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ANALOGY AND SYMMETRY: SOME HEURISTIC SPATIAL GENERALIZATIONS OF THE OCTAGON

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Abstract: *This paper uses a heuristic approach to reflect on the generalization in geometry by analogy from the plane to the space. Focusing in the regular octagon, several criteria are presented to answer of the question of which polyhedra would be "analogous" to the octagon. The presented construction procedures generate polyhedra which are similar to the octagon, and the symmetry of the obtained solids is used as key indicator of the level of analogy. The procedures can be extended to other shapes and planar patterns with octagonal symmetry.*

Keywords: Analogy; Generalization; Symmetry; Octagon; Polyhedra.

INTRODUCTION

In architecture, the regular octagon is used as a transition from the square to the circle. In fact, the ground planes of numerous buildings are bi-dimensional representations of constructive solutions (mainly by means of octagonal prisms and pyramidal frustums) for a three-dimensional transition from the cube (room) to a hemispherical dome (ceiling) (Figure 1 top left). So, in practice, it could be stated that architects are use spatial versions of the octagon. Also, the deep coffers designed in several coffered ceilings constructed during the early Renaissance period in Spain (Figure 1 top right) provide spatial patterns, which somehow resemble a planar pattern, in such a way that, in the space, certain polyhedra could play a similar role than the octagon in the plane (Redondo, 2022). Thus, it could be argued that the octagonal prism and the pyramidal frustum could be regarded as rough spatial generalizations of the octagon. What is more, we could expect that the generalizations of a geometric object may be similar to the generalized object, not only in the shape, but also in their geometric behaviour in the space. In other words, we desire a good three-dimensional generalization of the octagon that was not only similar to it (in the general meaning of resemblance), but also analogous.

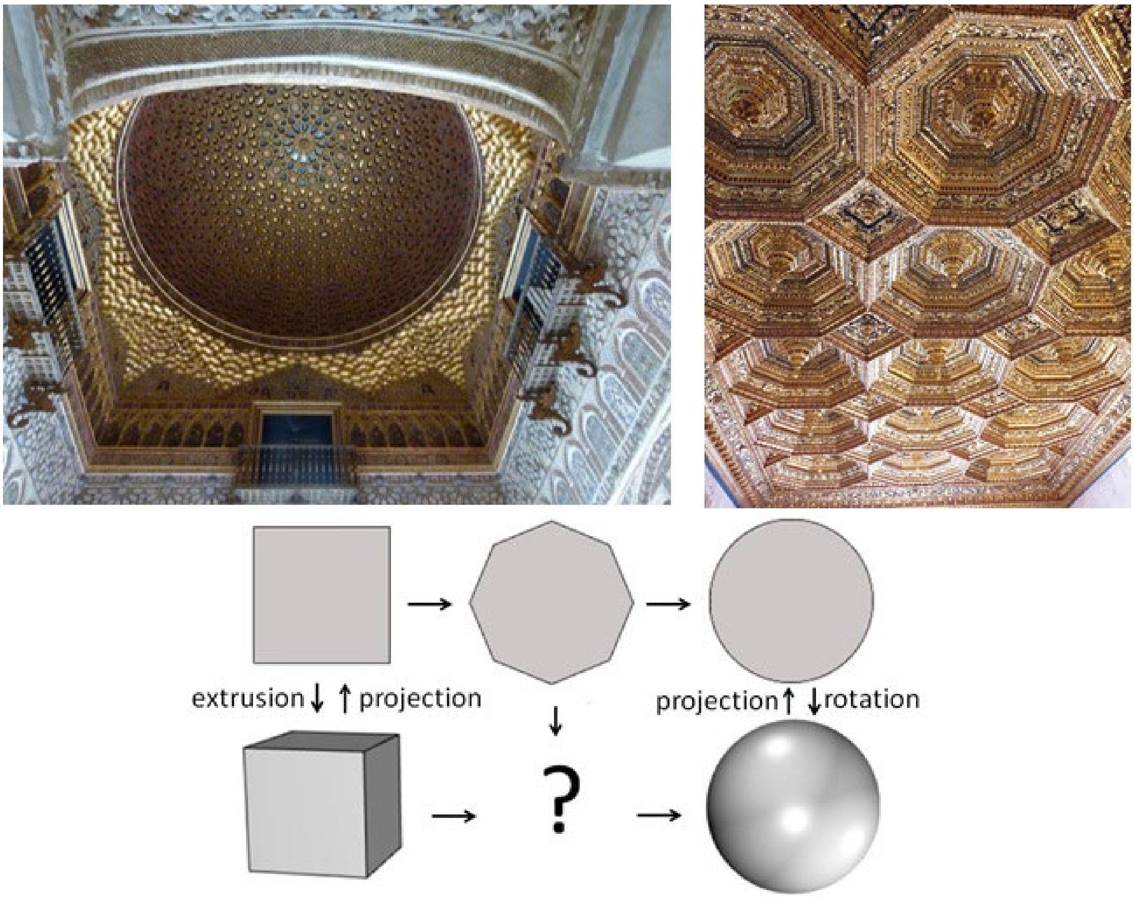

Figure 1 Dome of Salon de Embajadores in Alcazar de Sevilla (Top left).
Diagram of analogy plane/space (bottom). Coffered ceiling in Generalitat (Top right) (Photos by the author)

The aim of this paper is to answer two questions: "*Which are the polyhedra analogous to the regular octagon? What is the most analogous of them?*" So, we face a problem of *generalization* from the plane to space, that we will try to solve by means of *analogy* as a *heuristic strategy*.

ON ANALOGY IN GEOMETRY

Analogy can be defined as a type of similarity on a more conceptual level. According to Polya (1945, p. 37), two objects are analogous, if they agree in certain relations of their respective parts. The discovery of analogies between polygons and polyhedra may suggest that in the essence of both concepts something exists being invariant by passing from the plane to the space. So, according to the general meaning of symmetry*, analogous geometric objects are symmetric*. There are no known standard infallible procedures to recognize analogous geometric objects; in addition, the concept of geometric analogy is often ambiguous because different types and level of analogies can be reasonable. For instance, a triangle can be considered analogous to a tetrahedron, but also to any pyramid (Polya 1954, pp. 13-15).

In order to clarify the notion of analogous polyhedron, let us take a look at an intuitive commonly accepted analogy (Figure 1 bottom): A sphere is analogous to a circle. In fact, both objects share descriptive properties, as "*they have no corners/vertices nor sides/edges*", but also an ellipsoid. Here, we must give a clearly definable relation: "*all their points are located to a fixed distance to a given point*". But still it is no clearly enough, because the set of points which satisfies that condition depends on the considered norm, $\|\cdot\|_1$, $\|\cdot\|_2$ or $\|\cdot\|_\infty$ to define the distance (Figure 2 top). Figure 2 shows that in the plane we can find analogy's relations between the circle and the square: and in the space, among the sphere, the octahedron and the cube. So, the cube and the octahedron should be analogous as well[1]. Note that, the occult analogy derives from the three maximal cross-sections by the symmetry plans of both polyhedra, and the no-analogy emerges when we consider the way to fill the space (Figure 2 bottom).

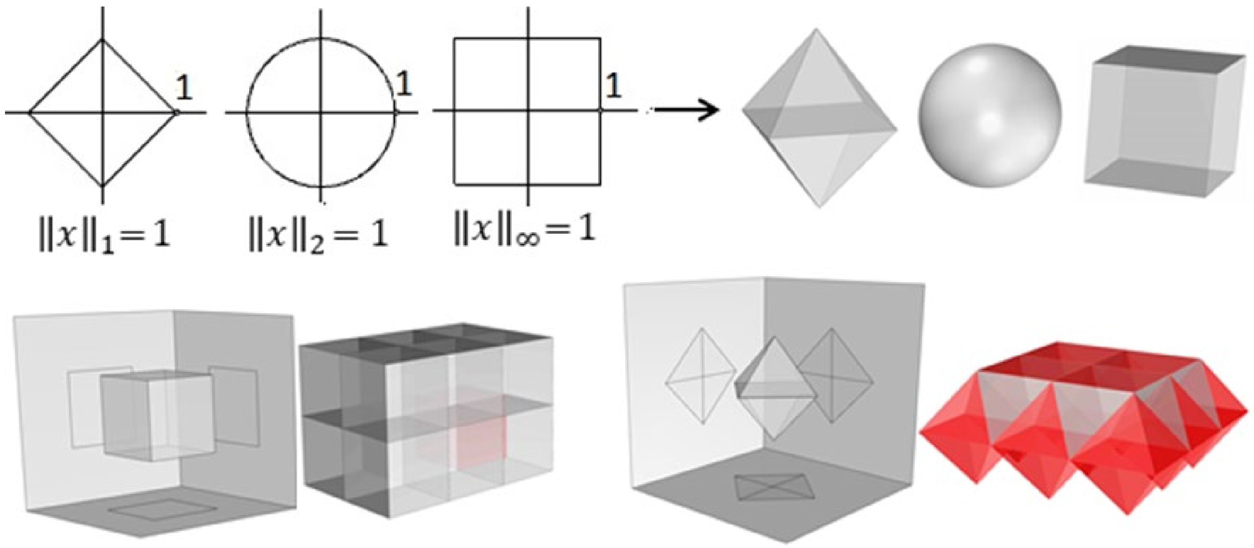

Figure 2 An analogy between "square & octahedron", "circle & sphere" and "square & cube) (top).
The projection of the maximal cross-sections of the cube and the octahedron; and the honeycombs (bottom)

The preceding reflections allow us to define our *notion* of polyhedron analogous to the octagon, and present construction procedures which guarantee analogy for the constructed polyhedron. For that we will translate the diverse ways to construct the regular octagon in the plane to space, by replacing their elements and relations with their corresponding analogous forms (e.g. square → cube, circle → sphere, side → face, line bisector → plane bisector,…) and including the "*extrusion*"+ the "convex hull procedure" and the "*rotation*" as heuristic tools. By nature of the general concept of analogy, a deductive proof is impossible to give, however in this context neither would it make any sense to provide it, because the aim of this paper is not the *proof*, but the *discovering* of plausible analogies between the polygon and the polyhedron.

The style and procedures that are presented in this paper configure a hybrid methodology, by combining the use of geometric and algebraic tools; however, the whole process is directed by the analogy as heuristic, so we can accept that we are heuristically reasoning. We will accept as control procedure of the generalization, a *looking back*, from the "*space to the plane*", by means of the contour of the cross-sections of the polyhedron; and the goodness of the achieved analogy will depend on the symmetry of the obtained polyhedron.

ON HEURISTIC

The word heuristic is used with several meanings. Originally, *heuristic* or "*ars inveniendi*" was the name given to the study of the discovering methods. In ancient Greece, attempts to construct a heuristic system are found in works of Euclides and Pappus. In the 17th century, the adjective heuristic appears involved in the discussions of philosophers-mathematicians (Descartes, Leibnitz, Tchishauss,…) regarding the abstract discovering of the Truth and the fundaments in mathematics. At that time, the term heuristic appears to be used as a synonym of discovering results by means of the algebraic calculations (*analysis*)[2], in contrast to the need of the validation of results by means of deductive proof (*synthesis*)[3]. Revived by G. Polya (1945, 1954, 1968) and A. Schoenfeld (1985), the modern heuristic focuses in the understanding of the processes of discovering and solving problems, *independently of the contents*, and the set of applicable rules, suggestions, tools and strategies to achieve provisional solutions, not definitive or complete, but plausible.

CONSTRUCTION OF A POLYHEDRON ANALOGOUS TO THE REGULAR OCTAGON

Taking into account the preceding observations, we can stablish that a polyhedron analogous to the regular octagon should be

> *I) A convex polyhedron with circumscribe sphere and octahedral symmetry.*
>
> *II) At least, three of its planes of symmetry produce octagonal cross-sections.*
>
> *III) The polyhedron generates a honeycomb analogous to some planar pattern by octagons.*

Note that requirements I and II allow the generalization of octagon's shape and requirement III guarantee that the octagon and the polyhedron fill the plane and the space in an analogous way, respectively. The analogy can be checked by means of the contour of its cross-sections and/or its orthogonal projections.

Construction 1:

The regular octagon can be obtained by intersection of two squares. In the space, the intersection of four cubes generates a chamfered cube (Figure 4 top) which satisfies I and II and generated a honeycomb analogous to the planar pattern by squares and elongated hexagons (Figure 3 left).

Construction 2:

In the plane, the regular octagon can be obtained by removing the vertices of the square (Figure 3 middle), then the truncated cube is analogous to the octagon. This polyhedron can be also obtained as the convex hull of the extrusions of an octagon along three orthogonal directions (Figure 4 centre). The honeycomb generated by truncated cubes and octahedra, is the trunc cubille of J. H. Conway (2008, p 296); with orthogonal projections that recover the pattern by octagons and squares, so the truncated cube satisfies III, but II is not satisfied. The octagon can be dissected into four polygons of type kite K_1 and K_2 (Figure 3 middle and right). Then the analogous polyhedron may to be dissected into eight analogous polyhedral kites.

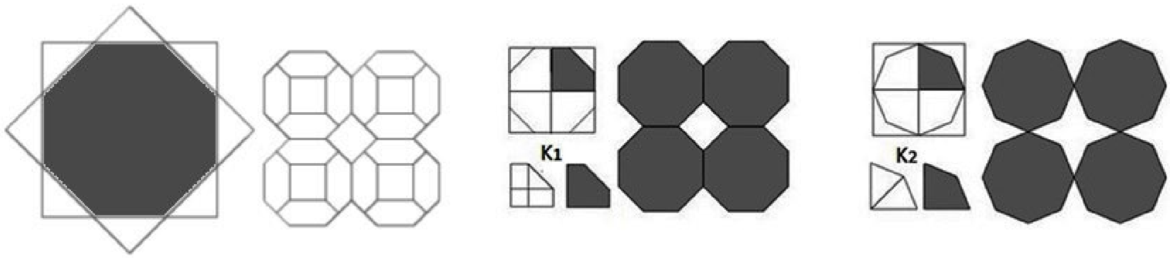

Figure 3 Octagon as intersection of squares and pattern by squares and elongated hexagons (left). Semiregular pattern by octagons and squares (middle). Pattern by octagons and four-pointed stars (right)

Construction 3:

The construction of a polyhedral kite analogous to K_1 can be made by means of the convex hull of three mutually perpendicular extrusions (Figure 4 bottom). The resulting polyhedron is the rhombicuboctahedron. This polyhedron satisfies I and II and is present in several honeycombs with cross-sections or orthogonal projections which reproduce the planar pattern by octagons and squares shown in Figure 3 (middle). For instance, the cantellated cubic honeycomb (rhombicuboctahedron, cuboctahedron and cube), called 2-RCO-trille by J. H. Conway (2008, p 297). Then, also satisfies condition III.

Construction 4:

A polyhedral generalization of the kite K_2 can be obtained by means of three mutually perpendicular extrusions "$K_2 \rightarrow$Two triangular prisms" (Figure 5 top). The union of eight of these polyhedral kites would be a distorted deltoidal icositetrahedron, which by means of the triangulation of its faces, can be converted into a variation of the disdyakis dodecahedron. The arrangement in row and columns "vertex to vertex" of this polyhedron, determines a grid and also a complementary grid of star polyhedra; then the four-pointed star is generalized too (Figure 5 top). Taking as unity the length of the perpendicular sides of the planar kite, we can use Cartesian coordinates. By considering the centre of the polyhedron as the origin of the system, the vertices *A, B, C, D, E* and *F* of the polyhedral kite can be given by $(1,0,0)$, $(\sqrt{2}/2, \sqrt{2}/2,\ 0)$, $(0,1,0)$, $(0, \sqrt{2}/2, \sqrt{2}/2)$, $(0,0,1)$ and $(\sqrt{2}/2\,, 0, \sqrt{2}/2)$.The intersection of the extrusions gives point $P(\sqrt{2}/2\,, \sqrt{2}/2\,, \sqrt{2}/2)$ (see Figure 5 top) which does not belong to the sphere determined by A, B, C, D, E and F. However, condition I can also be satisfied by moving point P on the right line $x = y = z$, to the point $(\sqrt{3}/3\,, \sqrt{3}/3\,, \sqrt{3}/3)$.

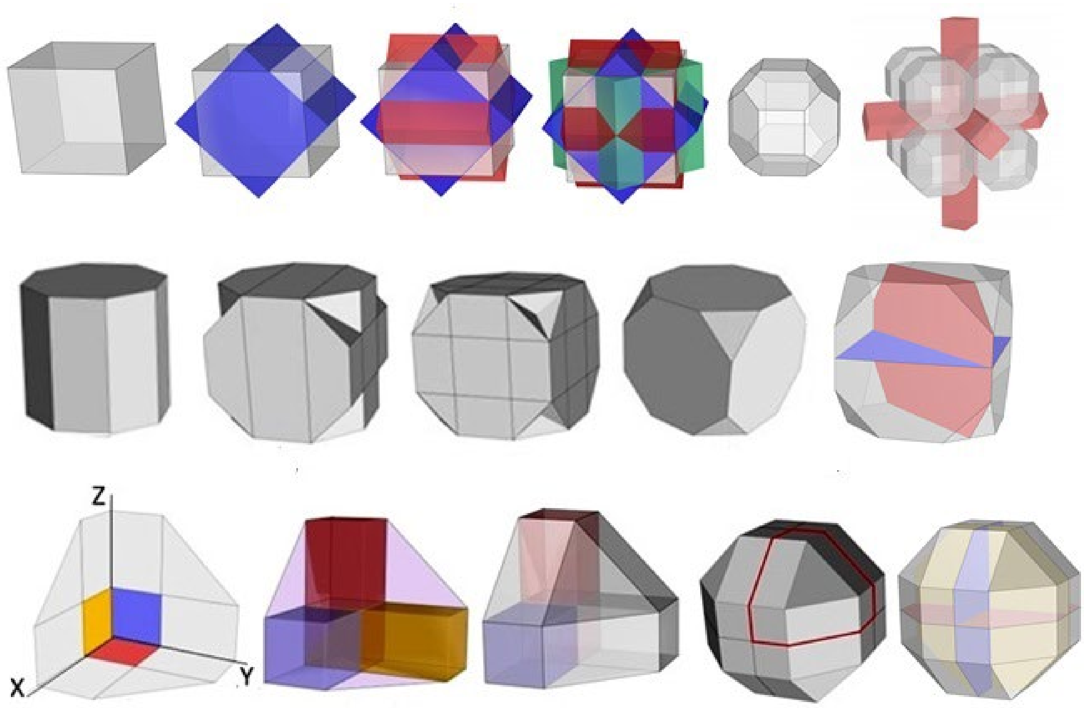

Figure 4 Constructions of the chamfered cube and the honeycomb (top), the truncated cube (centre) and the construction of the rhombicuboctahedronl as union of eight polyhedral kites K1 (bottom)

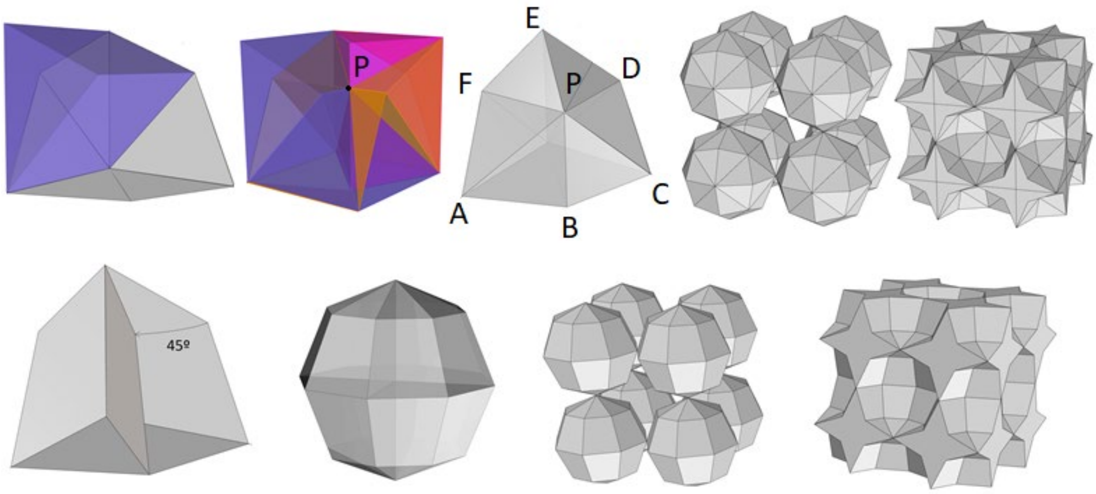

Figure 5 Construction of the polyhedral kite analogous to K_2, and grids of disdyakis dodecahedra (variation) and generalized four-pointed stars (top). Polyhedral kite by rotation of K_2, generated 32-faced polyhedron and generated grids.

Construction 5:

The second polyhedral kite analogous to K_2 could be obtained by rotation procedure of K_2 (Figure 5 bottom). The result is a 32-faced polyhedron, just a variation of the Campanus sphere, called *septuaginta duarum basium* by Pacioli (Pacioli, 1509, pl XXXIX). In both cases, the generated honeycomb is analogous to the pattern by octagons and four-pointed stars shown in Figure 3 right.

Different but equivalent constructions of the two preceding polyhedra can be seen in Redondo (2022).

Note that the key to the preceding construction procedures is the use of a quarter of the octagon, to obtain an eighth of the polyhedron, then by successive symmetries, the obtained polyhedron necessarily has octagonal symmetry. The method can be extended to generalize other figures as four-pointed stars and planar pattern with the same symmetry than the regular octagon. For instance, Figure 6 shows a spatial pattern analogous to the planar pattern by regular hexagons, squares and four-pointed stars of angle 30ºat the points.

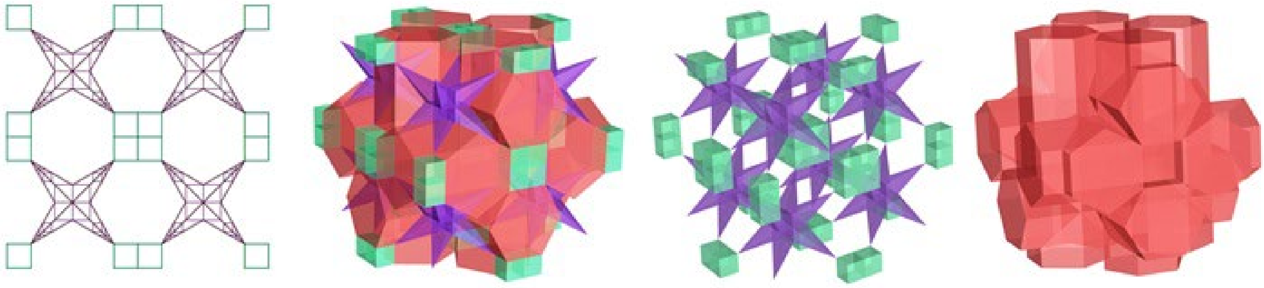

Figure 6 Honeycomb analogous to the uniform tiling $(4.6.4^*_{\pi/6}.6)$

CONCLUSION

We present an example of generalization by analogy in geometry. Several polyhedral analogous of the regular octagon are constructed. In concordance with the said before, the most symmetric polyhedron should be the most analogous as well. So, the analogy of the constructed polyhedral could be ordered as follows: truncated cube < chamfered cube ≤ rhombicuboctahedron ≤ "disdyakis dodecahedron" (variation) < 32-faced polyhedron. The construction procedure can be extended to other shapes and planar patterns with the same symmetry than the regular octagon.

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[1] "A general heuristic: where there is a structural similarity between two mathematical expressions, seek a corresponding structural similarity between the mathematical matters they express" (Larvor p. 201).

[2] Directly related with the search of a universal language.

[3] In his *Medicina mentis* (1695) Tschirnhauss emphasizes the necessity of combine synthesis and analisys.

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Antonia Redondo Buitrago has a PhD in Functional Analysis (complex powers of operators). She became interested in the field of continued fractions and geometry. She is specialized in the study of proportions (golden section, metallic means, cordovan proportion, plastic number), non-constructible polygons and Spanish historic carpentry. Her contributions are mainly focused on the interdisciplinary aspects in the domain of the art, design, architecture, and the relationship between them.