

PLAYING SYMMETRIES

PORTUGUESE SIDEWALKS

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Carvalho, A., Santos, C. P., Silva, J. N., Teixeira, R. C. (2016). *Playing Symmetries: Sidewalks of Lisbon*, Proceedings of Recreational Mathematics Colloquium IV.

Abstract: *Part of the beauty of the Azores lies under the feet of anyone who walks in the archipelago. The traditional Portuguese pavement decorates public spaces with a very typical aesthetic. These examples illustrate different types of rosettes, friezes, and patterns. The rosettes are limited planar configurations, usually presented in circular frames. The friezes, due to their unidimensional nature, are found on sidewalks. The patterns, which can cover plane surfaces, show up in plazas. All the seven friezes, cyclic rosettes, dihedral rosettes, and some of the seventeen types of wallpapers were detected in the Azores islands. In this work, a card deck about the subject is presented. This deck suggests a walk through the 9 islands of the Azores and the symmetries of its pavements.*

Keywords: Symmetries; Portuguese Pavements; Azores Islands; Patterns.

INTRODUCTION

Repetitive designs in one or two directions can be classified mathematically by the types of symmetries they possess, and this classification gives rise to seven frieze patterns and to seventeen wallpaper patterns. Rosettes are other types of patterns in which the repetition of the design occurs about a single point, within a limited region of the plane. Rosettes are either dihedral or cyclic, depending on the presence or absence of mirror symmetries. Many Portuguese pavements are beautiful artistic works: all the seven friezes, cyclic rosettes, dihedral rosettes, and some of the seventeen types of wallpapers were detected in the Azores islands. In this work, a card deck about the subject is presented.

PORTUGUESE PAVEMENT

Portuguese Pavements are unique because of the designs, stone production, and mastery of the pavers (*calceteiros*). There are several motifs: geometric, figurative, or specificities (Henriques *et al.*, 2009). The Portuguese Pavements are a heritage of Roman culture from which many examples still exist in Portugal.

It was with King John II (1455–1495) that the commercial profitability in the cities of Lisbon and Oporto brought the opportunity for the construction of “Ruas Novas” (“New streets”) at the places where the main stores were concentrated.

With the earthquakes of 1531 and 1551, a “forced” motivation to make new streets naturally occurred. However, the most important moment happened after the Lisbon earthquake of 1755; a huge project started, not only for the reconstruction of the streets but also for opening new ones. One of the most important figures of this process was Sebastião de Carvalho e Melo (1699–1782), Marquis of Pombal.

It was in Lisbon that for the first time, in 1848, decorative paving was used in urban spaces: the Mar Largo project, a composition shaped like waves, was built at the D. Pedro IV Square (Rossio nowadays). But six years before this project, by an initiative of Lieutenant-General Eusébio Pinheiro Furtado, the narrow streets leading to the São Jorge Castle were paved with white (limestone) and dark (basalt) stones. Therefore, Portuguese Pavements are objects with rich history and tradition. Limestones often show a smooth and bright surface.

The most typical Portuguese Pavements are made with limestone and basalt. The settlement of paving units is done by technicians (*calceteiros*), which place stones over a layer of fine material with a small hammer. Some of the designs are rich in amazing symmetry. A good example is the Wave motif, “Mar largo”, repeated all over the world. Part of the beauty of the Azores lies under the feet

of anyone who walks in the archipelago (Teixeira, 2015). The traditional Portuguese pavement decorates public spaces with a very typical aesthetic.

SYMMETRIES

In mathematics, to “measure” the symmetry of a figure, it is important to identify the transformations that don't change that figure. The motif is not the most important thing, but how the invariant transformations occur.

Three mathematical objects are very common artistic pieces:

- *friezes*: motifs that repeat infinitely in one direction;
- *wallpapers*: motifs that repeat infinitely in two directions;
- *rosettes*: designs about a single point, within a limited region.

Many Portuguese pavements are beautiful friezes, wallpapers, or rosettes. To better understand the examples, although not exhaustively, it is important to mention some fundamental concepts for the mathematical classification of these three objects. Considering the two-dimensional case, there are four types of symmetry:

- *reflection symmetry* or *mirror symmetry* (related to a line, called the axis of symmetry);
- *rotational symmetry* (related to a point, called the rotation centre, and to an oriented angle);
- *translational symmetry* (related to a vector);
- *glide reflection symmetry* (resulting from the composition of a reflection with a translation of a vector parallel to the line that defines the reflection).

Friezes, wallpapers, and rosettes can be classified using these four types of symmetry. We remember that a half-turn is a rotational symmetry of 180 degrees. Also, we refer to a horizontal reflection, when the axis of symmetry has the same direction as the frieze, and to a vertical reflection, when the axis of symmetry is perpendicular to the frieze.

There are several notations for the symmetry classifications. William Thurston (1946-2012) proposed the interesting *Orbifold notation*. This notation has two advantages: it is reasonably simple to understand, and it is related to the *Magic Theorem*, which we will detail later. As an example, let us discuss the Orbifold signature of Avenida Miguel Bombarda in Lisbon (Figure 1a).

First, one identifies the *invariant rotations*, i.e., rotations that leave the wallpaper unchanged. In the proposed example, the wallpaper accepts two different rotations of 120° . The centres are marked in Figure 1b. The symbols «3» indicate the angles of rotation (third turns).

Second, one identifies the *axial reflections*. The symbol «*» is used to point out that there are symmetries of that type. If a wallpaper does not accept axial reflections, the signature has no «*».

Third, the centres of rotation *may belong (or not) to axes of reflection symmetries*. Thurston's notation is positional in the sense that the rotations whose centres do not belong to axes are placed to the left of the «*» and the centres that belong to axes are placed to the right of the «*». In our example, the signature is «3*3» since one centre belongs to a mirror axis and the other centre does not belong to any axis (Figure 1c).

Fourth, one identifies the *sliding reflections*, using the symbol «x» to denote them. When a wallpaper has no symmetry other than the translational symmetry, its signature is «0». The signature «**» is used when the wallpaper admits *exactly* two distinct axial reflections - by convention, that is the only case where the symbol «*» appears twice in the signature.

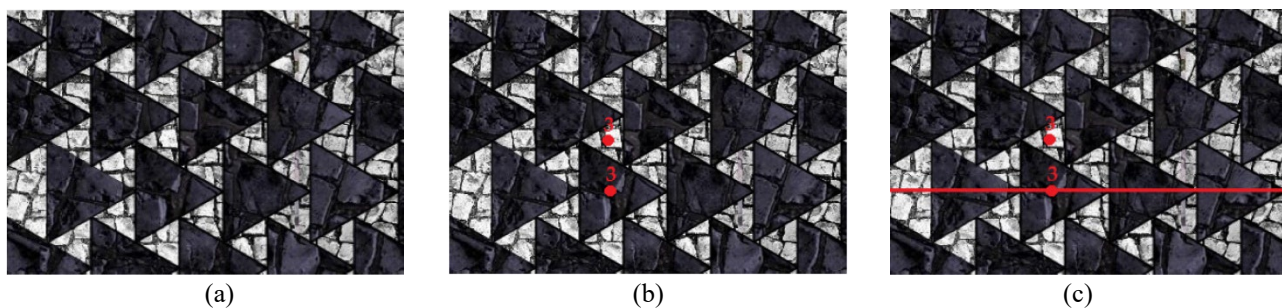


Figure 1 Wallpaper in Avenida Miguel Bombarda, Lisbon, with signature «3*3».

Regarding friezes, it is important to observe that there is always at least one centre of rotation. Looking at Figure 2, it is possible to visualize that the further away the centre is, the more the rounded band adapts to the original frieze. Due to that fact, a frieze always accepts the *point at infinity* (∞) as a centre of rotation.

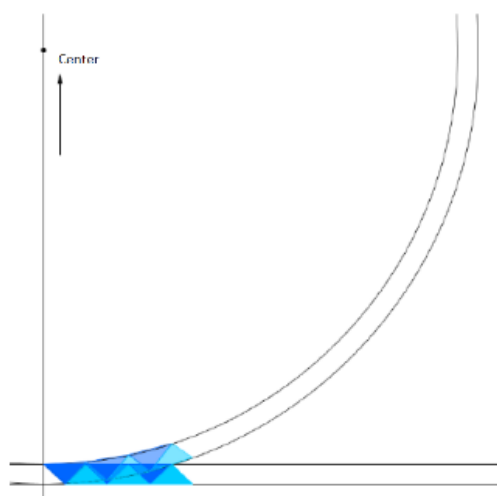


Figure 2 Centre of rotation of a frieze.

If a frieze does not accept any horizontal symmetry (neither axial reflection, nor sliding reflection, nor half-turn), then it accepts *two points at infinity*. Figure 3 shows the seven friezes in Lisbon.

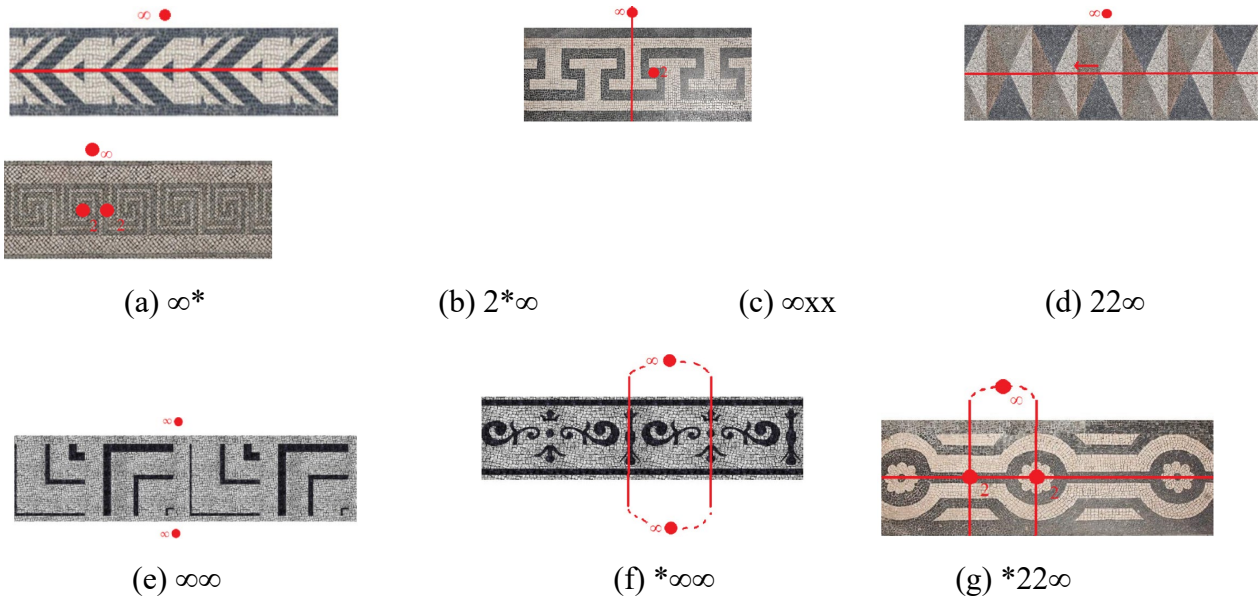


Figure 3 Several friezes in Lisbon.

Assigning values to the signature symbols accurately, the *Magic Theorem* states the existence of an invariant sum.

1. Symbols «2», «3», «4», «6», and «∞» are assigned the values $\frac{n-1}{n}$ if they are to the left of «*» and $\frac{n-1}{2n}$ if they are to the right of «*». In the case of a point at infinity, $\frac{n-1}{n} \rightarrow 1$ and $\frac{n-1}{2n} \rightarrow \frac{1}{2}$.
2. Symbols «*» and «x» are assigned the value 1.
3. Symbol «o» is assigned the value 2.

No matter what the signature is, *the sum of the symbol values is 2* — that is the *Magic Theorem*. If we make a list with all possible combinations of symbols whose sum is 2, there are exactly 24 possibilities: 17 wallpapers and 7 friezes.

Table 1. Summary of all possible combinations (Orbifold notation).

17 wallpapers					7 friezes		
*632	632	*442	442	*333	*22∞	22∞	2*∞
333	*2222	2222	4*2	3*3	2*22	*∞∞∞	∞∞∞
22*	**	*x	xx	22x	O	∞*	∞x

A flowchart to classify the frieze patterns is presented in Figure 4, a similar flowchart of the wallpapers can be seen in Carvalho *et al.*, 2016b. Considering again our initial example, «3*3» results in the sum $2/3+1+1/3=2$. The proof of the *Magic Theorem* is outside the scope of this text (Conway *et al.*, 2008).

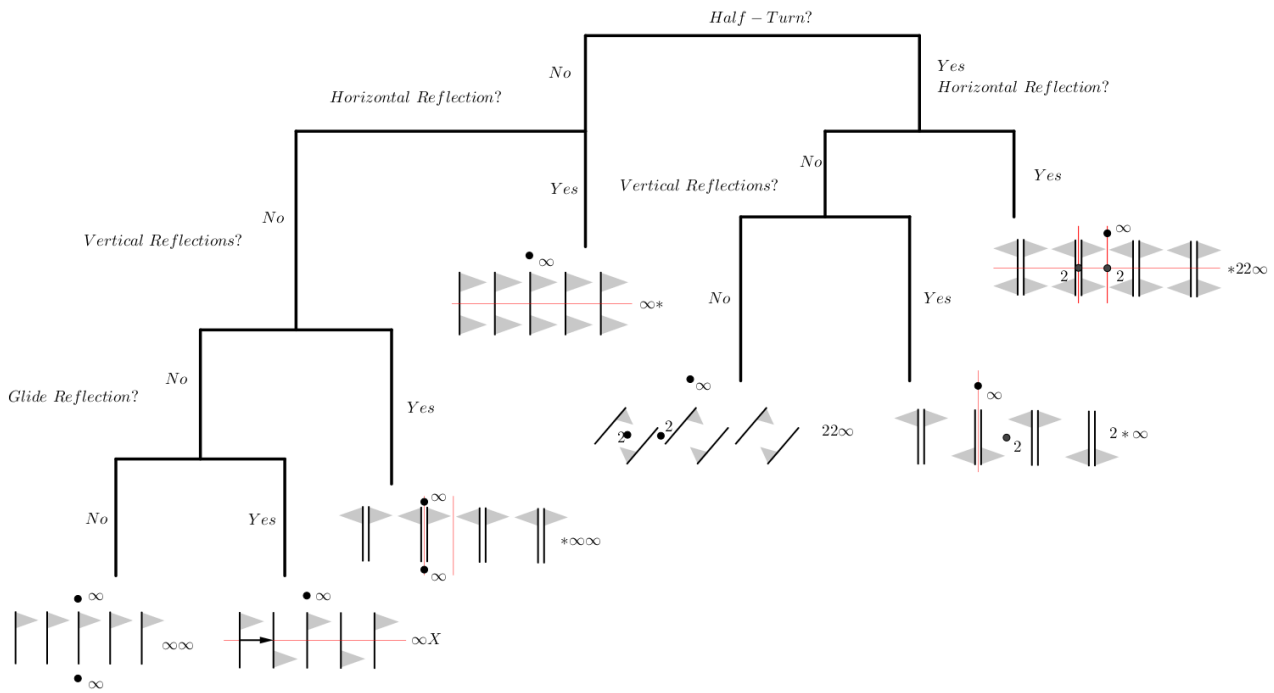


Figure 4 Flowchart of the frieze patterns (Carvalho *et al.*, 2016)

Rosettes are figures with rotational symmetries and, in some cases, mirror symmetries. It can be proved (Martin, 1982) that only two situations can occur: cyclic rosettes (figures with n rotational symmetries, C_n) or dihedral rosettes (figures with n rotational symmetries and n mirror symmetries, D_n). The rotational symmetries have all the same centre and are related to amplitudes of $360/n$ degrees and their multiples. The axes of symmetry, when existing, all go through the rotation centre.

After identifying the basic motif that is repeated around the rotation centre, all we must do is check for rotational symmetries alone (C) or whether there are also mirror symmetries (D) (Figure 5). A rosette C_1 is called asymmetric (because it lacks symmetry), since the only way it can match the initial position is by the trivial rotation of $360/1 = 360$ degrees (or 0 degrees). A rosette D_1 , besides the trivial rotation, presents a mirror symmetry.

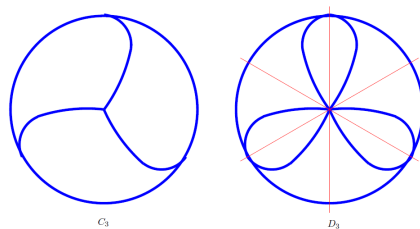


Figure 5 Cyclic rosette (on the left);
 Dihedral rosette (on the right).

THE AZOREAN DECK OF CARDS

The sidewalks of Lisbon are artistic works under our feet. An interesting, related project is the *Simetria Passo a Passo* supported by the Calouste Gulbenkian Foundation and organized by Ana Cannas da Silva in 2010 (Cannas da Silva,

2016). Until June 23, 2010, on the sidewalks of Lisbon, all the seven friezes, cyclic rosettes, dihedral rosettes, and eleven of the seventeen types of wallpapers were detected. According to Thurston’s orbifold notation, only missing 632, 333, *333, 22x, 4*2, and O.

In Carvalho *et al.*, 2016b, the authors found one example classified as O so, nowadays, five are missing. Under a joint project, Ludus Association – University of Lisbon, a card deck about the subject was designed *Playing Symmetries: Sidewalks of Lisbon* (Carvalho *et al.*, 2016a).

After the *Playing Symmetries: Sidewalks of Lisbon*, the authors launch the *Azorean Sidewalks Deck* (Figure 6). This deck suggests a walk through the 9 islands of the Azores and the symmetries of its pavements. The deck has problem cards (spades and hearts) and solution cards (clubs and diamonds). The problem cards contain two challenges: (1) identify the place where the pavement shown can be found; (2) identify its symmetries. Some examples of problem and solution cards are presented in Figure 7.



Figure 6 Deck as a map for the Azorean Sidewalks.

All the seven friezes, cyclic rosettes, dihedral rosettes, and some of the seventeen types of wallpapers were detected in the Azores islands and the main goal was to represent the nine islands (Figure 6). The cards have four wallpapers (**, *2222, 22*, and *442), fourteen friezes (two examples of each), and eight rosettes (four of each). Also, the deck box has an ambigram and mirror Jokers to identify simple reflections. The friezes, due to their unidimensional nature, are found on sidewalks; the patterns, which can cover plane surfaces, show up in plazas. The rosettes are limited planar configurations, usually presented in circular frames.

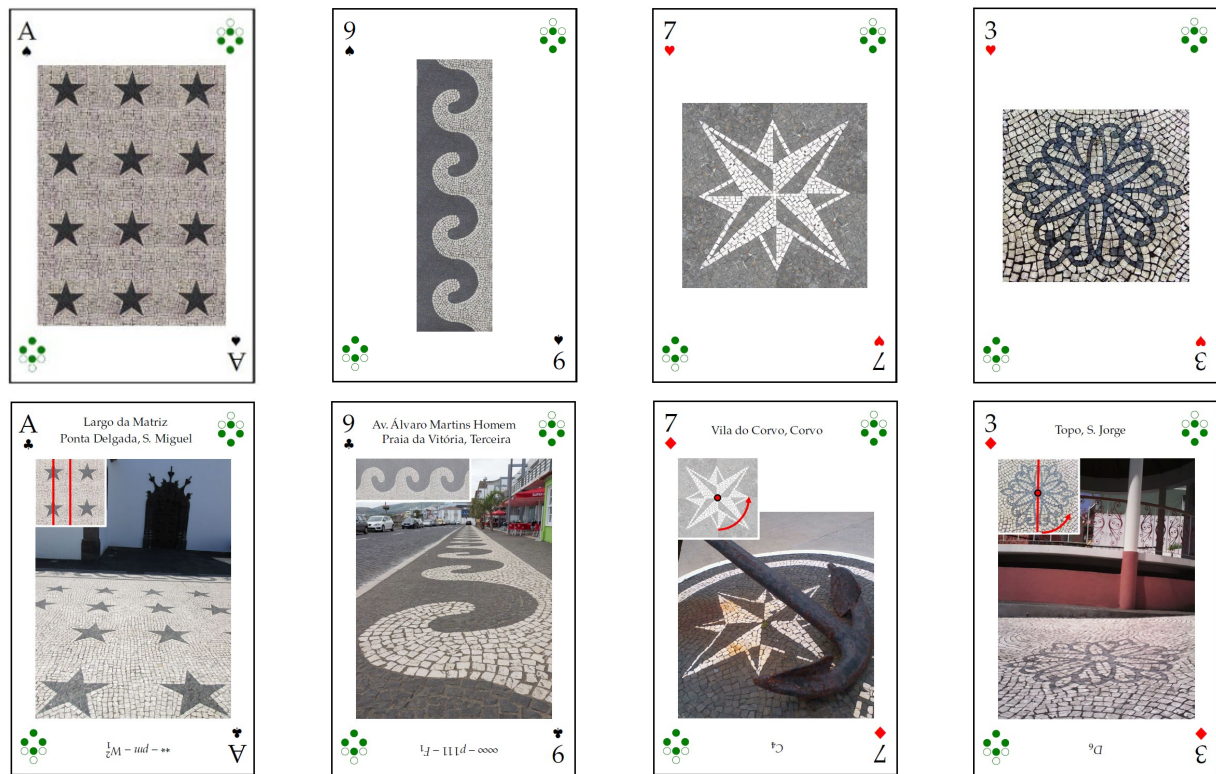


Figure 7 Some problem and solution cards (one wallpaper, one frieze, and two rosettes).

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