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ON THE UBIQUITY OF SYMMETRY IN LOGICAL GEOMETRY

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Abstract: *The research framework of Logical Geometry investigates two major sets of logical relations holding between formulas in a logical fragment or between concepts in a lexical field. On the one hand, there is the classical set of Aristotelian relations of contradiction, (sub)contrariety and subalternation/implication. On the other hand, there is the set of Duality relations of internal, external and dual negation. Networks of logical relations are then given a visual representation by means of logical diagrams, the most well-known among which is no doubt the so-called Square of Oppositions. In this paper we investigate the omnipresent role of symmetry in Logical Geometry. This role can, first of all, be considered both from a logical/semantic perspective and from a geometrical/visual perspective. Secondly, symmetry turns up both on the first-order level of individual logical relations/ diagrams and on the second-order level of sets of logical relations/diagrams. This yields the four steps in our analysis: (i) symmetry within logical relations: logical & first order, (ii) symmetry between logical relations: logical & second-order, (iii) symmetry within logical diagrams: geometrical & first-order, and (iv) symmetry between logical diagrams: geometrical & second-order.*

Keywords: Symmetry; Logical Geometry; Aristotelian/Duality Relations; Opposition/Implication Relations; Aristotelian Diagrams

INTRODUCTION

In this paper we investigate the omnipresent role of symmetry in the research framework of Logical Geometry (LG). This framework (https://www.logicalgeometry.org) investigates logical relations of opposition/negation and implication/consequence holding between formulas ϕ and ψ in a logical fragment or between concepts in a lexical field. One classical set of logical relations is that of the so-called Aristotelian relations:

1.	contradiction	CD(ϕ,ψ)	ϕ and ψ cannot be true together and ϕ and ψ cannot be false together
2.	contrariety	CR(ϕ,ψ)	ϕ and ψ cannot be true together but ϕ and ψ can be false together
3.	subcontrariety	SCR(ϕ,ψ)	ϕ and ψ can be true together but ϕ and ψ cannot be false together
4.	subalternation	SA(ϕ,ψ)	ϕ implies ψ but ψ does not imply ϕ

Networks of logical relations are then given a visual representation by means of logical diagrams. In the case of the Aristotelian relations, the most well-known diagram is no doubt the so-called Square of Oppositions in Figure 1(a), but more complex hexagonal/octagonal extensions and even three-dimensional diagrams have been proposed and studied in LG.

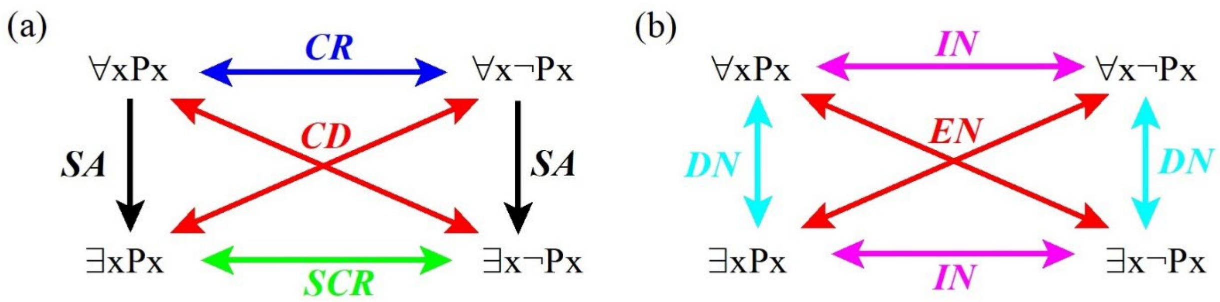

Figure 1 (a) classical Aristotelian square (b) classical Duality Square

Not surprisingly, the role of symmetry in LG can, first of all, be considered both from a logical/semantic perspective and from a geometrical/visual perspective. Secondly, symmetry turns up both on the first-order level of individual logical relations/diagrams and on the second-order level of sets of logical relations/diagrams. Cross-cutting these two bipartitions yields a four-step analysis:

(i)	symmetry *within logical relations*:	logical & first-order
(ii)	symmetry *between logical relations*:	logical & second-order
(iii)	symmetry *within logical diagrams*:	geometrical & first-order
(iv)	symmetry between logical diagrams:	geometrical & second-order

SYMMETRY WITHIN LOGICAL RELATIONS

Considering symmetry from a logical perspective on the first-order level of the four Aristotelian relations individually, we first of all observe a fundamental distinction between the three symmetric opposition relations of contradiction, contrariety and subcontrariety and the non-symmetric implication relation of subalternation: CD(ϕ,ψ) if and only if CD(ψ,ϕ) and (S)CR(ϕ,ψ) if and only if (S)CR(ψ,ϕ), but *not* SA(ϕ,ψ) if and only if SA(ψ,ϕ). This set of four Aristotelian relations has been argued to be hybrid between a homogeneous set of four opposition relations (all four of which are symmetric) and a homogeneous set of four implication relations (only two of which are symmetric).

In addition, LG draws a fundamental distinction between the four Aristotelian relations on the one hand and the so-called Duality relations on the other hand. The latter consist of external negation (EN), internal negation (IN) and dual negation (DN) – i.e., the combination of external and internal negation – in Figure 1(b). In many cases, a given fragment of four formulas – such as that in Figure 1 – simultaneously exhibits an Aristotelian and a Duality constellation, which is why the two sets of relations have often been confused or conflated (Westerstahl, 2012). In LG, however, at least three arguments are given for the conceptual independence of Aristotelian and Duality relations (Demey & Smessaert, 2017/2018/2020; Smessaert, 2012; Smessaert & Demey, 2017). Firstly, all duality relations are symmetric – EN(ψ,ϕ)/IN(ψ,ϕ)/DN(ϕ,ψ) if and only if EN(ψ,ϕ)/IN(ψ,ϕ)/DN(ψ,ϕ) – but not all Aristotelian relations are, as observed above. Secondly, all duality relations are functional – any formula can only have one EN, IN or DN formula – but not all Aristotelian relations are functional – e.g., formulas can have more than one (S)CR formula. And thirdly: the duality relation IN corresponds to Aristotelian CR and/or SCR, thus breaking the one-to-one relationship between the two sets.

SYMMETRY BETWEEN LOGICAL RELATIONS

Moving to the second-order level of sets of logical relations, the two homogeneous sets of four opposition relations and of four implication relations – introduced above – first of all reveal a very nice parallel/symmetrical structure in terms of degree of informativity, which gives rise to a new type of (second-order) bilattice diagram (Smessaert & Demey, 2014). Secondly, when studying the interaction between Aristotelian and duality relations in square and octagonal diagrams, in particular when measuring the degree of convergence/divergence between the two sets, various patterns of (partial) symmetry arise, which can again be captured by means of a new type of (second-order) multigraph diagrams (Demey & Smessaert, 2020).

SYMMETRY WITHIN LOGICAL DIAGRAMS

The fragments of formulas considered in LG are usually assumed to be closed under negation: whenever formula ϕ belongs to the fragment, its negation ¬ϕ also belongs to the fragment. Furthermore, such pairs of contradictory formulas (PCDs) ϕ/¬ϕ are systematically located at diametrically opposed vertices of the Aristotelian diagrams (see Figure 1(a)). In other words, the Aristotelian relation of contradiction is represented by means of central symmetry, with the PCDs systematically yielding the diagonals of the diagrams. Also, when moving from 2D to 3D visual representations for more complex fragments, this same principle of central symmetry for the PCDs is maintained as much as possible. Due to the one-to-one correspondence between the Aristotelian relation of contradiction and the Duality relation of external negation, the Duality square in Figure 1(b) reveals exactly the same central symmetry with external negation visualised as the diagonals of the diagram. In addition, both the Aristotelian and the Duality squares in Figure 1 exhibit a perfect left to right mirror symmetry around a vertical axis through the centre of the diagram. Due to the symmetry of the dual negation, the Duality square in Figure 1(b) furthermore exhibits a perfect top to bottom mirror symmetry around the horizontal axis through the centre. The latter symmetry is lacking with the Aristotelian square in Figure 1(a) because of the non-symmetry of the subalternation relation.

Because they are closed under negation, Aristotelian diagrams (squares/hexagons/octagons) standardly contain an even number of vertices. Exceptionally, however, also pentagonal diagrams show up in which one formula ϕ is added to a standard square without including its negation ¬ϕ as well. In such cases, central symmetry is lost but the left to right mirror symmetry around the vertical axis is maintained. A similar situation arises when considering smaller, triangular substructures within Aristotelian diagrams from the perspective of the Cognitive Potentials of Free Ride and Derivative meaning (Shimojima, 2015; Smessaert *et al.*, 2020/2021). Such triangles cannot be closed under negation in principle, so they do not observe central symmetry, but they do observe the left to right mirror symmetry around the vertical axis, with the two SA triangles in the hexagon in Figure 2(a), or top to bottom around the horizontal axis with the CR/SCR triangles in Figure 2(b):

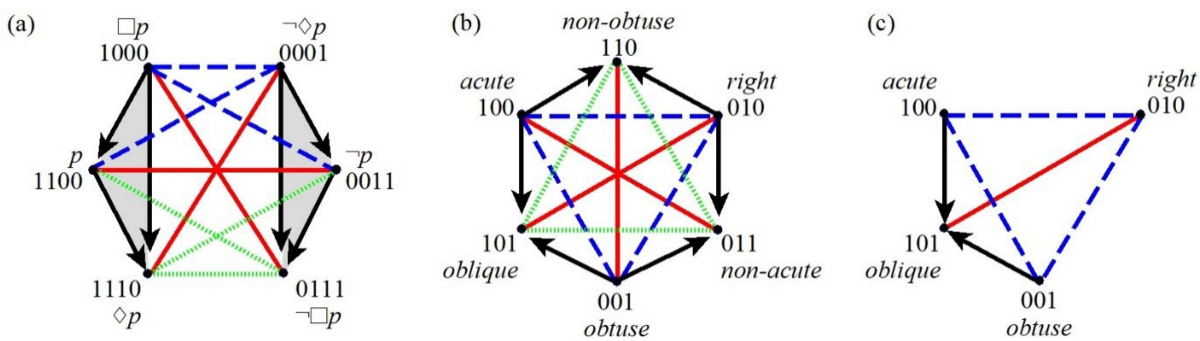

Figure 2 (a) Aristotelian SC-hexagon (b) Aristotelian JSB hexagons (c) kite structure

A final case in which the overall symmetry of Aristotelian diagrams is reduced is when the linguistic notion of markedness is considered in order to analyse the lexicalisation patterns for the various vertices of the diagrams. In the case of the hexagon in Figure 2(b), the vertices that receive the 'non-natural', complex lexicalisation are eliminated, thus yielding the so-called *kite*-structure in Figure 2(c) (Seuren & Jaspers, 2014; Smessaert & Demey, 2022). With the kite, central symmetry is lost since two of the three original PCDs are given up, and so is the mirror symmetry around the vertical axis. Only the mirror symmetry around the single PCD is maintained.

SYMMETRY BETWEEN LOGICAL DIAGRAMS

Finally, considering symmetry from a geometrical perspective on the second-order level of sets of diagrams, LG studies alternative visualisations for a given fragment of formulas and the Aristotelian relations holding of it. The two hexagons in Figure 2(a-b) belong to radically different Aristotelian families. Nevertheless, on a more abstract level they can systematically be related by "exchanging" the triangular SA and (S)CR constellations. The (dis)advantages of possible alternative representations are captured in terms of cognitive processing principles such as Congruence and Apprehension (Tversky, 2011; Smessaert *et al*., 2021). These same principles are invoked to account for the phenomenon of subdiagram embedding. The hexagons in Figure 2(a) and Figure 2(b) both contain three Aristotelian squares, but the rotational symmetry involved in the former is much more readily perceivable than that in the latter. Questions of symmetry also arise when standard Aristotelian diagrams are compared to the more iconic representation format of Logical Space diagrams, in which the cognitive mechanism of Free Ride (Shimojima, 2015; Smessaert *et al*., 2020) is more directly observable: diagrams that are informationally equivalent need not be computationally equivalent (Larkin & Simon, 1987).

CONCLUSION

In this paper we have investigated the omnipresent role of symmetry in the research framework of Logical Geometry. First of all, this role has been considered both from a logical/semantic perspective and from a geometrical/visual perspective. Secondly, symmetry has turned up both on the first-order level of individual logical relations/diagrams and on the second-order level of sets of logical relations/diagrams.

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