A NOTE ON PSEUDOVARIETIES OF COMPLETELY REGULAR SEMIGROUPS

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Abstract. A paper of Almeida and Trotter on completely regular semigroups makes essential use of free profinite semigroupoids over profinite graphs with infinitely many vertices. It has since been shown that such structures must be handled with great care. In this note, it is verified that the required properties hold for the profinite graphs considered by Almeida and Trotter, and thus the gap is filled.

1. Introduction

In [4] one finds a systematic study of relatively free profinite categories and relatively free profinite semigroupoids generated by profinite graphs with a finite number of vertices. The case of profinite graphs with an infinite number of vertices is more delicate, as highlighted in [1]. The main problem is that in the infinite-vertex case, the topological closure of the subsemigroupoid generated by the graph may not be a subsemigroupoid of the corresponding free profinite semigroupoid. This problem was overlooked in [3] and [2], where infinite-vertex free profinite semigroupoids are seriously considered for the first time as a tool to be used in the study of relatively free profinite semigroups. While their role in [3] can be considered marginal, in [2] some specific relatively free profinite semigroupoids with an infinite number of vertices play a key role in the proofs of the main results remain valid by establishing that, for the infinite-vertex graphs which are used in the proofs, the semigroupoids generated by the graphs are dense in the corresponding relatively free profinite semigroupoids.

2. The free profinite semigroupoid generated by $\partial_X P$

For general background on pseudovarieties of semigroups and of semigroupoids, and on their relatively free profinite structures, see [6]. We also adopt the notation of [2], which we recall here for the reader's benefit. Let A be a finite alphabet and let V be a pseudovariety of semigroups. The free pro-V semigroup generated by A is denoted by $\overline{\Omega}_A\mathsf{V}$, and the subsemigroup of $\overline{\Omega}_A\mathsf{V}$ generated by A is denoted by $\Omega_A\mathsf{V}$. Suppose that V contains the pseudovariety Sl of finite semilattices. This hypothesis enables us to consider the *content*, which is a continuous homomorphism c from $\overline{\Omega}_A\mathsf{V}$

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to the union-semilattice of subsets of A that maps each letter $a \in A$ to the singleton set $\{a\}$. The content function extends to $(\overline{\Omega}_A \mathsf{V})^1$ by letting $c(1) = \emptyset$. Suppose also that $\overline{\Omega}_A \mathsf{V}$ has 0, $\bar{0}$, 1, $\bar{1}$ functions, which means that, for every $w \in \overline{\Omega}_A \mathsf{V}$, there are factorizations $w = w_0 a w' = w'' b w_1$ in $(\overline{\Omega}_A \mathsf{V})^1$ such that $a, b \in A$ and

$$c(w) = c(w_0) \uplus \{a\} = \{b\} \uplus c(w_1),$$

where the factors w_0, a, b, w_1 are unique. We put $0(w) := w_0$, $1(w) := w_1$, $\bar{0}(w) := a$, and $\bar{1}(w) := b$. Note that the functions 0, $\bar{0}$, 1, $\bar{1}$ are continuous.

Let X be a subset of A with at least two elements. Consider a subset P of $\overline{\Omega}_A \mathsf{V}$ such that $c(u) \subseteq X$ for every $u \in P$. We define a graph[1] $\partial_X P$ on the set of all factors $v \in (\overline{\Omega}_A \mathsf{V})^1$ of elements of P such that $|X \setminus c(v)| \in \{1, 2\}$. The vertices are those v such that $|X \setminus c(v)| = 2$ and the edges are the remaining elements of $\partial_X P$. The adjacency functions are $\alpha = 0$ and $\omega = 1$, where, in a graph, we respectively denote by $\alpha(u)$ and $\omega(u)$ the source and the target of an edge u. This definition of $\partial_X P$ is taken from [2, Subsection 3.1], where the necessary hypothesis that the content of all elements of P is contained in X is not made explicit, but is implicitly used.

Profinite graphs are defined in [2] as being the inverse limits of finite graphs, in the category of topological graphs, with finite graphs having the discrete topology. It is folklore that the profinite graphs are precisely the topological graphs whose topology is a Boolean space (a proof can be found in [7].) It follows immediately from this characterization that the graph $\partial_X P$ is profinite if P is a closed subset of $\overline{\Omega}_A \mathsf{V}$, which happens in particular when $P = \overline{\Omega}_X \mathsf{V}$, the case considered in the main results of [2].

Let W be a pseudovariety of finite semigroupoids. We adopt the definition from [1] of a *pro*-W semigroupoid as being a compact semigroupoid which is *residually in* W in the sense that, for every pair u, v of distinct elements of S, there is a continuous semigroupoid homomorphism $\varphi\colon S \to F$ into a semigroupoid F of W satisfying $\varphi(u) \neq \varphi(v)$. For the case where W is the pseudovariety Sd of all finite semigroupoids, one uses *"profinite"* as a synonym of "pro-W". We remark that there is an unpublished example due to G. Bergman (mentioned in [5]) of an infinite-vertex semigroupoid which is profinite according to this definition, but which is not an inverse limit of finite semigroupoids.

Let Γ be a profinite graph and let W be a pseudovariety of semigroupoids. The free pro-W semigroupoid generated by Γ, denoted $\overline{\Omega}_\Gamma \mathsf{W}$, is a pro-$\mathsf{W}$ semigroupoid, together with a continuous graph homomorphism $\iota\colon \Gamma \to \overline{\Omega}_\Gamma \mathsf{W}$, with the following property: for every continuous graph homomorphism $\varphi\colon \Gamma \to F$ into a semigroupoid of W, there is a unique continuous semigroupoid homomorphism $\hat{\varphi}\colon \overline{\Omega}_\Gamma \mathsf{W} \to F$ such that $\hat{\varphi} \circ \iota = \varphi$. It is easy to show that this semigroupoid is indeed unique, up to isomorphism of compact semigroupoids, and a proof of its existence is made in [1] by a reduction to the finite-vertex case treated in [4]. Moreover, if W contains nontrivial semigroups, we may assume (as we do from hereon) that ι is the inclusion mapping [1]. The subsemigroupoid of $\overline{\Omega}_\Gamma \mathsf{W}$ generated by $\iota(\Gamma)$ is denoted by $\Omega_\Gamma \mathsf{W}$. If Γ has a finite number of vertices then $\Omega_\Gamma \mathsf{W}$ is dense in $\overline{\Omega}_\Gamma \mathsf{W}$ [4]. However, in general, that property fails, and it is only by iterating transfinitely algebraic and topological closures that one reachs the profinite semigroupoid $\overline{\Omega}_\Gamma \mathsf{W}$ [1].

[1] Throughout this paper, by graph we mean a directed graph.

As remarked in [4], if Γ is a finite graph, then $\overline{\Omega}_\Gamma \mathsf{W}$ is metrizable. The following is a generalization of this fact, which will be used later on.

Proposition 2.1. *Let Γ be a profinite graph and let W be a pseudovariety of semigroupoids. If Γ is metrizable, then so is $\overline{\Omega}_\Gamma \mathsf{W}$.*

For the proof of Proposition 2.1 one uses the next folklore result, for which we do not have a direct reference. However, a proof is implicit in the proof of [8, Corollary 1.1.13] (there the reference is made to profinite second countable spaces, but by Urysohn's metrization theorem [9, Theorem 23.1], a profinite space is second countable if and only if it is metrizable.) In the statement we use the notation in [8, Chapter I] for inverse systems.

Proposition 2.2. *Let the topological space X be the inverse limit of an inverse system $\{X_i, \varphi_{ij}, I\}$ of finite discrete spaces. Then X is metrizable if and only if for some countable totally ordered subset J of I (with order type equal to that of ω) the space X is the inverse limit of the restricted inverse system $\{X_i, \varphi_{ij}, J\}$.*

Proof of Proposition 2.1. By Proposition 2.2, the profinite graph Γ is, in the category of profinite graphs, an inverse limit $\varprojlim_{n\geq 1} \Gamma_n$ for some inverse sequence of finite graphs Γ_n. By the construction of $\overline{\Omega}_\Gamma \mathsf{W}$ given in [1], the semigroupoid $\overline{\Omega}_\Gamma \mathsf{W}$ embeds as a closed subsemigroupoid of $\varprojlim_{n\geq 1} \overline{\Omega}_{\Gamma_n} \mathsf{W}$. Since Γ_n is finite, the profinite semigroupoid $\overline{\Omega}_{\Gamma_n} \mathsf{W}$ is metrizable. As the inverse limit of a sequence of metrizable spaces is metrizable (cf. [9, Theorem 22.3]), it follows that $\varprojlim_{n\geq 1} \overline{\Omega}_{\Gamma_n} \mathsf{W}$ and $\overline{\Omega}_\Gamma \mathsf{W}$ are metrizable. □

Corollary 2.3. *The profinite semigroupoid $\overline{\Omega}_{\partial_X \overline{\Omega}_A \mathsf{V}} \mathsf{W}$ is metrizable, for every finite alphabet A and every subset X of A with at least two elements.*

Proof. Since A is finite, $\overline{\Omega}_A \mathsf{V}$ is metrizable, and therefore so is $\partial_X \overline{\Omega}_A \mathsf{V}$. □

The relatively free profinite semigroupoids which intervene in the main results in [2] are of the form $\overline{\Omega}_{\partial_X \overline{\Omega}_A \mathsf{V}} \mathsf{W}$. We remark that in (the applications of) Theorem 2.5 of [2], the graph $\partial_X \Omega_A \mathsf{V}$ is identified with the graph $\partial_X(A^+)$ and the subsemigroupoid $\langle \partial_X \Omega_A \mathsf{V} \rangle$ of $\overline{\Omega}_{\partial_X \overline{\Omega}_A \mathsf{V}} \mathsf{W}$ generated by $\partial_X \Omega_A \mathsf{V}$ is identified with the free semigroupoid $(\partial_X(A^+))^+$. These identifications hold in many cases, namely when V and W contain all finite nilpotent semigroups: we then have $\Omega_A \mathsf{V} \cong A^+$ and $\Omega_\Gamma \mathsf{W} \cong \Gamma^+$ (cf. [1, Theorem 3.16].) But, for example, denoting by Ab the pseudovariety of finite Abelian groups, we know that $\Omega_A \mathsf{Ab}$ is not isomorphic to A^+. To fix this problem it suffices to replace $\partial_X(A^+)$ by $\partial_X \Omega_A \mathsf{V}$ and $(\partial_X(A^+))^+$ by $\langle \partial_X \Omega_A \mathsf{V} \rangle$ when appropriate in [2].

Another problem in [2] that needs to be fixed stems from the fact that in the proof of Theorem 2.5 in [2] it is assumed that the subsemigroupoid of $\overline{\Omega}_{\partial_X \overline{\Omega}_A \mathsf{V}} \mathsf{W}$ generated by $\partial_X \Omega_A \mathsf{V} \subseteq \partial_X \overline{\Omega}_A \mathsf{V}$ is dense in $\overline{\Omega}_{\partial_X \overline{\Omega}_A \mathsf{V}} \mathsf{W}$. As already mentioned, after the publication of [2], examples of profinite graphs Γ such that $\Omega_\Gamma \mathsf{W}$ is not dense in $\overline{\Omega}_\Gamma \mathsf{W}$ were given in [1]. Therefore, one needs to verify if the denseness assumption made in the proof of Theorem 2.5 in [2] really holds. In the next proposition, we show that it does hold when V is a pseudovariety of semigroups such that $\mathsf{Sl} \subseteq \mathsf{V} \subseteq \mathsf{CR}$, where CR denotes the pseudovariety of completely regular semigroups, and such that $\overline{\Omega}_A \mathsf{V}$ has 0, $\bar{0}$, 1, $\bar{1}$ functions for every finite set A.

This suffices to guarantee the main results in Sections 3 to 6 of [2], all about pseudovarieties satisfying these conditions.

Theorem 2.4. *Let* V *be a pseudovariety of semigroups such that* $\mathsf{Sl} \subseteq \mathsf{V} \subseteq \mathsf{CR}$ *and let* W *be any pseudovariety of semigroupoids containing some nontrivial semigroup. Let* A *be a finite alphabet, and let* X *be a subset of* A *with at least two elements. Suppose that* $\overline{\Omega}_A\mathsf{V}$ *has* 0, $\bar{0}$, 1, $\bar{1}$ *functions. Then the subsemigroupoid* $\langle \partial_X \Omega_A \mathsf{V}\rangle$ *of* $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ *is dense. In particular,* $\Omega_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ *is dense in* $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$.

Proof. To avoid confusion between the multiplication in $\overline{\Omega}_A\mathsf{V}$ and the edge multiplication in $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$, we denote the multiplication of two composable elements e, f of $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ by $e \circ f$.

Note that $\partial_X\overline{\Omega}_A\mathsf{V} = \overline{\partial_X\Omega_A\mathsf{V}}$, so that, in particular, $\overline{\langle \partial_X\Omega_A\mathsf{V}\rangle}$ contains $\partial_X\overline{\Omega}_A\mathsf{V}$. Therefore, since $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ is the unique closed subsemigroupoid of $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ containing $\partial_X\overline{\Omega}_A\mathsf{V}$, to show that $\langle \partial_X\Omega_A\mathsf{V}\rangle$ is dense in $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ it suffices to show that $\overline{\langle \partial_X\Omega_A\mathsf{V}\rangle}$ is a subsemigroupoid of $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$.

Thanks to Corollary 2.3, what we want to show translates into proving that if two edges e and f of $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$ are limits of sequences $(e_n)_n$ and $(f_n)_n$ of elements of $\langle \partial_X\Omega_A\mathsf{V}\rangle$ such that $e \circ f$ is defined, then $e \circ f$ is also the limit of a sequence of elements of $\langle \partial_X\Omega_A\mathsf{V}\rangle$. This property is established if one shows that it is possible to replace each f_n by another element $\tilde{f}_n$ of $\langle \partial_X\Omega_A\mathsf{V}\rangle$ such that $e_n \circ \tilde{f}_n$ is defined and $(\tilde{f}_n)_n$ converges to f. The claim follows by continuity of edge multiplication in $\overline{\Omega}_{\partial_X\overline{\Omega}_A\mathsf{V}}\mathsf{W}$. Actually, the replacement needs only to be done on a subsequence, so we may take subsequences of $(e_n)_n$ and $(f_n)_n$ as convenient.

The only relevant part of e is its end vertex u. We denote by u_n the end vertex of e_n and by u'_n the beginning vertex of f_n. We then have $\lim u_n = u = \lim u'_n$. Since the content function is continuous, by taking subsequences we may assume that $c(u_n) = c(u) = c(u'_n)$ for every n. Since $f_n \in \langle \partial_X\Omega_A\mathsf{V}\rangle$, there is a factorization $f_n = s_n \circ t_n$ such that $s_n \in \partial_X\Omega_A\mathsf{V}$ and $t_n \in \langle \partial_X\Omega_A\mathsf{V}\rangle \uplus \{I_n\}$, where I_n denotes an adjoined local identity. Again taking subsequences, we may assume that $(s_n)_n$ converges to s and $(t_n)_n$ converges to t (where t may be an adjoined local identity I.) Note that, by continuity of $\circ$, we have $f = s \circ t$. Consider factorizations of the form

$$s_n = 0(s_n)\bar{0}(s_n)x_n = y_n\bar{1}(s_n)1(s_n),$$

with $x_n, y_n \in (\overline{\Omega}_A\mathsf{V})^1$. We have $0(s_n) = \alpha(s_n) = u'_n$ and $\omega(s_n) = 1(s_n) = \alpha(t_n)$. Once again by taking subsequences, we may suppose that $(x_n)_n$ and $(y_n)_n$ respectively converge to some elements x and y of $(\overline{\Omega}_A\mathsf{V})^1$. Consider the following element of $\overline{\Omega}_A\mathsf{V}$:

$$r_n = u_n\bar{0}(s_n)x_n \cdot s_n^{n!}.$$

Since $c(r_n) = c(s_n)$, we know that r_n is an edge of $\partial_X\Omega_A\mathsf{V}$ such that $1(r_n) = 1(s_n)$, and we also have $0(r_n) = u_n$ due to the equality $c(u_n) = c(0(s_n))$. Therefore, we may define the element $e_n \circ r_n \circ t_n$ of $\langle \partial_X\Omega_A\mathsf{V}\rangle$. By continuity of $\bar{0}$, we have

$$\lim u_n\bar{0}(s_n)x_n = u\bar{0}(s)x = \lim 0(s_n)\bar{0}(s_n)x_n = s.$$

On the other hand, the sequence $(s_n^{n!})_n$ converges to the idempotent s^ω. Therefore, we have $\lim r_n = s^{\omega+1} = s$, where the last equality holds because $\mathsf{V} \subseteq \mathsf{CR}$. Hence, we obtained a sequence $(e_n \circ r_n \circ t_n)_n$ of elements of $\langle \partial_X\Omega_A\mathsf{V}\rangle$ converging to $e \circ s \circ t = e \circ f$. □

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