delta + 1$$

and so  $\text{Im}(\xi) \subseteq D_{\delta+1}(\text{Im}(\xi'))$ .

(iii)  $\Rightarrow$  (iv). If we consider a geodesic triangle  $[[\xi_0, \xi_1, \xi_2]]$  in  $(X, d)$  and view  $\text{Im}(\xi_0) \cup \text{Im}(\xi_1)$  as the image of a single path, coterminal with  $\xi_2$ , it follows from condition (iii) that  $(X, d)$  is  $\varepsilon$ -hyperbolic. Therefore, by Theorem 2.1, there exists some  $\mu \geq 0$  such that

$$\text{mesh}(\Delta) \leq \mu$$

for every geodesic triangle  $\Delta$  in  $(X, d)$ .

Let  $\Delta' = [[\xi'_0, \xi'_1, \xi'_2]]$  be a triangle in  $(X, d)$ . Consider a geodesic triangle  $\Delta = [[\xi_0, \xi_1, \xi_2]]$  where  $\xi_i$  and  $\xi'_i$  are coterminal for  $i = 0, 1, 2$ . Since  $\text{mesh}(\Delta) \leq \mu$ , there exist  $p_i \in \text{Im}(\xi_i)$  such that

$$\text{diam}(\{p_0, p_1, p_2\}) \leq \mu.$$

Now, also by condition (iii), there exist  $q_i \in \text{Im}(\xi'_i)$  such that  $d(p_i, q_i) \leq \varepsilon$  for  $i = 0, 1, 2$ . Therefore, for all  $i, j \in \{0, 1, 2\}$ , we get

$$d(q_i, q_j) \leq d(q_i, p_i) + d(p_i, p_j) + d(p_j, q_j) \leq 2\varepsilon + \mu.$$

Therefore  $\text{mesh}(X, d) \leq 2\varepsilon + \mu$  and (iv) holds.

(iv)  $\Rightarrow$  (i). Assume that  $\text{mesh}(X, d) = \mu < +\infty$ . Let  $\Pi = [[\xi_0, \dots, \xi_n]]$  be a geodesic polygon in  $(X, d)$  with geodesics  $\xi_i : [0, s_i] \rightarrow X$  for  $i = 0, \dots, n$ . Write  $x_i = 0\xi_i = s_{i-1}\xi_{i-1}$  for  $i = 0, \dots, n$ . Let  $p \in \text{Im}(\xi_n)$ , say  $p = y\xi_n$ , and let  $Z = \text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1})$ . We prove that  $d(p, Z) \leq \frac{3\mu+1}{2}$ .

We may of course assume that  $d(p, x_0), d(p, x_n) \geq \frac{\mu+1}{2}$ . We can build three paths  $\xi'_0, \xi'_1, \xi'_2$  satisfying the following conditions:

- $\text{Im}(\xi'_0) = [y + \frac{\mu+1}{2}, s_n]\xi_n \cup Z$  and has endpoints  $(y + \frac{\mu+1}{2})\xi_n$  and  $x_n$ ;
- $\text{Im}(\xi'_1) = [0, y - \frac{\mu+1}{2}]\xi_n$  and has endpoints  $x_n$  and  $(y - \frac{\mu+1}{2})\xi_n$ ;
- $\text{Im}(\xi'_2) = [y - \frac{\mu+1}{2}, y + \frac{\mu+1}{2}]\xi_n$  and has endpoints  $(y - \frac{\mu+1}{2})\xi_n$  and  $(y + \frac{\mu+1}{2})\xi_n$ .

Let  $\Delta = [[\xi'_0, \xi'_1, \xi'_2]]$  (see Figure 2). Since  $\text{mesh}(\Pi) \leq \mu$ , there exist  $u_i \in \text{Im}(\xi'_i)$  such that  $\text{diam}(\{u_0, u_1, u_2\}) \leq \mu$ . Suppose that  $u_0 \notin Z$ . Then  $u_0 \in [y + \frac{\mu+1}{2}, s_n]\xi_n$ . Since  $u_1 \in [0, y - \frac{\mu+1}{2}]\xi_n$  and  $\xi_n$  is a geodesic, this contradicts  $d(u_0, u_1) \leq \mu$ . Hence  $u_0 \in Z$  and we get

$$d(p, Z) \leq d(p, u_0) \leq d(p, u_2) + d(u_2, u_0) \leq \frac{\mu+1}{2} + \mu = \frac{3\mu+1}{2}$$

as claimed. Therefore  $(X, d)$  is polygon  $\frac{\mu+1}{2}$ -hyperbolic.  $\square$

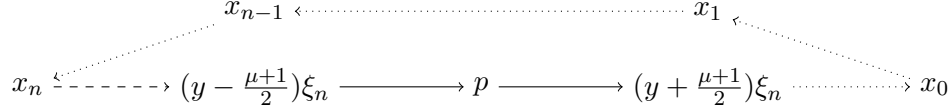


Figure 1: The triangle  $\Delta = [[\xi'_0, \xi'_1, \xi'_2]]$

## 4 Virtually free groups

We present next several equivalent geometric characterizations of virtually free groups. We only have found out that the equivalence (i)  $\Leftrightarrow$  (iv) had been previously proved by Antolin in [1] after we had written our own proof of the theorem, so we decided to keep our alternative proof of the equivalence (i)  $\Leftrightarrow$  (ii).

We denote by  $(g|h)_p^A$  the Gromov product of  $g, h \in G$  with basepoint  $p$  in  $(\bar{\Gamma}_A(G), d_A)$ .

**Theorem 4.1** *Let  $G = \langle A \rangle$  be a finitely generated group. Then the following conditions are equivalent:*

- (i)  $G$  is virtually free;
- (ii)  $\bar{\Gamma}_A(G)$  is polygon hyperbolic;
- (iii) there exists some  $\delta' \geq 0$  such that

$$(g_0|g_n)_p^A \geq \min\{(g_0|g_1)_p^A, \dots, (g_{n-1}|g_n)_p^A\} - \delta'$$

for all  $n \geq 2$  and  $g_0, \dots, g_n, p \in G$ ;

- (iv) there exists some  $\varepsilon \geq 0$  such that, for all coterminal geodesic  $\xi$  and path  $\xi'$  in  $(\bar{\Gamma}_A(G), d_A)$ ,

$$\text{Im}(\xi) \subseteq D_\varepsilon(\text{Im}(\xi'));$$

- (v)  $\text{mesh}(\bar{\Gamma}_A(G), d_A) < +\infty$ .

**Proof.** (i)  $\Rightarrow$  (ii). Since  $G$  is virtually free and finitely generated,  $G$  has a finite index free subgroup  $F$  of finite rank. By [7, Proposition 1.11], there exists a finite generating set  $B$  of  $F$  such that  $\bar{\Gamma}_A(G)$  is quasi-isometric to  $\bar{\Gamma}_B(F)$ . Let  $C$  be a basis of  $F$ . As mentioned in Section 2,  $\bar{\Gamma}_B(F)$  is quasi-isometric to  $\bar{\Gamma}_C(F)$ . Since  $\bar{\Gamma}_C(F)$  is a tree, it is trivially polygon 0-hyperbolic. Therefore, by Lemma 3.1(i),  $\bar{\Gamma}_A(G)$  is polygon hyperbolic.

(ii)  $\Rightarrow$  (i). Since  $\bar{\Gamma}_A(G)$  is polygon hyperbolic, it is in particular hyperbolic. Thus  $G$  is a hyperbolic group. Suppose that  $G$  is not virtually free. By a theorem of Bonk and Kleiner [2, Theorem 1], the hyperbolic plane  $\mathbb{H}^2$  admits a quasi-isometric embedding into  $\bar{\Gamma}_A(G)$ . In view of Lemma 3.1(i), to reach the required contradiction it suffices to show that  $\mathbb{H}^2$  is not polygon hyperbolic.

We shall use the Poincaré disk model of  $\mathbb{H}^2$  (see [9, p.232] for details). We can view  $\mathbb{H}^2$  as the open disk  $\{z \in \mathbb{C} \mid |z| < 1\}$ . Geodesic lines are either diameters of the disk or arcs of a circle whose ends are perpendicular to the disk boundary. We denote by  $d$  the Poincaré metric on  $\mathbb{H}^2$ .

Let  $\varepsilon > 0$ . Using a diameter and a sufficiently large number of small circles (see Figure 2),



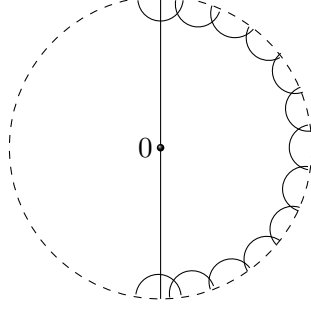


Figure 2: A geodesic 14-gon in the Poincaré disk

we can build a geodesic polygon  $[[\xi_0, \dots, \xi_n]]$  such that  $\text{Im}(\xi_n)$  is contained in the diameter and  $|z| > 1 - \varepsilon$  for every  $z \in \text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1})$ . Consider  $0 \in \text{Im}(\xi_n)$ . Then

$$d(0, z) = \left| \ln \frac{1 - |z|}{1 + |z|} \right|$$

(see [9, p.248]). Since the euclidean distance from 0 to  $\text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1})$  tends to 1 when  $\varepsilon$  tends to 0, it follows that  $d(0, \text{Im}(\xi_0) \cup \dots \cup \text{Im}(\xi_{n-1}))$  tends to  $+\infty$  when  $\varepsilon$  tends to 0. Therefore  $\mathbb{H}^2$  is not polygon hyperbolic.

(ii)  $\Rightarrow$  (iii) and (ii)  $\Leftrightarrow$  (iv)  $\Leftrightarrow$  (v). By Theorem 3.2.

(iii)  $\Rightarrow$  (ii). If  $(X, d)$  is a metric space and  $x, y, z, x', y', z' \in X$  are such  $d(x, x'), d(y, y'), d(z, z') \leq \frac{1}{2}$ , then

$$\begin{aligned} (x'|y')_{z'} &= \frac{1}{2}(d(z', x') + d(z', y') - d(x', y')) \\ &\leq \frac{1}{2}(d(z', z) + d(z, x) + d(x, x') + d(z', z) + d(z, y) + d(y, y')) \\ &\quad + d(x, x') - d(x, y) + d(y', y) \\ &\leq \frac{1}{2}(d(z, x) + d(z, y) - d(x, y) + 3) = (x|y)_z + \frac{3}{2}. \end{aligned}$$

Let  $n \geq 2$  and  $x_0, \dots, x_n, q \in \bar{\Gamma}_A G$ . Then there exist  $g_0, \dots, g_n, p \in G$  such that  $d_A(x_0, g_0), \dots, d_A(x_n, g_n), d_A(q, p) \leq \frac{1}{2}$ . By the hypothesis and the inequality proved above, we get

$$\begin{aligned} (x_0|x_n)_q^A &\geq (g_0|g_n)_p^A - \frac{3}{2} \geq \min\{(g_0|g_1)_p^A, \dots, (g_{n-1}|g_n)_p^A\} - \delta' - \frac{3}{2} \\ &\geq \min\{(x_0|x_1)_q^A, \dots, (x_{n-1}|x_n)_q^A\} - \delta' - 3. \end{aligned}$$

Thus we may apply Theorem 3.2 and obtain condition (ii).  $\square$

We remark that the hyperbolicity condition for a geodesic metric space (all geodesic triangles being  $\delta$ -thin) is equivalent to all geodesic  $m$ -gons being  $(m - 2)\delta$ -thin when we fix  $m \geq 4$ : for the direct implication we triangulate the polygon, and the converse follows from adding trivial geodesics to a triangle. Thus what distinguishes virtually free groups among arbitrary hyperbolic groups is the capacity of using the same  $\delta$  for all polygons simultaneously.

With respect to bigons, we must recall a remarkable theorem of Papasoglu [15, Theorem 1.4], which proved that a finitely generated group  $G = \langle A \rangle$  is hyperbolic if and only if there exists some  $\delta \geq 0$  such that all geodesic bigons in  $\bar{\Gamma}_A(G)$  are  $\delta$ -thin.

As a matter of fact, Papasoglu uses in his statement a synchronous version of thinness (the fellow travel property). We say that two coterminal geodesics  $\xi, \xi' : [0, s] \rightarrow \bar{\Gamma}_A(G)$  fellow travel with bound  $\delta$  if  $d_A(t\xi, t\xi') \leq \delta$  for every  $t \in [0, s]$ . Clearly, this implies  $\text{Im}(\xi) \subseteq D_\delta(\text{Im}(\xi'))$ .

On the other hand, suppose that  $\text{Im}(\xi) \subseteq D_\varepsilon(\text{Im}(\xi'))$  and let  $t \in [0, s]$ . Then  $d_A(t\xi, t'\xi') \leq \varepsilon$  for some  $t' \in [0, s]$ . It is easy to see that  $t' > t + \varepsilon$  contradicts  $\xi'$  being a geodesic (we may use  $\xi$  and a geodesic from  $t\xi$  to  $t'\xi$  to produce a shorter alternative to  $\xi'$ ), and by symmetry we get  $|t' - t| \leq \varepsilon$ . Hence

$$d_A(t\xi, t'\xi') \leq d_A(t\xi, t'\xi) + d_A(t'\xi, t'\xi') \leq \varepsilon + |t' - t| \leq 2\varepsilon$$

and so the two geodesics follow travel with bound  $2\varepsilon$ . This confirms that the two notions of thinness are indeed equivalent.

However, we must note that Papasoglu's theorem does not hold for arbitrary geodesic metric spaces. Indeed, geodesic bigons are 0-thin in any space where there exists a unique geodesic connecting a given pair of points. This is for instance the case of the Euclidean plane, as noted in [15]. Another counterexample would be  $\bar{\Gamma}_A(G)$  when  $A = \{a_1, a_2, \dots\}$  and  $G$  is the group presented by  $\langle A \mid a_n^{2n+1} \ (n \geq 1) \rangle$  (the free product of cyclic groups of all odd orders).

We end with an application of Theorem 4.1.

**Corollary 4.2** *Let  $G$  be a finitely generated virtually free group and let  $H \leq G$ . Then the following conditions are equivalent:*

- (i)  $H$  is quasi-convex;
- (ii)  $H$  is finitely generated.

**Proof.** (i)  $\Rightarrow$  (ii). Since every quasi-convex subgroup of a hyperbolic group is finitely generated [3, Lemma III.Γ.3.5].

(ii)  $\Rightarrow$  (i). Assume that  $G = \langle A \rangle$  and  $H = \langle B \rangle$  with  $A, B$  finite. Let

$$M = \max\{d_A(1, b) \mid b \in B\}.$$

By Theorem 4.1, there exists some  $\varepsilon \geq 0$  such that, for all coterminal geodesic  $\xi$  and path  $\xi'$  in  $(\bar{\Gamma}_A(G), d_A)$ ,

$$\text{Im}(\xi) \subseteq D_\varepsilon(\text{Im}(\xi')).$$

Let  $h, h' \in H$  and let  $\eta : [0, s] \rightarrow \bar{\Gamma}_A(G)$  be a geodesic with  $0\eta = h$  and  $s\eta = h'$ . Write  $h' = hb_1 \dots b_n$  for some  $b_1, \dots, b_n \in \tilde{B}$ . Let  $\eta'$  be a path in  $\bar{\Gamma}_A(G)$  concatenating geodesics with endpoints  $hb_1 \dots b_{i-1}$  and  $hb_1 \dots b_i$  for  $i = 1, \dots, n$ . Then  $\text{Im}(\eta) \subseteq D_\varepsilon(\text{Im}(\eta'))$ . It is immediate that  $\text{Im}(\eta') \subseteq D_M(H)$ , hence

$$\text{Im}(\eta) \subseteq D_{\varepsilon+M}(H)$$

and  $H$  is  $(\varepsilon + M)$ -quasi-convex.  $\square$

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