Abstract of a Lecture

Lição de Sintese

The geometry and dynamics of complex ordinary differential equations

Geometria e Dinâmica de Equações Diferenciais Ordinárias Complexas

Provas de Agregação

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1 Introduction

Roughly speaking my research concerns the dynamics and the geometry of complex ordinary differential equations. More precisely, a good part of my research has been focused on local and global aspects of holomorphic vector fields and/or foliations on 3-dimensional (regular) manifolds.

Since a significant part of my work concerns singularities of vector fields/holomorphic foliation in dimensions greater than or equal to 3, let us begin by singling out some global difficulties arising in these problems that have no 2-dimensional counterpart. In what follows, unless otherwise mentioned, (singular) holomorphic foliations are always of dimension 1. In other words, these foliations are locally given by the (local) orbits of a holomorphic vector field having singular set of codimension at least 2.

It is well known that the study of singularities of holomorphic foliations in dimension at least 3 is much more involved than the analogous problem in dimension 2. One of the main difficulties comes from the fact that these singularities encode some *global dynamics* on the divisors naturally associated to them. To explain the role played by these dynamics, we may think of the one-point blow-up of a "generic" homogeneous vector field on $\mathbb{C}^3$. While the singularities of the blown-up foliation become "simple", the understanding of the initial singularity clearly requires the understanding of the global foliation induced on the projective space identified with the exceptional divisor by the homogeneous vector field in question. In general, this foliation possesses a very complicated dynamics leaving no algebraic "object" invariant. This phenomenon does not occur in dimension 2 since the foliation induced on the projective line consists of the union of a unique leaf with finitely many singular points. Thus the dynamics obtained on the divisor is rather trivial.

Another well known additional difficulty in problems involving singularities in dimension greater than 2 is the absence of a desingularization procedure as effective as Seidenberg's theorem valid in dimension 2. In fact, according to Seidenberg, for every holomorphic foliation on a complex surface, there exists a finite sequence of one-point blow-ups such that the corresponding transform of the initial foliation possesses only *elementary singular points*. Recall that a singular point is said to be *elementary* if the foliation admits at least one eigenvalue different from zero at it. It turns out, however, that a faithful analogue of Seidenberg's result for foliations on 3-manifolds cannot exist: there are some non-simple singularities that are persistent under blow-up transformations (cf. [9] for details). Nonetheless different sorts of *final models* for certain "desingularization" procedures were described for example in the following papers [9], [26], [28] and, more recently, in [38].

We can also include in this "list of additional difficulties" some problems related to divergent normal forms (irregular singularities). In the case of saddle-node singularities in n-dimensional manifolds, where $n \geq 3$, the foliation may admit two or more eigenvalues equal to zero. In this case not only their formal normal forms are poorly understood but also the resummation techniques are much less developed. Naturally, the same problem occurs for every other singularity with highest rank of resonance relations.

My research presented in this text is contained in the papers [24], [29], [31], [32], [33], [35], [36], [38], [39] and [40]. Many of these papers solve long standing problems in the area, including the existence of invariant analytic surfaces for commuting vector fields, the topological type of leaves associated with Arnolds A^{2n+1} singularities, the proof that complete integrability is not a topological invariant of 1-dimensional foliations on $(\mathbb{C}^3, 0)$, a topological characterization of virtually solvable subgroups of $\text{Diff}\,(\mathbb{C}^2, 0)$ and, more recently, the proof of a sharp resolution theorem for singularities of complete vector fields in dimension 3.

2 Invariant analytic sets for (Lie algebras of) vector fields

The problem of existence of "invariant manifolds" has always been a central theme in the theory of dynamical systems. Among others, these "invariant manifolds" usually provide reductions on the dimension of the corresponding phase-space. For example, in the general theory of hyperbolic systems, the so-called stable manifolds are examples of invariant manifolds and, in fact, their existence form a cornerstone of the hyperbolic theory.

In the local theory of vector fields a hyperbolic singular point of the vector field is an example of a hyperbolic set. The existence of stable manifolds for such points is a consequence of the general theory and ensured by the well-known Stable Manifold Theorem. Stable invariant manifolds, however, may fail to exist if the singular point is no longer hyperbolic. For example, the integral curves associated to the (non-hyperbolic) vector field

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

are circles centered at the origin of $\mathbb{R}^2$ and the stable manifold is clearly empty in that case. Besides, even in the case where the singular point is hyperbolic the stable manifold may not provide reduction on the dimension of the corresponding phase-space. To find examples, it is sufficient to think of a planar vector field whose hyperbolic singular point has two conjugated non-real eigenvalues, i.e. two non-real and non pure imaginary eigenvalues. In fact, in this case, the corresponding integral curves are spirals and the stable/unstable manifold contains a neighborhood of the singular point.

The general problem of existence of "invariant manifolds" may also be considered in holomorphic dynamics. In this case, however, there arise some important differences with the real counterpart. For example, in the holomorphic setting we look for (proper) "invariant manifolds" that are analytic, which is a much stronger regularity condition. We allow, however, the analytic manifolds to be singular in the sense of analytic sets, i.e. they are "invariant varieties" as opposed to manifolds. In the sequel, the word "manifold" will be saved for smooth objects.

Briot and Bouquet were the first to consider the mentioned problem for holomorphic vector fields defined on a neighborhood of the origin of $\mathbb{C}^2$. Basically they looked for the existence of the so-called separatrices, i.e. of (germs of) analytic curves passing through the singularity and invariant by the vector field in question. In [2], Briot and Bouquet claimed the existence of separatrices for all holomorphic vector fields on $(\mathbb{C}^2, 0)$. Their proof, however, contained a gap and their classical work was completed only much later by Camacho and Sad. In fact, in their remarkable paper [5], Camacho and Sad prove the following.

Theorem 1 *[5] Let $\mathcal{F}$ be a singular holomorphic foliation defined on a neighborhood of the origin of $\mathbb{C}^2$. Then there exists an analytic invariant curve passing through $(0,0)$ and invariant by $\mathcal{F}$.*

Recall that in dimension 2 the singularities of every holomorphic foliation are necessarily isolated. Yet, the above Theorem applies equal well to holomorphic vector fields. In fact, if X is a holomorphic vector field on $(\mathbb{C}^2, 0)$ with a curve of singular points, then its components admit a non-invertible common factor. Up to eliminating these common factors, we obtain a vector field Y everywhere parallel to X and with isolated singular points. The Theorem can then be applied to Y and this yields an invariant curve for X as well. Alternatively, even the curves of zero of X might be thought of as an invariant curve for X whether or not it is invariant for the underlying foliation. In any case, the above argument applied to the vector field Y shows that there always exist a curve invariant by both X and the underlying foliation.

It is surprising that separatrices for holomorphic vector fields on $(\mathbb{C}^2, 0)$ always exist, despite the condition of analyticity for the invariant curve. Note that the invariant curves for the holomorphic vector field $y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$ mentioned above are given by the two straight lines $y = \pm ix$, which are totally contained in the non-real part of $\mathbb{C}^2$ (up to the singular point itself). In general, however, it is natural to allow the invariant curves to be singular at the singular point of the vector field, otherwise no general existence statement would hold. Indeed, as simple example, consider the holomorphic vector field $2y\partial/\partial x + x^3\partial/\partial y$. Since this vector field admits $f(x,y) = x^3 - y^2$ as first integral, it immediately follows that the only separatrix of X is the cusp of equation $\{x^3 - y^2 = 0\}$, which is clearly not smooth at the origin. Fortunately, allowing separatrices to be singular is not a problem, as the classical works of resolution of singularities of Hironaka can be used to desingularize them, as well as general invariant analytic sets.

Unfortunately, the existence of separatrices is no longer a general phenomenon in dimension 3. To begin with, note that when we move to dimension 3, it becomes necessary to distinguish between foliations of *dimension* 1 and foliations *of dimension* 2 *(or of codimension* 1*)*. In the case of 1-dimensional foliations, a separatrix still is an (germ of) analytic curve passing through the origin and invariant by the foliation. In the context of codimension 1 foliations, however, a separatrix should be understood as a germ of surface (i.e. 2-dimensional analytic set) passing through the origin and invariant by the foliation. In fact, in dimension 3, the existence of separatrices is no longer a general phenomenon, regardless of the dimension of the foliation, as it will made clear below.

After the work of Briot and Bouquet (whose results were regarded as complete for many years), Thom asked if invariant subvarieties always exist for codimention 1 foliations on $\mathbb{C}^3$. A counterexample to his question was given by Jouanolou [17] on $\mathbb{C}^3$. As it will become clear in the course of the discussion below, although the problem on the existence of separatrices in higher dimensions is *a priori* a local problem, the conterexample provided by Jouanolou is of *global nature* (more details below). Concerning 1-dimensional foliations, examples of 1-dimensional foliations without separatrices on 3-manifolds were found by Gomez-Mont and Luengo, [14], some years later.

1. Gomez-Mont and Luengo counterexample

The example provided by Gomez-Mont and Luengo relies on a simple idea though its implementation requires significant computational effort, which is carried with computer assistance. Yet, we may quickly describe the structure of their construction which relies on two simple observations. Consider then the foliation $\mathcal{F}$ on $(\mathbb{C}^3, 0)$ given by a holomorphic vector field satisfying the following consitions

(1) The origin $(0,0,0) \in \mathbb{C}^3$ is an isolated singularity of X

(2) $J^1X(0,0,0) = 0$ but $J^2X(0,0,0) \neq 0$, where $J^kX(0,0,0)$ stands for the jet of order k of X at the origin ($k = 1, 2$).

(3) The quadratic part X^2 of X at $(0,0,0)$ is a vector field whose singular set has codimension 2. Also X^2 is not a multiple of the Radial vector field $x\partial/\partial x + y\partial/\partial y + z\partial/\partial z$.

Assume that $\mathcal{F}$ has a separatrix C and consider the blow-up $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ centered at the origin. Denote by π the blow-up map so that $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ and let $\pi^{-1}(0)$ denote the exceptional divisor isomorphic to $\mathbb{CP}(2)$. Since X^2 is not a multiple of the Radial vector field, there follows that $\pi^{-1}(0)$ is invariant by $\widetilde{\mathcal{F}}$. Hence the restriction of $\widetilde{\mathcal{F}}$ to $\pi^{-1}(0)$ can be seen on a foliation of degree 2 on $\mathbb{CP}(2)$ (cf. item (3)).

Since $\pi^{-1}(0)$ is invariant by $\widetilde{\mathcal{F}}$, it follows that the transform $\pi^{-1}(C)$ of the separatrix C can only intersect $\pi^{-1}(0)$ at singular points of $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$. In other words, $\pi^{-1}(C)$ must be a separatrix (not contained in $\pi^{-1}(0)$) for one of the singular points of $\widetilde{\mathcal{F}}$.

Now, the second ingredient is as follows: as a foliation of degree 2 on $\mathbb{CP}(2)$, $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$ has at most (and generically) 7 singular points. Since it is hard to control the position of 7 points in $\mathbb{CP}(2)$, the authors of [14] proceed as follows.

(A) Let the singular points "collide" so as to have only 3 of them (position is then easily controlled)

(B) Each of the 3 singular points will have an eigenvalue equal to zero in the direction transverse to $\pi^{-1}(0)$. The 3 singular points are therefore saddle-node singularities (in dimension 3).

(C) Furthermore, arrange for the saddle-node singularities to have two equal (and non-zero) eigenvalues tangent to $\pi^{-1}(0)$. In other words, the singular points in question are (codimension 1) resonant saddle-nodes with weak direction transverse to $\pi^{-1}(0)$.

(D) As it well-known, it is easy to produce saddle-node singular points with no separatrix not contained in the invariant 2-plane associated with the non-zero eigenvalues

The remainder of the proof of [14] consists of showing that it is, indeed, possible to prescribe a quadratic X^2 and a cubic X^3, homogeneous components for the vector field X, so that all of the preceding conditions are satisfied.

Note that conditions (A), (B) and (C) depend only on the quadratic part X^2. The role played by the appropriated chosen cubic parte X^3 can be summarized as follows.

- it ensures each of the singular points of $\widetilde{\mathcal{F}}$ are isolated singular points coinciding with the corresponding singular points of $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$. Here we note that the homogeneous foliation associated with X^2 has zeros all along the fibers of $\widetilde{\mathbb{C}}^3 \to \widetilde{\pi}^{-1}(0)$ passing through the singular points of $\widetilde{\mathcal{F}}|_{\pi^{-1}(0)}$. Thus some higher order perturbation to X^2 is already needed to have isolated singular points.
- having made sure the singular points are isolated, the cubic part X^3 of X also takes care of condition (D)

As mentioned, the verification that all these conditions are compatible is conducted in [14] with the assistance of formal computations programs.

2. Jouanoulu conterexample

The example of a codimension 1 foliation admitting no separatrix provided by Jouanolou is the foliation $\mathcal{D}_n$ defined as the kernel of the (integrable) 1-form

$$\Omega_n = (yx^n - z^{n+1})\, dx + (zy^n - x^{n+1})\, dy + (xz^n - y^{n+1})\, dz\,,$$

with $n \in \mathbb{N}$. It can easily be checked that the kernel of Ω_n always contains the radial direction so that it naturally induces a line field, and hence a (1-dimensional) foliation $\mathcal{F}_n$, on $\mathbb{CP}(2)$. The 1-dimensional foliation $\mathcal{F}_n$ can also be viewed as follows: take the blow-up of $\mathbb{C}^3$ centered at the origin and let $\widetilde{\mathcal{D}}_n$ stands for the transformed foliation of $\mathcal{D}_n$. Since the Radial vector field

$$R = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

is tangent to $\mathcal{D}_n$, the leaves of $\widetilde{\mathcal{D}}_n$ are (generically) transverse to the exceptional divisor which, in turn, is isomorphic to $\mathbb{CP}(2)$. The intersection of the leaves of $\widetilde{\mathcal{D}}_n$ with the exceptional divisor are then the leaves of the 1-dimensional foliation $\mathcal{F}_n$ mentioned above.

The main result of Jouanolou states that $\mathcal{F}_n$ leaves no algebraic curve invariant. This implies that $\mathcal{D}_n$ cannot admit a separatrix. In fact, if $\mathcal{D}_n$ admits a separatrix S, i.e. an analytic surface S that is invariant by $\mathcal{D}_n$, then the intersection of the strict transform of S by the mentioned blow-up at the origin with the exceptional divisor would be an invariant algebraic curve for $\mathcal{F}_n$, contradicting the result of Jouanolou.

3. Many other conterexamples - a construction

There are many other examples of codimension 1 foliations on $\mathbb{C}^3$ without separatrix. Let me briefly explain how numerous similar examples can be constructed. Consider a homogeneous polynomial vector field Z defined on $\mathbb{C}^3$ and having an isolated singularity at $(0,0,0) \in \mathbb{C}^3$. Unless Z is a multiple of the Radial vector field R, it induces a 1-dimensional holomorphic foliation $\mathcal{F}$ on $\mathbb{CP}(2)$. Conversely every 1-dimensional foliation on $\mathbb{CP}(2)$ is induced by a homogeneous vector field on $\mathbb{C}^3$. Next we consider the 2-dimensional distribution of planes on $\mathbb{C}^3$ which is spanned by Z and R. The Euler relation (i.e. the equality $[R,Z] = (d-1)Z$, where d is the degree of Z) shows that Z, R generates a Lie algebra isomorphic to the Lie algebra of the affine group. The corresponding distribution is therefore integrable and hence yields a codimension 1 foliation that is going to be denoted by $\mathcal{D}$. Clearly the punctual blow-up $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ does not leave the exceptional divisor $E \simeq \mathbb{CP}(2)$ invariant (since the Radial vector field is tangent to $\mathcal{D}$) and thus $\widetilde{\mathcal{D}}$ induces a 1-dimensional foliation on $E \simeq \mathbb{CP}(2)$. This 1-dimensional foliation naturally corresponds to the intersections of the leaves of $\widetilde{\mathcal{D}}$ with E. However, by construction, it also coincide with the leaves of $\mathcal{F}$, the foliation induced by the homogeneous vector field Z. Just to give an example, the foliation induced on $\mathbb{CP}(2)$ by the Jouanolou foliation $\mathcal{D}_n$ is the one induced by the polynomial homogeneous vector field

$$Z = y^n \frac{\partial}{\partial x} + z^n \frac{\partial}{\partial y} + x^n \frac{\partial}{\partial z} \, .$$

As far as the existence of separatrices for $\mathcal{D}$ is concerned, the upshot of the preceding construction is as follows: if $\mathcal{D}$ possesses a separatrix, the tangent cone of this separatrix yields an algebraic curve in $E \simeq \mathbb{CP}(2)$ which must be invariant under $\mathcal{F}$. Nonetheless, today it is known that, in a very strong sense, most choices of Z leads to a foliation $\mathcal{F}$ that does not leave any proper analytic set invariant (cf. for example [20], [22]). As a result the codimension 1 foliation obtained by means of Z, R, for a generic choice of Z, does not have separatrices. We also note that, for these examples, no separatrix can be produced by adding "higher order terms" to $\mathcal{D}$. Jouanolou example fits this pattern.

This well known phenomena have led the experts (such as F. Cano, D. Cerveau and L. Stolovitch among others) to wonder that the "correct" generalization of Camacho-Sad theorem would involve codimension 1 foliations spanned by a pair of commuting vector fields (not everywhere parallel). The theorem below confirms their intuition and affirmatively answers it. The proof can be found in [31].

Theorem 2 *[31] Consider holomorphic vector fields X, Y defined on a neighborhood of the origin of $\mathbb{C}^3$. Suppose that they commute and are linearly independent at generic points (so that they span a codimension* 1 *foliation denoted by $\mathcal{D}$). Then $\mathcal{D}$ possesses a separatrix.*

The existence of separatrices for codimension 1 foliations in general was also the object of some remarkable papers such as [7] where it is proved, in particular, that a non-dicritical codimension 1 foliation on a neighborhood of $\mathbb{C}^3$ always possesses a separatrix. Recall that a codimension 1 foliation is called *dicritical* if there is a finite sequence of blow-ups with invariant centers such that the total exception divisor possesses an irreducible component that is not invariant for the corresponding transform of the initial foliation. However, as it follows from the preceding discussion, the set of foliations that fail to be *non-dicritical* is not negligible.

The main difficulty in establishing the existence of a separatrix for a dicritical codimension 1 foliation lies in controlling the dynamics of the 1-dimensional foliations induced on the non-invariant, i.e. dicritical, components of the exceptional divisor obtained after a suitable sequence of blow-ups.

The key to prove the above theorem in our case was to observe that these foliations always possess certain invariant algebraic curves provided that they are spanned by commuting vector fields. It is the existence of these algebraic curves that leads to the existence of separatrices. As it was to be expected, in the proof of our main result, it was needed to discuss the effect of the blow-up procedures of [7] and [8] on the initial vector fields X, Y and the fundamental desingularization results of these papers for codimension 1 foliations played a role in our argument.

Let me be more precise concerning the idea of the proof of Theorem 2. Essentially, we have to show that the phenomenon described above (i.e. in the case of a foliation generated by the Radial vector field and a homogeneous holomorphic/meromorphic vector field) cannot take place in our context, unless the 1-dimensional foliation induced on $\mathbb{CP}(2)$ admits certain invariant curves. To do that we shall consider the intersection of our codimension 1 foliation $\mathcal{D}$ spanned by the commuting vector fields X, Y with a given component of the exceptional divisor. Unless this component is invariant by the codimension 1 foliation, this intersection defines a foliation of dimension 1 on it. Except for some rather special situations that are already "linear" in a suitable sense, we are going to show that all the leaves of the latter foliation are properly embedded (in particular they are compact provided that the mentioned component of the exceptional divisor is so). This statement is, indeed, equivalent to saying that the corresponding foliation admits a non-constant meromorphic first integral as it follows from [18]. In general we shall directly work with the existence of a non-constant meromorphic first integral for foliations as above.

In view of the result proved by Cano and Cerveau, we have assumed the codimension 1 foliation $\mathcal{D}$ to be dicritical. We assumed first that a non-irreducible component appears immediately after a single one-point blow-up along the assumption that the first jet of X and Y at the point where we have centered the blow-up is zero. Since we are assuming the arising exceptional divisor not to be invariant by the transformed foliation, we have that there exists a holomorphic vector field Z tangent to $\mathcal{D}$ and such that its first non-zero homogeneous component Z^H of Z is multiple of the Radial vector field. There must then exist holomorphic functions f, g and h such that

$$fX + gY = hZ \,,$$

By exploiting the commutativity of the vector fields X, Y and the assumptions of "non-linearity" of X, Y we are able to prove that the transformed foliation of $\mathcal{D}$ induces a 1-dimensional foliation on the exceptional divisor with a holomorphic/meromorphic first integral. In fact, it is the first non-trivial homogeneous components of X, Y (denoted by X^H, Y^H, respectively) that will play a role. Since X, Y commute, so does X^H, Y^Y. We can prove by using the above relation that none of X^H, Y^H is a multiple of the Radial vector field and, consequently, they induce a 1-dimensional foliation on $\mathbb{CP}(2)$ by means of a punctual blow-up at the origin. The foliations induce by X^H and by Y^H must coincide since $\mathbb{CP}(2)$ is not invariant by $\widetilde{\mathcal{D}}$, the transform of $\mathcal{D}$. Furthermore, they must coincide with the restriction of $\widetilde{\mathcal{D}}$ to $\mathbb{CP}(2)$. It remains to prove that X^H possesses a (non-constant) first integral and details are provided in the paper.

Next we are led to consider the special situations of "linear" foliations that may not possess any non-constant first integral. Fortunately, in these cases the existence of a separatrix can be established by more direct methods. An example of a "linear case" would consist of a pair of vector fields X, Y with X linear and Y equal to the Radial vector field. These two vector fields commute and span a codimension 1 foliation whose (one-point) blow-up at the origin does not leave the corresponding exceptional divisor invariant. Furthermore the foliation induced on the corresponding exceptional divisor by the mentioned blown-up foliation coincides with the foliation induced on $\mathbb{CP}(2)$ by X. In particular X can be chosen so that the "generic" leaf is not compact. However, in this situation the foliation induced by X on $\mathbb{CP}(2)$ still has a compact leaf which "immediately" leads to the existence of the desired separatrix.

Recall that the singular set of foliations on 3-manifolds may have irreducible components of dimension 1. We have then deduced an analogue of the above mentioned results for blow-ups centered at a (smooth and irreducible) curve. Also in this case we have considered separately the "linear" and "non-linear" case. We had to adapt the notion of "linearity" in this case. To explain the need of this adaption (that will be explicitly defined below), let me describe an example that was pointed out to us by D. Cerveau and that illustrates the problem about the existence of first integrals as above as also some intermediary results which are crucial for establishing the existence of these first integrals in the case that a blow-up along a (smooth) curve is considered. This example goes as follows. Consider the pair of vector fields X, Y given by

$$X = zy\frac{\partial}{\partial y} + z^2\frac{\partial}{\partial z} \qquad \text{and} \qquad Y = x^2\frac{\partial}{\partial x} + axy\frac{\partial}{\partial y}\,.$$

These two vector fields commute and span a codimension 2 foliation denoted by $\mathcal{D}$. They also leave the axis $\{y = z = 0\}$ invariant. Consider the blow-up of $\mathcal{D}, X, Y$ centered over $\{y = z = 0\}$. The transform $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ does not leave the exceptional divisor invariant. Furthermore the leaves of the foliation induced on the non-compact exceptional divisor by intersecting it with the leaves of $\widetilde{\mathcal{D}}$ are themselves non-compact. The explanation for this phenomenon is that the blow-up of Y is regular at generic points of the exceptional divisor. Indeed, X is already regular at generic points of the axis $\{y = z = 0\}$. Hence this case must be considered as "linear" (indeed even "regular"). It then follows that the appropriate notion of order of a vector field relative to a curve is such that the resulting order for X as above is *zero*. In practice it works as follows.

Let X be a holomorphic vector field with a singular point at the origin. Consider the Taylor expansion of X at the origin. The order of X at the origin is said to be the degree of the first non-zero homogeneous component of its Taylor expansion. This can also be viewed as the integer d for which

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Gamma_\lambda^* X$$

is a non-zero holomorphic vector field, where $\Gamma_\lambda^* X$ denotes the pull-back of X by the homothety $\Gamma_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, \lambda z)$. Note that the limit above corresponds to the first non-zero homogeneous component on the Taylor's expansion. Suppose now that $C = \{y = z = 0\}$ is contained in the singular set of X so that the blow-up centered along this curve of singular points will be considered. The order of X with respect to C is now defined as the integer d for which

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{d-1}} \Lambda_\lambda^* X$$

is a non-zero holomorphic vector field, where now $\Lambda_\lambda^* X$ denotes the pull-back of X by the homothety $\Lambda_\lambda : (x, y, z) \mapsto (x, \lambda y, \lambda z)$. The limit above for the appropriate d is said to be the first non-trivial homogeneous component of X with respect to the variables x, y. We say the the vector fields are "linear" if the appropriate orders are equal to 1. Note then that in the example presented by Cerveau, although the vector field Y has order 2 at the origin, its order with respect to C is zero (it is regular at generic points of C).

Theorem 2 established the existence of separatrices for the codimension 1 foliations spanned by two commuting vector fields. We can ask what happens to the vector fields X, Y themselves or to the foliations induced by them. In this context, Theorem 2 was complemented by another result in [35]. More precisely, the following has been proved.

Theorem 3 *[35] Let X and Y be two holomorphic vector fields defined on a neighborhood U of $(0, 0, 0) \in \mathbb{C}^3$ which are not linearly dependent on all of U. Suppose that X and Y vanish at the origin and that one of the following conditions holds:*

- $[X, Y] = 0$;
- $[X, Y] = cY$, *for a certain* $c \in \mathbb{C}^*$.

Then there exists a germ of analytic curve $\mathcal{C} \subset \mathbb{C}^3$ *passing through the origin and simultaneously invariant under* X *and* Y.

First of all, we should note that the above theorem applies not only to the commutative Lie algebras but also to the Lie algebra of the affine group. Recall, however, that the analogue of Theorem 2 in the case of affine actions is known to be false since the classical work of Jouanolou. Whereas Theorem 3 holds interest in its own right as a theorem claiming the existence of invariant manifolds (curves in this case), our paper [36] also contains a non-trivial application of this result (Cf. Theorem 6).

Furthermore, Theorem 3 states that X and Y possess a common invariant curve without mentioning if the curve in question is invariant for the associated foliations (recall, for example, that $\{x = 0\}$ is invariant by $x\partial/\partial x$ but not by the corresponding foliation). It is however easy to check that in the particular case that we consider X as being a homogeneous vector field and Y as a multiple of the Radial vector field, the existence of a common separatrix for $\mathcal{F}_X$ and $\mathcal{F}_Y$ can easily be deduced. In fact, the leaves of $\mathcal{F}_Y$ are simply the radial lines. Concerning $\mathcal{F}_X$, since it is not a multiple of the Radial vector field, it induces a 1-dimensional foliation on $\mathbb{CP}(2)$ by means of the one-point blow-up of $\mathbb{C}^3$ at the origin. The foliation in question possesses isolated singular points and it can easily be check that the radial line naturally associated with any of these singular points is invariant by $\mathcal{F}_X$ as well. We believe that the existence of a common separatrix for $\mathcal{F}_X$ and $\mathcal{F}_Y$ in the general case can also be established.

To finish this section we are just going to give an idea of the proof of Theorem 3. Let then $\mathcal{D}$ stands for the codimension 1 foliation spanned by X and Y. We have that $\operatorname{codim}(\operatorname{Sing}(\mathcal{D})) \geq 2$. In other words, $\operatorname{Sing}(\mathcal{D})$ is of one of the following types: the union of a finite number of irreducible curves, a single point (the origin), or simply empty (i.e. $\mathcal{D}$ is regular). Since $\operatorname{Sing}(\mathcal{D})$ is naturally invariant by X and Y, the result immediately holds if $\dim(\operatorname{Sing}(\mathcal{D})) = 1$. Hence we can assume without loss of generality that $\operatorname{Sing}(\mathcal{D})$ has codimension at least 3.

Since the singular set of $\mathcal{D}$ has codimension at least 3, $\mathcal{D}$ possesses a holomorphic first integral f thanks to Malgrange Theorem. Let then $S = f^{-1}(0)$. We have that S is an invariant surface for $\mathcal{D}$ and, consequently, for X and Y. Furthermore, S can be assumed to be irreducible. In fact, if it was not, then the intersection of any two irreducible components of S is an invariant curve for both X, Y and the conclusion holds. The surface S can be assumed either regular or having an isolated singularity at the origin (in fact, if the singular set of S contains a singular curve, then the singular curve is a common separatrix for X, Y). The following can be noted

- In the case that S is smooth, both foliations possess a separatrix through the origin by Camacho-Sad Theorem. We have however to check that the imposed conditions implies that at least one of their separatrices coincide.
- In the case of singular surfaces, there are examples of holomorphic vector fields without separatrix (cf. [4]). This phenomenon needs to be ruled out in the case in question.

Consider then the restrictions of X and Y to S along with the corresponding tangency locus (which is non-empty since the origin is a common singular point of X, Y). Since the tangency locus $\operatorname{Tang}(X|_S, Y|_S)$ is invariant by both X and Y, the result immediately holds in the case where its dimension equals 1. So, we shall consider separately the case where $\operatorname{Tang}(X|_S, Y|_S) =$ and $\operatorname{Tang}(X|_S, Y|_S) = S$

Assuming that $\operatorname{Tang}(X|_S, Y|_S) =$, we get that S is a surface with singular set of codimension at least 2 and equipped with two linearly independent vector fields. This implies that tangent sheaf to

S is locally trivial which, in turn, implies that S is smooth. However, being S smooth, we have that S is locally equivalent to C^2 and the tangency locus of two vector fields on there cannot be reduced to a single point. The contradiction excludes this case.

We should then assume that $\mathrm{Tang}\,(X|_S, Y|_S) = S$, i.e. X and Y coincide up to a multiplicative function on S. The existence of the desired common separatrix is then ensured in the case where S is smooth. It remains to consider the case where S is singular at the origin. The argument in this case relies on proving that the (1-dimensional) foliation induced on S by either X or Y possesses a non-constant holomorphic first integral. The level curve of this first integral containing the origin then yields the desired separatrix. Details can be found in the paper in question.

3 Vector fields with univalued solutions - Global dynamics

Recall that a solution φ of a real ordinary differential equation (ODE) always possesses a maximal domain of definition contained in $\mathbb{R}$. In other words, fixed an initial condition, the solution φ is defined on a maximal interval $I_0 \subseteq \mathbb{R}$ in the sense that if $\{t_i\} \subseteq I_0$ is a sequence converging to an endpoint of $I_0 (\neq \pm\infty)$, then the sequence $\{\varphi(t_i)\}$ leaves every compact set in M as $i \to \infty$. Unlike the real case, the solution of a complex ODE (i.e. those where the time is a complex variable) does not admit in general a maximal domain of definition contained in $\mathbb{C}$. In fact, those admitting a maximal domain of definition are somehow “rare” among all complex ODE’s. The absence of maximal domains of definition for solutions of complex ODE’s is closed related to standard “monodromy” phenomena arising when extending holomorphic functions along paths. In view of this, complex ODE whose solutions admit maximal domains of definition are, roughly speaking, those whose solutions are univalued.

The understanding of the mentioned equations, which is a far classical and important problem with many interesting applications, is one of the topics in my research. It was probably Painlevé who first paid attention to these problems while studying equations not admitting movable singular points. Painlevé’s motivations were mostly concerned with the theory of special functions and they remain an active area of research nowadays. These and other problems connected to special function theory also constitute a motivation for my own past and future work.

On the other hand, in algebraic/complex geometry, there is also a fundamental problem of describing the “pairs” consisting of a holomorphic vector field defined on compact manifold. Since we are dealing with compact manifolds, vector fields as above are necessarily *complete* in the sense that their solutions are defined on all of $\mathbb{C}$. In some more specific cases (e.g. affine geometry, groups of birational automorphisms etc), one also pays attention to the problem of classifying complete holomorphic vector fields (on open manifolds). In both cases, a wealth of information is encoded in the nature of the singular points of the corresponding vector fields. A surprising connection between the study of the mentioned singular points and the existence of maximal domains of definitions for solutions of complex ODEs was realized by Rebelo who introduced the notion of *semicomplete singularity* in [30]. The idea is that the solutions of these (local) vector fields must be univalued since they have a “realization” as a complete vector field on some complex manifold. The condition of being “semicomplete” for a (germ of) vector field turned out to be very non-trivial and, indeed, to capture almost all of the “intrinsic nature” of germs of vector fields that actually admit global realizations as complete vector fields. The “classification” of semicomplete singularities of vector fields has then become an important problem which, a *posteriori*, has also show a number of interesting connections with integrable systems and certain remarkable kleinian groups, as will be seen later.

To begin with, let us introduce some definitions and results that will be useful in the sequel. So, let X be a holomorphic vector field defined on a possibly open complex manifold M and let U be an open subset of M.

Definition 1 *The holomorphic vector field X is said to be semicomplete on U if for every $p \in U$ there exists a connected domain $V_p \subset \mathbb{C}$ with $0 \in V_p$ and a map $\phi_p : V_p \to U$ satisfying the following conditions:*

- $\phi_p(0) = p$
- $\phi'_p(T) = X(\phi_p(T))$, *for every $T \in V_p$.*
- *For every sequence $\{T_i\} \subset V_p$ such that $\lim_{i\to\infty} T_i = \hat{T} \in \partial V_p$ the sequence $\{\phi_p(T_i)\}$ escapes from every compact subset of U.*

In this way $\phi : V_p \subset \mathbb{C} \to U$ is a maximal solution of X in a sense similar to the notion of "maximal solutions" commonly used for real differential equations. A semicomplete vector field on U gives rise to a semi-global flow Φ on U. A useful criterion to detect semicomplete vector fields can be stated as follows. First consider a holomorphic vector field X on U and note that the local orbits of X define a singular foliation $\mathcal{F}$ on U. A regular leaf L of $\mathcal{F}$ is naturally a Riemann surface equipped with an Abelian 1-form dT_L which is called the *time-form* induced on L by X. Indeed, at a point $p \in L$ where $X(p) \neq 0$, dT_L is defined by setting $dT_L(p).X(p) = 1$. Now, we have the following

Proposition 1 *[30] Let X be a holomorphic vector field defined on a complex manifold M. Assume that X is semicomplete on M. Then*

$$\int_c dT_L \neq 0$$

for every open (embedded) path $c : [0,1] \to L$.

This proposition allows us to easily check that the vector field $X = x^3\partial/\partial x$ is not semicomplete on any neighborhood of the origin of $\mathbb{C}$. In fact, for every given neighborhood U of the origin, there exists $\varepsilon > 0$ such that the ball B_ε with center at the origin and radius ε is contained in U. Consider then the open path contained in U and given by $c(t) = \frac{\varepsilon}{2}e^{\pi i t}$, $0 \leq t \leq 1$. Since the time-form on the (unique) regular leaf associated to X is given by

$$dT = \frac{dx}{x^3},$$

it becomes clear that the integral of the time form along the open path in question is zero and the claim immediately follows. We can also check that the vector field in question is not semicomplete on any neighborhood of the origin by noticing that the solution the differential equation associated to it

$$x(T) = \frac{x_0}{\sqrt{1 - 2x_0^2 T}}$$

is multivalued, where $x_0 = x(0)$.

It should be mentioned that if X is semicomplete on U and $V \subset U$, the restriction of X to V is semicomplete as well. Thus the notion of "semicomplete singularity" (or germ of semicomplete vector field) is well defined. It immediately follows that if X is globally defined on a compact manifold M then X is semicomplete at every singular point. In this sense, semicomplete vector fields can be viewed as the "local version" of complete ones. In fact, a singularity that is not semicomplete cannot be realized by a complete vector field. In particular, it cannot be realized by a globally defined holomorphic vector field on a compact manifold. Yet, the same definition applies also to more global context since the set U need not be "small". This is especially meaningful in the context of rational/polynomial vector fields that may be semicomplete away from their pole divisors. In these situation we shall use the terminology *uniformizable vector field* so as to save the phrase "semicomplete vector field" for situations where we shall be working on a neighborhood of a singular point.

In dimension 2, semicomplete singularities of vector fields (whether or not isolated) were fully classified by Ghys and Rebelo and this classification was, in particular, strongly used in the description of pairs of compact complex surfaces equipped with a globally defined holomorphic vector field obtained in [11]. The extension of their results to higher dimensions is, however, a very challenging and wide open problem, already in dimension 3. In fact, the study of semicomplete vector fields in dimensions ≥ 3 was initiated by A. Guillot [15], [16] who sought to classify quadratic semicomplete vector fields on $(\mathbb{C}^3, 0)$ (since the vector fields are homogeneous, they are semicomplete on a neighborhood of the origin if and only if they are "uniformizable" on all of $\mathbb{C}^n$). The interest in homogeneous vector fields comes, in part, from the fact that semicompleteness is closed for the topology of uniform convergency on compact subsets and, consequently, if a given vector field is semicomplete then so is its first non-zero homogeneous components. Another motivation for Guillot's work stemmed from the evidence that among semicomplete vector fields one often finds especially interesting/remarkable examples of dynamical systems, an idea totally in line with Painlevé's point of view concerning equations without movable singular points. It is fair to say that some of the main outcomes in Guillot's work concern the description of certain examples exhibiting remarkable properties in a way or in another.

Some of my works are contributions to the study of uniformizable vector fields on higher dimensional manifolds (cf. [39], [40], [33], [35]). The results with global nature will be discussed below, while the results with local nature will be discussed in the next section.

Let us focus on the class of uniformizable vector fields from a definitely global point of view: the vector fields in question are polynomial on $\mathbb{C}^n$ and are supposed to be semicomplete on all of $\mathbb{C}^n$ (as above mentioned, given the global nature of the discussion we shall use the terminology "uniformizable" instead of "semicomplete"). As it will be pointed out later, the methods to be discussed in this section apply also to uniformizable rational vector fields (where uniformizable means semicomplete away from its pole divisor). Examples fitting in this context include complete polynomial vector fields but also certain uniformizable vector fields with solutions defined on hyperbolic domains (some of them defined on a disc) as it happens in the case of Halphen vector fields. The works of Ablowitz and his collaborators on evolutions equations - many of them appearing in fluid dynamics - also provides numerous examples of equations/vector fields to which our methods are applicable, see for example [1] and references therein.

This said, in the paper [33], a method to investigate the domain of definition of solutions for polynomial (or more generally rational) vector fields was introduced. The method is quite general in that it applies to arbitrarily high dimensions. Yet, it provides new results already in dimension 3. The mentioned paper fundamentally consists of two parts, the first one corresponding to a general setup along with the basic estimates/results whereas the second part provides some applications of it. To greater or lesser extent, the applications given there arise from following the solution of a (complex) polynomial/rational vector field over "special real paths going off to infinity". Recall that being the vector fields considered polynomial vector fields in $\mathbb{C}^n$, they admit a meromorphic extension to $\mathbb{CP}(n)$. In particular, they induce a holomorphic foliation $\mathcal{F}$ in $\mathbb{CP}(n)$. Let us denote by Δ_∞ the hyperplane at infinity of $\mathbb{CP}(n)$, i.e. let $\Delta_\infty = \mathbb{CP}(n) \setminus \mathbb{C}^n$. We have that Δ_∞ is invariant by $\mathcal{F}$ unless the top degree homogeneous component of X is a multiple of the Radial vector field.

Before describing the method in question, let us present some of the applications of it. The first application obtained concerns a confinement-type theorem for solutions of complete polynomial vector fields on $\mathbb{C}^n$.

Theorem 4 *[33] Suppose that X is a complete polynomial vector field of degree at least 2 on $\mathbb{C}^n$. Fix an arbitrarily small neighborhood V of $(\mathrm{Sing}\,(\mathcal{F}) \cap \Delta_\infty) \cup \mathrm{Sing}\,(X)$ in $\mathbb{CP}(n)$ and suppose we are given a point $p \in \mathbb{C}^n$, $X(p) \neq 0$, and an angle $\theta \in (-\pi/2, \pi/2)$. Denote by L_p the leaf of $\mathcal{F}$ through p and consider the parametrization of L_p by $\mathbb{C}$ (possibly as a covering map) which is given by $\Phi_p(T) = \Phi(T, p)$. Then* *there exists a separating curve* $c : (-\infty, \infty) \to \mathbb{C}$, $\Phi_p(c(0)) = p$, *and an unbounded component* $\mathcal{U}^+$ *of* $\mathbb{C} \setminus \{c(t)\}$ *such that the following holds: the set* $\mathcal{T}_V \subset \mathcal{U}^+ \subset \mathbb{C}$ *defined by*

$$\mathcal{T}_V = \{T \in \mathcal{U}^+ \subset \mathbb{C} \; : \; \Phi(T, p) \in V\}$$

satisfies

$$\lim_{r \to \infty} \frac{\mathrm{Meas}\,(\mathcal{T}_V \cap B_r)}{\mathrm{Meas}\,(\mathcal{U}^+ \cap B_r)} = 1\,,$$

where Meas *stands for the usual Lebesgue measure of* $\mathbb{C}\,(\simeq \mathbb{R}^2)$ *and* B_r *the ball of radius* r *centered at* $0 \in \mathbb{C}$.

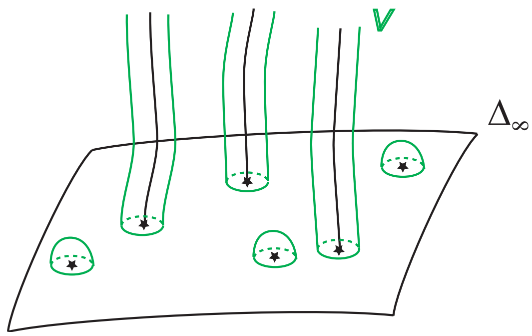

Figure 1: A neighborhood V of the singular set

The "separating curve" is actually a geodesic for some singular flat structure on $\mathbb{C}$ having bounded coefficients with respect to the standard flat structure. As a consequence of this fact, it follows that $\mathrm{Meas}\,(\mathcal{U}^+ \cap B_r)$ is actually comparable to the measure of the large discs B_r. This can naturally be seen as a confinement phenomenon since the solutions of a complete polynomial vector field spend a significant "part of their existence in arbitrarily small regions of the phase space" and hence "are highly non-ergodic". This phenomenon of "strong non-ergodicity" becomes more clear after the following corollary of the preceding theorem.

Corollary 1 *[33] Let us keep the notations of the Theorem 4 and let* $\mathcal{T}_V(r)$ *be the set of*

$$\mathcal{T}_V(r) = \{T \in B_r \subset \mathbb{C}\,;\; \Phi(T, p) \in V\}\,.$$

Then, there exists $\delta > 0$ *uniform (i.e. not depending on the neighborhood* V*) such that*

$$\liminf_{r \to \infty} \frac{\mathrm{Meas}\,(\mathcal{T}_V(r))}{\mathrm{Meas}\,(B_r)} \geq \delta > 0\,.$$

Since the above inequality remains valid if we reduce V, we see that the frequency with which the point p visits the neighborhood V is "far" from being proportional to the size of V.

The statement of Theorem 4 and of the corresponding corollary indicate that the structure of the singularities of $\mathcal{F}$ lying in Δ_∞ must hold significant information on the global dynamics of corresponding vector fields. This is a principle similar to the "heuristics" involved in the classical Painlevé test: the local behavior at singular points have strong influence on the global dynamics of the system. To put this principle to test, we have assumed that the singularities of $\mathcal{F}$ lying in Δ_∞ are "simple" (see definition below for the precise conditions). The idea is that these singularities can be understood in detail and hence we should be able to derive strong consequences of the global behavior of the corresponding polynomial vector field. Theorem 5 then fully vindicates our principle. Before stating the theorem, let us just mention that by "simple" singularity, it is meant the following types of singular points $q \in \Delta_\infty$ for $\mathcal{F}$:

(1) Non-degenerate singularities: this means that $\mathcal{F}$ can locally be represented by a vector field having non-degenerate linear part at q (i.e. the Jacobian matrix of X at q is invertible, equivalently, it possesses n eigenvalues different from zero). Besides, since resonances may arise, we assume that q is not of Poincaré-Dulac type, i.e. if all the eigenvalues of $\mathcal{F}$ at q belong to $\mathbb{R}^{*+}$, then $\mathcal{F}$ must be locally linearizable about q.

(2) Codimension 1 saddle-nodes: these are singularities of $\mathcal{F}$ lying in Δ_∞ whose eigenvalue associated to the direction transverse to Δ_∞ is equal to zero whereas it has $n-1$ eigenvalues different from zero and corresponding to directions contained in Δ_∞. Again we require that the singularity for the $(n-1)$-dimensional foliation induced on the plane Δ_∞ should not be a singularity of Poincaré-Dulac type.

Thus we have the following.

Theorem 5 *[33] Let X be a complete polynomial vector field on $\mathbb{C}^n$ whose singular set has codimension at least 2. Suppose that all singularities of $\mathcal{F}$ lying in Δ_∞ are simple. Then all leaves of $\mathcal{F}$ can be compactified into a rational curve (i.e. $\mathcal{F}$ can be pictured as a "non-linear rational pencil").*

Before proceeding and mentioning another application of these techniques (also provided in [33]), let me describe the method we introduced in the mentioned paper and give the main ideas for the proves of the theorems stated above.

So, let X be a polynomial semicomplete vector field on all of $\mathbb{C}^n$. Our method relies on estimating the "speed" of the vector field X near Δ_∞, the hyperplane at infinity (which coincides with the divisor of poles since X is polynomial). This is done in two steps. The first step consists of eliminating the unbounded factor of X over Δ_∞ so as to obtain a "local regular vector field" about every regular point $p \in \Delta_\infty$ of $\mathcal{F}_\infty$, where $\mathcal{F}_\infty$ stands for the foliation induced by X at Δ_∞. Recall that being X polynomial, it admits a meromorphic extension to the plane at infinity inducing, in particular, a holomorphic foliation on that. However, it turns out that these locally defined vector fields, obtained by eliminating the unbounded factor, depend to some extent on the choice of local coordinates so that they do not patch together in a "foliated" global vector field. Nonetheless, two local representatives obtained through overlapping coordinates differ only by a multiplicative constant. This means that this collection of local vector fields defines a global affine structure on every leaf of $\mathcal{F}_\infty$. The interest of the mentioned affine structure lies in the fact that it lends itself well to provide estimates for the flow of X as long as accurate estimates for the "distance" from the orbit in question to Δ_∞ are available.

The second ingredient of our construction is a quantitative measure of "the rate of approximation" of a leaf of $\mathcal{F}$ to Δ_∞. Because $\Delta_\infty \subset \mathbb{C}P(n)$ and the Fubini-Study metric on $\mathbb{C}P(n)$ has positive curvature, it is well known that complex submanifolds always "bend themselves towards Δ_∞". In our case, this implies that the distance (relative to the Fubini-Study metric) of a leaf L of $\mathcal{F}$ to Δ_∞ can never reach a local minimum unless this minimum is zero. Our mentioned second ingredient is reminiscent from this remark though, in the mentioned paper, the euclidean metric on suitably chosen affine coordinates, as opposed to the globally defined Fubini-Study metric, was chosen. The choice is however a relatively minor technical point due to the fact that the euclidean metric is better adapted to work with the above mentioned affine structure. Besides, by exploiting the fact that the submanifolds in questions are actual leaves of a foliation, a quantitative version of the rate of approximation of a leaf to Δ_∞ is derived. The phenomenon goes essentially as follows. At each regular point p of a leaf L

of $\mathcal{F}$ there is the "steepest descent direction of L towards Δ_∞", namely the negative of the gradient of the distance function restricted to L. This yields a singular real one-dimensional oriented foliation $\mathcal{H}$ on L. Roughly speaking, an exponential rate of approximation for L to Δ_∞ over the trajectories of $\mathcal{H}$ can then be obtained. Since L is endowed with a conformal structure, it makes sense to define also foliations $\mathcal{H}^\theta$ whose (oriented) trajectories makes an angle θ with the oriented trajectories of $\mathcal{H}$ ($\theta \in [-\pi/2, \pi/2]$). For $\theta \in]-\pi/2, \pi/2[$ an exponential rate of approximation for L to Δ_∞ over the trajectories of the associated real foliation can also be obtained (note that the foliation $\mathcal{H}^{\pi/2}$, which is orthogonal to $\mathcal{H}$, is constituted by level curves for the above mentioned distance function). Finally, the estimates on the exponential rate of approximation combines to the "uniform" estimates related to the foliated affine structure to produce accurate estimates for the time taken by the flow of X over trajectories of $\mathcal{H}$.

To better explain the method, assume for simplicity that X is a (polynomial) homogeneous vector field of degree $d \geq 2$ on $\mathbb{C}^3$. Assume, in addition, that X is not a multiple of the Radial vector field. Let us consider $\mathbb{CP}(3)$, the compactification of $\mathbb{C}^3$ by adjunction of the plane at infinity Δ_∞, and let M stands for the manifold obtained from $\mathbb{CP}(3)$ through a punctual blow-up at the origin. The manifold M can be viewed as a fiber bundle by projective lines equipped with two natural projections, namely

$$\mathcal{P}_0 : \widetilde{\mathbb{C}}^3 \to \Delta_0$$
$$\mathcal{P}_\infty : \widetilde{\mathbb{C}}^3 \to \Delta_\infty \, ,$$

where $\widetilde{\mathbb{C}}^3$ stands for the blow-up of $\mathbb{C}^3$ at the origin and Δ_0 represents the divisor obtained by the punctual blow-up of $\mathbb{CP}(3)$ at the origin.

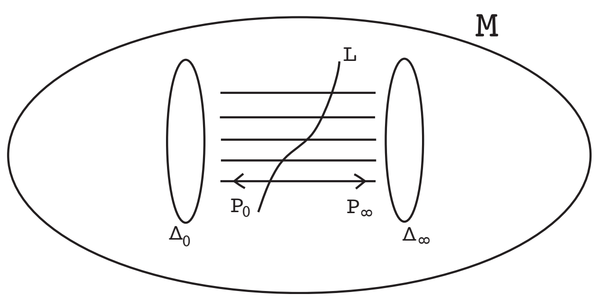

Figure 2: The manifold M and the corresponding bundle projections

Since the vector field X is polynomial, it admits a meromorphic extension to M, where Δ_0 corresponds to the zero divisor and Δ_∞ to the pole divisor. The vector field X induces a holomorphic foliation $\widetilde{\mathcal{F}}$ on M and, since we are assuming X not to be a multiple of the Radial vector field, $\widetilde{\mathcal{F}}$ leaves the two divisors Δ_0 and Δ_∞ invariant. Note that, since X is homogeneous, the projection of every leaf L onto Δ_0 (resp. Δ_∞), $\mathcal{P}_0(L) = L_0$ (resp. $\mathcal{P}_\infty(L) = L_\infty$), is clearly a leaf of $\widetilde{\mathcal{F}}_0$ (resp. $\widetilde{\mathcal{F}}_\infty$), the restriction of $\mathcal{F}$ to Δ_0 (resp. Δ_∞). Let then L be non-algebraic leaf of $\widetilde{\mathcal{F}}$ not contained in $\Delta_0 \cup \Delta_\infty$. We have that the restriction of $\mathcal{P}_0$ (resp. $\mathcal{P}_\infty$) to L realizes L as an Abelian covering of L_0 (resp. L_∞). It then follows that the non-compact leaves L, L_0, L_∞ have all the same nature: either they are all covered by $\mathbb{C}$ or they are all covered by the unit disc D. Furthermore L_0, L_∞ are isomorphic as Riemann surfaces while L is an Abelian covering of L_0, L_∞.

In order to study the behavior of the solutions nearby the infinity (i.e. away from compact subsets of $\mathbb{C}^3$), let M be equipped with affine coordinates (x, y, z) such that

(i) $\{z = 0\} \subset \Delta_\infty$, $(x, y) \in \mathbb{C}^2$, $z \in \mathbb{C}$.

(ii) the transformed $\widetilde{X}$ of the vector field X on M is given by

$$\widetilde{X} = \frac{1}{z^{d-1}}\left[F(x,y)\frac{\partial}{\partial x} + G(x,y)\frac{\partial}{\partial y} + zH(x,y)\frac{\partial}{\partial z}\right] \tag{1}$$

where F, G are polynomials of degree either d or $d+1$ and H is a polynomial of degree d (the independence of F, G and H on the variable z is a consequence of the homogeneous character of X).

(iii) The projection $\mathcal{P}_\infty : M \to \Delta_\infty$ in the above coordinates becomes $(x,y,z) \mapsto (x,y)$.

Also, it can be assumed that the line at infinity $\Delta_\infty^{(x,y)}$ over the plane at infinity $\Delta_\infty \simeq \{z = 0\}$ is not invariant by $\widetilde{\mathcal{F}}$.

Let $L_\infty \subseteq \Delta_\infty$ be a leaf of the foliation $\widetilde{\mathcal{F}}_\infty$. Denote by S the cone over L_∞, i.e. let $S = \mathcal{P}_\infty^{-1}(L_\infty)$. The mentioned cone is a 2-dimensional immersed singular surface clearly invariant by $\mathcal{F}$. In particular, S is naturally equipped with a holomorphic foliation, denoted by $\widetilde{\mathcal{F}}_S$, having a transversely conformal structure.

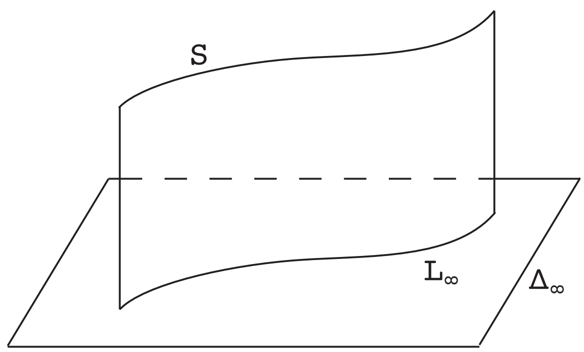

Figure 3: The invariant cone over L_∞

Let then L be a leaf of $\mathcal{F}$ contained in the cone over L_∞. The first step of our method consists on having quantitative estimates on the "speed" with which L approaches Δ_∞. As mentioned before, since $\Delta_\infty \subseteq \mathbb{CP}(n)$ and the Fubini-Study metric on $\mathbb{CP}(n)$ has positive curvature, it is well known that complex submanifolds always bend towards Δ_∞ (see for example [21]). So, to keep a "good control" of the directions over which the leaf L approach the infinity we proceed as follows. We shall equip L with an Abelian 1-form ω_1 related to the holonomy of the leaf in question. To be more precise, suppose that L_∞ is locally parameterized by

$$x \to (x, y(x), 0)\,.$$

Then L is parameterized by $x \to (x, y(x), z(x))$, where $z = z(x)$ satisfies

$$\frac{dz}{dx} = \frac{H(x,y(x))}{F(x,y(x))}dx\,.$$

It then follows that

$$z = z_0 \exp\left(\int_{x_0}^{x} \frac{H(x,y(x))}{F(x,y(x))}dx\right).$$

We have then that the Abelian 1-form

$$\omega_1 = \frac{H(x,y(x))}{F(x,y(x))}dx$$

is the logarithmic derivative of the holonomy in the sense that if c is a path on L_∞ then

$$(\mathrm{Hol}(c))'(c(0)) = e^{-\int_c \omega_1}.$$

Fix the a point $p \in L_\infty$. We claim that there exist real trajectories on L_∞ having contractive holonomy. The trajectories in question correspond to the leaves of the real oriented foliation on L_∞ defined by

$$\mathcal{H}: \ \{\mathrm{Im}(\omega_1) = 0\},$$

where $\mathrm{Im}(\omega_1)$ stands for the imaginary part of ω_1 and the orientation is such that

$$\mathrm{Re}(\omega_1.\phi'(t)) = \omega_1.\phi'(t) > 0.$$

In fact, if $c:[0,1] \to l$ is the parametrization of a leaf l of $\mathcal{H}$, $l \subseteq L_\infty$, then

$$\left|\mathrm{Hol}(c)'\right| = e^{-\int_c \omega_1} = e^{-\mathrm{Re}\left(\int_c \omega_1\right)} < 1.$$

Note that the trajectories defined above are not the only trajectories having a contractive holonomy. Is fact, for every fixed $\theta \in]-\pi/2, \pi/2[$, the oriented real foliation $\mathcal{H}_\theta$ making an angle θ with $\mathcal{H}$ is such that the holonomy with respect to their leaves is contractive.

Fix then a leaf L of $\mathcal{F}$ contained in the cone over L_∞. Fix a point $p \in L_\infty$ and let q be a point projecting on p. Let $l_p \subseteq L_\infty$ be a leaf of $\mathcal{H}$ and let l_q stands for the lift of the mentioned leaf to L. A first remark that can be made is that our lift does not leave the affine coordinates (x,y,z) above. In fact, it can be checked that points in the line al infinity $\Delta_\infty^{(x,y)}$ of the plane at infinity Δ_∞ provide singularities for $\mathcal{H}$ of source type. To prove Theorem 4 we have to control the "high" (i.e. the distance of L to L_∞) and the time that we pass away from a fixed neighborhood V of the singular set of $\mathcal{F}$.

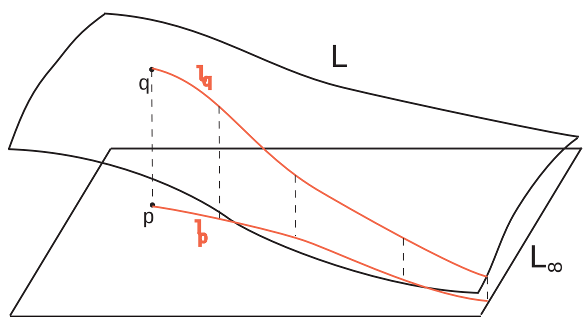

Figure 4: A leaf L, its projection L_∞ and the leaves of $\mathcal{H}$ passing through the points p and q.

To begin with note that away from V the 1-form ω_1 is bounded from below by a positive constant α (up to consider the parametrization $y \to (x(y), y, 0)$ instead of the considered one). Note that although we are away from a fixed neighborhood of the singular set of $\mathcal{F}$, the domain of definition may contain singularities of the real foliation $\mathcal{H}$. Singularities of $\mathcal{H}$ may be of three types: sinks, sources or saddles. The two first ones provide a minimum or a maximum to the (local) distance from L_∞ to L, respectively. If a minimum is attained, then we have "arrived" to Δ_∞ since, as mentioned before, we cannot have a minimum unless it is zero. Furthermore, it is clear that sources are not reached by the ("positive" direction) of our oriented leaves. Finally, concerning sources, it can be proved that we can exclude an arbitrarily small neighborhood of it and still keep the contractive holonomy by following the leaves of $\mathcal{H}_\theta$ for some θ belonging to $]-\pi/1+\delta, \pi/2-\delta[$, with $\delta > 0$.

To finish the idea of the proof of Theorem-4, let us just show some estimates. To be brief we will explain how to proceed in the case we stay away from a fixed neighborhood V of the singular set of foliation. The main idea is to prove that the time passed in $M \setminus V$ is finite. We recall that the singularities of $\mathcal{H}$ at points in $\Delta_\infty^{(x,y)}$ (the line at infinity of the hyperplane at infinity) are "source-like" so that an oriented trajectory of $\mathcal{H}$ cannot actually intersect $\Delta_\infty^{(x,y)}$. Though these trajectories of $\mathcal{H}$ may come "close" to $\Delta_\infty^{(x,y)}$, owing to Lemma 3.10 of [33] we know that every sufficiently long segment of l_p has "most of its length" contained in a fixed compact part of the affine $\mathbb{C}^2$ associated to the coordinates (x,y). Let then a compact part K of the mentioned affine copy of $\mathbb{C}^2$ be fixed and let us precise the estimates we need on this compact part - the estimates of Lemma 3.10 about the non-compact part allows us adapt the estimated we present below away from K.

So, let $c : [0,1] \to l_q$ be a parametrization of a connected path of l_q above l_p. We have that the "high" $z = z(t)$ along l_q satisfies

$$\begin{aligned}
|z| &= \left| z_0 e^{-\int_c \omega_1} \right| = |z_0| e^{-\mathrm{Re}(\int_c \omega_1)} \\
&= |z_0| e^{-\int_0^1 \mathrm{Re}(\omega_1(c(t)).c'(t))dt} = |z_0| e^{-\int_0^1 |\omega_1(c(t)).c'(t)|dt} \\
&\leq |z_0| e^{-\alpha.\mathrm{lenght}(c)} ,
\end{aligned}$$

where the last inequality comes from the fact that ω_1 is bounded from below by α away from V. We have then that if the length of c is greater than $\ln 2/\alpha$, then

$$|z_1| = |z(1)| \leq \frac{|z(0)|}{2} = \frac{|z_0|}{2} .$$

We have finally to control the time we take to cover the path l_q. Recall that the time-form associated to a leaf L is well-defined provided that L is not contained in the divisor of zeros and poles of X. If the vector field X is supposed to be semicomplete, then its restriction to L is everywhere holomorphic and the orders of its zeros cannot exceed 2. It follows at once that dT is meromorphic on all of L and it has no zeros. Furthermore, the poles of dT have order bounded by 2. Finally, recall also that given a curve $c : [0,1] \to L$ joining two points $c(0)$ and $c(1)$ in L satisfying $X(c(0)) \neq 0$ and $X(c(1)) \neq 0$, the integral $\int_c dT$ measures the time needed to traverse c from $c(0)$ to $c(1)$ following the flow of X as long as X is semicomplete. In fact, when a vector field is semicomplete the notion of time arising from its semi-global flow is well-defined.

Thus the integral $\int_{l_q} dT$ can be estimated as follows. The time-form on L is given, in local coordinates (x,y,z), by $dT = z^{d-1} dx/F(x,y)$. Since l_p, the projection of l_q by $\mathcal{P}_\infty$, is contained on a compact set not intersecting the singular set of $\widetilde{\mathcal{F}}_\infty$, the absolute value of $F(x,y)$ is bounded from below, i.e. $|F(x,y)| \geq \beta > 0, \quad \text{for all} \quad (x,y) \in \Delta_\infty \setminus V$. Otherwise we replace F by G (recall that we are dealing only with regular points of $\widetilde{\mathcal{F}}$ on Δ_∞). Hence, considering l_q as the concatenation of segments having length equal to $\ln(2)/\alpha$, $l_q = \sum_{i=0}^\infty l_{i,q}$, it follows that

$$\begin{aligned}
\left| \int_{l_q} dT \right| &= \left| \sum_{i=0}^\infty \int_{l_{i,q}} \frac{z^{d-1}}{F(x,y)} dx \right| \leq \sum_{i=0}^\infty \left| \int_0^1 \frac{z_{i,q}^{d-1}(t)}{F(x_{i,q}(t), y_{i,q}(t))} x'_{i,q}(t) dt \right| \\
&\leq \sum_{i=0}^\infty \int_0^1 \frac{|z_{i,q}(t)|^{d-1}}{|F(x_{i,q}(t), y_{i,q}(t))|} |x'_{i,q}(t)| dt \leq \sum_{i=0}^\infty \int_0^1 \frac{|z_0|^{d-1} (\frac{1}{2})^{i(d-1)}}{\beta} |l'_{i,p}(t)| dt \\
&\leq \frac{|z_0|^{d-1}}{\beta} \mathrm{length}(l_{i,p}) \sum_{i=0}^\infty \left(\frac{1}{2^{d-1}} \right)^i = \frac{|z_0|^{d-1} \ln(2)}{\alpha\beta} \frac{1}{1 - (\frac{1}{2})^{d-1}} < \infty
\end{aligned}$$

where $l_{i,q}(t) = (x_{i,q}(t), y_{i,q}(t), z_{i,q}(t))$, $t \in [0,1]$, is such that $l_q = \sum_{i=0}^\infty l_{i,q}$ and $\mathcal{P}_\infty(l_{i,q}) = l_{i,p}$. The conclusion follows.

4 Vector fields with univalued solutions - Local aspects

The interest of the notion of semicomplete vector fields (cf. Section 3) comes from the fact that the restriction of a complete holomorphic vector field, defined on a manifold M, to every open set $U \subseteq M$, is automatically semicomplete. Furthermore, given a semicomplete vector field on an open set U, its restriction to an open set $V \subseteq U$ is also semicomplete. In this sense, we are allowed to talk about germs of semicomplete vector fields. With abuse of notation, we can also talk about semicomplete singularities. By semicomplete singularities we mean a singular point associated to a semicomplete vector field on a small neighborhood of the point in question. Note however that a singularity may have more than one representative vector field and not all of them need to be semicomplete.

From the preceding it follows that semicomplete vector fields can be viewed as the "local version" of complete vector fields. In fact, a singularity that is not semicomplete cannot be realized by a complete vector field. In particular, it cannot be realized by a globally defined holomorphic vector field on a compact manifold. The understanding of semicomplete singularities is then important to the understanding of holomorphic vector fields (globally) defined on compact manifolds.

There is a long standing well-known question by E. Ghys that can be formulated in terms of semicomplete vector fields as follows: let X be a holomorphic vector field on $(\mathbb{C}^n, 0)$ that is semicompete and having an isolated singularity at the origin.Is it true that $J^2X(0) \neq 0$, i.e. must the second jet of X at the singular point be different from zero?

His motivation is, at least partially, related to problems about bounds for the dimension of automorphism groupd of compact complex manifolds. To be more precise, consider a compact complex manifold M and denote by $\mathrm{Aut}(M)$ the group of holomorphic diffeomorphisms of M. It is well-known that $\mathrm{Aut}(M)$ is a finite dimensional complex Lie group whose Lie algebra can be identified with $\mathfrak{X}(M)$, the space of all holomorphic vector fields defined on M. It is known that the dimension of the automorphism group of M cannot be bounded by the dimension of M, in general: it is sufficient to consider the family of Hirzbruch surfaces $\{F_n\}$, whose dimension of automorphism group equals $n+5$, for $n \geq 1$. However the same question can be formulated for special classes of manifolds. We have for example for the class of projective manifolds with Picard group isomorphic to $\mathbb{Z}$, Hwuang and Mok asked if $\mathbb{CP}(n)$ is the one having bigger automorphism group.

Let is briefly explain how an affirmative answer to Ghys conjecture can help us in the above mentioned problems. Suppose that M is a compact complex manifold of dimension n and fix a point $p \in M$ and $k \in \mathbb{N}$. We have the following short exact sequence

$$\mathfrak{X}_p^k(M) \to \mathfrak{X}(M) \to J_p^k(M),$$

where $\mathfrak{X}_p^k(M)$ stands for the set of holomorphic vector fields with vanishing k-jet at p and $J_p^k(M)$ denote the space of k-jets. Thus, we have

$$\dim \mathfrak{X}(M) \leq \dim \mathfrak{X}_p^k(M) + \dim J_p^k(M).$$

Concerning the space of jets, there exist effective bounds for $\dim J_p^k(M)$ in terms of $n = \dim(M)$. If $\dim \mathfrak{X}_p^k(M)$ has bounds in terms of $\dim(M)$ for a certain $p \in M$ and $k \in M$, then $\dim \mathfrak{X}(M)$ and, consequently, $\dim(\mathrm{Aut}(M))$ has such bounds as well. For example, suppose that we happen to know that for a certain class of of manifolds every singularity of a vector field is necessarily isolated. Then, if Ghys conjecture hold then $\dim \mathfrak{X}_p^3(M) = 0$ and then we obtain the desired bounds for $\dim \mathfrak{X}$.

So, let us focus on Ghys conjecture. In the paper [30], where the notion of semicomplete vector field was introduced, it has been proved that a semicomplete holomorphic vector field X on $(\mathbb{C}^2, 0)$ with an isolated singular point at the origin must satisfy $J_0^2 X \neq 0$. The proof rellies on the fact that the foliation associated to X possesses at least one separatrix, i.e. a germ of analytic curve that is invariant by the foliation in question. Being the singular set of X reduced to the origin, it follows that the restriction of X to the mentioned invariant curve does not vanish identically. Furthermore, this restriction is still a semicomplete vector field. Considering then the restriction of X to a separatrix the problem is essentially reduced to an one-dimensional situation (albeit the separatrix may be singular). This one-dimensional case is treated in the same paper by direct methods.

Later, semicomplete vector fields with an isolated singular point at the origin and vanishing linear part have been characterized (cf. [13]). A geometric study of the above mentioned vector fields/corresponding foliations has been done allowing the authors to prove that all those vector fields are integrable in the sense that they admit a (non-trivial) holomorphic or meromorphic first integral. Furthermore, the vector field X is conjugate to its first non-trivial homogeneous component thus providing a neat classification theorem.

The question of whether or not the above results still hold for semicomplete vector fields (at isolated singular points) in higher dimensional manifolds is a natural one. However, many new difficulties involved in any attempt at generalizing the preceding results to higher dimensional manifolds can immediately be pointed out. For example, unlike the case of holomorphic foliations on $(\mathbb{C}^2, 0)$, there exist holomorphic foliations on $(\mathbb{C}^3, 0)$ with an isolated singular point but no separatrix (i.e. invariant analytic curve through the singular point, cf. Section 2). To prove, for example, these vector fields without separatrix fail to be semicomplete is a challenging problem. Another ingredient that played an important role on the classification of semicomplete holomorphic vector field in dimension 2 is the resolution theorem of Seidenberg. The lack of a faithfully analogous procedure for reducing the singularities of vector fields in dimensions 3 and higher (cf. SectionSec:resolution for recent results) adds therefore to the difficulty of the problem. Furthermore, even in the case we are given a holomorphic foliation admitting a simple reduction of singularities in the sense of Seigenberg (where the linear part of the blown-up foliation has at least a non-vanishing eigenvalue at each singular point) additional difficulties are expected if we compare with the two-dimensional case. In fact, in the three-dimensional case saddle-nodes of codimension 2 (i.e. with two eigenvalues equal to zero) may appear and these singularities are still poorly understood.

The first deep investigations involving semicomplete vector fields in higher dimensions were conducted by A. Guillot in [15], and [16]. These investigations soon confirmed that the case of, say dimension 3, was far more subtle than its two-dimensional version. First, the mentioned papers by Guillot contain a huge variety of examples exhibiting a isolated singularities where the first jet of the vector field vanishes. An exhaustive classification of all possibilities analogous to the classification given in [13] is therefore unlikely or, at least, not very useful. In addition, among these examples there are also vector fields with complicated dynamics and admitting no holomorphic/meromorphic first integral.

Whereas the papers [30] and [13] deal with isolated singularities of a $\mathbb{C}$-action or, equivalently, of a complete vector field X on a complex two-dimensional ambient space, there was significant evidence that a "natural" extension of the methods and results obtained in these papers in dimension involved might be reached by considering two commuting vector fields or, more precisely, $\mathbb{C}^2$-actions of rank 2 (again the reader will note that a singularity of a globally defined $\mathbb{C}^2$-action is automatically semicomplete).

Let us focus on the problem about the vanishing order of a semicomplete vector field at an isolated singular point. The general principle to Ghys conjecture is, as previously mentioned, the existence of a separatrix through the isolated singular point. Recall that a holomorphic foliation by curves on a complex surface always admits separatrices through its singular points but this no longer holds when the ambient manifold is of dimension 3 or greater. However, as mentioned in Section 2, it has been proved in [35] that in the case we are given two holomorphic vector fields immersed in a representation of a Lie algebra of dimension 2, that are in addition linearly independent up to a set of codimension at least 2, then X, Y possess a common separatrix. As an important consequence of this result we were able to prove in the same paper that Ghys conjecture holds for vector fields on 3-dimensional compact manifolds whose automorphism group has dimension at least 2. More precisely, the following has been proved:

Theorem 6 *[35] Consider a compact complex manifold M of dimension 3 and assume that the dimension of $\mathrm{Aut}(M)$ is at least 2. Let Z be an element of $\mathfrak{X}(M)$ and suppose that $p \in M$ is an isolated singularity of Z. Then*

$$J^2(Z)(p) \neq 0$$

i.e., the second jet of Z at the point p does not vanish.

Note that M is not assumed to be algebraic in the above theorem. In fact, if M is algebraic, or more generally Kähler, then the statement holds in arbitrary dimensions and regardless of the condition on the dimension of $\mathrm{Aut}(M)$.

Our technique to derive Theorem 6 from Theorem 3 also yields another interesting result. In fact, the assumption that M is compact is not fully indispensable in many cases. For example, suppose that N is a Stein manifold of dimension 3 and suppose that N is effectively acted upon by a finite dimensional Lie group G. Then the Lie algebra $\mathfrak{G}$ of G embeds into the space $\mathfrak{X}_{\mathrm{comp}}(N)$ of *complete* holomorphic vector fields on N. The study of these complete holomorphic vector fields on Stein manifolds is a topic of interest having its roots in a classical work of Suzuki [42]. In this direction, our techniques yield:

Theorem 7 *Let N denote a Stein manifold of dimension 3 and consider a finite dimensional Lie algebra $\mathfrak{G}$ embedded in $\mathfrak{X}_{\mathrm{comp}}(N)$ (the space of complete holomorphic vector fields on N). Assume that the dimension of $\mathfrak{G}$ is at least 2. If Z is an element of $\mathfrak{G} \subseteq \mathfrak{X}(M)$ possessing an isolated singular point $p \in N$, then the linear part of Z at p cannot vanish, i.e. p is a non-degenerate singularity of Z.*

5 Generic pseudogroups on $(\mathbb{C}, 0)$ and the topology of leaves

In the study of some well-known problems about singular holomorphic foliations, we usually experience difficulties concerning to greater or lesser extent the topology of their leaves. Yet, most of these problems are essentially concerned with pseudogroups generated by certain local holomorphic diffeomorphisms defined on a neighborhood of $0 \in \mathbb{C}$ (recall definition below). In this sense, results about pseudogroups of $\mathrm{Diff}(\mathbb{C}, 0)$ generated by a finite number of local holomorphic diffeomorphisms are crucial for the understanding of certain singular foliations defined about the origin of $\mathbb{C}^2$. Furthermore, for most of these problems, it is necessary to consider classes of pseudogroups with a distinguished generating set all of whose elements have fixed conjugacy class in $\mathrm{Diff}(\mathbb{C}, 0)$. These statements will be explained below using a standard example.

Contrary to the previous sections, we consider here a singular holomorphic foliation defined about the origin in $\mathbb{C}^2$ and recall that these foliations are obtained by means of holomorphic vector fields having isolated singular points. The study of these singularities and of their deformations, paralleling Zariski problem, led to the introduction of the *Krull topology* in the space of these foliations. In this topology, a sequence of foliations $\mathcal{F}_i$ is said to converge to $\mathcal{F}$ if there are representatives X_i for $\mathcal{F}_i$ and X for $\mathcal{F}$ such that X_i is tangent to X, at the origin, to arbitrarily high orders (modulo choosing i large enough). It should be noted that, given a foliation $\mathcal{F}$, its resolution depends only on a finite jet of the Taylor series of X at the singular point in the sense that if $\mathcal{F}'$ is close to $\mathcal{F}$ in the Krull topology, both foliations will admit the same resolution map. Furthermore, the position of the singularities of the resolved foliations $\widetilde{\mathcal{F}}$, $\widetilde{\mathcal{F}}'$ will also coincide as well as the corresponding eigenvalues.

As an example, consider a nilpotent foliation $\mathcal{F}$ associated to Arnold singularity A^{2n+1}, i.e. a nilpotent foliation admitting a unique separatrix that happens to be a curve analytically equivalent to the cusp of equation $\{y^2 - x^{2n+1} = 0\}$. An important remark from what precedes is that whenever $\mathcal{F}'$ is sufficiently close to $\mathcal{F}$ in the Krull topology, $\mathcal{F}'$ is also a nilpotent foliation of type A^{2n+1}. In other words, the class of Arnold singularities is closed for small perturbations in the Krull topology. It is then natural to wonder what type of dynamical behavior can be expected from these foliations, or more precisely, from a "typical" foliation in this family. The following are examples of long-standing problems in the area:

Question: Does there exist a nilpotent foliation $\mathcal{F}$ in A^{2n+1} whose leaves are simply connected (apart maybe from a countable set)? Is the set of these foliations dense in the Krull topology, i.e. given a nilpotent foliation $\mathcal{F}$ in A^{2n+1}, does there exist a sequence of foliations $\mathcal{F}_i$ converging to $\mathcal{F}$ in the Krull topology and such that every $\mathcal{F}_i$ has simply connected leaves (with possible exception of a countable set of leaves)?

Our methods in [24] are powerful enough to affirmatively settle both questions above. Moreover, in the paper [32] we also establish that the countable set is, indeed, infinite and that the non-simply connected leaves are topologically cylinders. More precisely, in these two papers it is proved the following:

Theorem 8 *[24, 32]. Let $X \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ be a vector field with an isolated singularity at the origin and defining a germ of nilpotent foliation $\mathcal{F}$ of type A^{2n+1}. Then, for every $N \in \mathbb{N}$, there exists a vector field $X' \in \mathfrak{X}_{(\mathbb{C}^2,0)}$ defining a germ of foliation $\mathcal{F}'$ and satisfying the following conditions:*

(a) $J_0^N X' = J_0^N X$.

(b) $\mathcal{F}$ and $\mathcal{F}'$ have S as a common separatrix.

(c) there exists a fundamental system of open neighborhoods $\{U_j\}_{j\in\mathbb{N}}$ of S, inside a closed ball $\bar{B}(0,R)$, such that the following holds for every $j \in \mathbb{N}$:

(c1) The leaves of the restriction of $\mathcal{F}'$ to $U_j \setminus S$, $\mathcal{F}'|_{(U_j \setminus S)}$ are simply connected except for a countable number of them.

(c2) The countable set constituted by non-simply connected leaves is, indeed, infinite.

(c3) Every leaf of $\mathcal{F}'|_{(U_j \setminus S)}$ is either simply connected or homeomorphic to a cylinder.

These problems are related with pseudogroups generated by certain elements of Diff $(\mathbb{C},0)$, as it will be explained in the next paragraph. In this sense, let us start by recalling the notion of pseudogroup. Consider the group Diff $(\mathbb{C},0)$ of germs of holomorphic diffeomorphisms fixing $0 \in \mathbb{C}$, where the group law is induced by composition. Assume that G is actually a subgroup of Diff $(\mathbb{C},0)$ generated by the elements $h_1,\dots,h_k$. Then, consider a small neighborhood V of the origin where the local diffeomorphisms $h_1,\dots,h_k$, along with their inverses $h_1^{-1},\dots,h_k^{-1}$, are all well defined diffeomorphisms onto their images. The *pseudogroup* generated by $h_1,\dots,h_k$ (or rather by $h_1,\dots,h_k,h_1^{-1},\dots,h_k^{-1}$ if there is any risk of confusion) on V is defined as follows. Every element of this pseudogroup has the form $F = F_s \circ \dots \circ F_1$ where each F_i, $i \in \{1,\dots,s\}$, belongs to the set $\{h_i^{\pm 1}, i = 1,\dots,k\}$. The element F should be regarded as an one-to-one holomorphic map defined on a subset of V. Indeed, the domain of definition of $F = F_s \circ \dots \circ F_1$, as an element of the pseudogroup, consists of those points $x \in V$ such that for every $1 \le l < s$ the point $F_l \circ \dots \circ F_1(x)$ belongs to V. Since the origin is fixed by the diffeomorphisms $h_1,\dots,h_k$, it follows that every element F in this pseudogroup possesses a non-empty open domain of definition. This domain of definition may however be disconnected. Whenever no misunderstanding is possible, the pseudogroup defined above will also be denoted by G and we are allowed to shift back and forward from G viewed as pseudogroup or as group of germs.

So, let us explain how the above problems are concerned with pseudogroups generated by certain local elements of Diff $(\mathbb{C}, 0)$. To do this we will restrict ourselves to the particular case of a nilpotent foliation $\mathcal{F}$ associated to Arnold singularity A^3, i.e. a nilpotent foliation admitting a unique separatrix S that happens to be a curve analytically equivalent to the cusp of equation $\{y^2 - x^3 = 0\}$. For this type of foliation, the map associated to the desingularization of the separatrix $E_S : M \to \mathbb{C}^2$ reduces also the foliation $\mathcal{F}$ (see Figure 5 for the corresponding resolution). So, let us describe the resolution in question.

To begin with, let us consider standard coordinates (x, y) for $(\mathbb{C}^, 0)$ where the separatrix if given by $\{y^2 - x^3 = 0\}$. The origin of the mentioned coordinates is the (unique) singular point of $\mathcal{F}$. So, let us first consider the one-point blow-up of $\mathcal{F}$ centered at the origin. The strict transform of the separatrix is tangent to the resulting component of the exceptional divisor, denoted by C_1, at some point. This point of tangency is the unique singular point for the strict transform foliation and it is a degenerate singular point. So, consider now the punctual blow-up of the transformed foliation at this new singular point. Let C_2 be the resulting irreducible component of the exceptional divisor. Since C_1 and S were tangent, it follows that C_1, C_2 and S intersect all each other at the same point. This intersection is however transverse at this point. Nonetheless the eigenvalues of the foliation at this intersection and singular point are both equal to zero and so we need to perform a punctual blow-up at this point. Let us then perform a third one-point blow-up, centered at this intersection point and let C_3 be the new irreducible component of the exceptional divisor. Now, since C_1, C_2 and S were transverse we have that their strict transforms intersect C_3 at distinct points. We denote by s_1 the intersection point of C_1 with C_3, by s_2 the intersection point of C_2 with C_3 and by s_0 the intersection point of the separatrix with C_3.

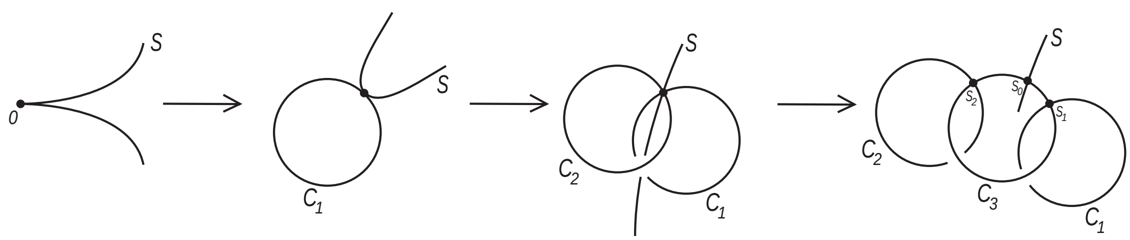

Figure 5: The desingularization diagram of the foliation associated with A^3

All those singular points s_0, s_1 and s_2 are non-degenerate singular points for $\mathcal{F}$. In fact, we have that both eigenvalues of the transformed foliation, still denoted by $\mathcal{F}$, at each one of those singular points are different from zero. For example, it can easily be checked that

1. the eigenvalues of $\mathcal{F}$ at s_1 are $1, -3$;

2. the eigenvalues of $\mathcal{F}$ at s_2 are $1, -2$.

The eigenvalues of $\mathcal{F}$ at s_0 can also be deduced by using the index formula.

Clearly, every component of the exceptional divisor is invariant by the foliation in question. So, $C_1 \setminus \{s_1\}$ is a leaf of $\mathcal{F}$ and so is $C_2 \setminus \{s_2\}$. We have that $C_1 \setminus \{s_1\}$ is isomorphic to $\mathbb{C}$ and thus, it is simply connected. This means that the holonomy of $\mathcal{F}$ with respect to this leaf is the identity. Now, recalling that the quotient of the eigenvalues of $\mathcal{F}$ at s_1 is negative real, it follows from Mattei-Moussu that $\mathcal{F}$ is linearizable in a sufficiently small neighborhood of s_1. Applying the same argument to $C_2 \setminus \{s_2\}$, we get that $\mathcal{F}$ is also linearizable in a small neighborhood of s_2 as well.

Denoting by h_{σ_1} and by h_{σ_2} the holonomy map of $\mathcal{F}$ with respect to small loops on $\mathbb{C}_3$ around s_1 and s_2, respectively, we have that h_{σ_1} is periodic of period 3 and h_{σ_2} is periodic of period 2. So, h_{σ_1} is analytically conjugated to a rotation of angle $2\pi/3$ while h_{σ_2} is analytically conjugated to a rotation of angle π. So, being σ_0, σ_1 and σ_2 loops on C^3 around s_0, s_1 and s_2, respectively, and recalling that they satisfy the relation

$$\sigma_0 \circ \sigma_1 \circ \sigma_2 = \text{id}\,,$$

it follows that the fundamental group of C_3 minus the three singular points is generated by σ_1 and σ_2 and, then, the global holonomy of $\mathcal{F}$ is generated by h_{σ_1} and h_{σ_2}. In other words, the global holonomy or, more precisely, the holonomy pseudogroup is generated by the (local) holonomy maps h_{σ_1} and h_{σ_2} at s_1 and s_2 w.r.t. the irreducible component intersecting the strict transform of the separatrix. Furthermore, these holonomy maps are of finite orders (2 and 3) and hence are linearizable, *though not necessarily in the same coordinate.*

From the above paragraph, it follows that every foliation associated to the Arnold singularity A^3 gives rise to a pseudogroup generated by two elements of $\text{Diff}\,(\mathbb{C},0)$: one having order 2 and another having order 3. In [24], we proved that the converse still holds. On other words, we prove that if we are given two diffeomorphisms f and g, one being conjugated to a rotation of order 2 and the other one conjugated to a rotation of order 3, there exists a foliation as above realizing f and g as generators of the global holonomy (of holonomy pseudogroup). In this sense the study of the foliations in question is "equivalent" to the study of pseudogroups of $\text{Diff}\,(\mathbb{C},0)$. The proofs of the above theorems are thus reduced to analogous statements for finitely generated pseudogroups of $\text{Diff}\,(\mathbb{C},0)$.

More generally, for a nilpotent foliation associated to Arnold singularity A^{2n+1}, we still have that the holonomy pseudo-group is generated by two diffeomorphisms f and g, one being conjugated to a rotation of order 3 but the other one conjugated to a rotation of order $2n+1$. In fact, the resolution diagram for such foliation is the same for the corresponding separatrix and is represented in Figure 6.

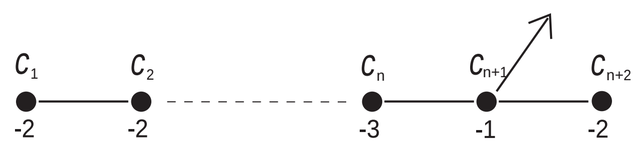

Figure 6: The desingularization diagram of the foliation associated with A^{2n+1}

The vertices of this graph correspond to the irreducible components of the resulting exceptional divisor. The weight of each irreducible component equals its self-intersection. In turn, the edges correspond to the intersection of two irreducible components whereas the arrow corresponds to the intersection point of the (unique) component C_{n+1} of self-intersection -1 with the transform of the separatrix S. The component C_{n+1} contains, as in the previous case, three singular points that we still denote by s_0, s_1 and s_2 and where s_0 is still the point determined by the intersection of C_{n+1} with the separatrix. Finally s_1 (resp. s_2) is the intersection point of C_{n+1} with C_{n+2} (resp. C_n).

The singular points of $\mathcal{F}$ are the intersection points of two consecutive components in the chain $C_1, \ldots, C_{n+2}$ along with the point s_0. All these singular points are simple in the sense that they possess two eigenvalues different from zero. The corresponding eigenvalues can precisely be determined by using the weights of the various components of the exceptional divisor. The argument applied in the case of the Arnold singularity A^3 allows us to say that the holonomy of $\mathcal{F}$ associated to the component C_{n+2} (i.e. the holonomy map associated to the regular leaf $C_{n+2} \setminus \{S_1\}$ of $\mathcal{F}$) coincides with the identity (the leaf in question is simply connected). Therefore the germ of $\mathcal{F}$ at s_1 admits a holomorphic first integral and since the corresponding eigenvalues are $1, 2$, we conclude that the local holonomy map g associated to a small loop around s_1 and contained in C_{n+1}, has order equal to 2. A similar discussion applies to the component C_1 and leads to the conclusion that the local holonomy map f associated to a small loop around s_2 and contained in C_{n+1} has order equal to $2n+1$. Since $C_{n+1} \setminus \{s_0, s_1, s_2\}$ is a regular leaf of $\mathcal{F}$, we conclude that the (image of the) holonomy representation of the fundamental group of $C_{n+1} \setminus \{s_0, s_1, s_2\}$ in $\mathrm{Diff}(\mathbb{C}, 0)$ is nothing but the group generated by f, g. Conversely, given two local diffeomorphism f, g of orders respectively $2, 2n+1$, they can be realized (up to simultaneous conjugation) as the holonomy of the corresponding component C_{n+1} for some foliation associated to Arnold singularity A^{2n+1}. This is done through a well-known gluing procedure explained in [24].

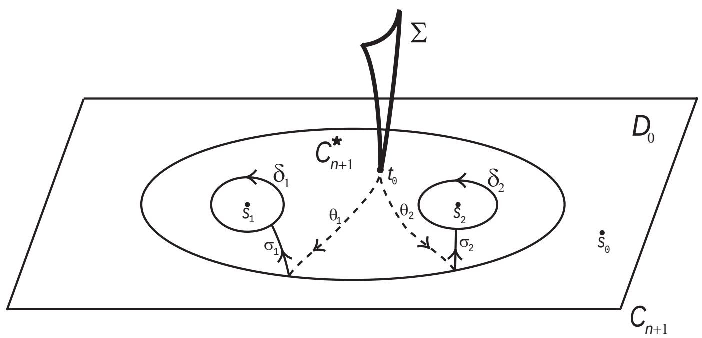

Figure 7: The holonomy representation

Finally, note that the above conclusion depends only on the configuration of the reduction tree which, in turn, is determined by some finite order jet of X. Hence, if the coefficients of Taylor series of the vector field X are perturbed starting from a sufficiently high order, the new resulting vector field X' will still give rise to a foliation whose singularity is reduced by the same blow-up map associated to the divisor of Figure 6. In particular, the holonomy representation of the fundamental group of $C_{n+1} \setminus \{s_0, s_1, s_2\}$ in $\mathrm{Diff}(\mathbb{C}, 0)$, obtained from this new foliation, is still generated by two elements of $\mathrm{Diff}(\mathbb{C}, 0)$ having finite orders respectively equal to 2 and to $2n+1$. Since every local diffeomorphism of finite order is conjugate to the corresponding rotation, it follows that the mentioned perturbations are made inside the conjugacy classes of f and g. This also justifies the fact that in Theorems 9 and 10 below only perturbations of local diffeomorphisms that do not alter the corresponding conjugation classes were allowed.

So, let $\mathrm{Diff}(\mathbb{C}, 0)$ be equipped with the so-called analytic topology which, unlike the Krull topology, has the Baire property. Next, consider a k-tuple of local holomorphic diffeomorphisms $f_1, \ldots, f_k$ fixing $0 \in \mathbb{C}$. Here we impose the condition that the local diffeomorphisms f_i can be perturbed only inside their conjugacy classes so as to be able to recover results for the initial foliations in the case where they have finite orders (and hence are all conjugate to a fixed rational rotation).

Fixed $\alpha \in \mathbb{N}$, in the sequel we denote by $\mathrm{Diff}_\alpha(\mathbb{C}, 0)$ the subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ whose elements are tangent to the identity to the order α. Finally, we have proved the following.

Theorem 9 *[24] Fixed $\alpha \in \mathbb{N}$, let $f_1, \ldots, f_k$ be given elements in $\mathrm{Diff}(\mathbb{C}, 0)$ and consider the cyclic groups $G_1, \ldots, G_k$ that each of them generates. There exists a G_δ-dense set $\mathcal{V} \subset (\mathrm{Diff}_\alpha(\mathbb{C}, 0))^k$ such that, whenever $(h_1, \ldots, h_k) \in \mathcal{V}$, the following holds:*

1. *The group generated by $h_1^{-1} \circ f_1 \circ h_1, \ldots, h_k^{-1} \circ f_k \circ h_k$ induces a group in $\mathrm{Diff}(\mathbb{C}, 0)$ that is isomorphic to the free product $G_1 * \cdots * G_k$.*

2. *Let $f_1, \ldots, f_k$ and $h_1, \ldots, h_k$ be identified to local diffeomorphisms defined about $0 \in \mathbb{C}$. Suppose that none of the local diffeomorphisms $f_1, \ldots, f_k$ has a Cremer point at $0 \in \mathbb{C}$. Denote by Γ^h the pseudogroup defined on a neighborhood V of $0 \in \mathbb{C}$ by the mappings $h_1^{-1} \circ f_1 \circ h_1, \ldots, h_k^{-1} \circ f_k \circ$*

h_k, where $(h_1,\ldots,h_k) \in \mathcal{V}$. Then V can be chosen so that, for every non-empty reduced word $W(a_1,\ldots,a_k)$, the element of Γ^h associated to $W(h_1^{-1}\circ f_1\circ h_1,\ldots,h_k^{-1}\circ f_k\circ h_k)$ does not coincide with the identity on any connected component of its domain of definition.

Theorem 9 implies items (a), (b) and (c) of Theorem 8. It should be noted that perturbing the foliation inside the class of Arnold singularities of type A^3 is equivalent to keep the conjugacy class of the generators of their local holonomy maps fixed. Furthermore, the analytic topology allows us to obtain information on the coefficients of the representative vector fields so as to be able to derive information concerning the Krull topology. Items (d) and (e) follow from the following result proved in [32]:

Theorem 10 *[32] Suppose we are given f, g in $\mathrm{Diff}\,(\mathbb{C},0)$ and denote by D an open disc about $0\in\mathbb{C}$ where f, g and their inverses are defined. Assume that none of the local diffeomorphisms f, g has a Cremer point at $0\in\mathbb{C}$. Then, there is a G_δ-dense set $\mathcal{U}\subset \mathrm{Diff}_\alpha(\mathbb{C},0)\times\mathrm{Diff}_\alpha(\mathbb{C},0)$ such that, whenever (h_1,h_2) lies in $\mathcal{U}$, the pseudogroup Γ_{h_1,h_2} generated by $\tilde{f}=h_1^{-1}\circ f\circ h_1$, $\tilde{g}=h_2^{-1}\circ g\circ h_2$ on D satisfies the following:*

1. *The stabilizer of every point $p\in D$ is either trivial or cyclic.*

2. *There is a sequence of points $\{Q_i\}$, $Q_i\neq 0$ for every $i\in\mathbb{N}^*$, converging to $0\in\mathbb{C}$ and such that every Q_n is a hyperbolic fixed point of some element $W_i(\tilde{f},\tilde{g})\in\Gamma_{h_1,h_2}$. Furthermore the orbits under Γ_{h_1,h_2} of Q_{n_1}, Q_{n_2} are disjoint provided that $n_1\neq n_2$.*

To conclude, we should only mention that Theorems 9 and 10 can be applied to much larger classes of foliations. In fact, they can be applied to every class of foliations that are stable under perturbations in the Krull topology such as, for example, those singularities whose resolution tree has only hyperbolic singular points.

6 Integrability of foliations and related problems on $\mathrm{Diff}(\mathbb{C}^n,0)$

In the context of singularities of holomorphic foliations in dimension 2, the topological nature of the foliation and the existence of non-constant holomorphic first integrals possesses a surprisingly strong connection which was put forward in the seminal paper [23]. In fact, the existence of first integrals for the foliations in question can be read off natural necessary topological conditions. More precisely, the following is proved in the mentioned paper:

Theorem 11 *[23] Consider a holomorphic foliation $\mathcal{F}$ defined on a neighborhood U of the origin of $\mathbb{C}^2$ and having an (isolated) singularity at $(0,0)$. The foliation $\mathcal{F}$ has a non-constant holomorphic first integral $f: U\to\mathbb{C}$ if and only if the following two conditions are satisfied:*

1. *only a finite number of leaves accumulates on $(0,0)$;*

2. *the leaves of $\mathcal{F}$ are closed on $U\setminus\{(0,0)\}$.*

It immediately follows from what precedes that the existence of a non-constant holomorphic first integral for a singular foliation on $(\mathbb{C}^2,0)$ is a topological invariant. In other words, consider two local foliations by curves $\mathcal{F}_1$, $\mathcal{F}_2$ about $(0,0)\in\mathbb{C}^2$ that are topologically equivalent in the sense that there is a homeomorphism h defined about $(0,0)\in\mathbb{C}^2$ and taking the leaves of $\mathcal{F}_1$ to the leaves of $\mathcal{F}_2$. Then $\mathcal{F}_1$ admits a non-constant holomorphic first integral if and only if so does $\mathcal{F}_2$. In the sequel, the phrase "first integral" will always mean a non-constant first integral.

Possible generalizations of the above mentioned phenomenon have long attracted interest. First, a classical example attributed to Suzuki and discussed in [10] shows that the existence of meromorphic first integrals is no longer a topological invariant. However, in dimension 3, many experts have wondered whether the existence of two "independent" holomorphic first integrals would constitute a topological invariant of the singularity. Recently, in [29], this question was answered in the negative. Indeed, we proved:

Theorem 12 *[29] Denote by $\mathcal{F}$ and $\mathcal{D}$ the foliations associated to the vector fields X and Y, respectively, given by*

$$X = 2xy\frac{\partial}{\partial x} + (x^3 + 2y^2)\frac{\partial}{\partial y} - 2yz\frac{\partial}{\partial z}\,,$$
$$Y = x(x - 2y^2 - y)\frac{\partial}{\partial x} + y(x - y^2 - y)\frac{\partial}{\partial y} - z(x - y^2 - y)\frac{\partial}{\partial z}\,.$$

The foliations $\mathcal{F}, \mathcal{D}$ are topological equivalent. Nonetheless $\mathcal{F}$ admits two independent holomorphic first integrals while $\mathcal{D}$ does not.

Our construction is inspired from Suzuki's example based on a simple observation that the existence of two independent holomorphic first integrals may give rise to a meromorphic first integral for the restriction of the foliation to certain invariant surfaces. In fact, let $\mathcal{F}$ be a foliation on $(\mathbb{C}^3, 0)$ admitting two (necessarily) non-constant and independent holomorphic first integrals F and G. Consider the decomposition of F and G into irreducible factors

$$\begin{aligned} F &= f_1^{m_1} \cdots f_k^{m_k} \\ G &= g_1^{n_1} \cdots g_l^{n_l}\,. \end{aligned}$$

Suppose that F and G have no common irreducible factor, modulo multiplication by nowhere vanishing functions. Then the restriction of G to, for example, $\{f_1 = 0\}$ is a non-constant holomorphic first integral for the restriction of $\mathcal{F}$ to the surface in question. In particular, the restriction of the foliation $\mathcal{F}$ to $\{f_1 = 0\}$, viewed as a singular foliation defined on a (possibly singular) surface, admits finitely many separatrices. In this case, all leaves of $\mathcal{F}|_{\{f_1=0\}}$ are "fully identified" by G in the sense that the restriction of G to $\{f_1 = 0\}$ provides a non-constant holomorphic first integral for $\mathcal{F}|_{\{f_1=0\}}$. Assume now that f_1 is a common irreducible factor for F and G. Then the restrictions of both F and G to $\{f_1 = 0\}$ vanish identically. In this case, the leaves of $\mathcal{F}|_{\{f_1=0\}}$ cannot be distinguished by either F or G. Nonetheless, it is possible to obtain a non-constant first integral for the restriction of $\mathcal{F}$ to $\{f_1 = 0\}$ as a function of F and G. To be more precise, there exists positive integers n_1, m_1 such that the function

$$\frac{F^{n_1}}{G^{m_1}} = \frac{f_2^{m_2 n_1} \cdots f_k^{m_k n_1}}{g_2^{n_2 m_1} \cdots g_l^{n_l m_1}} \tag{2}$$

is a non-constant first integral of $\mathcal{F}_{|_{\{f_1=0\}}}$. However, in general, this first integral is meromorphic rather than holomorphic as shown by the simple example below.

Example 1 *Consider the holomorphic functions $F = xy$ and $G = xz$ which clearly define two independent holomorphic first integrals for the foliation associated to the vector field $X = x\partial/\partial x - y\partial/\partial y - z\partial/\partial z$. Both F, G vanish identically on the invariant manifold $\{x = 0\}$. Nonetheless, the function $F/G = y/z$ provides a meromorphic first integral for the restriction of $\mathcal{F}$ to this invariant manifold.*

In view of the above observation, and recalling that the existence of a meromorphic first integral is not a topological invariant, the definition of the foliations $\mathcal{F}$ and $\mathcal{D}$ in Theorem 12 is itself inspired from the Suzuki and Cerveau-Mattei examples in the following sense. The plane $\{z = 0\}$ is invariant by both $\mathcal{F}$, $\mathcal{D}$ and the restriction of $\mathcal{F}$ (resp. $\mathcal{D}$) to this invariant manifold coincides with the foliation provided by Cerveau-Mattei (resp. Suzuki). Furthermore $\mathcal{F}$ and $\mathcal{D}$ were chosen so that the image of each leaf of $\mathcal{F}$ (resp. $\mathcal{D}$) under the projection map $\mathrm{pr}_2(x, y, z) = (x, y)$ is still a leaf of $\mathcal{F}$ (resp. $\mathcal{D}$) and, in addition, a sort of "saddle behaviour" for their leaves with respect to the third axes was introduced (by "saddle behaviour" it is meant that as the variable x on the local coordinates of a leaf decreases to zero, the variable z increases monotonically to exit a fixed neighborhood of the origin). The "saddle behaviour" was carefully chosen so that the topological equivalence between the restrictions of $\mathcal{F}$, $\mathcal{D}$ to the invariant plane $\{z = 0\}$ can be extended to an entire neighborhood of the origin.

From the above construction, the foliation $\mathcal{F}$ possesses $\overline{F} = (y^2 - x^3)z^2$ and $\overline{G} = xz$ as independent holomorphic first integrals. Furthermore, it is clear that the foliation $\mathcal{D}$ cannot admit two independent holomorphic first integrals. In fact, if this were the case, then the quotient between suitable powers of their first integrals would provide a meromorphic first integral for the restriction of $\mathcal{D}$ to the invariant plane $\{z = 0\}$. As already mentioned, it is known that such meromorphic first integral does not exist.

It can be noted that the singular set of the foliations considered in Theorem 12 is not reduced to a single point and this might suggest that the "correct" generalization of Mattei-Moussu theorem involves isolated singularities. This is actually not the case. As seen from the above construction, which often constitutes an essential obstruction for the topological invariance of "complete integrability" is the existence of invariant surfaces over which the correspondent foliation is "dicritical" (the foliation over a surface of said dicritical if it possesses infinitely many separatrices). Furthermore, there is vast evidence that completely integrable foliations, on 3-dimensional manifolds, possessing an isolated singular point must admit an invariant surface over which the correspondent foliation is dicritical. In fact, in the same paper, the following result was proved.

Theorem 13 *[29] Let $\mathcal{F}$ be a foliation by curves on $(\mathbb{C}^3, 0)$ having an isolated singularity at the origin and admitting two independent holomorphic first integrals. Suppose that $\widetilde{\mathcal{F}}$, the transform of $\mathcal{F}$ by the one-point blow-up centered at the origin, has only isolated singularities which, in addition, are simple. Then $\mathcal{F}$ possesses an invariant surface over which the induced foliation is dicritical.*

Concerning the role played by the above mentioned invariant surfaces, recall that a deep study of topological properties of foliations on $(\mathbb{C}^2, 0)$ possessing meromorphic first integrals was conducted by M. Klughertz in [19]. Her techniques yield several examples where "topological invariance" for the existence of meromorphic first integrals fails. Relatively simple adaptations of the proof of Theorem 12 then enable us to obtain several other examples of foliations on $(\mathbb{C}^3, 0)$ for which the "topological invariance" of the existence of two independent holomorphic first integrals is not verified. We conjecture, however, that if we are given two foliations by curves on $(\mathbb{C}^3, 0)$, $\mathcal{F}_1$, $\mathcal{F}_2$, that are topologically equivalent and do not admit invariant surfaces over which the induced foliations are dicritical, then $\mathcal{F}_1$ admits two holomorphic first integrals if and only if so does $\mathcal{F}_2$.

Recall that the Seidenberg desingularization theorem plays a major role in the study of singular foliations in dimension 2 and, in particular, it is used in the topological characterization of integrable foliations. A completely faithful generalization of the Seidenberg result for foliations on 3-manifolds cannot exist, since some non-simple singularities are persistent under blow-ups (cf. Section 7). Nonetheless final models on a desingularization process of foliations on 3-manifolds have been described on different papers such as [9], [26], [28] and [38]. In a first moment, the idea to remove the generic condition from Theorem 13, and/or to eventually prove the above mentioned conjecture, consists of showing that this "special type" of singular points cannot appear in the desingularization procedure of $\mathcal{F}$ provided that $\mathcal{F}$ is completely integrable. Following some discussions with D. Panazzolo, this assertion can probably be established by building on the material of the mentioned above papers.

Motivated by the above examples, we conducted a more in-depth study of Mattei-Moussu's results as well as their possible generalizations. There, it should be noted that the Mattei-Moussu argument also states that the existence of these first integrals can be detected at the level of the topological dynamics associated with the holonomy pseudogroup of the foliation in question. Let us make it precise.

Consider a holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^2, 0)$ and assume it admits a holomorphic first integral. Then so does the holonomy pseudogroup of the foliation in question. More precisely, the holonomy pseudogroup of the given foliation correspond (up to a change of coordinates) to a group of rotations being, in particular, finite. It then follows that every element of the mentioned pseudogroup has finite orbits. In fact, the pseudogroup itself has finite orbits. Recall that we say that a group G has finite orbits, if there exists a sufficiently small neighborhood V of the origin such that the set $\mathcal{O}_V^G(p)$ is finite for every $p \in V$, where

$$\mathcal{O}_V^G(p) = \{q \in V \; : \; q = h(p), \; h \in G \text{ and } p \in \mathrm{Dom}_V(h)\}.$$

The central point of the proof of Mattei-Moussu theorem is a converse for the previous statement which is valid for subgroups of $\mathrm{Diff}\,(\mathbb{C}, 0)$, namely:

Proposition 2 *[23] Let G be a finitely generated pseudogroup of* $\mathrm{Diff}\,(\mathbb{C}, 0)$*. Assume that G has finite orbits. Then G is itself finite.*

Being the holonomy pseudogroup of $\mathcal{F}'$ finite, it immediately follows that the pseudogroup in question is cyclic and consequently, it admits a holomorphic first integral. Finally, they also proved that the foliation has a holomorphic first integral as well, by extending the holomorphic first integral of the holonomy pseudogroup through the saturated of the leaves.

With respect to the extension of the first integral of the holonomy pseudogroup through the saturated of leaves, the following should be noted. The resolution of a foliation as $\mathcal{F}$ is such that every singular point in the final model is in the Siegle domain, i.e. if λ_1, λ_2 are the eigenvalues at a singular point then both eigenvalues are different from zero and $\lambda_1/\lambda_2 \in \mathbb{R}^-$. One such singular point is such that the corresponding foliation possesses exactly two separatrices. Mattei proved in an unpublished manuscript that the saturated of a transversal section to any one of the two separatrices, at a point sufficiently close to the the singularity, together with the other separatrix contains a neighbourhood of the singularity. As a consequence for the initial foliation $\mathcal{F}$, the saturated of the transversal section to one of the separatrices of $\mathcal{F}$, together with the other separatrices, constitutes a neighbourhood of the origin.

Extensions of all of the previous results to foliations on higher dimensional manifolds where provided, at least under suitable conditions. Let us start by stating the extention obtained with respect to this last result, that is the result ensuring that the saturated of a transversal section to a separatrix through a singular point in the Siegle domain, together with the other separatrix, constitutes a neighborhood of the origin. In higher dimensions, the result becomes:

Proposition 3 *[39, 37] Let $\mathcal{F}$ be a singular foliation associated to a holomorphic vector field X with an isolated singularity at the origin of $\mathbb{C}^n$. Suppose that the origin belongs to the Siegel domain and satisfy the following condition:*

(a) The eigenvalues $\lambda_1, \ldots, \lambda_n$ of the linear part of X at $0 \in \mathbb{C}^n$ are all different from zero and there exists a straight line through the origin, in the complex plane, separating (for example) λ_1 from the remainder eigenvalues.

(b) Up to a change of coordinates, $X = \sum_{i=1}^n \lambda_i x_i (1 + f_i(x)) \partial/\partial x_i$, where $x = (x_1, \ldots, x_n)$ and $f_i(0) = 0$ for all i.

Then the saturated of a transversal section to the separatrix associated with the eigenvalue λ_1 at a point sufficiently close to the origin, together with the invariant manifold transverse to the mentioned separatrix contains a neighborhood of the origin.

To begin with, it should be noted that if X is a vector field on $\mathbb{C}^3$ with an isolated singularity at the origin and of "strict Siegel type" (i.e. the convex hull of $(\lambda_1, \lambda_2, \lambda_3)$ contains a neighbourhood of the origin), then conditions (a) and (b) are immediately satisfied (cf. [3] for item (b); with respect to item (a), it is clear there exists at least one eigenvalue λ_i such that the angle between λ_i and the other eigenvalues is greater than $\pi/2$). This result is then "generic" in dimension 3. There are, however, examples of vector fields whose origin belongs to the boundary of the convex hull of $(\lambda_1, \lambda_2, \lambda_3)$ does not satisfying item (b) (cf. [6]). Nonetheless, the existence of the three invariant hyperplanes plays a role in the proof of the theorem in question.

Let us say a few words about the proofs. In dimension 2 the proof is based on the following. Consider standard coordinates (x_1, x_2) where X takes on the form of item (b). Fix the separatrix S given in the present coordinates as the x_1-axis and let Σ stands for a transversal section to S at a point sufficient close to the origin. Assume, without loss of generality, that $\Sigma \subseteq \{x_1 = \varepsilon\}$ for some arbitrarily small $\varepsilon \in \mathbb{R}$. Then

- consider on S the loop given as $x_{1,l}(t) = \varepsilon e^{2\pi i t}$, with $0 \leq t \leq 1$. A "kind" of solid torus around the origin can be obtained by taking the lift of the loop $x_{1,l}$ for all points on Σ;
- next, consider on S the "radial directions" given by $x_{1,A}(t) = Ae^{-t}$, $t > 0$, for every $A \in \mathbb{C}$ with $|A| = \varepsilon$, and take the lift of the mentioned path through every single point of the previous "solid torus".

So, fixed a point p on the above "solid torus", let $A = pr_1(p)$, where pr_1 stands for the projection map of $\mathbb{C}^2$ on the first component, i.e. in local coordinates (x, y) we have $p_r(x, y) = x$. By providing precise estimates, Mattei proved that if $\varphi(t) = (x_{1,A}(t), y(t))$ stands for the lift of the path $(x_{1,A}(t), 0)$ through $(A, 0)$ along the leaf passing through p, then $y(t)$ escapes from any small neighborhood of the origin. Essentially, the lift of the mentioned path has a saddle behavior and it is this saddle behavior that ensures that the saturated of the transversal section must contain a neighborhood of the origin, up to joining the other separatrix.

In higher dimensions, extra conditions had to be imposed to ensure that the leaves of the foliation present a similar behavior. Before accurately indicating the role of the imposed conditions in the proof of Proposition 3, let us make some remarks. To begin with, consider a vector field X on $(\mathbb{C}^3, 0)$ taking on the form $X = \sum_{i=1}^{3} \lambda_i x_i (1 + f_i(x))\, \partial/\partial x_i$ with $(\lambda_1, \lambda_2, \lambda_3) = (1, 1+i, -2-i)$, so that the origin is a singular point of strict Siegel type. Furthermore there exists a straight line through the origin, in the complex plane, separating λ_1 from the remainder eigenvalues. Therefore, Proposition 3 states that the saturated of a transversal section to the separatrix S given as the x_1-axis through a point arbitrarily close to the origin, jointly with the invariant hyperplane $\{x_1 = 0\}$, constitutes a neighborhood of the origin. It can easily be checked however that the lift of the "radial directions" given by $x_{1,A}(t) = Ae^{-t}$, $t > 0$, for every $A \in \mathbb{C}$ with $|A| = \varepsilon$, through any point along $\Sigma_A = \{x_1 = A\}$ does not present a "saddle behaviour". To be more precise, if $\varphi(t) = (x_{1,A}(t), x_2(t), x_3(t))$ stands for the lift of the path $(x_{1,A}(t), 0, 0)$ through $(A, 0, 0)$ along the leaf passing through $p \in \Sigma_A$, then $x_2(t)$ goes to zero as t goes to infinity. This happens since the angle between λ_1 and λ_2 is strictly less than $\pi/2$. Essentially we were able to establish the saddle behavior by taking the lift along paths distinct from the "radial lines" on S but still accumulating at the origin. Let us precise the path in question.

Assume without of generality that $\lambda_1 = 1$. So, in the complex plane, let l be a straight line through the origin separating λ_1 from the remaining eigenvalues. Consider then the straight line orthogonal to l at the origin and denote be L the part of this straight line that is contained in the left half-plane (negative real part). Finally, denote by $\bar{L}$ the complex conjugate of L. Suppose that $v = \alpha + i\beta$, with $\alpha > 0$, is a directional vector of L. Then let

$$T = \{z \in \mathbb{C} : z = x + iy, x \in \bar{L}, -\pi < y \leq \pi\}$$

(cf. figure 8). It can easily be checked that the image of T by the application map $\phi : z \to \varepsilon e^z$ covers $\{z : |z| \leq \varepsilon\} \setminus \{0\}$, being the map in question one-to-one. Moreover, the image by ϕ of the elements $z = iy$, with $-\pi < y \leq \pi$, corresponds to the circle in S of radius ε and centered at the origin.

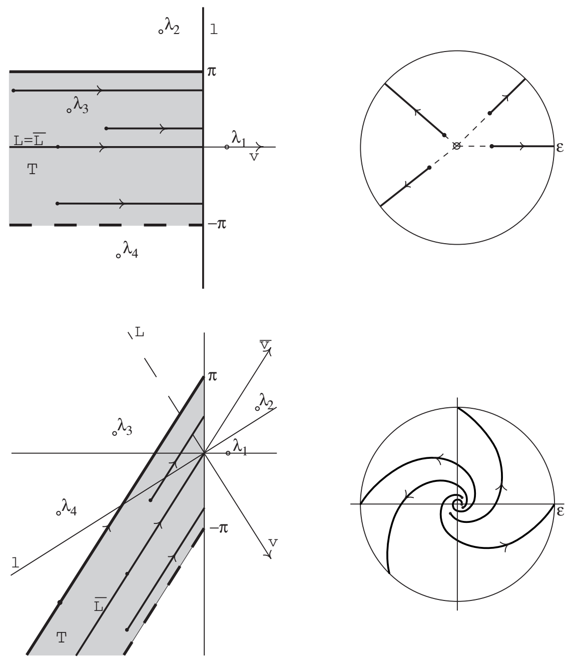

Figure 8:

For every $y \in]-\pi, \pi]$ we fix, let $c_y(t)$ be the path on the complex plane defined by

$$c_y(t) = iy + \frac{1}{v} t \,,$$

with v as above, for $t \in]-\infty, 0]$. Consider the logarithmic spiral curve contained in the x_1-axis given by $r_y(t) = (\varepsilon e^{c_y(t)}, 0, \ldots, 0)$ for $t \in]-\infty, 0]$. The spiral curve is such that $|\varepsilon e^{c_y(0)}| = \varepsilon$ and $\varepsilon e^{c_y(t)}$ goes to zero as t goes to $-\infty$. Fix now and element $z \in \{x_1 = \varepsilon e^{iy}\}$ and let r_z be the lift of r_y through the lift through z. The corresponding lift has a saddle behavior in the sense that the modulus of every component of r_y increases as the modulus of the first component decreases. In fact, if $\varphi(t) = (\varepsilon e^{c_y(t)}, x_2(t), \ldots, x_n(t))$ stands for the mentioned lift, then $x_2(t), \ldots, x_n(t)$ satisfies

$$\begin{cases} \frac{dx_2}{dt} = \frac{\lambda_2}{v} x_2 \left(1 + A_2(\varepsilon e^{c_y(t)}, x_2, \ldots, x_n)\right) \\ \vdots \\ \frac{dx_n}{dt} = \frac{\lambda_n}{v} x_n \left(1 + A_n(\varepsilon e^{c_y(t)}, x_2, \ldots, x_n)\right) \end{cases}$$

for some holomorphic functions $A_2, \ldots, A_n$. It is the fact that the angle between v and λ_i is greater than $\pi/2$ that ensures that $|x_i|$ increases as t goes to $-\infty$ (cf. [37] for precise calculations on the estimates).

Let us return to Proposition 2, the central point of the Mattei-Moussu Theorem.

From the proposition above, it follows that the holonomy pseudogroup of $\mathcal{F}$ is conjugated to a finite group of rotations. It immediately follows that it admits a first integral taking on the form z^n, for a certain $n \in \mathbb{N}^*$. This fist integral may then be globalized following the leaves of $\mathcal{F}$ leading us to Mattei-Moussu's Theorem.

The above proposition is, in fact, quite important and similar characterizations of pseudogroups having finite orbits (or more generally locally finite orbits) has many applications. In the paper [34] the finiteness of a pseudogroup having finite orbits was established in $\text{Diff}(\mathbb{C}^n, 0)$ under rather restrictive conditions involving isolated fixed points. Also several non-trivial examples of infinite pseudogroups having *finite* orbits were provided already in $\text{Diff}(\mathbb{C}^2, 0)$. Yet, a far more relevant question was to figure out, in the general case, which type of algebraic conditions a subgroup of $\text{Diff}(\mathbb{C}^n, 0)$ possessing finite orbits should verify. Differential Galois theory, as well as Morales-Ramis theory cf. [27], suggests that a pseudogroup having finite orbits may be (virtually) solvable. In the paper [36], we confirmed this suggestion for subgroups of $\text{Diff}(\mathbb{C}^2, 0)$. More precisely, the following was proved:

Theorem 14 *[36] Suppose that G is a finitely generated pseudosubgroup of* $\text{Diff}(\mathbb{C}^2, 0)$ *with locally discrete orbits. Then G is virtually solvable.*

Since solvable groups are associated to systems "integrable by quadratures", Theorem 14 states that the "integrability of the group" can be detected through its topological dynamics. Note that the assumption of having locally discrete orbits (as opposed to finite orbits) was introduced so as to allow more general types of first integrals including meromorphic ones.

It should be noted that the condition "virtually", on the statement of Theorem 14, is natural in the sense that the assumption of having locally discrete orbits is stable under finite extensions of the group. Furthermore, this condition is always necessary since the icosahedron group can be realized in $\text{Diff}(\mathbb{C}^2, 0)$.

Whereas the statement of Theorem 14 appears to be sharp, it can considerably be strengthened in the case of groups of diffeomorphisms tangent to the identity. Let then $\text{Diff}_1(\mathbb{C}^2, 0)$ denote the subgroup of $\text{Diff}(\mathbb{C}^2, 0)$ consisting of diffeomorphisms tangent to the identity. Recall also that a point p is said to be *recurrent* if its orbit under G accumulates non-trivially on p itself (this definition excludes points having "periodic" orbit). The following result provides strong quantitative information concerning the set of recurrent points.

Theorem 15 *[36] Suppose that $G \subseteq \text{Diff}_1(\mathbb{C}^2, 0)$ is non-solvable. Then there exists a neighborhood of the origin U and a countable union $K \subset U$ of proper analytic subsets of U such that every point in $U \setminus K$ is recurrent for G (in particular the set of recurrent points has full Lebesgue measure).*

7 Resolution of singularities for 1-dimensional foliations and for vector fields on 3-manifolds

In this section it will be presented the main results obtained in the paper [38] along with the basic notions needed to make their statements intelligible. A discussion about the place of these results in the current state-of-art in the area will also be presented.

Recall first that a singular, one-dimensional holomorphic foliation $\mathcal{F}$ on $(\mathbb{C}^n, 0)$ is nothing but the (singular) foliation defined by the local orbits of a holomorphic vector field defined on a neighborhood of the origin and having zero-set of codimension at least 2. A simple consequence of Hilbert nullstellensatz is that, up to multiplying vector fields by a meromorphic function, every meromorphic vector field X

on $(\mathbb{C}^3, 0)$ induces a singular holomorphic foliation on a neighborhood of the origin. This foliation will be called the foliation associated with X. Clearly two (meromorphic) vector fields have the same associated foliation if and only if they differ by a multiplicative (meromorphic) function. Conversely, a vector field X inducing a given foliation $\mathcal{F}$ will be called a representative of $\mathcal{F}$ if X is holomorphic and the set of zeros of X has codimension at least two. In other words, a representative vector field of $\mathcal{F}$ is any holomorphic vector field tangent to $\mathcal{F}$ and having a zero-set of codimension at least 2.

There follows from the preceding that there is no point in considering "singular meromorphic foliations" since all foliations in this category would, in fact, be holomorphic. Similarly, (singular) holomorphic foliations have empty zero-divisor since their singular sets have codimension at least 2. In other words, whenever we are exclusively concerned with foliations, we can freely eliminate any (meromorphic) common factor between the components of a vector field tangent to the foliation to obtain a representative vector field. Naturally this cannot be done if we are focusing on an actual fixed vector field X as it so often happens.

In the above mentioned context of singular points, *resolution theorems* - also known as *desingularization theorems* - are geared towards foliations in that we are "free" to eliminate non-trivial common factors between the components of a vector field whenever these common factors arise from transforming a representative vector field by a birational map. To further clarify these issues, we may recall that the prototype of all "resolution theorems" for foliations is provided by Seidenberg's theorem (cf. [41]) which is valid for foliations defined on a two-dimensional ambient. More precisely, if $\mathcal{F}$ denotes a singular holomorphic foliation defined on a neighborhood of $(0,0) \in \mathbb{C}^2$, then Seidenberg's theorem asserts the existence of a finite sequence of one-point blow-up maps, along with transformed foliations $\mathcal{F}_i$ $(i = 1, \ldots, n)$

$$\mathcal{F} = \mathcal{F}_0 \overset{\Pi_1}{\longleftarrow} \mathcal{F}_1 \overset{\Pi_2}{\longleftarrow} \cdots \overset{\Pi_l}{\longleftarrow} \mathcal{F}_n$$

such that the following holds:

- Each blow-up map Π_i $(i = 1, \ldots, n)$ is centered at a singular point of $\mathcal{F}_{i-1}$.
- All singular points of $\mathcal{F}_n$ are *elementary*, i.e. $\mathcal{F}_n$ is locally given by a representative vector field X_n whose linear part at the singular point in question has at least one eigenvalue different from zero (cf. below).

Whereas Seidenberg's theorem is directly concerned with foliations, it is also very effective when applied to vector fields defined on complex surfaces. The general principle to use Seidenberg theorem to study vector fields - as opposed to foliations - consists of applying Seidenberg theorem to the associated foliation while also keeping track of the divisor of zeros/poles of the transformed vector field. In line with this point of view, Seidenberg's theorem is equally satisfying: the structure of the resolution map (the composition of the blow-ups Π_i) is such that the transform of holomorphic vector fields retains its holomorphic character (here the reader is reminded that the transform of a holomorphic vector field by a birational map is, in general, a meromorphic vector field). More generally, Seidenberg's procedure allows for an immediate computation of the zero-divisor of the transformed vector field. For example, if we blow-up a vector field X having an isolated singularity at $(0,0) \in \mathbb{C}^2$ and denote by k the degree of the first non-zero homogeneous component of the Taylor series of X at $(0,0)$, then the zero-divisor of the blow-up of X coincides with the exceptional divisor and has multiplicity $k-1$ (unless X is actually a multiple of the radial vector field $x\partial/\partial x + y\partial/\partial y$ in which case the multiplicity is k).

In dimension 2, the classical Seidenberg Theorem provides an optimal algorithm for simplifying the singularities of a foliation. On the other hand, when one moves to dimension 3, the situation is no longer so simple. The well-known example of Sancho and Sanz shows the existence of foliations in $(\mathbb{C}^3,0)$ that cannot be reduced by standard blow-up centered at the singular set of the foliation in question. To be more precise, they have shown that the foliation associated with the vector field

$$X = x\left(x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y} - \beta z\frac{\partial}{\partial z}\right) + xz\frac{\partial}{\partial y} + (y - \lambda x)\frac{\partial}{\partial z}$$

possesses a strictly formal separatrix $S = S_0$ through the origin such that the singular point p_n (selected by the transformed separatrices S_n in the sense that they correspond to the intersection of S_n with the excetional divisor) is a nilpotent singular point for the corresponding foliation, for all $n \in \mathbb{N}$. In fact, the representative of the singular point p_n is given by a vector field on the above 3-parameter family. The fact that every separatrix S_n is stricty formal says that even in the case we allow blow-ups to be centered at analytic invariant curves that *are not* necessarily contained in singular set of the foliation, a resolution procedure still does not exist.

The generalization of Seidenberg's theorem to foliations on $(\mathbb{C}^3,0)$ is a very subtle problem. A very satisfactory answer is provided in [25], [26] and it relies heavily on a previous result by Panazzolo in [28]. Slightly later, the topic was revisited from the point of view of valuations in [9]. The "final models" in the resolution theorem proved in [9] are, however, not as accurate as those in [26]. The paper [38] grew out of an attempt to use the mentioned results to obtain a sharper resolution result which *would hold for the special class of holomorphic foliations* which is associated with semicomplete vector fields (cf. Theorem 17 below). Whereas the class of foliations associated with semicomplete vector fields is rather special, it contains the underlying foliations of all complete vector fields as well as many foliations arising in the context of Mathematical Physics and the importance of these examples justifies the interest in a sharper (or "simpler") resolution statement valid only for this class of foliations.

The resolution theorem in [9] was not really suited to our needs because the corresponding "final models" were not accurate enough. As to the resolution theorem in [26], we were unsure of the behavior of vector fields - as opposed to foliations - under their procedure. Basically, we did not know if the weighted blow-ups on Panazzolo's algorithm [28] always transform holomorphic vector fields on holomorphic vector field rather than meromorphic ones (question that does not arise in the context of foliations, as explained above). In fact, it is convenient to point out that, in full generality, the transform of a holomorphic vector field by a birational map is a meromorphic vector field. To provide an explicit example, consider the holomorphic vector field $X = F(x,y,z)\partial/\partial x + G(x,y,z)\partial/\partial y + H(x,y,z)\partial/\partial z$ where $F(x,y,z) = y$ and G and H are such that the z-axis $\{x = y = 0\}$ is contained in the singular set of X. Let (x,t,z) be coordinates for the weighted blow-up (of weight 2) centered at the z-axis in which the corresponding projection map Π is given by $\Pi(x,t,z) = (x^2, tx, z)$. A direct inspection shows that the corresponding transform $\Pi^* X$ of X is given by

$$\begin{aligned}\Pi^* X &= \frac{1}{2x}F(x^2,tx,z)\frac{\partial}{\partial x} + \left[-\frac{t}{2x^2}F(x^2,tx,z) + \frac{1}{x}G(x^2,tx,z)\right]\frac{\partial}{\partial t} + \\ &\quad + H(x^2,tx,z)\frac{\partial}{\partial z}\,.\end{aligned}$$

Clearly $F(x^2,tx,z)/2x$ and $G(x^2,tx,z)/x$ are both holomorphic but $tF(x^2,tx,z)/2x^2$ is *strictly meromorphic*. Therefore $\Pi^* X$ is meromorphic with poles over the exceptional divisor.

Although checking whether or not Panazzolo's algorithm in [28] is such that the transforms of holomorphic vector fields retain their holomorphic character should be straightforward, the algorithm itself is rather involved with many different cases so that we were very grateful to the referee of our paper [38] for confirming that this is, in fact, the case. In other words, holomorphic vector fields are transformed into holomorphic vector fields by the algorithm in [28]. Still, when studying the papers in question, we felt it would be nice to try and complete the work of Cano-Roche-Spivakovsky [9] by deriving "final models" similar to those of [26], which are described in Theorem 16 below.

Theorem 16 *[38] Let $\mathcal{F}$ denote a (one-dimensional) singular holomorphic foliation defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$. Then there exists a finite sequence of blow-up maps along with transformed foliations*

$$\mathcal{F} = \mathcal{F}_0 \xleftarrow{\Pi_1} \mathcal{F}_1 \xleftarrow{\Pi_2} \cdots \xleftarrow{\Pi_l} \mathcal{F}_n \tag{3}$$

satisfying all of the following conditions:

(1) The center of the blow-up map Π_i is (smooth and) contained in the singular set of $\mathcal{F}_{i-1}$, $i = 1, \ldots, n$.

(2) The singularities of $\mathcal{F}_n$ are either elementary or persistently nilpotent.

(3) The number of persistently nilpotent singularities of $\mathcal{F}_n$ is finite and each of them can be turned into elementary singular points by performing a single weighted blow-up of weight 2.

As it is implicit in the above statement, persistent nilpotent singular points is a special type of nilpotent singular point whose definition and characterization is present on Section 4 of [38] (cf. Theorem 3 of the mentioned paper). They also play a special role in the resolution theorem of [26]. Namely, they appear as singularities associated with a special type of $\mathbb{Z}/2\mathbb{Z}$-orbifold which, incidentally, require a weight 2 blow-up to be turned into elementary ones. It is also worth pointing out that both statements are sharp in the sense that the well known example by Sancho and Sanz shows the use of a weight 2 blow-up cannot be avoided (cf. Sections 2 and 4 of the same paper).

In particular, both Theorem 16 and the resolution theorem, Theorem 2, in [26] asserts the existence of a birational model for $\mathcal{F}$ where all singularities of $\mathcal{F}$ are elementary except for finitely many ones that can be turned into elementary singular points by means of a single blow-up of weight 2. In this sense, differences between these two theorems are down to the way in which these rational models are constructed. Alternatively, Theorem 16 can simply be regarded as a new proof of the resolution theorem in [26].

In terms of the construction of the mentioned rational models, we briefly mention that McQuillan and Panazzolo work in the category of *weighted blow up*, along with the corresponding orbifolds, while in Theorem 16 we restrict ourselves as much as possible to the use of standard (i.e. unramified) blow-ups. Once again, additional information on these strategies can be found in Section 2 of [38].

At this point, it is convenient to introduce some terminology. Throughout this section the term *blow-up* will refer to *standard (i.e. homogeneous) blow-ups*. This applies, in particular, to the statement of Theorem A. As to blow-ups with weights (i.e. non-homogeneous or ramified blow-ups) which are inevitably also involved in the discussion, *these will be explicitly referred to as weighted (or ramified) blow-ups*.

Also, we will say that a (germ of) foliation $\mathcal{F}$ can be *resolved* if there is a sequence of blowing-ups as in (3) leading to a foliation $\mathcal{F}_n$ all of whose singularities are elementary. Similarly, a sequence of blowing-ups as in (3) will be called a *resolution of $\mathcal{F}$* if all the singular points of $\mathcal{F}_n$ are elementary. Whenever sequences of *weighted blow-ups* leading to a foliation having only elementary singular points are considered, they may be referred to as a *weighted resolution of $\mathcal{F}$*. With this terminology, while every germ of foliation on $(\mathbb{C}^3, 0)$ admits a weighted resolution, as follows from [26] or Theorem A, the mentioned examples of Sancho and Sanz show that not all of them admit a resolution. Section 2 of [38] contains a detailed discussion on the mutual interactions involving [9], [26], and our discussion revolving around Theorem 16.

We can now go back towards our initial motivation, namely to germs of foliations $\mathcal{F}$ on $(\mathbb{C}^3, 0)$ that are associated with a semicomplete vector field. Since the notion of semicomplete singularity was introduced along with its first applications to the (global) study of complex vector fields ([30]), it has been natural to ask whether all foliations in this class admit a resolution. A special instance of this problem which is of interest in the study of complex Lie group actions consists of asking whether the underlying foliation of a complete holomorphic vector field (on some complex manifold of dimension 3) can be transformed into a foliation all of whose singular points are elementary by means of a sequence of blow-ups as in (3).

To state our results concerning this special class of foliations, let us place ourselves once and for all in the context of semicomplete vector fields. First, it is convenient to recall that a singularity of a holomorphic vector field X is said to be *semicomplete* if the integral curves of X admit a maximal domain of definition in $\mathbb{C}$, cf. Section 3. In particular, whenever X is a *complete vector field* defined on a complex manifold M, every singularity of X is automatically semicomplete. The answer to the above question is hen provided by the following theorem:

Theorem 17 *[38] Let X be a semicomplete vector field defined on a neighborhood of the origin in $\mathbb{C}^3$ and denote by $\mathcal{F}$ the holomorphic foliation associated with X. Then one of the following holds:*

1. *The linear part of X at the origin is nilpotent (non-zero).*
2. *There exists a finite sequence of blow-ups maps along with transformed foliations*

$$\mathcal{F} = \mathcal{F}_0 \overset{\Pi_1}{\longleftarrow} \mathcal{F}_1 \overset{\Pi_2}{\longleftarrow} \cdots \overset{\Pi_r}{\longleftarrow} \mathcal{F}_r$$

such that all of the singular points of $\mathcal{F}_r$ are elementary. Moreover, each blow-up map Π_i is centered in the singular set of the corresponding foliation $\mathcal{F}_{i-1}$. In other words, the foliation $\mathcal{F}$ can be resolved.

Let us emphasize that item 1 in Theorem 17 means that the linear part of X is (nilpotent) *non-zero from the outset*. In other words, if the foliation $\mathcal{F}$ associated with X cannot be resolved, then X has a non-zero nilpotent linear part and this property is "universal" in the sense that it does not depend on any sequence of blow-ups/blow-downs carried out. In particular, we can choose a "minimal model" for our manifold and the corresponding transform of X will still have non-zero nilpotent linear part at the corresponding point. Moreover, from Theorem 3 on [38] about "persistent nilpotent singularities", it is easy to obtain accurate normal forms for the vector field X.

Also, the statement of Theorem 17 involves the linear part of the vector field X rather than the linear part of the associated foliation $\mathcal{F}$. This makes for a stronger statement which is better emphasized by Corollary 2 below:

Corollary 2 *[38] Let X be a semicomplete vector field defined on a neighborhood of $(0,0,0) \in \mathbb{C}^3$ and assume that the linear part of X at the origin is equal to zero. Then item (2) of Theorem B holds.*

More precisely, Theorem 17 asserts that foliations associated with semicomplete vector fields in dimension 3 can be resolved by a sequence of blow-ups centered in the singular set except for a very specific case in which the vector field X (and hence the foliation $\mathcal{F}$) has a "universal" non-zero nilpotent linear part. As mentioned, these statements have the advantage of involving the vector field and not only the underlying foliation. To clarify the meaning of this sentence, consider a holomorphic (semicomplete) vector field X having the form $X = fY$, where Y is another holomorphic vector field and f is a holomorphic function. Whereas X and Y induce the same singular foliation $\mathcal{F}$, an immediate consequence of Corollary 2 is that $\mathcal{F}$ must be as in item (2) of Theorem 17 *provided that f vanishes at the origin*: in fact, if f and Y are as indicated, then the linear part of X vanishes at the origin at that $\mathcal{F}$ is, indeed, singular (clearly there is nothing to be proved if $\mathcal{F}$ is regular). In other words, if $X = fY$ as above with $f(0,0,0) = 0$ and X semicomplete, then the foliation associated with X can certainly be resolved even if Y has a nilpotent singular point at the origin.

A few additional comments are needed to fully clarify the role of item (1) in Theorem B. First note that more accurate normal forms are available for the vector fields in question: indeed, Theorem 3 of [38] provides accurate normal forms for all persistent nilpotent singular points. In addition, *not all* nilpotent vector fields giving rise to persistent nilpotent singularities are semicomplete and, in this respect, the normal form provided by the mentioned Theorem will further be refined.

Next, taking into account the global setting of complete vector fields, it is natural to wonder if there is, indeed, *complete vector fields* inducing a foliation with singular points that cannot be resolved. As a consequence of Theorem 17, such vector fields would definitely be pretty remarkable since they must have a (non-zero) "universal" nilpotent singular point. To confirm that these global situations do exist, however, it suffices to note that the polynomial vector field

$$Z = x^2\partial/\partial x + xz\partial/\partial y + (y - xz)\partial/\partial z$$

can be extended to a complete vector field defined on a suitable open manifold (details on Section 6 of [38]). As will be seen, the origin in the above coordinates constitutes a nilpotent singular point of Z that cannot be resolved by means of blow-ups as in item (2) of Theorem 17, albeit this nilpotent singularity can be resolved by using a blow-up centered at the (invariant) x-axis.

Finally, the question raised above about the existence of singularities as in item (1) of Theorem 17 in global settings can also be asked in the far more restrictive case of holomorphic vector field defined on *compact manifolds* of dimension 3. Owing to the compactness of the manifold, every such vector field is automatically complete. In this setting, the methods used in the proof of Theorem 17 easily yield:

Corollary 3 *[38] Let $\mathcal{F}$ be the foliation associated with a vector field X globally defined on some compact manifold M of dimension 3. Then every singular point of $\mathcal{F}$ can be resolved.*

Let us close this section with a couple of remarks inspired by some questions asked to us by A. Glutsyuk. Essentially his questions concern resolution strategies with minimal number of (weighted) blow ups which can also be seen as an analogue of some questions previously considered in the context of Hironaka's theorem. In this respect, it is clear that being able to work with weighted blow ups, as opposed to standard ones, increases the chances of reducing the number of blow ups to resolve a given foliation. Indeed, it is easy to produce examples of this phenomenon already in dimension 2 and in the context of Seidenberg's theorem. Hence, there is no chance that the strategy used in the proof of Theorem 16 will in general minimize the number of blow ups required to resolve a given foliation. However, we ignore if Panazzolo's algorithm [28] has minimizing properties in the preceding sense.

A similar question directly motivated by the fact that in dimension 2 standard blow ups suffice to resolve any foliation, consists of trying to minimize the number of weighted blow ups needed to obtain the resolution. In this case, and at least for generic foliations, Theorem refteo:A seems to provide a satisfactory answer. Let us try to sketch an argument in this direction. As it follows from Theorem 3 of [38], persistent nilpotent singular points are naturally associated with certain formal separatrices (i.e. formal invariant curves) having some special properties. Their "position" in the exceptional divisor obtained after finitely many blow ups is thus determined by the corresponding formal separatrices. In particular, it is possible to talk about these singularities being in "general position" for a given germ in an intrinsic way, i.e. independently of the use of any sequence of (standard) blow ups. At least when these singularities are in "general position" for a foliation $\mathcal{F}$, then Theorem 16 should minimize the number of weighted blow ups needed to turn $\mathcal{F}$ into a foliation all of whose singular points are elementary. Indeed, each such singularity requires at least one weighted blow up to be turned into elementary singular points and each such blow up can non-trivially affect only one of these singularities thanks to the "general position" assumption. Thus the number of weighted blow ups needed cannot be smaller than the number of persistent nilpotent singularities and the later is matched by the procedure in Theorem A. We ignore, however, if the "general position assumption" is really needed for this statement. Note that if there is a foliation $\mathcal{F}$ that can be resolved by using less weighted blow ups than those prescribed in Theorem A, then $\mathcal{F}$ should conceal at least two persistent nilpotent singularities so "close" to each other that they can both be turned into elementary singular points by means of a same weighted blow up.

8 Closing remarks and future directions

The problems we plan to attack in the near future have different natures. In fact, a characteristic of the area of complex ODEs is an almost always present interaction between dynamics and geometry and this interaction has also been illustrated at several instances of the previous sections. At several points, dynamical results/techniques were used to approach geometric problems, and conversely.

Also, it should be noted that a good part of the problems involving continuous dynamics (represented by complex ODEs/foliations) boil down on iterative dynamics through their transverse dynamics which are described by the correspondent holonomy pseudogroups. These pseudogroups are akin to the objects belonging to the domain of iterative dynamics up to one fundamental issue: in general the maps are not globally defined. In other words, there is no "holonomy group" but only "holonomy pseudogroup". This presents us with a double challenge: we need to study the dynamics of pseudogroups containing maps that are not globally defined and, on the other hand, we should try in certain special situations to "globalize" these holonomy maps as much as possible so as to be able to call on the arsenal of iterative dynamics that are available for globally defined dynamical systems. These two aspects of continuous dynamical systems will appear again in the subsequent discussion.

Finally, we give a list of goals, which we hope to achieve in the near future. Naturally, many of the results and techniques on the papers discussed over this text will have some role to play in our approach to the questions listed blow.

1. We intend on considering the Hwang-Mok problem (cf. Section 4) providing a partial answer to it. More precisely, we believe that the dimension of the automorphism group of algebraic manifolds of dimension 3 with Picard group isomorphic to $\mathbb{Z}$ is at most 30. In this direction we have recently proved that Ghys conjecture holds in a class of compact manifolds containing Käler (and so, in particular, algebraic) manifolds of any dimension - this is done by controlling the dynamics of the orbits of holomorphic vector fields on the manifolds in question (to be more precise by proving that the orbits are compactifiable - a folkloric result in the algebraic case) and then to prove the existence of a separatrix through an isolated singular point. We have still to deal with the non-isolated singularities

2. Also, there are some distinguished equations, or systems of equations, that play important roles in several aspects of Mathematics, often reaching out to Physics and applications. We will focus on Airy, Painlevé equations PI through PV (notation: PI-PV). All of this in both classical and generalized contexts. Whereas these systems are the object of a massive body of literature, many important questions remain to be answered. Among those, I mention integrabilibity problems and computation of Malgrange pseudogroup, classification of special solutions - including eternal ones (i.e. complete ones), asymptotic behavior/estimates, the (possible) occurrence of chaotic behavior and its quantification. The standard methods used for these systems - to greater or lesser extent revolving around integrable systems techniques - have come short of providing enough insight in the above problems. In this direction, my aim is to shed new light in these questions by bringing into the discussion methods of

continuous and discrete complex dynamical systems complemented by specific geometric techniques, such as desingularization results.

3. Halphen equation have thoroughly been studied by A. Guillot. In terms of vector fields they are given by a degree 2 homogeneous polynomial vector field X on $\mathbb{C}^3$ satisfying $[C,X]=2R$, where R stands for the Radial vector field and C is a constant vector field. Since we also have the relations $[R,X]=X$ and $[R,C]=C$, the triple X,R,C constitutes a Lie algebra isomorphic to $\mathfrak{psl}_2\mathbb{C}$. Guillot proved in [16] that Halphen vector fields contain examples of semi-complete vector fields whose solutions have maximal domain of definition covered by the disc and with complicated dynamics (none of these phenomenon happens for semicomplete vector fields in dimension 2). He also proves that the mentioned vector fields can be extended to complete vector fields on suitable complex 3-manifolds that he constructed. In [33] we have proved that these manifolds are not Kähler. In fact, the method introduced in [33] allows us to recover a good part of the results obtained by Guillot.

Halphen vector fields are determined by triangular groups. In fact, they induce Riccati foliations with base $\mathbb{CP}(1)$ minus three points, being the triangular group strictly related with the monodromy of the foliations. By introducing a meromorphic variant of Halphen vector fields (not only for 3-manifolds), we will be able to obtain semicomplete vector fields associated to far more general Kleinian groups. This allows us, in particular, to consider deformation spaces for these equations. Also we will investigate the problem of completing the solutions of the corresponding meromorphic Halphen systems so as to obtain a more systematic method of constructing non-Käler complex manifolds. A rather general theory should be developed here relying on fine aspects of the dynamics of Kleinian groups along with non-Kahler complex manifolds.

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